A Review of General Relativity

From the Equivalence Principle to Einstein's Field Equations

Dr. Nontapat Wanwieng

National Astronomical Research Institute of Thailand (NARIT)

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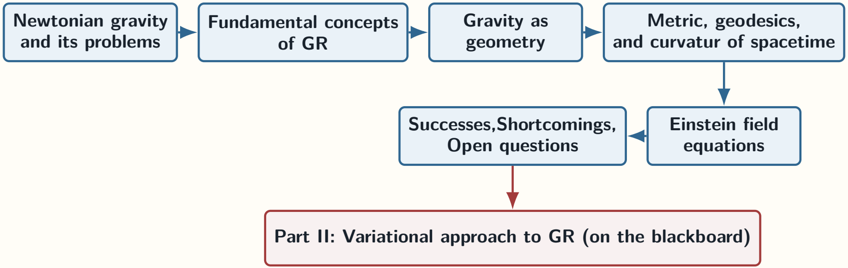
Roadmap of This lecture



Two questions connect the whole review

How does gravity affect matter? **How does matter generate gravity?**

- **Part I:** physical logic with minimal mathematical detail.
- **Part II:** the same endpoint from the principle of least action.

The Newtonian Picture of Gravity

Fixed background

Space is a fixed three-dimensional Euclidean arena, time is universal, and gravity is a scalar potential $\Phi(t, \mathbf{x})$.

How does gravity affect matter?

$$\mathbf{a} = -\boldsymbol{\nabla}\Phi$$

The potential determines acceleration.

How does matter generate gravity?

$$\nabla^2\Phi = 4\pi G\rho$$

Mass density sources the potential.

Matter creates the potential, and the potential determines motion—while space and time remain fixed.

Why Newtonian Gravity Must Be Replaced

Newtonian gravity assumes

- instantaneous gravitational influence,
- separate, absolute space and time,
- mass density ρ as the only source.

Special Relativity requires

- no physical influence faster than c,
- observer-dependent spacetime and simultaneity,
- energy, momentum, pressure, and stress as possible sources.

$$E = mc^2 \qquad \Longrightarrow \qquad \text{mass is one form of energy}$$

The conflict is conceptual

A relativistic theory needs a new description of both the source of gravity and the gravitational field itself.

Universality of Free Fall

Ordinary interactions

- Motion depends on properties of the body.
- In electromagnetism, trajectories depend on q/m.
- Different particles can follow different paths in the same field.

Gravity

- m_i: resistance to acceleration,
- m_g: coupling to gravity,
- experiments find $m_g = m_i$ to very high precision.

$$m_i\mathbf{a} = m_g\mathbf{g}, \qquad m_g = m_i \quad \Longrightarrow \quad \boxed{\mathbf{a} = \mathbf{g}}$$

Weak Equivalence Principle

Freely falling test bodies with the same initial position and velocity follow the same trajectory, independent of mass, composition, or internal structure.

The free-fall trajectory appears to depend on spacetime, not on the particular object.

The Einstein Equivalence Principle

Modern statement

In a sufficiently small freely falling laboratory, all local nongravitational laws of physics reduce to those of Special Relativity.

What can be removed locally

- the effects of a uniform gravitational field,
- equivalently, the effects of uniform acceleration,
- for all local nongravitational experiments.

What cannot be removed

- variation of gravity over an extended region,
- relative acceleration of nearby free-fall particles,
- tidal effects produced by spacetime curvature.

The word *local* is essential

The experiment must be small enough that tidal variations can be neglected.

The Principle of Relativity in the Presence of Gravity

Local principle of relativity

All local nongravitational laws of physics are the same in every local inertial frame—everywhere and everywhen in the universe.

Operational meaning

- Use small, freely falling laboratories.
- Repeat the same local experiment under identical local conditions.
- Different observers may use different coordinates.
- The physical result and the form of the laws must agree.

Geometric implication

- Each sufficiently small region looks like Minkowski spacetime.
- There need not be one global Minkowski frame.
- The variation of local inertial frames from point to point is described by curvature.

Special Relativity survives locally, even when spacetime is globally curved.

Gravity as the Geometry of Spacetime

Mathematical arena

- a four-dimensional differentiable manifold,
- equipped with a Lorentzian metric $g_{\mu\nu}$,
- with spacetime interval

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}.$$

Since $g_{\mu\nu} = g_{\nu\mu}$, the metric has ten algebraically independent components in four dimensions.

Physical role of the metric

- measures proper time and spatial distance,
- determines light cones and causal structure,
- governs free fall of matter and light,
- encodes gravitational redshift and tidal effects.

A dynamical field

The metric is not a fixed arena. It is the gravitational field itself.

Newton: gravity is Φ on spacetime. **Einstein: gravity is the metric $g_{\mu\nu}$ of spacetime.**

General Covariance and Minimal Coupling

General covariance

- Coordinates are labels chosen by an observer.
- Physical laws should not depend on a particular choice of coordinates.
- Tensor equations retain the same form under coordinate transformations.

From flat to curved spacetime

1. Write the Special-Relativistic law in tensorial form.
2. Apply the minimal-coupling prescription:

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}, \qquad \partial_\mu \longrightarrow \nabla_\mu.$$

Why replace ordinary derivatives?

The partial derivative of a vector is generally not a tensor; the covariant derivative corrects for the variation of the coordinate basis.

Tensorial form removes coordinate dependence; minimal coupling promotes a flat-spacetime law to curved spacetime.

Free Fall as Spacetime Geodesics

Flat Minkowski spacetime

$$\frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = 0$$

Using $u^\nu = dx^\nu/d\tau$ and the chain rule,

$$\frac{du^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\,\partial_\nu u^\mu = u^\nu \partial_\nu u^\mu = 0.$$

Curved spacetime

$$\partial_\nu \longrightarrow \nabla_\nu \qquad \Longrightarrow \qquad \boxed{u^\nu \nabla_\nu u^\mu = 0}.$$

Equivalently,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0.$$

Physical meaning

The four-velocity is parallel transported along the particle's worldline: the particle moves as straight as possible through curved spacetime.

Equivalent definition—exercise

For a massive particle, a geodesic extremizes the proper time between two events: $\delta\int d\tau = 0$.

Tidal Gravity and the Riemann Tensor

Newtonian: for two particles separated by ξ^i,

$$\frac{d^2\xi^i}{dt^2} = -\frac{\partial^2\Phi}{\partial x^i \partial x^j}\,\xi^j = \frac{\partial g^i}{\partial x^j}\,\xi^j.$$

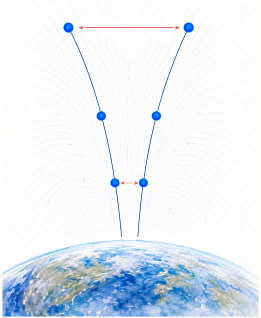
Tidal gravity comes from the spatial variation of **g**, not **g** itself.

Relativistic: nearby freely falling worldlines obey the geodesic-deviation equation,

$$\boxed{\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\alpha\beta}\,u^\nu \xi^\alpha u^\beta}$$

which in the local rest frame reduces to $\dfrac{d^2\xi^i}{dt^2} \simeq -c^2 R^i{}_{0j0}\,\xi^j$, giving the correspondence

$$\frac{\partial g^i}{\partial x^j} \longleftrightarrow -c^2 R^i{}_{0j0}.$$



- Each particle follows its own geodesic.
- Riemann controls their *relative* acceleration.
- The relativistic tidal field — an operational measure of curvature.

A single geodesic describes free fall; the Riemann tensor governs the relative acceleration of nearby geodesics.

Constructing the Geometric Side

Matter side

- Source: energy–momentum tensor $T_{\mu\nu}$.
- Contains energy density, momentum density, pressure, and stress.
- Satisfies local conservation:

$$\nabla_\mu T^{\mu\nu} = 0.$$

Therefore the geometric side must also be a divergence-free rank-two tensor.

Curvature contractions

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}.$$

Einstein tensor

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}$$

The contracted Bianchi identity gives

$$\nabla_\mu G^{\mu\nu} = 0.$$

The conservation property of $G_{\mu\nu}$ matches the conservation property of $T_{\mu\nu}$.

Einstein's Field Equations

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Geometry

- spacetime curvature,
- cosmological constant,
- nonlinear equations for $g_{\mu\nu}$.

Matter and energy

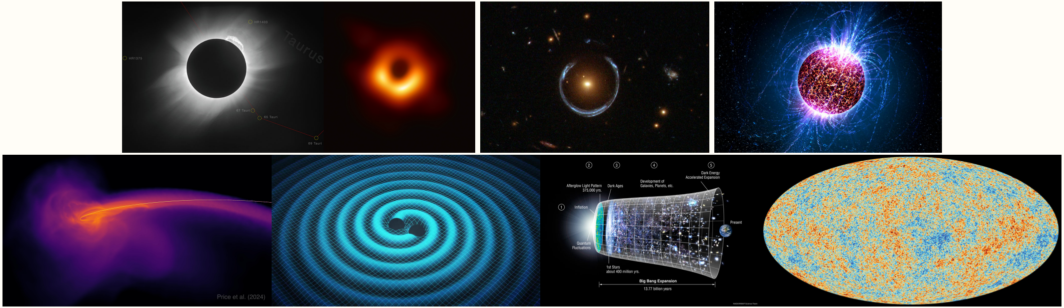
- energy and momentum,
- pressure and stress,
- all encoded in $T_{\mu\nu}$.

The two questions answered

Matter generates gravity through the field equations.
Gravity affects matter through geodesic motion in the resulting geometry.

Matter tells spacetime how to curve; spacetime tells matter how to move.

Triumphs of General Relativity



Classical and astrophysical tests

- Mercury's anomalous perihelion advance,
- gravitational redshift,
- light bending and gravitational lensing,
- Shapiro time delay,
- relativistic stars and black holes.

Radiation and cosmology

- binary-pulsar orbital decay,
- direct detection of gravitational waves,
- compact-object mergers and accretion,
- framework for modern cosmology.

Limitations and Open Problems

General Relativity is a low-energy effective theory

GR is classical and describes the leading low-energy, low-curvature behavior of a deeper theory. It cannot be extrapolated without limit to arbitrarily high energies or curvatures.

Where GR breaks down ⇒ quantum gravity

- Quantum effects of spacetime become unavoidable near the Planck scale, $E_{\mathrm{Pl}} \sim 10^{19}$ GeV, $\ell_{\mathrm{Pl}} \sim 10^{-35}$ m.
- Perturbative quantization of GR is non-renormalizable.
- Singularities—at black-hole centers and at the origin of the Universe—mark where the classical description fails.

Cosmological puzzles ⇒ modified gravity

- Dark matter: missing gravitating mass, or a change in the gravitational law?
- Dark energy: cosmic acceleration and the cosmological-constant problem.
- Possible tensions among cosmological observations.