Renormalisation process for disconnected operators from Feynman-Hellmann techniques in lattice QCD

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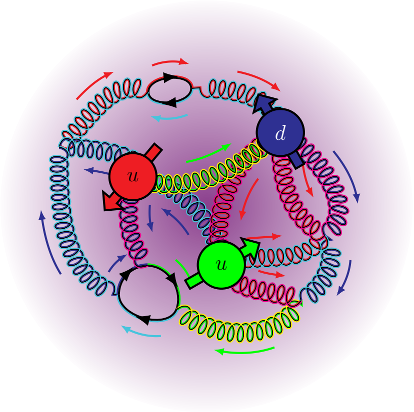
Hadronic Matrix Elements

- Hadronic observables are predicted from operator matrix elements in field theories.

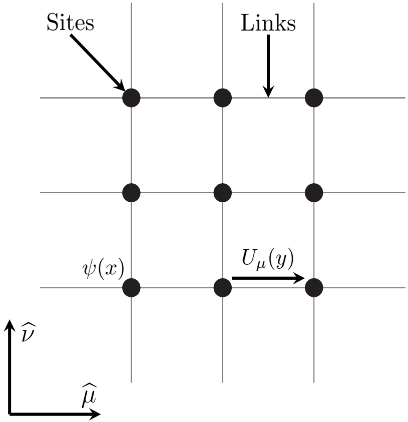
$$\langle H' | \mathcal{O} | H \rangle$$

- Such results provide access to form factors, transition amplitudes, structure functions, inputs to distribution functions, and so on...



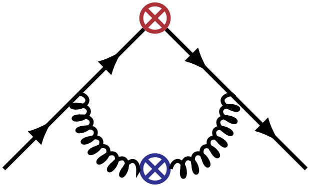
Non-perturbative Methods

- In strongly coupled regimes, traditional perturbative methods break down
  - Must turn to non-perturbative alternatives
- Lattice methods provide a numerical based alternative to directly compute relevant operator matrix elements



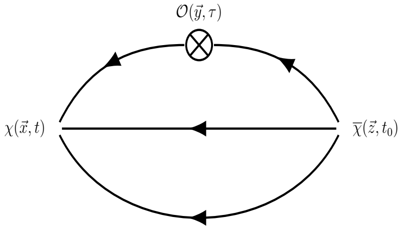
Connected vs. Disconnected Operators

- In lattice QCD, operators may be considered in two broad categories; connected operators, and disconnected operators.
- Connected operators are coupled to a fermion source and/or sink through fermion propagators.
- Disconnected operators are only coupled through intermediate gluon states, either fermion or gluon coupling operators.



Connected Operators

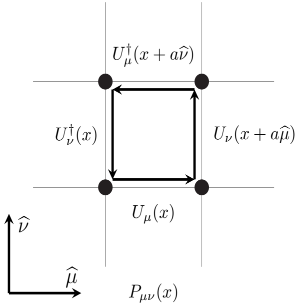
- Connected operator matrix elements are computed via contractions with fermion propagators and relevant projectors.
- Propagators are obtained through matrix inversions of the Dirac operator.
- This results in exact evaluations of the fermion path integral on Monte Carlo sampled gauge field configurations.



Disconnected Operators

- Disconnected operator matrix elements are computed from gauge link variables themselves.
- Calculations typically suffer from strong statistical noise, often require large amounts of compute time (e.g., all-to-all propagators).
- Techniques to reduce noise: gradient/Wilson flows, smearing etc., often affect coupling to hadrons or remove relevant physics (e.g., short range effects).

- Alternative means of accessing these operators is of great interest, to alleviate these issues.

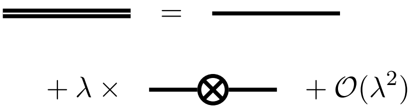


Feynman-Hellmann Methods

- Operator of interest is included as a background field in the QCD action.

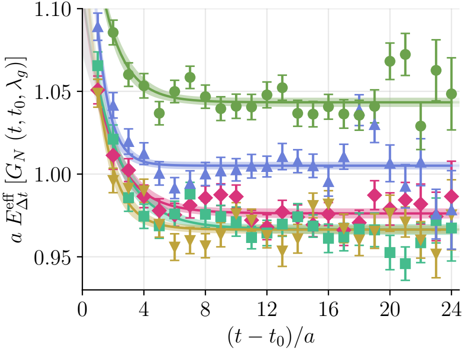
$$S_E[U] \to S_E[U] + \lambda \mathcal{O}[U]$$

- This creates a new Feynman rule.
- Propagators couple to the background field proportionate to field strength λ.
- Perturbatively expanding in λ allows access to matrix element of relevant power of the operator.



Effect on Hadrons

- This extra Feynman rule emerges as a shift in the hadron ground state energy.
- Extract the matrix element from the relevant order of shift in energy.
- Be careful of higher order artefacts.



Analysis/Simulation Parameters

- Operators considered are components of the fermion and gluon energy-momentum tensor
- Matrix elements are proportional to $\langle x \rangle_f$, the momentum fractions/first Mellin moments of the PDFs

Quenched

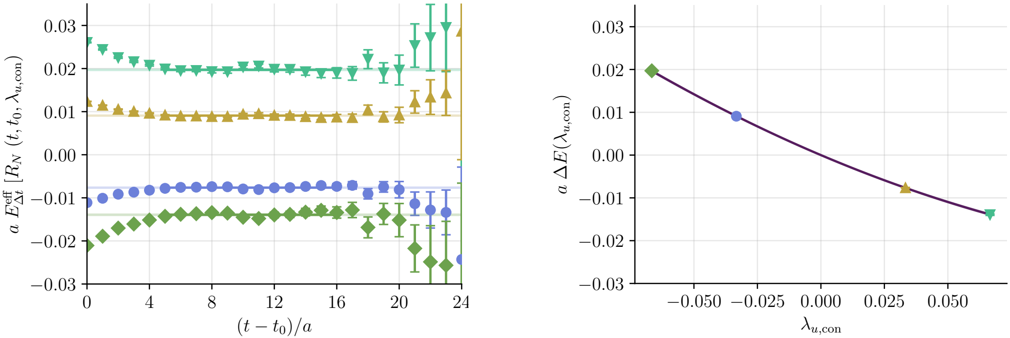
V_4	β	λ_g	N_{cfg}
$24^3 \times 48$	6.00	0	1000
		-0.0666	1000
		-0.0333	1000
		$+0.0333$	1000
		$+0.0666$	1000

$N_f = 2$

V_4	β	N_f	κ	λ_g	$\lambda_{q,\text{dis}}$	N_{cfg}
$24^3 \times 48$	5.29	2	0.1355	0	0	951
				-0.0666	0	1021
				-0.0333	0	1537
				$+0.0333$	0	921
				$+0.0666$	0	1074
				0	-0.0666	1128
				0	-0.0333	1367
				0	$+0.0333$	1371
				0	$+0.0666$	1384

Connected Results

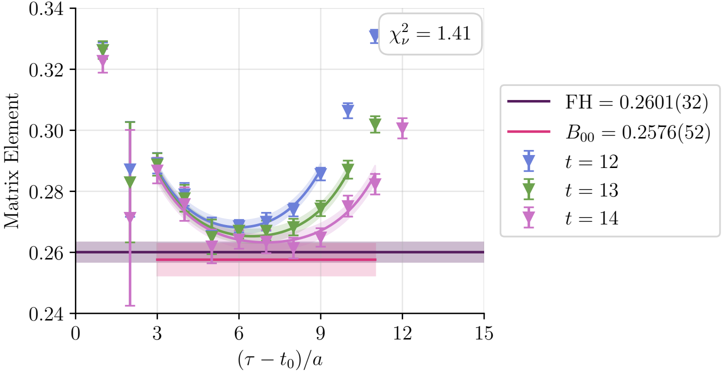
$N_f = 2$, fermion operator inserted on doubly-represented quark in baryon



- Ratio of 2-point correlators provide ΔE_0, shift relative to unperturbed point.

Comparison to 3-point Results

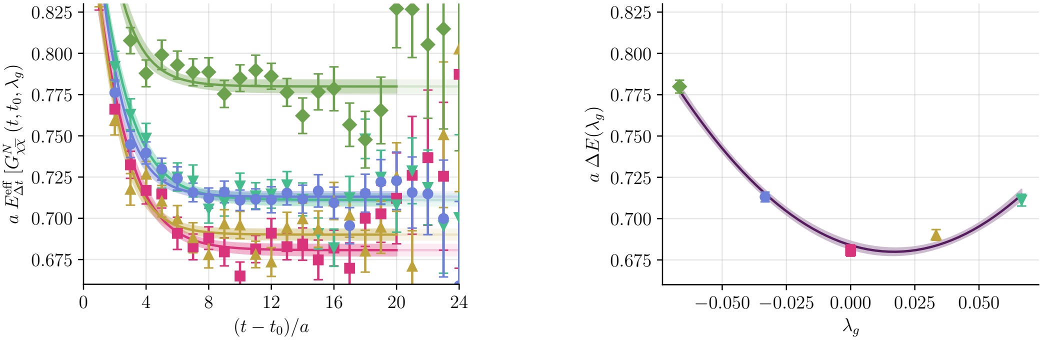
Quenched, fermion operator inserted on doubly-represented quark in baryon



Comparing to direct evaluation of same operator, methods agree to 1σ.

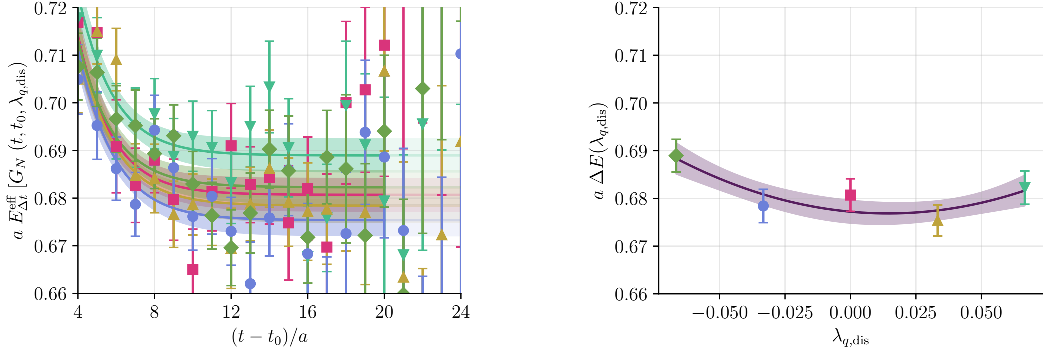
Disconnected Gluon Results

$N_f = 2$, gluon operator inserted in baryon

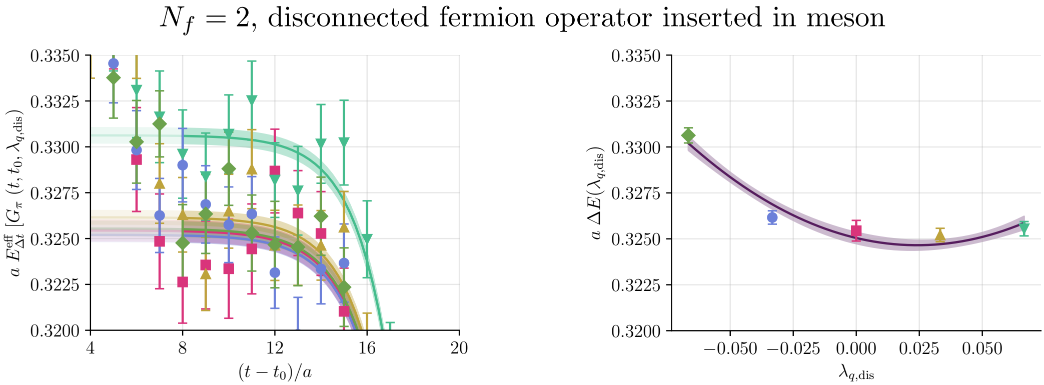


Disconnected Quark Results

$N_f = 2$, disconnected fermion operator inserted in baryon



Disconnected Quark Results



Obtained Values

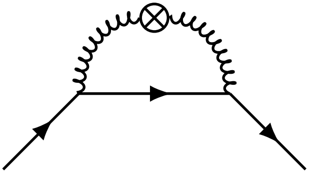
$$N_f = 2$$

Hadron	$\langle x \rangle_g$	$\langle x \rangle_{u,\mathrm{con}}$	$\langle x \rangle_{d,\mathrm{con}}$	$\langle x \rangle_{q,\mathrm{con}}$	$\langle x \rangle_{q,\mathrm{dis}}$
Nucleon	0.926(66)	0.3699(33)	0.1630(19)	0.5329(50)	0.074(49)
Pion	0.983(19)			0.5228(16)	0.100(10)

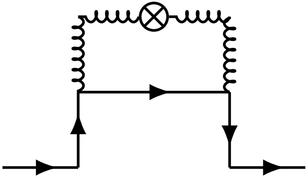
- These values are defined in the lattice renormalisation scheme, must be matched to other schemes for evaluation

Renormalisation Procedure

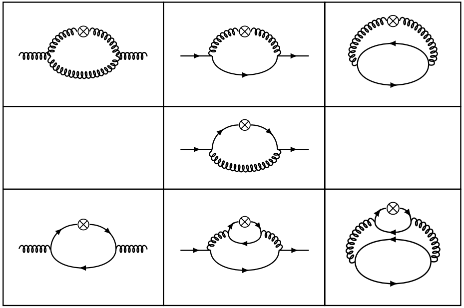
- Operators computed on a lattice carry explicit regulator dependence
- Matching factors are required to match onto continuum based renormalisation schemes
  - where expected sum rules may be recovered



$= Z_\psi Z_\mathcal{O} \times$



Mixing Matrix



- Connected and disconnected operators mix when renormalising between schemes
- Mixing renormalisation factors obtained from computing one species operator insertion on another species
- May obtain operator insertion from quark/gluon propagators under FH perturbation

RI′ – MOM Procedure

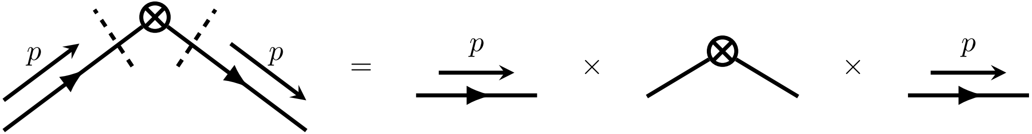
- Avoiding explicit regulator dependence, renormalisation factors computed to rescale propagators/vertex functions at matching kinematic points

$$1 = Z_{ff'}(\mu) Z_f^{-1}(\mu) \frac{1}{\mathcal{N}} \mathrm{Tr}_{cs} \left\{ \left(\Gamma^{\mathrm{Born}}_{ff'}(p^2) \right)^{-1} \Gamma^{\mathrm{Lat}}_{ff'}(p^2) \right\} \Big|_{p^2=\mu^2}$$

- Z_f similarly renormalises the propagator of species f
- Additional concerns emerge with gluon vertex, must make sure to avoid BRST-dependent/EoM proportionate terms

RI′ – MOM Procedure

- Get to vertex functions from 3-point correlators of fermions/gluons



$$C_{\psi\mathcal{O}\overline{\psi}}(p) = S(p) \times \Gamma_{\psi\mathcal{O}\overline{\psi}}(p) \times S(p)$$

- Which we get from FH perturbations!

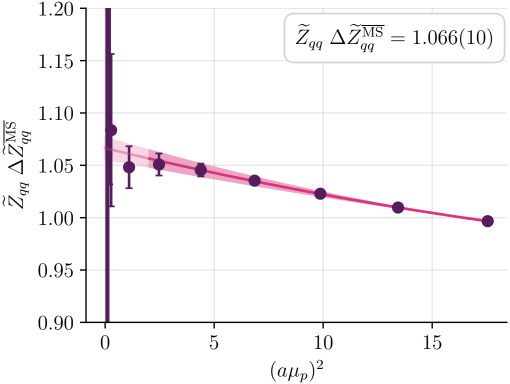
$\frac{\partial}{\partial\lambda}($  $) =$  $+\,\mathcal{O}(\lambda)$

Inverse Mixing Matrix

- As mixings involve sums of fermion & gluon terms at different kinematics, not very practical
- Much more workable to compute the *inverse* renormalisation matrix
- Mixed renormalised operator matrix elements (e.g., $\langle q|\, \mathcal{O}_g \,|q\rangle^R$ vanish)
- Results in set of independent conditions, individually computed

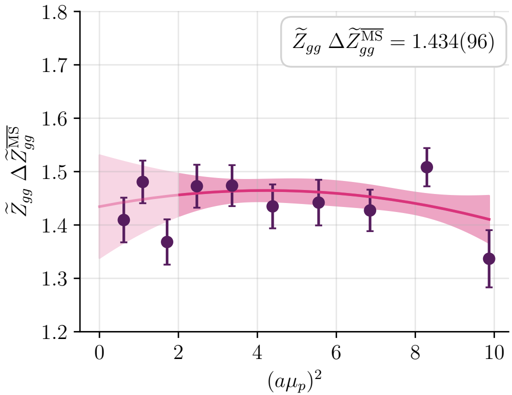
$$\begin{pmatrix} \mathcal{O}_g \\ \mathcal{O}_{q,\mathrm{con}} \\ \mathcal{O}_{q,\mathrm{dis}} \end{pmatrix}^{\mathrm{Lat}} = \begin{pmatrix} \widetilde{Z}_{gg} & \widetilde{Z}_{gq} & \widetilde{Z}_{gq} \\ 0 & \widetilde{Z}_{qq} & 0 \\ \widetilde{Z}_{qg} & \delta\widetilde{Z}_{qq} & (\widetilde{Z}_{qq} + \delta\widetilde{Z}_{qq}) \end{pmatrix} \begin{pmatrix} \mathcal{O}_g \\ \mathcal{O}_{q,\mathrm{con}} \\ \mathcal{O}_{q,\mathrm{dis}} \end{pmatrix}^{R}$$

Diagonal Quark Factors



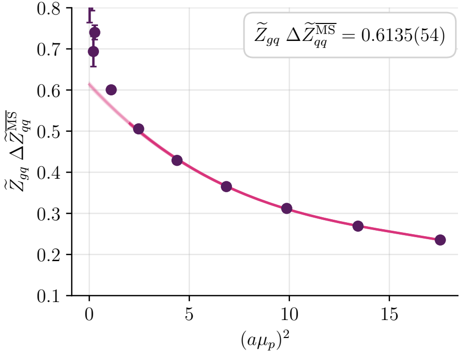
- Statistically well-behaved, only needs ~ 20 configurations for a precise signal

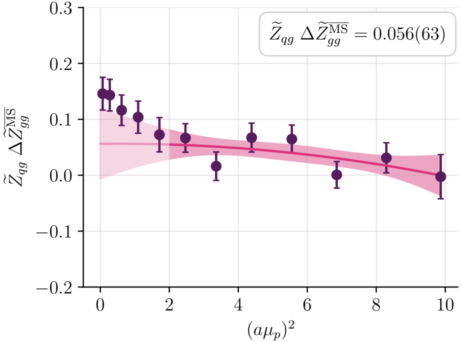
Diagonal Gluon Factors



- Less precise, though reasonably flat in kinematic scale

Sector Mixing Factors





- Insignificant signal in glue-in-quark contribution, but strong result for quark-in-glue
  - Inverse matrix components, not immediately obvious how this manifests

Renormalised Results

Quenched:

$$\begin{pmatrix} Z_{gg}^{\overline{\mathrm{MS}}} & Z_{gq}^{\overline{\mathrm{MS}}} \\ 0 & Z_{qq}^{\overline{\mathrm{MS}}} \end{pmatrix} = \begin{pmatrix} 0.725(17) & -0.247(12) \\ 0 & 0.9530(35) \end{pmatrix}$$

$N_f = 2$:

$$\begin{pmatrix} Z_{gg}^{\overline{\mathrm{MS}}} & Z_{gq}^{\overline{\mathrm{MS}}} & Z_{gq}^{\overline{\mathrm{MS}}} \\ 0 & Z_{qq}^{\overline{\mathrm{MS}}} & 0 \\ Z_{qg}^{\overline{\mathrm{MS}}} & \delta Z_{qq}^{\overline{\mathrm{MS}}} & Z_{qq}^{\overline{\mathrm{MS}}} + \delta Z_{qq}^{\overline{\mathrm{MS}}} \end{pmatrix} = \begin{pmatrix} 0.662(23)(2) & -0.345(14)(49) & -0.345(14)(49) \\ 0 & 0.9379(90) & 0 \\ -0.032(39)(85) & -0.003(19)(4) & 0.935(21)(4) \end{pmatrix}$$

- Apply these matrices to our hadron matrix elements to renormalise

Renormalised Results

Quenched

κ	am_π	m_π(MeV)	$\langle x\rangle_g^{N,\overline{\mathrm{MS}}}$	$\langle x\rangle_u^{N,\overline{\mathrm{MS}}}$	$\langle x\rangle_d^{N,\overline{\mathrm{MS}}}$	$\langle x\rangle_q^{N,\overline{\mathrm{MS}}}$
0.1320	0.540	1066	0.432(23)	0.3696(19)	0.1715(10)	0.5412(29)
0.1333	0.411	812	0.496(33)	0.3635(22)	0.1624(12)	0.5259(34)
0.1342	0.300	592	0.483(46)	0.3643(48)	0.1558(38)	0.5201(83)

κ	am_π	m_π(MeV)	$\langle x\rangle_g^{\pi,\overline{\mathrm{MS}}}$	$\langle x\rangle_q^{\pi,\overline{\mathrm{MS}}}$
0.1320	0.540	1066	0.419(11)	0.5423(21)
0.1333	0.411	812	0.483(13)	0.5217(21)
0.1342	0.300	592	0.542(40)	0.478(21)

Renormalised Results

$$N_f = 2$$

κ	am_π	m_π(MeV)	$\langle x \rangle_g^N$	$\langle x \rangle_{u,\text{con}}^{N,\overline{\text{MS}}}$	$\langle x \rangle_{d,\text{con}}^{N,\overline{\text{MS}}}$	$\langle x \rangle_{q,\text{con}}^{N,\overline{\text{MS}}}$	$\langle x \rangle_{q,\text{dis}}^{N,\overline{\text{MS}}}$
0.1355	0.325	765	0.403(48)	0.3469(46)	0.1529(24)	0.4998(69)	0.038(55)

κ	am_π	m_π(MeV)	$\langle x \rangle_g^{\pi,\overline{\text{MS}}}$	$\langle x \rangle_{q,\text{con}}^{\pi,\overline{\text{MS}}}$	$\langle x \rangle_{q,\text{dis}}^{\pi,\overline{\text{MS}}}$
0.1355	0.325	765	0.435(20)	0.4903(50)	0.061(27)

Renormalised Results

Quenched

κ	am_π	m_π(MeV)	$\langle x \rangle_g^{N,\overline{\text{MS}}} + \langle x \rangle_q^{N,\overline{\text{MS}}}$
0.1320	0.540	1066	0.973(24)
0.1333	0.411	812	1.022(33)
0.1342	0.300	592	1.003(47)

κ	am_π	m_π(MeV)	$\langle x \rangle_g^{\pi,\overline{\text{MS}}} + \langle x \rangle_q^{\pi,\overline{\text{MS}}}$
0.1320	0.540	1066	0.961(10)
0.1333	0.411	812	1.004(12)
0.1342	0.300	592	1.020(40)

Renormalised Results

$$N_f = 2$$

Nucleon

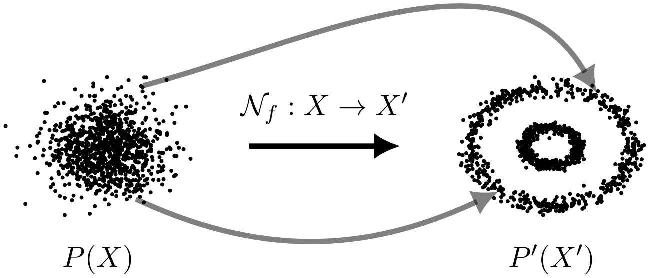
κ	am_π	m_π(MeV)	$\langle x \rangle_g^N + \langle x \rangle_{q,\mathrm{con}}^{N,\overline{\mathrm{MS}}} + \langle x \rangle_{q,\mathrm{dis}}^{N,\overline{\mathrm{MS}}}$
0.1355	0.325	765	0.941(53)

Pion

κ	am_π	m_π(MeV)	$\langle x \rangle_g^{\pi,\overline{\mathrm{MS}}} + \langle x \rangle_{q,\mathrm{con}}^{\pi,\overline{\mathrm{MS}}} + \langle x \rangle_{q,\mathrm{dis}}^{\pi,\overline{\mathrm{MS}}}$
0.1355	0.325	765	0.987(25)

- But where to next?

Normalising Flows



- Trainable function mapping a sample drawn from one distribution to a sample drawn from another
- Mechanism to embed FH perturbation,

$$P[U] = \exp(-S[U]) \;\rightarrow\; P'[U] = \exp(-S[U] - \lambda\mathcal{O}[U])$$

Normalising Flows

Variance reduction in lattice QCD observables via normalizing flows

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- Work published in July 2026 used normalising flows to embed hadronic operators in gauge ensembles
- Flows were trained on small geometries (4^4 to $4^3 \times 32$) then scaled to larger lattices (up to $16^3 \times 32$)

- Flows require training time and data, not an insignificant cost
- Interest in how FH may fit in; as training or validation data, probing the target distribution directly at larger lattice geometries

Summary

- We have demonstrated use of the Feynman-Hellmann methods to obtain hadronic matrix elements of a disconnected operator, coupling to both gluons and fermions.
- We have also applied these methods to obtain a full renormalisation mixing matrix non-perturbatively.
- The results obtained recover expected sum rules.
- These methods may combine with developing statistical methods to probe the disconnected sector further.