Review of General Relativity

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Problem Set

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Conventions

- Unless otherwise stated, we work in four spacetime dimensions with coordinates $x^\mu = (ct, \boldsymbol{x})$ and metric signature $(-,+,+,+)$. Greek indices $\mu, \nu, \ldots$ run over $0,1,2,3$, while Latin indices $i, j, \ldots$ denote spatial components. Repeated indices are summed, and indices are raised and lowered with the metric:

$$V_\mu = g_{\mu\nu} V^\nu, \qquad V^\mu = g^{\mu\nu} V_\nu, \qquad g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu{}_\nu.$$

  We use $\partial_\mu \equiv \partial/\partial x^\mu$ and keep G and c explicit unless stated otherwise. The index conventions for the $(4+d)$-dimensional geometry of Exercise 9 are specified there.

- The metric determinant and invariant volume element are denoted by

$$g \equiv \det(g_{\mu\nu}), \qquad \mathrm{d}V_g = \sqrt{-g}\,\mathrm{d}^4 x.$$

  With the adopted Lorentzian signature, $g < 0$. Parentheses denote symmetrization with weight one:

$$A^{(\mu\nu)} \equiv \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right).$$

- Unless explicitly stated otherwise, the connection is the Levi–Civita connection,

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right).$$

  It is torsion-free and metric-compatible:

$$\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}, \qquad \nabla_\lambda g_{\mu\nu} = \nabla_\lambda g^{\mu\nu} = 0.$$

  In Exercise 7, by contrast, the torsion-free connection and the metric are initially treated as independent fields.

- The covariant derivative acts on a vector, a covector, and a scalar as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda, \qquad \nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda{}_{\mu\nu} V_\lambda, \qquad \nabla_\mu \phi = \partial_\mu \phi.$$

  For a scalar field,

$$\nabla_\mu \nabla_\nu \phi = \partial_\mu \partial_\nu \phi - \Gamma^\lambda{}_{\mu\nu} \partial_\lambda \phi,$$

  and the covariant d'Alembertian (Laplace–Beltrami operator) is

$$\Box\phi \equiv \nabla^\mu \nabla_\mu \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\!\left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi\right).$$

- The Riemann curvature tensor is defined by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}.$$

  The Ricci tensor, Ricci scalar, and Einstein tensor are

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

- For use in the Palatini formalism, the covariant derivative of a contravariant rank-two tensor density $\mathcal{Q}^{\alpha\beta}$ of weight w is

$$\nabla_\lambda \mathcal{Q}^{\alpha\beta} = \partial_\lambda \mathcal{Q}^{\alpha\beta} + \Gamma^{\alpha}{}_{\lambda\rho}\mathcal{Q}^{\rho\beta} + \Gamma^{\beta}{}_{\lambda\rho}\mathcal{Q}^{\alpha\rho} - w\,\Gamma^{\rho}{}_{\rho\lambda}\mathcal{Q}^{\alpha\beta}.$$

  In Exercise 7, $\mathcal{Q}^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta}$ has weight $w = +1$.

- The divergence theorem in n dimensions is

$$\int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\, \nabla_\mu V^\mu = \oint_{\partial\mathcal{M}} \mathrm{d}^{n-1}x \sqrt{|h|}\, V^\mu n_\mu = \oint_{\partial\mathcal{M}} V^\mu \,\mathrm{d}\Sigma_\mu,$$

  where n_μ is the outward-pointing unit normal, h_{ij} is the induced metric on $\partial\mathcal{M}$, and $\mathrm{d}\Sigma_\mu = n_\mu \sqrt{|h|}\,\mathrm{d}^{n-1}x$. Consequently, for a scalar A,

$$\int_{\mathcal{M}} \sqrt{|g|}\, A \nabla_\mu V^\mu \,\mathrm{d}^n x = \oint_{\partial\mathcal{M}} A V^\mu \,\mathrm{d}\Sigma_\mu - \int_{\mathcal{M}} \sqrt{|g|}\, V^\mu \nabla_\mu A \,\mathrm{d}^n x.$$

  Boundary terms may be discarded when the variations have compact support or when the boundary conditions stated in the relevant exercise make them vanish.

- For a worldline action $S = \int L(x, \dot{x})\,\mathrm{d}\lambda$, with $\dot{x}^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$ and fixed endpoints, the Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0.$$

  For a scalar field Φ with

$$S = \int \mathrm{d}^4 x \sqrt{-g}\, \mathcal{L}(\Phi, \nabla_\mu \Phi, g_{\mu\nu}),$$

  they take the covariant form

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \nabla_\mu \left[\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi)}\right] = 0.$$

Exercise 1. Timelike geodesic as a path of extremal proper time

By introducing a specific but arbitrary coordinate system, show that among all timelike worldlines that a particle could take to get from event A to event B, the one or ones whose proper-time lapse is stationary under small variations of the path are the free-fall geodesics. In other words, an action principle for a timelike geodesic $x^\mu(\lambda)$, written in any coordinate system x^μ, is

$$\delta \int_A^B \mathrm{d}\tau = \frac{1}{c}\,\delta \int_0^1 \left(-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right)^{1/2} \mathrm{d}\lambda = 0,$$

where λ is an arbitrary parameter, which by construction ranges from 0 at A to 1 at B.

Exercise 2. Super-Hamiltonian for free-particle motion

(a) Show that, among all curves $x^\mu(\lambda)$ that could take a particle from event $A = x^\mu(0)$ to event $B = x^\mu(\lambda_1)$, those that satisfy the action principle

$$\delta\left[\frac{1}{2}\int_0^{\lambda_1} g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\,\mathrm{d}\lambda\right] = 0$$

are geodesics, and λ is an affine parameter along the geodesic related to the particle's four-momentum by $p^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\lambda$.

Note. In this action principle, in contrast with the previous exercise, the integration parameter is necessarily affine.

(b) The Lagrangian $\mathcal{L}(x^\mu, \mathrm{d}x^\mu/\mathrm{d}\lambda)$ associated with this action principle is

$$\mathcal{L} = \frac{1}{2}g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda},$$

where the coordinates $\{x^0, x^1, x^2, x^3\}$ appear in the metric coefficients and their derivatives appear explicitly. Using the standard principles of Hamiltonian mechanics, show that the momentum canonically conjugate to x^μ is

$$p_\mu = g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda},$$

which is in fact the covariant component of the particle's four-momentum. Show, further, that the Hamiltonian associated with the particle's Lagrangian is

$$\mathcal{H} = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta.$$

Explain why this guarantees that Hamilton's equations

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\alpha}$$

are satisfied if and only if $x^\alpha(\lambda)$ is a geodesic with affine parameter λ and p_α is the corresponding tangent covector.

(c) Show that Hamilton's equations guarantee that, if the metric coefficients are independent of some coordinate x^A, then p_A is a conserved quantity.

The quantity $\mathcal{H} = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta$ is often called the geodesic's **super-Hamiltonian**. It turns out that the easiest way to compute geodesics numerically—for example, for particle motion around a black hole—is often to solve the super-Hamiltonian's Hamilton equations.

Exercise 3: A Scalar field in curved spacetime

The action for a single real scalar field ϕ minimally coupled to gravity is

$$S_\phi = \int \Big[-\tfrac{1}{2}\, g^{\alpha\beta}(\nabla_\alpha\phi)(\nabla_\beta\phi) - V(\phi)\Big]\sqrt{-g}\,\mathrm{d}^4x. \tag{1}$$

(a) **Equation of motion.** Using the Euler–Lagrange equation stated in the Conventions, derive the equation of motion

$$\Box\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0.$$

(b) **Stress–energy tensor.** By varying the action with respect to the inverse metric, show that

$$T^{(\phi)}_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\alpha\beta}} = \nabla_\alpha\phi\,\nabla_\beta\phi - \tfrac{1}{2}\, g_{\alpha\beta}\, g^{\rho\sigma}\nabla_\rho\phi\,\nabla_\sigma\phi - g_{\alpha\beta}\, V(\phi).$$

> *Hint.* You may use without proof the variational identity $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\,\delta g^{\alpha\beta}$, which is proven in Exercise 5(b).

(c) **Conservation and the equation of motion.** Show by direct computation that

$$\nabla^\alpha T^{(\phi)}_{\alpha\beta} = \Big(\Box\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi}\Big)\nabla_\beta\phi,$$

Conclude that the equation of motion in part (a) implies covariant conservation. Conversely, show that conservation implies the scalar equation wherever $\nabla_\beta\phi$ is nonzero.

Exercise 4. Electromagnetism in curved spacetime

The Lagrange density for electromagnetism in curved spacetime is

$$\mathcal{L} = -\tfrac{1}{4}\, F^{\alpha\beta}F_{\alpha\beta} + A_\mu J^\mu, \qquad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2}$$

where J^μ is a conserved current, treated (like A_μ, with the index positions shown) as metric-independent.

(a) **Stress–energy tensor.** Derive the energy–momentum tensor by functional differentiation with respect to the metric:

$$T_{\alpha\beta} = F_\alpha{}^\lambda F_{\beta\lambda} - \tfrac{1}{4}\, g_{\alpha\beta}\, F^{\rho\sigma}F_{\rho\sigma} + g_{\alpha\beta}\, A_\lambda J^\lambda.$$

Show that the pure-field part ($J^\mu = 0$) is traceless, and comment on the physical meaning of tracelessness.

> *Hint.* Write $F^{\alpha\beta}F_{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}$ so that all the metric dependence is explicit, and use $T_{\alpha\beta} = -2\,\partial\mathcal{L}/\partial g^{\alpha\beta} + g_{\alpha\beta}\mathcal{L}$ (derived in Exercise 5(f)).

(b) **Maxwell's equations.** Vary (2) with respect to A_ν to obtain the curved-space Maxwell equations

$$\nabla_\mu F^{\nu\mu} = J^\nu.$$

(c) **A non-minimal coupling.** Consider adding a new term to the Lagrangian,

$$\mathcal{L}' = \beta\, R^{\mu\nu}\, g^{\rho\sigma}\, F_{\mu\rho}F_{\nu\sigma},$$

with β a constant. How are Maxwell's equations altered in the presence of this term? Show that they become

$$\nabla_\mu\big(F^{\nu\mu} + 2\beta\,\mathcal{F}^{\nu\mu}\big) = J^\nu, \qquad \mathcal{F}^{\nu\mu} \equiv R^{\lambda\mu}F_\lambda{}^\nu - R^{\lambda\nu}F_\lambda{}^\mu.$$

(d) **Einstein's equation and current conservation.** Explain how Einstein's equation is altered by $\mathcal{L}'$: which qualitatively new kinds of terms appear, and why? (The relevant identity for $\int \Theta^{\mu\nu}\,\delta R_{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x$ with a general symmetric weight $\Theta^{\mu\nu}$ generalizes the computation of Exercise 8(a); you are not asked to track every index.) Finally, show that the current is still conserved, $\nabla_\nu J^\nu = 0$.

Exercise 5. Deriving the Einstein field equations from a variational principle

We vary the action

$$S \;=\; \kappa\int_{\mathcal{M}}(R-2\Lambda)\sqrt{-g}\,\mathrm{d}^4x + S_{\mathrm{M}}, \qquad S_{\mathrm{M}} = \int \mathcal{L}_{\mathrm{M}}\sqrt{-g}\,\mathrm{d}^4x, \tag{3}$$

with respect to the inverse metric $g^{\alpha\beta}$ and require $\delta S = 0$ for arbitrary variations vanishing (together with their first derivatives) on $\partial\mathcal{M}$. Here $g = \det g_{\alpha\beta}$, Λ is the cosmological constant, and κ is a constant to be fixed. The sub-parts assemble the standard result piece by piece.

(a) **Relation between $\delta g_{\alpha\beta}$ and $\delta g^{\alpha\beta}$.** The metric and inverse satisfy $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha{}_\gamma$. By varying this identity, show that

$$\delta g_{\alpha\beta} = -\,g_{\alpha\mu}g_{\beta\nu}\,\delta g^{\mu\nu}, \qquad \text{equivalently} \qquad g_{\alpha\beta}\,\delta g^{\alpha\beta} = -\,g^{\alpha\beta}\,\delta g_{\alpha\beta}.$$

(b) **Variation of $\sqrt{-g}$.** Let $g \equiv \det(g_{\mu\nu})$ (where $g<0$ for a Lorentzian metric). Show that

$$\delta g = g\,g^{\mu\nu}\delta g_{\mu\nu}.$$

Hence derive the two equivalent identities

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}.$$

> *Hint.* Start from the matrix identity
>
> $$\ln|\det\mathbf{M}| = \mathrm{Tr}(\ln\mathbf{M}),$$
>
> and vary both sides. Then set $\mathbf{M} = (g_{\mu\nu})$ and use the result of part (a) to express the final answer in terms of $\delta g^{\mu\nu}$.

(c) **Palatini identity for $\delta R_{\alpha\beta}$.** Show that the variation of the Ricci tensor is a difference of total covariant derivatives,

$$\delta R_{\alpha\beta} = \nabla_\mu(\delta\Gamma^\mu{}_{\alpha\beta}) - \nabla_\beta(\delta\Gamma^\mu{}_{\alpha\mu}).$$

> *Hint.* (i) The difference of two connections is a tensor, so $\delta\Gamma^\mu{}_{\alpha\beta}$ is a genuine $(1,2)$ tensor. (ii) Write $\delta R_{\alpha\beta}$ from the definition and regroup the $\partial\,\delta\Gamma$ and $\Gamma\,\delta\Gamma$ terms into covariant derivatives.

(d) **The Ricci-scalar term is a boundary term.** Using $R = g^{\alpha\beta}R_{\alpha\beta}$ and $\nabla_\lambda g^{\alpha\beta} = 0$, show that

$$g^{\alpha\beta}\,\delta R_{\alpha\beta} = \nabla_\mu w^\mu, \qquad w^\mu = g^{\alpha\beta}\,\delta\Gamma^\mu{}_{\alpha\beta} - g^{\mu\beta}\,\delta\Gamma^\alpha{}_{\alpha\beta},$$

so that $\int g^{\alpha\beta}\,\delta R_{\alpha\beta}\sqrt{-g}\,\mathrm{d}^4x$ reduces to a boundary integral. Under the boundary conditions stated above, $\delta g^{\alpha\beta} = 0$ and $\nabla_\rho\delta g^{\alpha\beta} = 0$ on $\partial\mathcal{M}$, show that it vanishes and therefore does not contribute to the bulk equations. Comment briefly on the role of the Gibbons–Hawking–York term if only the induced boundary metric is fixed.

(e) **Assemble δS_{EH} and the Einstein tensor.** Combine parts (a)–(d) to show

$$\delta \int R\sqrt{-g}\,\mathrm{d}^4x = \int (R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R)\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}^4x, \qquad \frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{EH}}}{\delta g^{\alpha\beta}} = R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R \equiv G_{\alpha\beta}.$$

(f) **The cosmological and matter terms.** For the cosmological piece $S_\Lambda = -2\Lambda\int\sqrt{-g}\,\mathrm{d}^4x$, use part (b) to show $\frac{1}{\sqrt{-g}}\,\delta S_\Lambda/\delta g^{\alpha\beta} = \Lambda g_{\alpha\beta}$. Then, defining the Hilbert stress–energy tensor $T_{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}}\,\delta S_{\mathrm{M}}/\delta g^{\alpha\beta}$, show equivalently that $\delta S_{\mathrm{M}} = -\frac{1}{2}\int T_{\alpha\beta}\,\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}^4x$, and derive $T_{\alpha\beta} = -2\,\partial\mathcal{L}_{\mathrm{M}}/\partial g^{\alpha\beta} + g_{\alpha\beta}\mathcal{L}_{\mathrm{M}}$.

(g) **Einstein field equations with Λ.** Impose $\delta S = 0$ for arbitrary $\delta g^{\alpha\beta}$ on the full action (3) and show that the field equations take the form

$$G_{\alpha\beta} + \Lambda\,g_{\alpha\beta} = \frac{1}{2\kappa}\,T_{\alpha\beta}.$$

> *Discussion.* Note the coupling κ is still undetermined at this stage—the variational principle fixes the *form* of the equations but not the strength of gravity. That is the task of Exercise 6.

Exercise 6. Fixing κ from the Newtonian limit

The variational derivation leaves κ free. We pin it down by demanding that GR reduce to Newtonian gravity—Poisson's equation—in the weak-field, static, slow-motion regime. Throughout, Λ is negligible on these scales and is dropped (except in the optional final part).

(a) **Trace-reverse the field equations.** Starting from $G_{\alpha\beta} = \frac{1}{2\kappa}T_{\alpha\beta}$ (with $\Lambda = 0$), contract with $g^{\alpha\beta}$ to find $R = -\frac{1}{2\kappa}T$, and hence show

$$R_{\alpha\beta} = \frac{1}{2\kappa}\Big(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T\Big).$$

(b) **Weak static field and the Newtonian potential.** Write $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h| \ll 1$ and $\partial_0 h = 0$. By comparing the geodesic equation for a slow particle, $\ddot{\boldsymbol{x}} = -\nabla\Phi$, with the geodesic equation to linear order, show that consistency requires

$$g_{00} \approx -(1 + 2\Phi/c^2), \qquad \text{i.e.} \qquad h_{00} = -2\Phi/c^2.$$

(c) **Linearized R_{00} and the dust source.** (i) Show that to linear order in h, for a static field, $R_{00} \approx -\frac{1}{2}\nabla^2 h_{00} = +\frac{1}{c^2}\nabla^2\Phi$. (ii) For pressureless dust $T_{\alpha\beta} = \rho u_\alpha u_\beta$ with $u^\alpha \approx (c, \mathbf{0})$, show $T_{00} \approx \rho c^2$, $T \approx -\rho c^2$, and hence $T_{00} - \frac{1}{2}g_{00}T \approx \frac{1}{2}\rho c^2$.

(d) **Match to Poisson and solve for κ.** Insert parts (c)(i)–(ii) into the 00 component of the trace-reversed equation and compare with Poisson's equation $\nabla^2\Phi = 4\pi G\rho$ to show

$$\boxed{\kappa = \frac{c^4}{16\pi G}} \qquad \Longrightarrow \qquad G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}\,T_{\alpha\beta}.$$

> *Discussion.* This is where the abstract variational constant acquires physical meaning: $1/2\kappa = 8\pi G/c^4$ is the coupling that will appear in *every* field equation you write for the rest of the school. In modified-gravity theories (Exercise 8) this "constant" is exactly what gets promoted to a field.

(e) **Optional: the cosmological term in the Newtonian limit.** Restore Λ. (i) Show the Λ term is equivalent to a perfect-fluid vacuum with $\rho_\Lambda c^2 = \Lambda c^4/8\pi G$ and $p_\Lambda = -\rho_\Lambda c^2$ (equation of state $w = -1$). (ii) Show that carrying Λ through the Newtonian limit modifies Poisson's equation to $\nabla^2\Phi = 4\pi G\rho - \Lambda c^2$, giving a repulsive acceleration $\boldsymbol{g}_\Lambda = +\frac{1}{3}\Lambda c^2\,\boldsymbol{r}$ for $\Lambda > 0$.

Exercise 7. The Palatini formalism: metric compatibility as an equation of motion

In the metric formulation the Levi-Civita connection is *assumed*. In the **Palatini (first-order) formalism** we treat the connection ${\Gamma^\mu}_{\alpha\beta}$ and the metric $g_{\alpha\beta}$ as *independent* fields. The action is formally the same,

$$S[g^{\alpha\beta},\Gamma] = \int g^{\alpha\beta}\,R_{\alpha\beta}(\Gamma)\,\sqrt{-g}\,\mathrm{d}^4x, \tag{4}$$

but $R_{\alpha\beta}(\Gamma)$ is built *purely from the connection*, with no a priori relation to $g_{\alpha\beta}$. Varying with respect to $g^{\alpha\beta}$ returns $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0$. Here we vary with respect to Γ and show it is *forced* to be the Christoffel connection. Assume throughout that Γ is symmetric (torsion-free).

(a) **Variation of $R_{\alpha\beta}$ w.r.t. the connection.** Holding $g^{\alpha\beta}$ fixed, show the Palatini identity $\delta R_{\alpha\beta} = \nabla_\mu(\delta{\Gamma^\mu}_{\alpha\beta}) - \nabla_\beta(\delta{\Gamma^\mu}_{\alpha\mu})$, where ∇ uses the full (undetermined) connection Γ. Explain in one line why $\delta{\Gamma^\mu}_{\alpha\beta}$ is a genuine $(1,2)$ tensor.

(b) **Connection field equation.** Define the contravariant tensor density $Q^{\alpha\beta} \equiv \sqrt{-g}\,g^{\alpha\beta}$, which has weight $+1$. Insert part (a) into δS, integrate by parts using compactly supported variations, and remember that $\delta{\Gamma^\lambda}_{\alpha\beta} = \delta{\Gamma^\lambda}_{\beta\alpha}$ because the connection is torsion-free. Show that

$$\delta S = \int \mathrm{d}^4x\,\left[-\nabla_\lambda Q^{\alpha\beta} + {\delta_\lambda}^{(\beta}\nabla_\sigma Q^{\alpha)\sigma}\right]\delta{\Gamma^\lambda}_{\alpha\beta},$$

and hence obtain

$$\nabla_\lambda Q^{\alpha\beta} - {\delta_\lambda}^{(\beta}\nabla_\sigma Q^{\alpha)\sigma} = 0.$$

> *Hint.* For a density, include the density-weight term in its covariant derivative. Equivalently, perform the integration by parts first with ordinary derivatives and then regroup the connection terms covariantly.

(c) **Densitized metric compatibility.** Contract the field equation of part (b) on λ and β. Show that in $n > 1$ dimensions

$$\nabla_\sigma Q^{\alpha\sigma} = 0, \qquad \Longrightarrow \qquad \nabla_\lambda(\sqrt{-g}\,g^{\alpha\beta}) = 0.$$

(d) **Recover the Levi-Civita connection.** Expand the last equation using the covariant derivative of a weight-$+1$ tensor density. Contract with $g_{\alpha\beta}$ and show that in $n > 2$ dimensions its trace vanishes, leaving $\nabla_\lambda g^{\alpha\beta} = 0$. Together with the torsion-free assumption, conclude that

$${\Gamma^\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}).$$

State the physical conclusion in one sentence.

Exercise 8. Scalar–tensor gravity and the Brans–Dicke limit

In **scalar–tensor theories** a scalar field ϕ couples *directly to the curvature scalar*—a non-minimal coupling beyond what the Equivalence Principle alone would suggest. The action is

$S = S_g + S_\phi + S_{\mathrm{M}}$ with

$$S_g = \int f(\phi)\, R \sqrt{-g}\, \mathrm{d}^4x, \qquad S_\phi = \int \Big[-\tfrac{1}{2} h(\phi)\, g^{\alpha\beta} \partial_\alpha \phi\, \partial_\beta \phi - V(\phi) \Big] \sqrt{-g}\, \mathrm{d}^4 x,$$
$$S_{\mathrm{M}} = \int \mathcal{L}_{\mathrm{M}}(g_{\alpha\beta}, \psi) \sqrt{-g}\, \mathrm{d}^4x,$$

where f, h, V define the theory and $\mathcal{L}_{\mathrm{M}}$ depends on the metric and matter ψ but *not* on ϕ.

(a) **Metric variation of the non-minimal term.** Following the same steps as the Einstein–Hilbert variation but keeping $f(\phi)$ as a fixed weight, integrate by parts *twice* to move the derivatives onto f. Show that the two total-derivative terms that vanish for constant f now leave behind

$$\delta S_g = \int \sqrt{-g}\, \mathrm{d}^4x\ \left[f(\phi)\, G_{\alpha\beta} + g_{\alpha\beta}\, \Box f - \nabla_\alpha \nabla_\beta f \right] \delta g^{\alpha\beta}.$$

Identify which step generates the $\Box f$ and $\nabla_\alpha \nabla_\beta f$ terms.

(b) **The gravitational field equation.** With $T^{(i)}_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\, \delta S_i / \delta g^{\alpha\beta}$ for $i = \phi, \mathrm{M}$, show that

$$G_{\alpha\beta} = f^{-1}(\phi) \Big[\tfrac{1}{2} T^{\mathrm{M}}_{\alpha\beta} + \tfrac{1}{2} T^{\phi}_{\alpha\beta} + \nabla_\alpha \nabla_\beta f - g_{\alpha\beta} \Box f \Big],$$

with $T^\phi_{\alpha\beta} = h\, \nabla_\alpha \phi \nabla_\beta \phi - g_{\alpha\beta} [\tfrac{1}{2} h\, g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V]$. Interpret the coefficient of $T^{\mathrm{M}}_{\alpha\beta}$: what plays the role of Newton's constant, and what happens if ϕ varies slowly?

(c) **The scalar equation of motion.** Vary the full action with respect to ϕ (note R enters S_g through $f(\phi)$) to show

$$h\, \Box \phi + \frac{1}{2} \frac{\mathrm{d}h}{\mathrm{d}\phi}\, g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} + R\, \frac{\mathrm{d}f}{\mathrm{d}\phi} = 0.$$

Why is the $R\, \mathrm{d}f/\mathrm{d}\phi$ term the hallmark of a scalar–tensor (rather than minimally coupled) theory?

(d) **Brans–Dicke theory and its GR limit.** Using the conventional normalization in which $\phi = G_{\mathrm{eff}}^{-1}$, Brans–Dicke theory corresponds to $f = c^4 \phi / 16\pi$, $h = c^4 \omega / 8\pi\phi$, and $V = 0$. (i) Show the action becomes

$$S_{\mathrm{BD}} = \frac{c^4}{16\pi} \int \mathrm{d}^4x \sqrt{-g} \Big[\phi R - \frac{\omega}{\phi} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi \Big] + S_{\mathrm{M}}.$$

(ii) Identify the effective, spacetime-dependent Newton constant $G_{\mathrm{eff}}(\phi)$. (iii) Under regular boundary conditions, argue that as $\omega \to \infty$ the scalar freezes and ordinary GR is recovered. What does the ϕ equation say about $\nabla \phi$ in that limit? For context, massless Brans–Dicke theory is constrained by Solar-System tests to $\omega \gtrsim 4 \times 10^4$.

Exercise 9. Kaluza–Klein compactification and the radion

Another way to modify general relativity is to allow for the existence of extra spatial dimensions: models in which spacetime appears four-dimensional on large scales even though there are really $4 + d$ dimensions in total, the extra d dimensions being *compactified* on some small manifold. Models of this kind are known as **Kaluza–Klein theories**, and their physical consequences turn out to be closely related to those of the scalar–tensor theories of Exercise 8.

Let G_{ab} be the metric of a $(4+d)$-dimensional spacetime with coordinates X^a, where the indices a, b run from 0 to $d+3$. We adopt the compactification ansatz

$$\mathrm{d}s^2 = G_{ab}\,\mathrm{d}X^a\,\mathrm{d}X^b = g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + b^2(x)\,\gamma_{ij}(y)\,\mathrm{d}y^i\,\mathrm{d}y^j, \tag{5}$$

where x^α are coordinates on the four-dimensional spacetime, y^i $(i, j = 1, \ldots, d)$ are coordinates on the extra-dimensional manifold, and the scale factor $b(x)$ depends only on x^α. The internal metric γ_{ij} is taken to be maximally symmetric,

$$R[\gamma]_{ij} = k\,(d-1)\,\gamma_{ij} \qquad \Longrightarrow \qquad R[\gamma] = d(d-1)\,k,$$

with curvature parameter k (written k rather than κ to avoid a clash with the gravitational coupling of Exercises 5–6). The internal geometry is of course dynamical; (5) is a simplifying ansatz retaining only the overall-size mode $b(x)$, the lightest of the compactification modes. The action is the $(4+d)$-dimensional Hilbert action plus a matter term,

$$S = \int \mathrm{d}^{4+d}X\,\sqrt{-G}\,\Big[\frac{c^4}{16\pi G_{4+d}}\,R[G_{ab}] + \mathcal{L}_\mathrm{M}\Big], \tag{6}$$

where $G = \det G_{ab}$, $R[G_{ab}]$ is the Ricci scalar of G_{ab}, and the matter density $\mathcal{L}_\mathrm{M}$ (metric determinant factored out) is treated as a fixed scalar weight throughout.

(a) **Volume element.** Show that the metric determinant factorizes as

$$\sqrt{-G} = b^d\,\sqrt{\gamma}\,\sqrt{-g}, \qquad \gamma = \det\gamma_{ij}.$$

(b) **Connection of the ansatz.** Show that the only nonvanishing Christoffel symbols of (5) are

$$\Gamma^\alpha{}_{\beta\gamma}[g], \qquad \Gamma^i{}_{jk}[\gamma], \qquad \Gamma^\alpha{}_{ij} = -\,b\,(\nabla^\alpha b)\,\gamma_{ij}, \qquad \Gamma^i{}_{\alpha j} = b^{-1}(\nabla_\alpha b)\,\delta^i{}_j,$$

i.e. the remaining mixed components $\Gamma^\alpha{}_{\beta i}$ and $\Gamma^i{}_{\alpha\beta}$ vanish.

(c) **Ricci tensor.** Using part (b), show that

$$R_{\alpha\beta} = R[g]_{\alpha\beta} - d\,b^{-1}\nabla_\alpha\nabla_\beta b, \qquad R_{ij} = R[\gamma]_{ij} - \gamma_{ij}\big[b\,\Box b + (d-1)(\nabla b)^2\big],$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, $(\nabla b)^2 = g^{\mu\nu}(\nabla_\mu b)(\nabla_\nu b)$, and ∇_μ is the covariant derivative of $g_{\alpha\beta}$.

(d) **Ricci scalar.** Contract part (c) with the inverse metric of (5) to obtain

$$R[G_{ab}] = R[g_{\alpha\beta}] + b^{-2}R[\gamma_{ij}] - 2d\,b^{-1}\Box b - d(d-1)\,b^{-2}(\nabla b)^2.$$

(e) **Dimensional reduction.** Define the internal volume at $b = 1$ and the four-dimensional Newton constant by

$$\mathcal{V} = \int \sqrt{\gamma}\,\mathrm{d}^d y, \qquad \frac{1}{16\pi G_4} = \frac{\mathcal{V}}{16\pi G_{4+d}}.$$

Perform the integral over the extra dimensions in (6) (possible because b is independent of y^i) and integrate by parts to show

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\,\Big\{\frac{c^4}{16\pi G_4}\Big[b^d R[g] + d(d-1)\,b^{d-2}g^{\alpha\beta}(\nabla_\alpha b)(\nabla_\beta b) + d(d-1)\,k\,b^{d-2}\Big] + \mathcal{V}\,b^d\,\mathcal{L}_\mathrm{M}\Big\}.$$

(f) **A Brans–Dicke theory in disguise.** Setting $\Phi = b^d$, show that the gravitational sector of part (e) takes the Brans–Dicke form of Exercise 8(d),

$$\frac{c^4}{16\pi G_4}\Big[\Phi\,R[g] - \frac{\omega_\mathrm{BD}}{\Phi}\,g^{\alpha\beta}(\nabla_\alpha\Phi)(\nabla_\beta\Phi)\Big], \qquad \omega_\mathrm{BD} = -\,\frac{d-1}{d}.$$

Compare with the solar-system bound quoted in Exercise 8(d): can pure Kaluza–Klein gravity with a massless radion be viable?

(g) **Weyl transformation to the Einstein frame.** Let $\beta = \log b$ and perform the conformal transformation $\tilde{g}_{\mu\nu} = e^{d\beta} g_{\mu\nu}$. You may use without proof the standard four-dimensional identity: for $\tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}$,

$$R[g] = e^{2\omega}\big(R[\tilde{g}] + 6\,\tilde{\Box}\,\omega - 6\,\tilde{g}^{\mu\nu}\partial_\mu\omega\,\partial_\nu\omega\big), \qquad \tilde{\Box} = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu.$$

Dropping total derivatives, show that the reduced action becomes

$$S = \int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\,\Big\{\frac{c^4}{16\pi G_4}\Big[R[\tilde{g}] - \tfrac{1}{2}d(d+2)\,\tilde{g}^{\mu\nu}(\tilde{\nabla}_\mu\beta)(\tilde{\nabla}_\nu\beta) + d(d-1)\,k\,e^{-(d+2)\beta}\Big] + \mathcal{V}\,e^{-d\beta}\,\mathcal{L}_{\mathrm{M}}(e^{-d\beta}\tilde{g}_{\mu\nu},\psi)\Big\}.$$

(h) **The canonically normalized radion.** Define

$$\phi = \sqrt{\frac{d(d+2)}{2}}\,\bar{m}_{\mathrm{P}}\,\beta, \qquad \bar{m}_{\mathrm{P}}^2 \equiv \frac{c^4}{8\pi G_4}$$

(with $\hbar = c = 1$, $\bar{m}_{\mathrm{P}}$ is the usual reduced Planck mass; here it is simply shorthand for the constant $\sqrt{c^4/8\pi G_4}$). Show that

$$\begin{aligned} S = \int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\,\Bigg\{ &\frac{c^4}{16\pi G_4} R[\tilde{g}] - \frac{1}{2}\,\tilde{g}^{\mu\nu}(\tilde{\nabla}_\mu\phi)(\tilde{\nabla}_\nu\phi) \\ &+ \frac{1}{2}\,d(d-1)\,k\,\bar{m}_{\mathrm{P}}^2 e^{-\sqrt{2(d+2)/d}\,\phi/\bar{m}_{\mathrm{P}}} \\ &+ \mathcal{V}\,e^{-\sqrt{2d/(d+2)}\,\phi/\bar{m}_{\mathrm{P}}}\,\mathcal{L}_{\mathrm{M}}(e^{-d\beta}\tilde{g}_{\mu\nu},\psi)\Bigg\}, \end{aligned} \tag{7}$$

and read off the radion potential $V(\phi)$ (ignoring the matter term). The scalar ϕ is known as the **dilaton** or **radion**, and characterizes the size of the extra-dimensional manifold.

(i) **Radion (in)stability.** Ignoring $\mathcal{L}_{\mathrm{M}}$, analyze the behavior of the radion for the three cases $k = 0$, $k > 0$ and $k < 0$: where does ϕ roll, and what does this imply for the size b of the extra dimensions? Conclude in one or two sentences what is required for stable compactification.