Confinement, chiral symmetry and the Bridges of Königsberg

CRA[1&2] & Vanessa Robins[3]

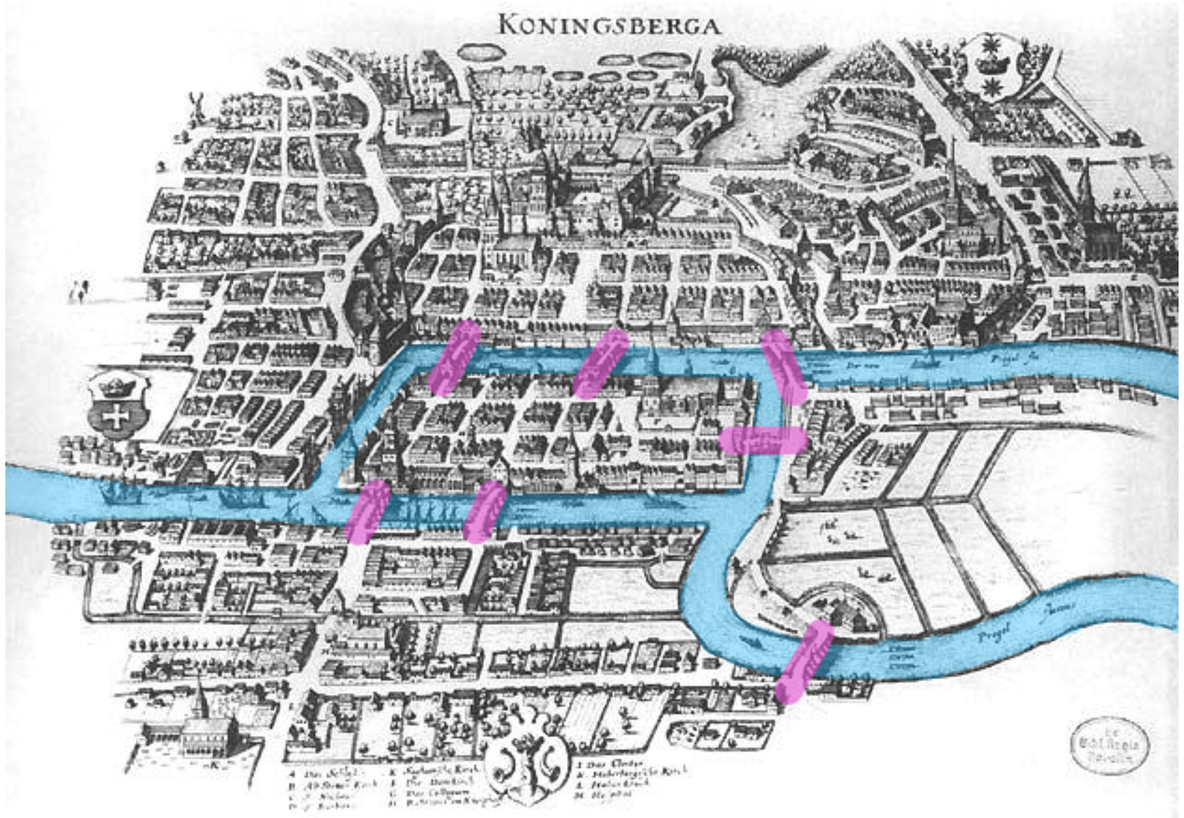
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Special Thanks to:

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4 Trinity College, Dublin, Ireland
5 Adelaide University, Australia

Bridges of Königsberg



Merian-Erben

Can you cross all seven bridges once?

Solved by Euler (1736)

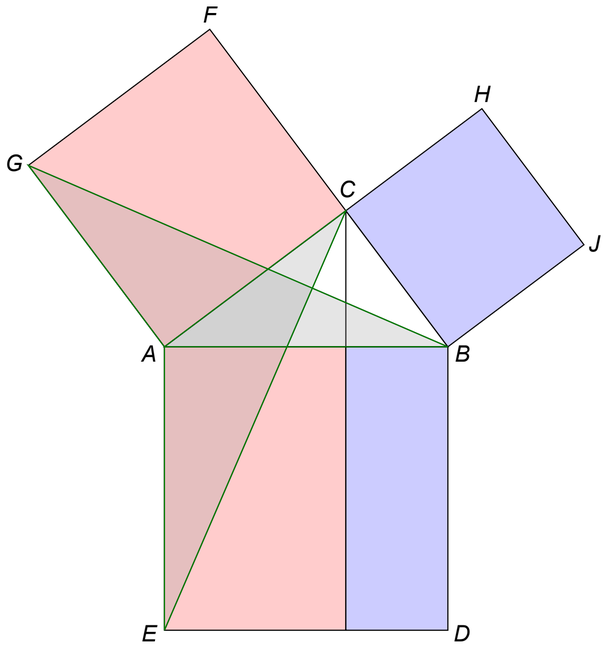
Basis of ***Topology***

The Study of Shapes

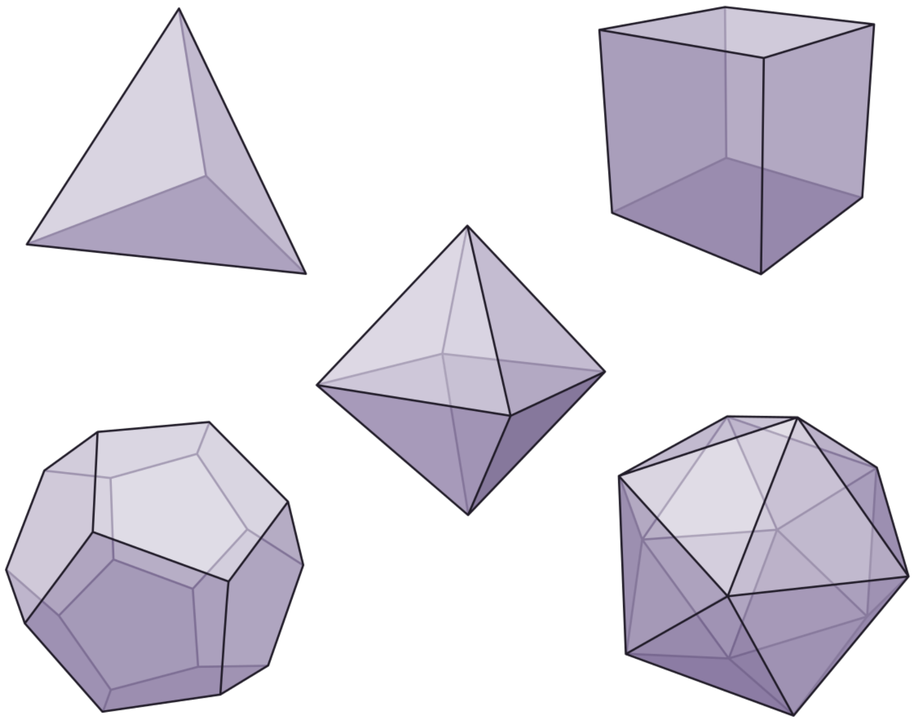


Geometry & Topology

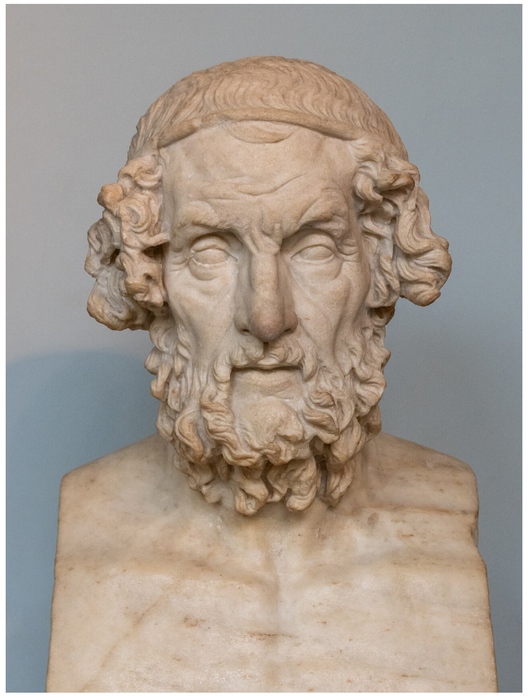
Ancient Greeks solved algebraic problems
using geometric shapes, areas & line lengths



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Homer's Odyssey



Photograph by Mike Peel
(www.mikepeel.net)

Path Integral (aka Partition Function)

Continuum

Lattice

$$Z = \int \mathscr{D}A\, \mathscr{D}\overline{\psi}\, \mathscr{D}\psi\ \exp(-S(A,\psi,\overline{\psi})) = \sum_{U_l} \sum_{\overline{\psi}_i} \sum_{\psi_j} \exp(-S(U_l,\psi_i,\overline{\psi}_j))$$

$$S = S_F + S_G$$

$$S_F = \int d^4x\, \overline{\psi}(x)\, [\, \not{D} + m\,]\, \psi(x) = \sum_{<i,j>} \overline{\psi}_i\, \mathscr{M}_{ij}\, \psi_j$$

$$S_G = \int d^4x\, \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) = \sum_p U_p$$

The Mystical Magic of the Fermion Integral

Form-changing… Circe…

Grassman Integration: $\int d\psi = 0$ & $\int d\psi\, \psi = 1$ & Diagonalising $\mathscr{M}_{ij}$

Continuum

$$\int \mathscr{D}\overline{\psi}\, \mathscr{D}\psi\, \exp\Big(-\int d^4x\, \overline{\psi}(x)\, [\, \not{D} + m\,]\, \psi(x) \Big) = \det[\, \not{D} + m\,]$$

Lattice

$$\sum_{\overline{\psi}_i} \sum_{\psi_j} \exp\Big(-\sum_{<i,j>} \overline{\psi}_i\, \mathscr{M}_{ij}\, \psi_j \Big) = \det[\, \mathscr{M}\,]$$

$\mathscr{M}_{ij}$ indices i and j run over lattice sites only (colour & spinor suppressed)

$\det$ of $N \times N$ matrix has $N!$ terms, each of which has N factors

For a $24^3 \times 64$ lattice, including colour & spinor indices:

$$N \sim 25{,}000{,}000$$

$$N! \sim 10^{180{,}000{,}000}$$

Geometrical Interpretation of Determinant

Consider the fermion matrix $\mathscr{M}_{ij}$

It is the fermion-gauge interaction $\longrightarrow$ it is local

i.e. it connects neighbouring lattice sites i and j

Associate:

row and column ***indices*** i and j of $\mathscr{M}_{ij}$ with the lattice sites

elements of $\mathscr{M}_{ij}$ with the interactions btw sites i and j

$\mathscr{M}_{ij} = 0$ when sites i and j are not nearest neighbours.

$\longrightarrow$ $\mathscr{M}_{ij}$ is ***very sparse***

Geometrical Interpretation of Determinant

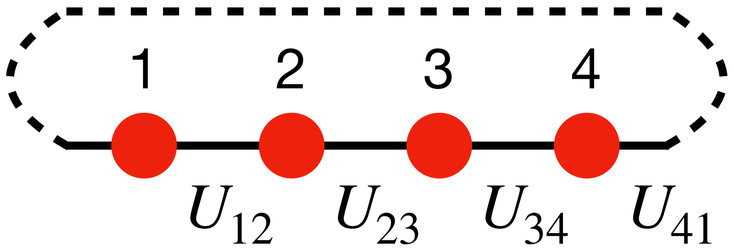
1-dim case

Consider 1-dim lattice

$\mathscr{M}$ is tri-diagonal

In d-dim $\mathscr{M}$ has $2d$ non-zero entries

for each row (and column)



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} m & U_{12} & 0 & U_{14} \\ U_{21} & m & U_{23} & 0 \\ 0 & U_{32} & m & U_{34} \\ U_{41} & 0 & U_{43} & m \end{pmatrix} \end{array}$$

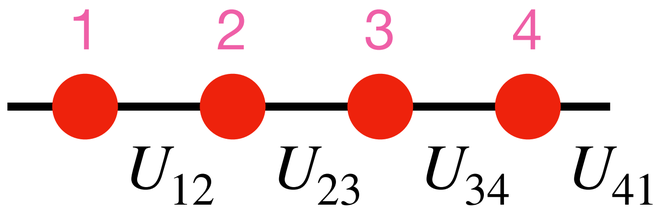
Geometrical Interpretation of Determinant

1-dim case

What is a determinant?

It is the sum of terms, and each term is a product

$$\det = \sum \text{terms} = \sum \prod \text{elements}$$



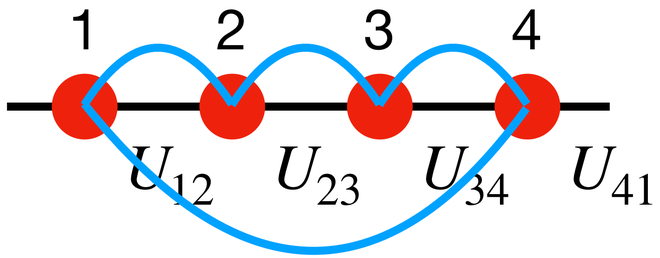
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} m & U_{12} & 0 & U_{14} \\ U_{21} & m & U_{23} & 0 \\ 0 & U_{32} & m & U_{34} \\ U_{41} & 0 & U_{43} & m \end{pmatrix} \end{array}$$

Algorithm:

1. choose any row i and col $j \rightarrow$ 1st element
2. erase that row & col and choose a new element
3. repeat from 2 until none left
4. multiply all elements $\rightarrow$ 1st term
5. repeat for all possible combinations

Geometrical Interpretation of Determinant

1-dim case

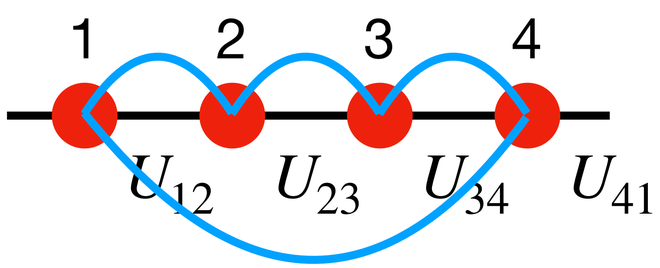


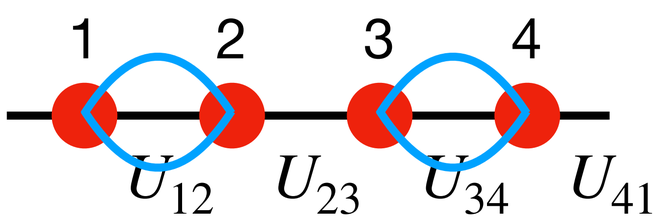
$$\begin{pmatrix} m & U_{12} & 0 & U_{14} \\ U_{21} & m & U_{23} & 0 \\ 0 & U_{32} & m & U_{34} \\ U_{41} & 0 & U_{43} & m \end{pmatrix}$$

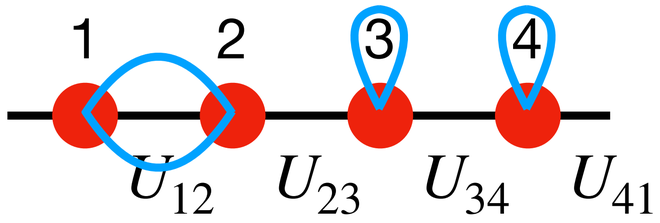
$$U_{12}U_{23}U_{34}U_{41}$$

Geometrical Interpretation of Determinant

Hamilton Cycles (Generalised Wilson Loops)







$$\begin{pmatrix} m & U_{12} & 0 & U_{14} \\ U_{21} & m & U_{23} & 0 \\ 0 & U_{32} & m & U_{34} \\ U_{41} & 0 & U_{43} & m \end{pmatrix}$$

$$\begin{pmatrix} m & U_{12} & 0 & U_{14} \\ U_{21} & m & U_{23} & 0 \\ 0 & U_{32} & m & U_{34} \\ U_{41} & 0 & U_{43} & m \end{pmatrix}$$

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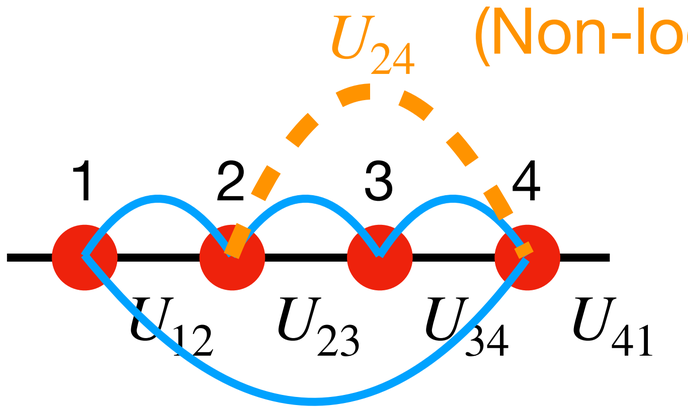
$$U_{12}U_{23}U_{34}U_{41}$$

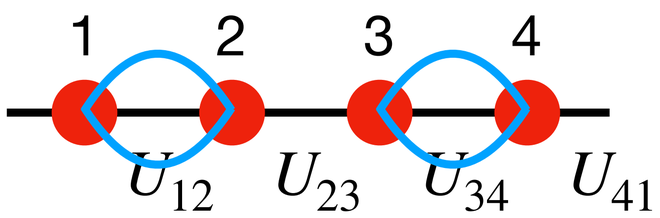
$$U_{12}U_{21}U_{34}U_{43}$$

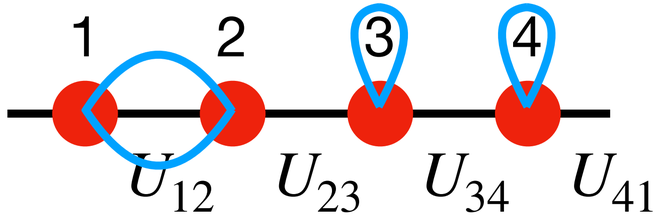
$$U_{12}U_{21}\,m^2$$

Geometrical Interpretation of Determinant

1-dim case







$$\begin{pmatrix} m & U_{12} & 0 & U_{14} \\ U_{21} & m & U_{23} & 0 \\ 0 & U_{32} & m & U_{34} \\ U_{41} & 0 & U_{43} & m \end{pmatrix}$$

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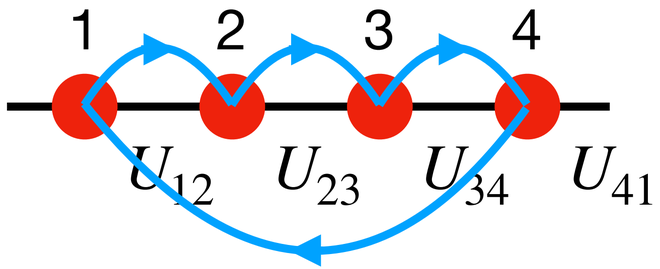
$$U_{12}U_{23}U_{34}U_{41}$$

$$U_{12}U_{21}U_{34}U_{43}$$

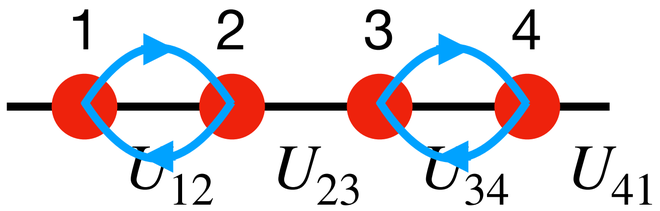
$$U_{12}U_{21}\,m^2$$

Geometrical Interpretation of Determinant

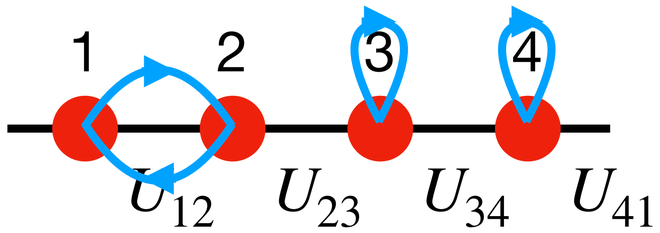
1-dim case



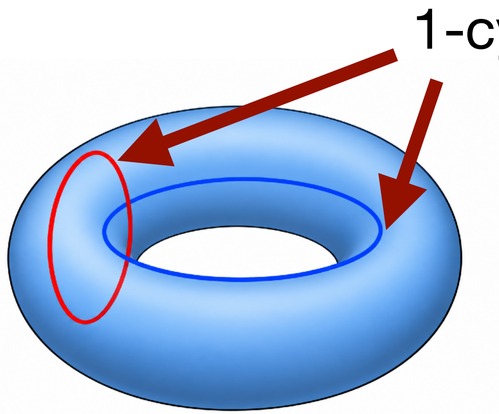
Single Connected Oriented 1-cycle



Many Connected Oriented 1-cycles



Many Connected Oriented "1-cycles"

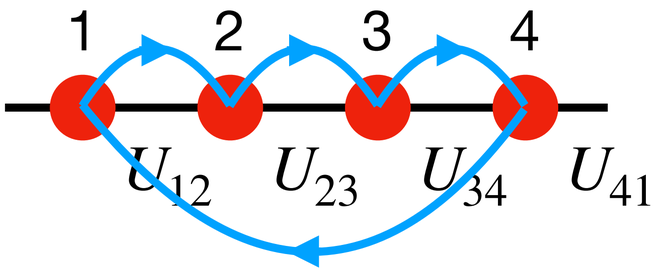


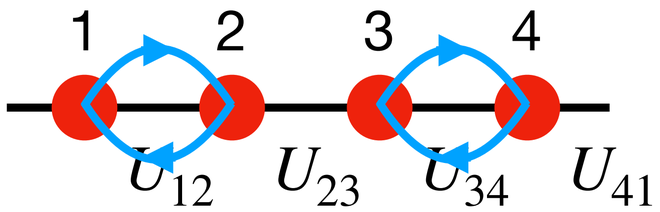
Each term in det has:

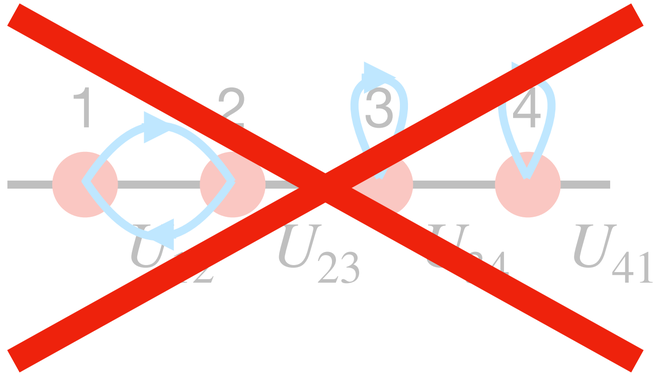
1. ***Every site*** included
2. Each with an ***incoming*** and ***outgoing*** path
3. Not every link included

Gauge invariant by construction

Determinant for $m = 0$







$$\begin{pmatrix} 0 & U_{12} & 0 & U_{14} \\ U_{21} & 0 & U_{23} & 0 \\ 0 & U_{32} & 0 & U_{34} \\ U_{41} & 0 & U_{43} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & U_{12} & 0 & U_{14} \\ U_{21} & 0 & U_{23} & 0 \\ 0 & U_{32} & 0 & U_{34} \\ U_{41} & 0 & U_{43} & 0 \end{pmatrix}$$

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$$U_{12}U_{23}U_{34}U_{41}$$

$$U_{12}U_{21}U_{34}U_{43}$$

$$U_{12}U_{21}\,m^2$$

Using the Geometrical Interpretation of Determinant

Cyclops…

Each term in the det corresponds to a set of 1-cycles which “fill” the lattice.

Call this set a ***Global 1-Set***, $\mathcal{G}_1$

An $N \times N$ matrix (i.e. N sites), has $N!$ different $\mathcal{G}_1$ (= no. of terms in det)

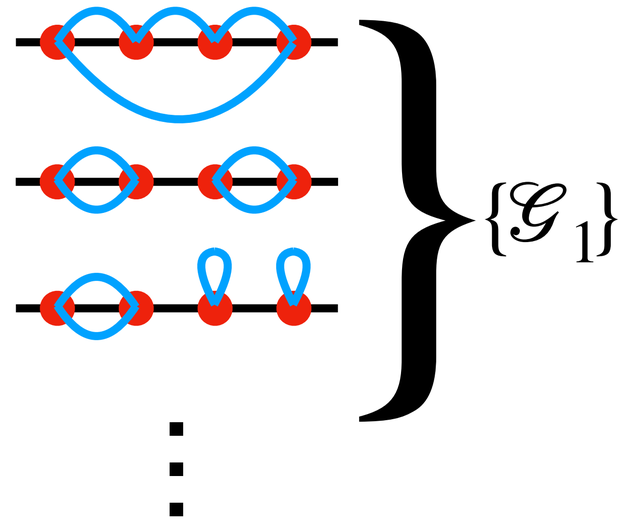
$$\det[\,\mathscr{M}\,] = \sum_{c \in \{\mathcal{G}_1\}} \text{term}(c) = \sum_{c \in \{\mathcal{G}_1\}} \prod_{l=1}^{N} \mathscr{M}_l$$

$$Z = \sum_{U_l} \sum_{\overline{\psi}_i} \sum_{\psi_j} \exp(-S_G - S_F)$$

$$= \sum_{U_l} \exp\Big(-\sum_p U_p\Big)\, \det[\,\mathscr{M}\,]$$

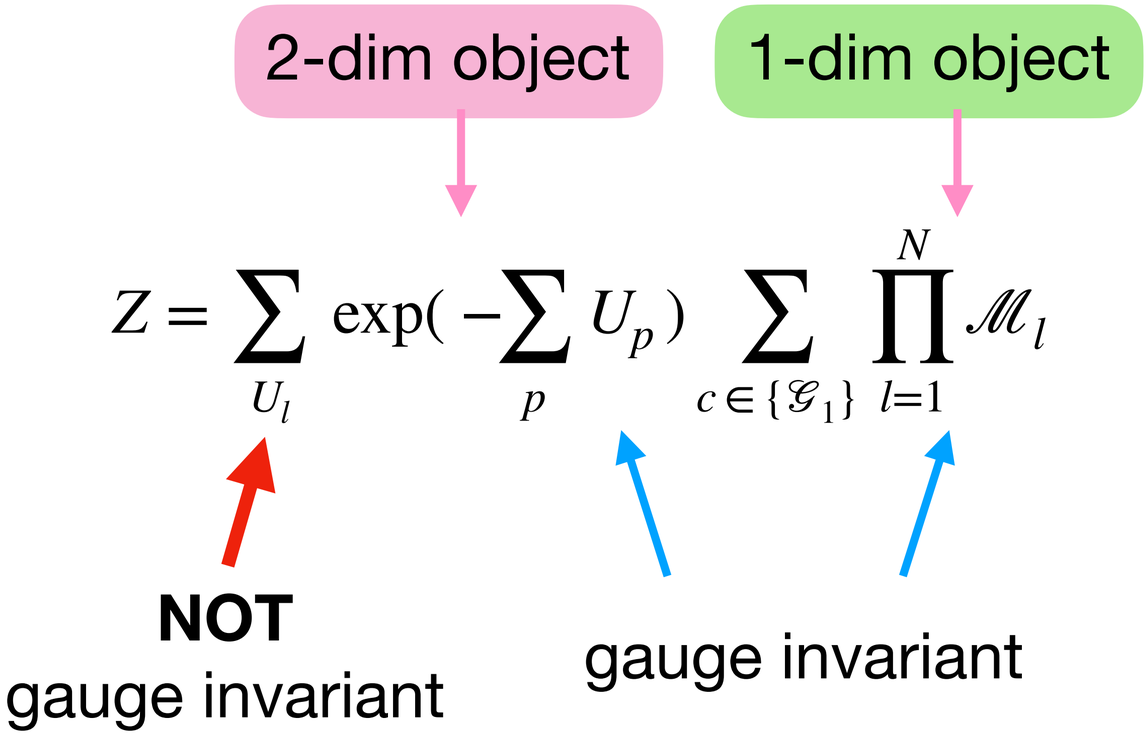
$$= \sum_{U_l} \exp\Big(-\sum_p U_p\Big) \sum_{c \in \{\mathcal{G}_1\}} \prod_{l=1}^{N} \mathscr{M}_l$$

$$\sum \text{term} =$$



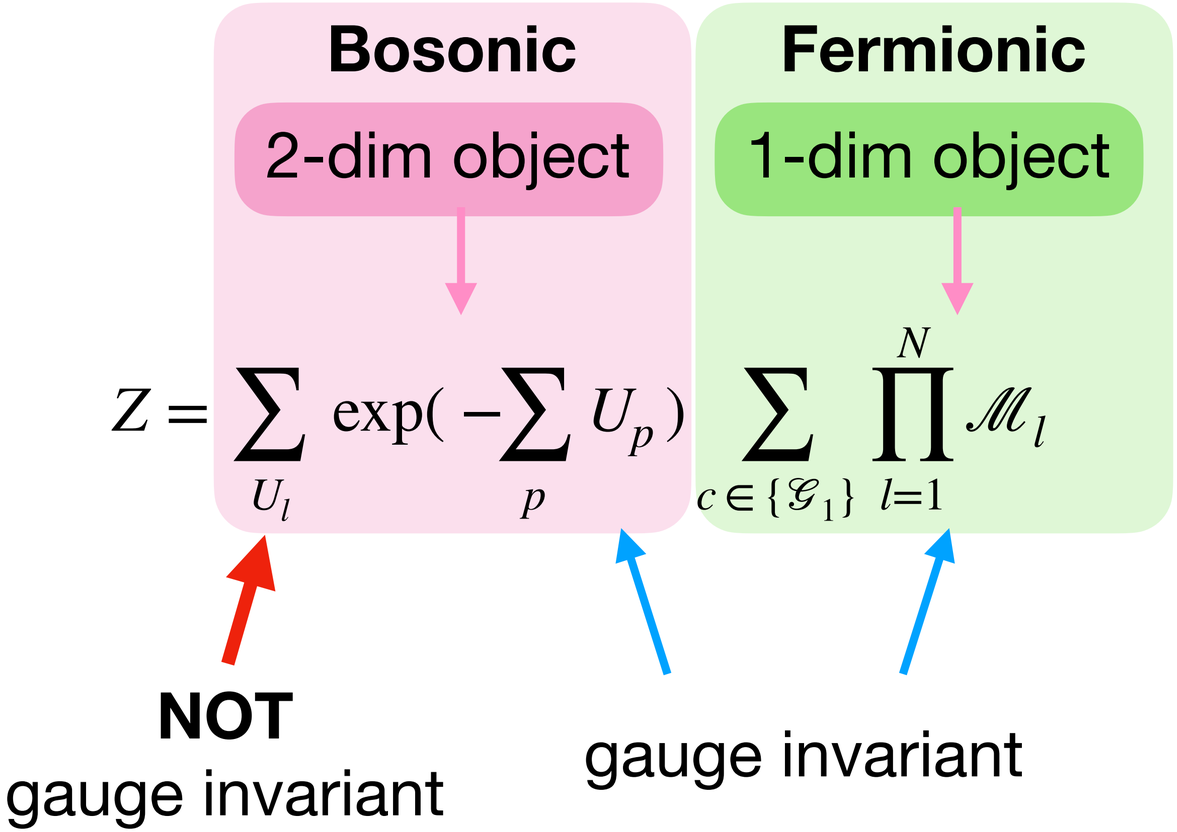
Geometrical Interpretation of Path Integral

Lotus-Eaters…



Geometrical Interpretation of Path Integral

Aeolus…



Geometrical Interpretation of Path Integral

Scylla and Charybdis…

Bosonic

2-dim object

Fermionic

1-dim object

$$Z = \sum_{U_l} \exp\Big(-\sum_{p} U_p\Big) \sum_{c \in \{\mathcal{G}_1\}} \prod_{l=1}^{N} \mathcal{M}_l$$

NOT
gauge invariant

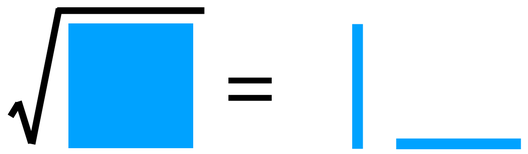
gauge invariant

Q. What is a fermion?

A. fermion $= \sqrt{\text{boson}}$ *(Dirac 1928)*

from: $E = \pm\sqrt{m^2 + p^2}$

$\sqrt{\text{plaquette}} =$ link1, link2



$\sqrt{\text{child}} =$ parent1, parent2

Geometrical Interpretation of Path Integral

Calypso…

$$Z = \sum_{U_l} \exp\Big(-\sum_{p} U_p\Big) \sum_{c_1 \in \{\mathcal{G}_1\}} \prod_{l \in c_1} \mathcal{M}_l$$

$$= \sum_{c_2 \in \{\mathcal{G}_2\}_{g.i.}} \prod_{p \in c_2} \exp(-U_p) \sum_{c_1 \in \{\sqrt{c_2}\}_{g.i.}} \prod_{l \in c_1} \exp(-iA_l)$$

$$= \sum_{c_2 \in \{\mathcal{G}_2\}_{g.i.}} \prod_{p \in c_2} |p| \sum_{c_1 \in \{\sqrt{c_2}\}_{g.i.}} \prod_{l \in c_1} |l|$$

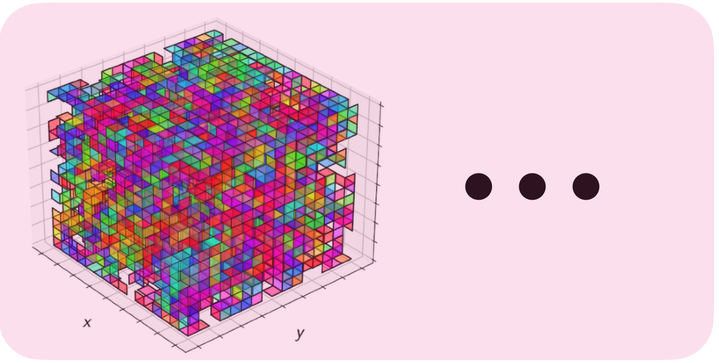
Geometrical Interpretation of Path Integral

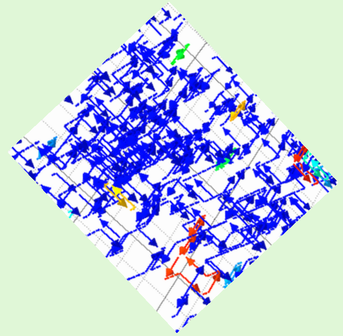
Shape-changing… Circe…

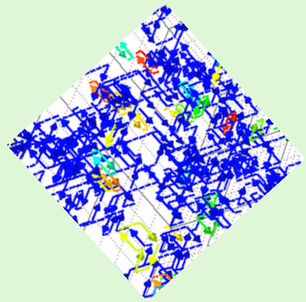
$$Z = \sum_{U_l} \exp\Big(-\sum_{p} U_p\Big) \sum_{c_1 \in \{\mathcal{G}_1\}} \prod_{l \in c_1} \mathcal{M}_l$$

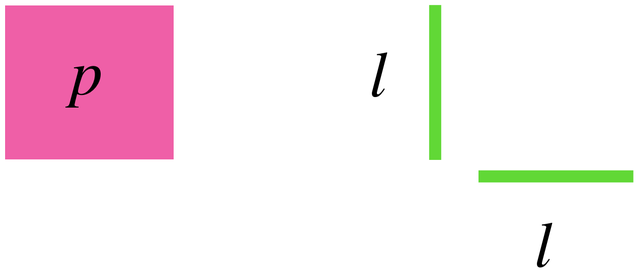
$$= \sum_{c_2 \in \{\mathcal{G}_2\}_{g.i.}} \prod_{p \in c_2} \exp(-U_p) \sum_{c_1 \in \{\sqrt{c_2}\}_{g.i.}} \prod_{l \in c_1} \exp(-iA_l)$$

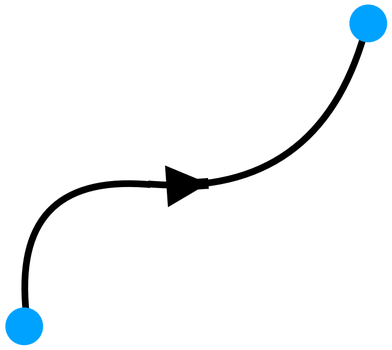
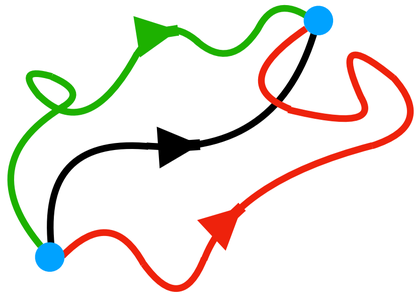
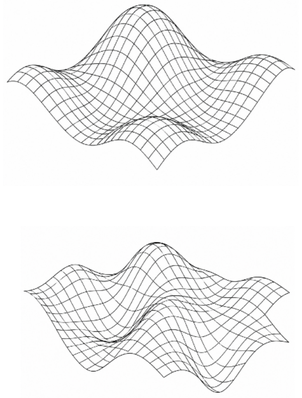
$$= \sum_{c_2 \in \{\mathcal{G}_2\}_{g.i.}} \prod_{p \in c_2} |p| \sum_{c_1 \in \{\sqrt{c_2}\}_{g.i.}} \prod_{l \in c_1} |l|$$

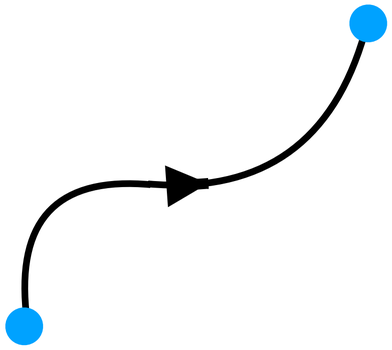
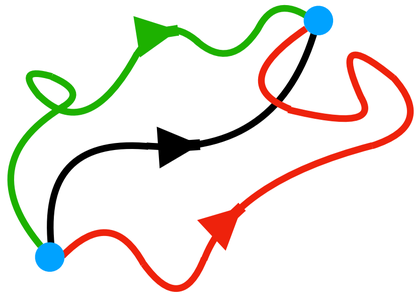
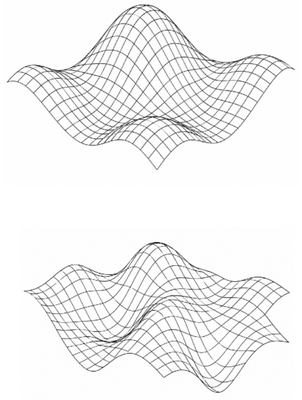








	Classical	Quantum Mechanics	Quantum Field Theory
Answer =	Path with Min Action	$Z = \sum_{\text{all poss paths}} \text{stuff} \times e^{-\text{Action}}$	$Z = \sum_{\text{all poss field cfgs}} \text{stuff} \times e^{-\text{Action}}$
		1st quantised	2nd quantised
Visualisation		Σ  ⋮	Σ  ⋮

	Classical	Quantum Mechanics	Quantum Field Theory	"Greek" Field Theory
Answer =	Path with Min Action	$Z = \sum_{\text{all poss paths}} \text{stuff} \times e^{-\text{Action}}$	$Z = \sum_{\text{all poss field cfgs}} \text{stuff} \times e^{-\text{Action}}$	$Z = \sum_{c_2 \in \{\mathcal{G}_2\}} P(c_2) \sum_{c_1 \in \{\sqrt{c_2}\}} P(c_1)$
		1st quantised	2nd quantised	3rd quantised
Visualisation		Σ  ⋮	Σ  ⋮	***"Quantise Everything at all dimensions !"***

Geometrical Interpretation of Path Integral

Underworld…

$$Z = \sum_{U_l} \exp\Big(-\sum_{p} U_p\Big) \sum_{c_1 \in \{\mathcal{G}_1\}} \prod_{l \in c_1} \mathscr{M}_l$$

$$= \sum_{c_2 \in \{\mathcal{G}_2\}_{g.i.}} \prod_{p \in c_2} \exp(-U_p) \sum_{c_1 \in \{\sqrt{c_2}\}_{g.i.}} \prod_{l \in c_1} \exp(-iA_l)$$

$$= \sum_{c_2 \in \{\mathcal{G}_2\}_{g.i.}} \prod_{p \in c_2} |p| \sum_{c_1 \in \{\sqrt{c_2}\}_{g.i.}} \prod_{l \in c_1} |l|$$

Maxwell's Eq'ns
2-d
bosons
spin 1

Dirac Eq'n
1-d
fermonic
spin 1/2

Extrapolating to higher dimensions

Beware The Sirens…

Maxwell's Eq'ns
2-d
bosons
spin 1

Dirac Eq'n
1-d
fermonic
spin 1/2

$$Z = \sum_{c_2 \in \{\sqrt{c_4}\}_{g.i.}} \prod_{p \in c_2} |p| \sum_{c_1 \in \{\sqrt{c_2}\}_{g.i.}} \prod_{l \in c_1} |l|$$

Extrapolating to higher dimensions

Ithaca…

GR ?
4-d
gravitons
spin 2

Maxwell's Eq'ns
2-d
bosons
spin 1

Dirac Eq'n
1-d
fermonic
spin 1/2

$$Z = \sum_{c_4 \in \{\mathcal{G}_4\}_{g.i.}} \prod_{b \in c_4} |b| \sum_{c_2 \in \{\sqrt{c_4}\}_{g.i.}} \prod_{p \in c_2} |p| \sum_{c_1 \in \{\sqrt{c_2}\}_{g.i.}} \prod_{l \in c_1} |l|$$

?

Discussion

Plot failures…

Q. *Why* $\sqrt{shape}$ and not ∂ shape? (i.e. *boundary* of shape)

Note using boundary operator, dimension: $d \to d-1 \to d-2 \ldots$

A1. Dirac: fermion $= \sqrt{\text{boson}} \neq$ boson in d-1 dimensions

A2. ∂ Plaq = ***4*** links $\quad\to\quad$ overcounting..

Q. Fundamental bosons?

A1. No… $\sqrt{1} \neq 0$

A2. Klein Gordan: $\int \mathscr{D}\phi \exp[-(D\phi)^2] = 1/\sqrt{\det}$ has no geometric analogy

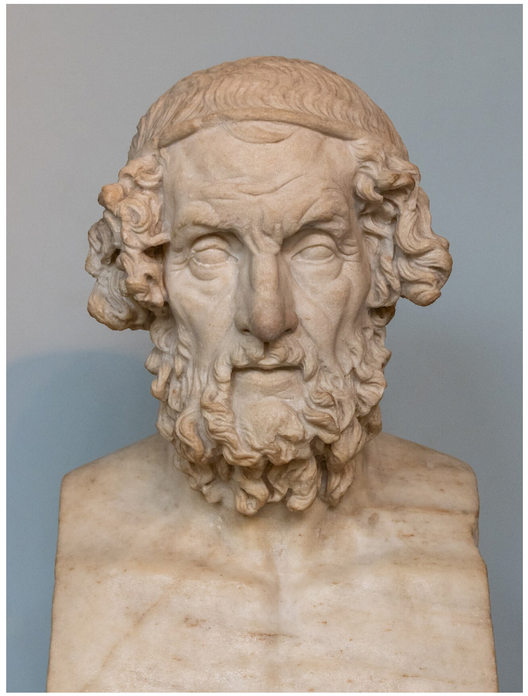
Determinant represented by spaghetti of 1-cycles (generalised Wilson loops) filling space-time

Notation: U_l also includes Dirac γ-matrices

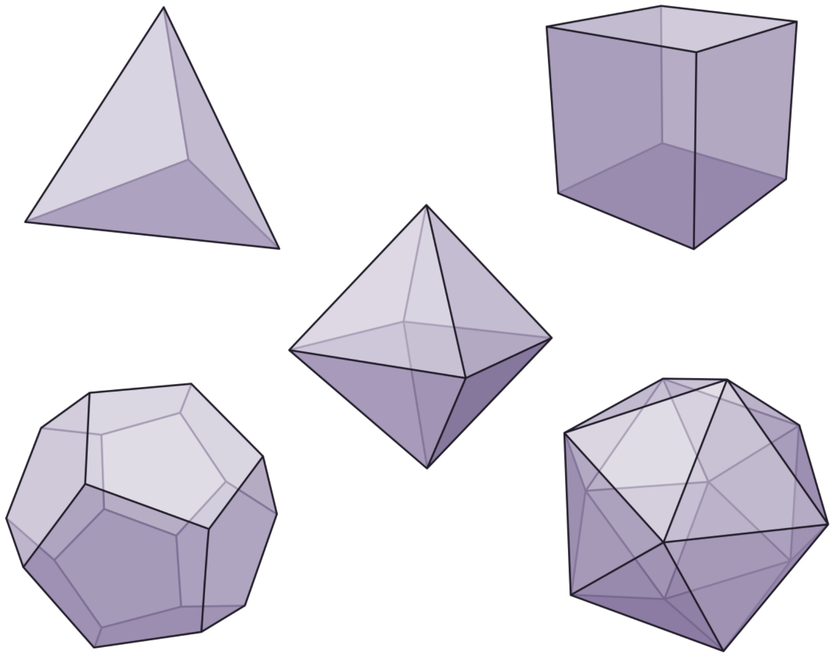
Physics (confinement & χSB) caused by preferential geometry (orientation) of quantised fields

Summary

QFT partition f'n —fermion integration→ determinant → det interpreted as 1-cycles



Photograph by Mike Peel
(www.mikepeel.net)



↓

Z = each d-dim shape
decomposed into d/2-dim shape
Quantise over everything

↓

Z = nested sum over all
dimensions and all
possibilities with weights


All for one and one for all
OH WHAT A FEELING
OH WHAT A FEELING
CALTEX
XXXX
TOYOTA

Back-up slides

QCD partition f'n and Peierls Argument

Entropy versus Energy

Q. Why does one, large, percolating vortex sheet dominate?

A. Infinite sum of all possible configs is dominated by a particular topology

$$Z = \sum_{\{i\}} \exp(-E_i) = \sum_{E_i} n(E_i)\, \exp(-E_i) \approx n(dominant)\, \exp(-E_{dominant})$$

- Energy = E_i ~ Area, Boltzmann weight exponentially ***decreases*** with ***Area***
- Multiplicity = $n(E_i)$ = number of micro states ~ (entropy)

  For 2D sheets embedded in 4D, Multiplicity exponentially ***increases*** with ***Area***
- When $N_\tau \to 0$, microstates (vortex sheets) geometrically constrained along temporal direction, periodicity means they must return to original position

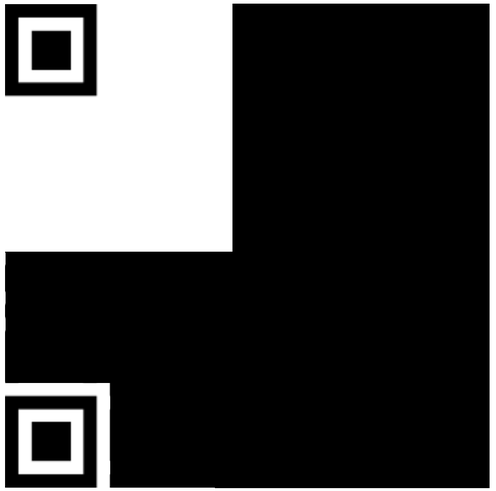
  $\longrightarrow$ **temporal alignment**

E.g. QR Codes

Typical



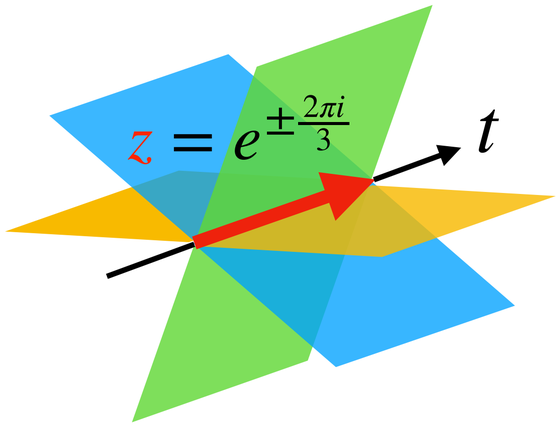
Rare



Direct vs Dual Lattice:

Overcoming gauge constraint?

Direct Lattice



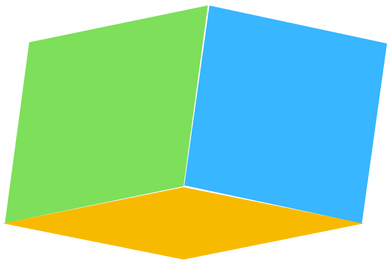
1 charged link in t-direction

→ **6** pierced plaquettes

in $t-x,\ t-y,\ t-z$ planes

Dual Lattice

(half integer sites)



1 cube

with **6** faces

in $y-z,\ z-x,\ x-y$ planes

