Lattice QCD constraints on pion electroproduction off a nucleon

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arXiv:2407.05334v2 / Phys. Rev. D 110, 094015 (2024)
arXiv:2603.06055v1

Contributors: Yu Zhuge, Zhan-Wei Liu, Derek B. Leinweber, Anthony W. Thomas

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I. N^* structure within HEFT

II. Extension to pion photoproduction

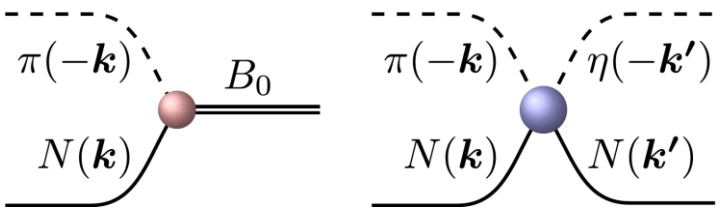
III. Extension to pion electroproduction

IV. Summary

N^* structure within HEFT

- Nucleon excited states have been studied within Hamiltonian effective field theory (HEFT)

$$G_\alpha^{N_i}(k) = \frac{\sqrt{3} g_\alpha^{N_i}}{2\pi f_\pi} \sqrt{\omega_{\mathrm{M}_\alpha}(k)} u(k)\,, \qquad V_{\alpha\beta}(k,k') = \frac{3 v_{\alpha\beta}}{4\pi^2 f_\pi^2} \tilde{u}_\alpha(k) \tilde{u}_\beta(k')$$



$N^*(1535), N^*(1650)$

Zhan-Wei Liu et al., Phys. Rev. Lett. 116, 082004 (2016)

C. D. Abell et al., Phys. Rev. D 108, 094519 (2023)

The Roper: $N^*(1440)$

Zhan-Wei Liu et al., Phys. Rev. D 95, 034034 (2017)

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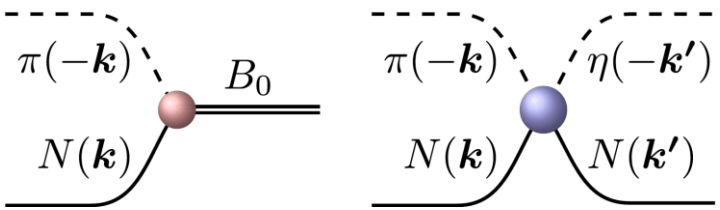
Shiryo Owa et al., Phys.Rev.D 109 11, 116022 (2024)

other resonances...

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- HEFT explores resonance structure through a joint analysis of
  - Experimental data: phase shifts, inelasticities, cross sections ...

$$T_{\alpha,\beta}(k,k';E) = \tilde{V}_{\alpha,\beta}(k,k';E) + \sum_\gamma \int q^2 dq\, \tilde{V}_{\alpha,\gamma}(k,q;E) \frac{1}{E - \omega_\gamma(q) + i\epsilon} T_{\gamma,\beta}(q,k';E)$$

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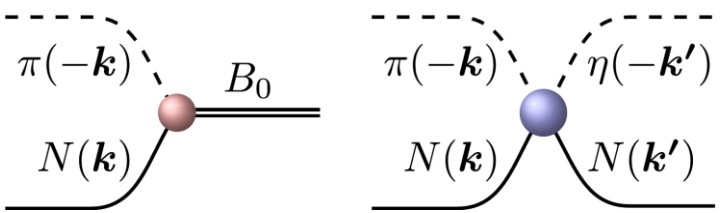
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  - Lattice QCD spectra
- The joint analysis constrains the model parameters

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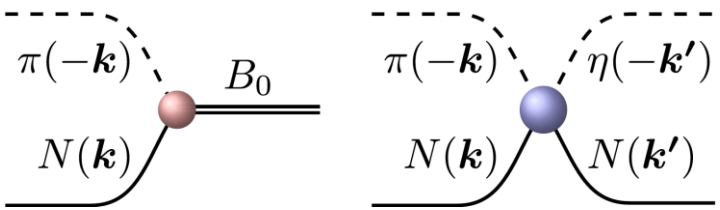
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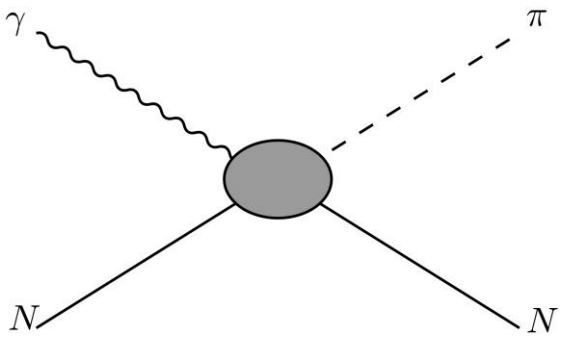
- Two upcoming talks on HEFT:
  - 25/09 Fang-Chao Han: Structure of $\Omega(2012)$ with Hamiltonian Effective Field Theory
  - 26/09 Kang Yu: Nonperturbative Hamiltonian Framework for General N-Body Finite-Volume Systems: three- and four-particle spectra

II. Extension to pion photoproduction

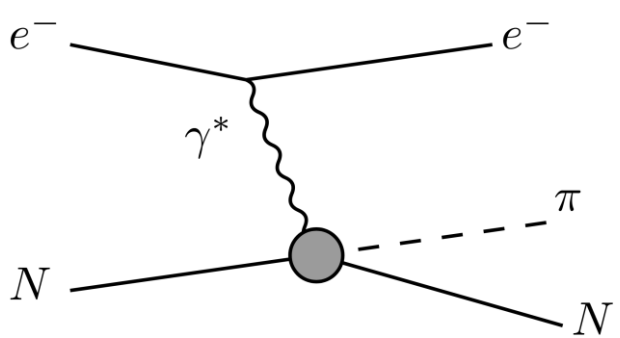
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Pion photo- and electroproduction

- Excited nucleon states also play an important role in pion photo- and electroproduction



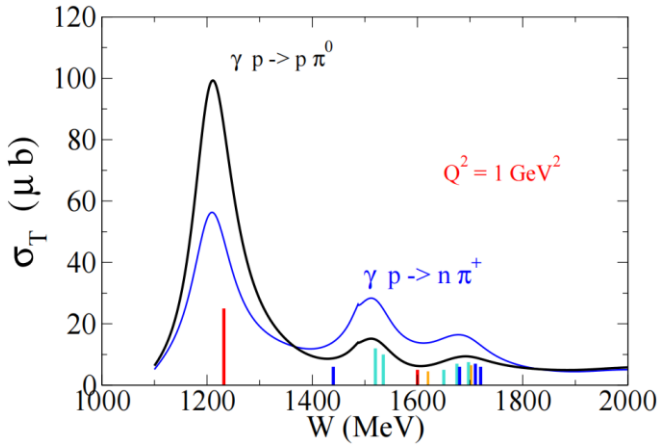
pion photoproduction



pion electroproduction

- These processes are important inputs to partial-wave analyses in the light-hadron resonance region

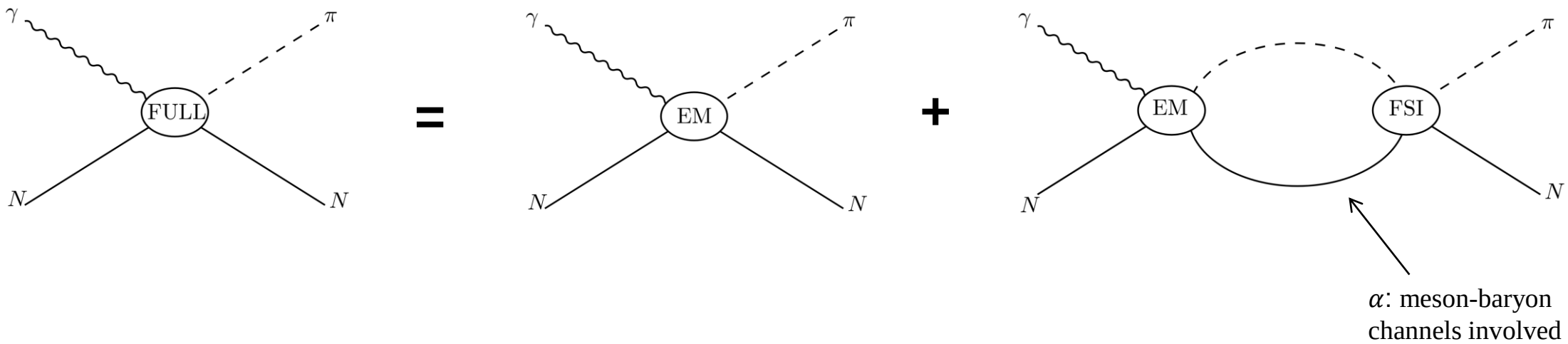
Transverse cross sections from MAID 2007
in the 1st, 2nd, and 3rd *N* and Δ resonance regions



Extension to pion photoproduction

- $\gamma + N \rightarrow \pi + N$

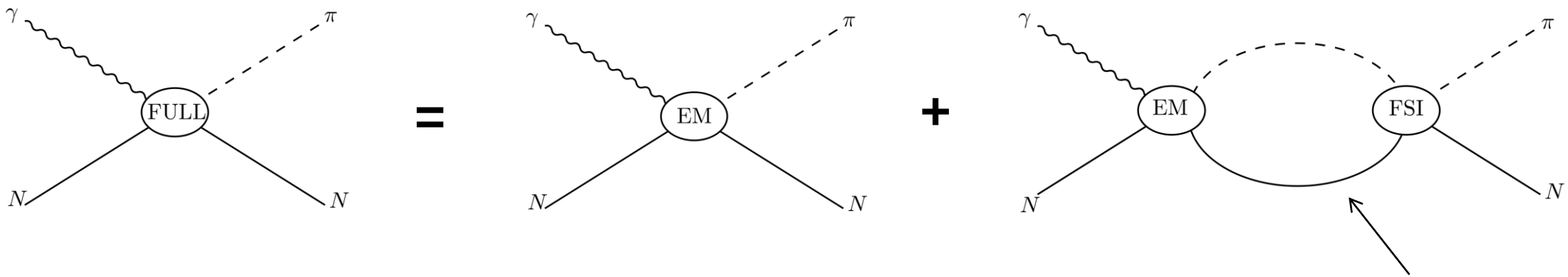
$$T^{\lambda_\gamma,\lambda_N}_{\pi N,\gamma N}(k,q;E) = V^{JLS;\lambda_\gamma,\lambda_N}_{\pi N,\gamma N}(k,q) + \sum_\alpha \int \mathrm{d}k' k'^2 V^{JLS;\lambda_\gamma,\lambda_N}_{\alpha,\gamma N}(k',q) \frac{1}{E-\omega_\alpha(k')+i\epsilon} T_{\pi N,\alpha}(k,k';E)$$



Extension to pion photoproduction

- $\gamma + N \to \pi + N$

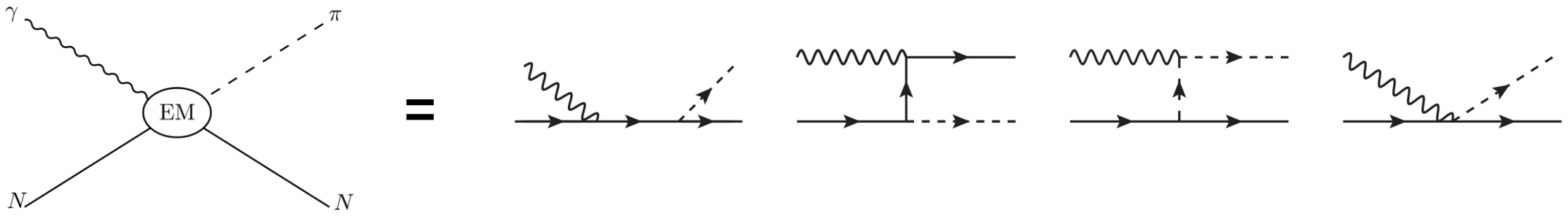
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- Two low-lying odd-parity $N(1/2^-)$ excited states $N^*(1535)$ / $N^*(1650)$
  - $S_{11}(L_{2I2J})$ partial wave
  - α: πN / ηN / $K\Lambda$
- The first $N(1/2^+)$ excited state $N^*(1440)$
  - P_{11} partial wave
  - α: πN / $\pi\Delta$ / σN

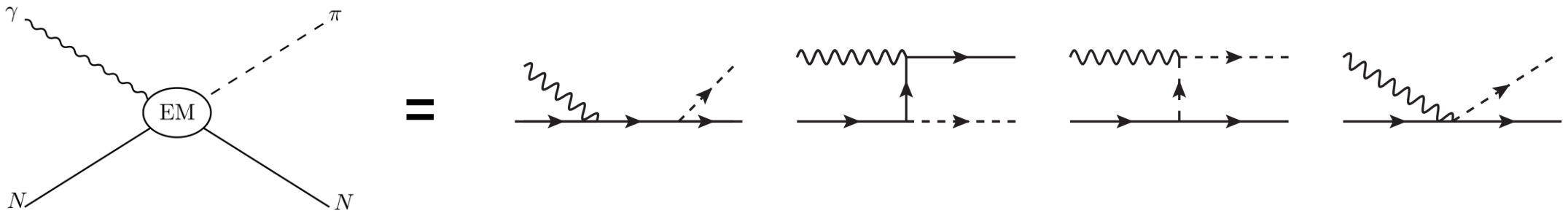
Electromagnetic transitions

- Electromagnetic transitions



Electromagnetic transitions

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- The Lagrangians and coupling constants are taken from the ANL-Osaka dynamical coupled-channel model

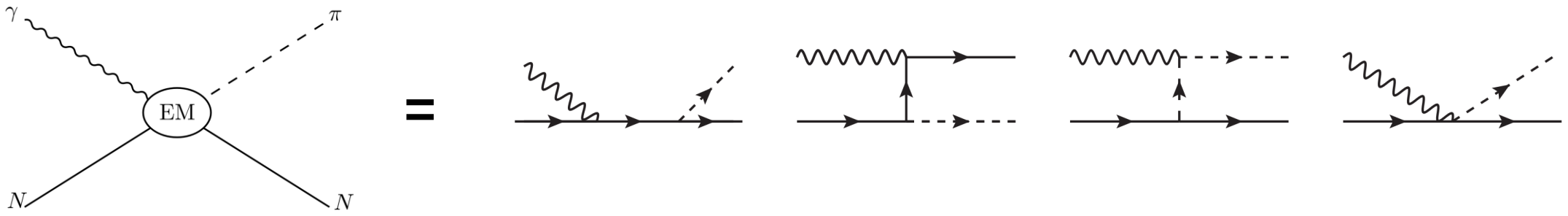
https://www.phy.anl.gov/theory/research/anl-osaka-pwa/

A. Matsuyama, T. Sato, and T. S. H. Lee, Phys. Rep. 439, 193 (2007)

H. Kamano, S. X. Nakamura, T. S. H. Lee, and T. Sato, Phys. Rev. C 88, 035209 (2013)

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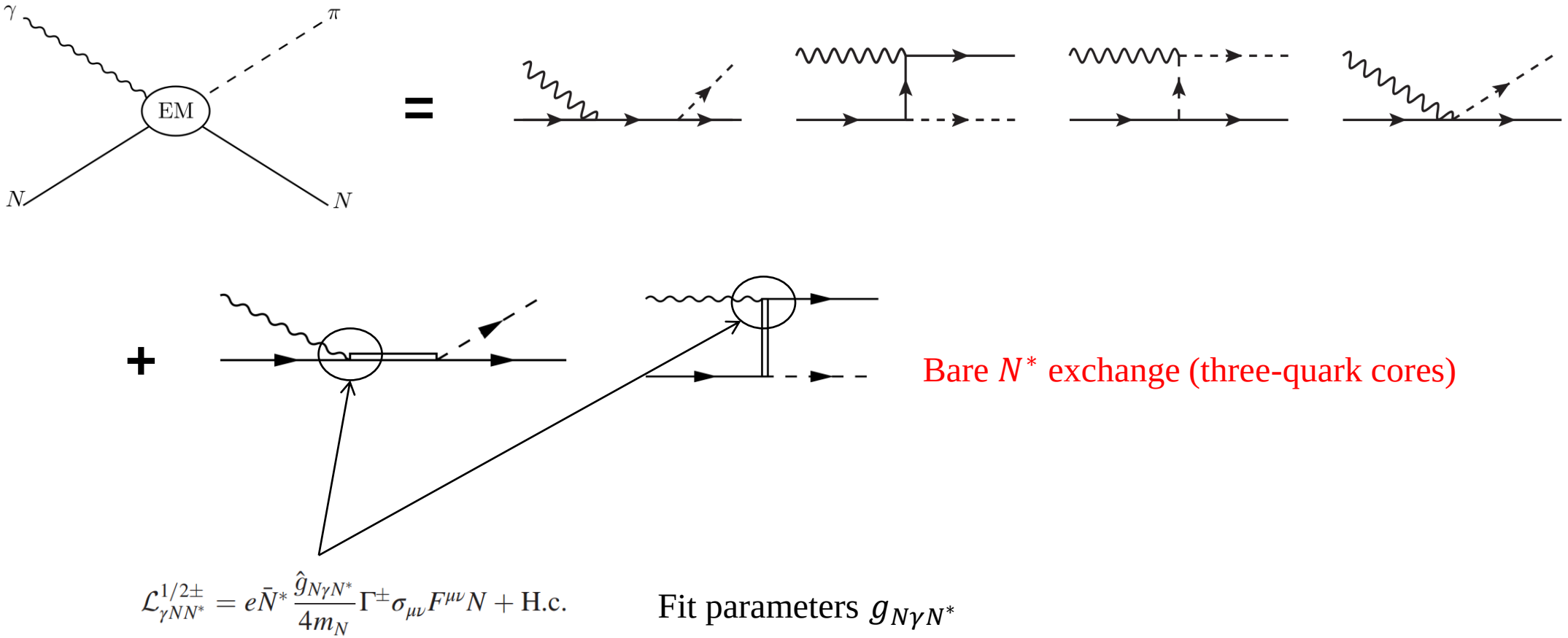
- Partial-wave decomposition:

Suh Urk Chung, Spin formalisms, BNL-76975-2006-IR

$$V^{JLS;\lambda_\gamma\lambda_N}_{\alpha,\gamma N}(k,q) = 4\pi\sqrt{\frac{2J+1}{4\pi}}\sum_{\substack{mm_s\\ m_1m_2}}(lmsm_s|JM)(s_1m_1s_2m_2|sm_s)\int d\Omega Y^*_{lm}(\theta,\phi)\mathcal{M}_{\alpha,\gamma N}(s_z'^{B_\alpha},\lambda_N,\lambda_\gamma;\vec{k},\vec{q})$$

Electromagnetic transitions

- Electromagnetic transitions



Bare N^* exchange (three-quark cores)

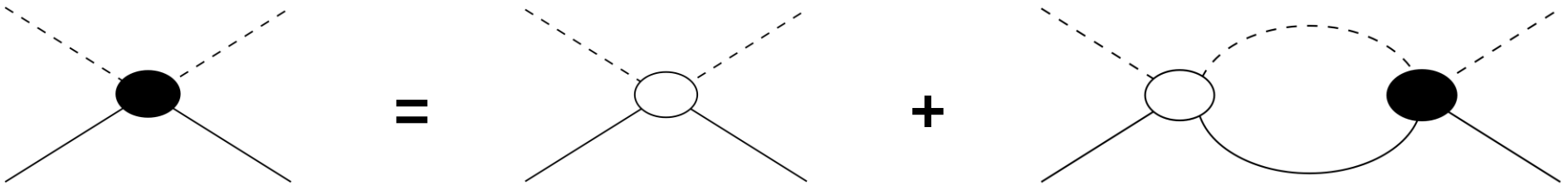
$$\mathcal{L}^{1/2\pm}_{\gamma NN^*} = e\bar{N}^* \frac{\hat{g}_{N\gamma N^*}}{4m_N} \Gamma^{\pm} \sigma_{\mu\nu} F^{\mu\nu} N + \text{H.c.}$$

Fit parameters $g_{N\gamma N^*}$

Final-state interactions

- Final-state interactions (FSI)

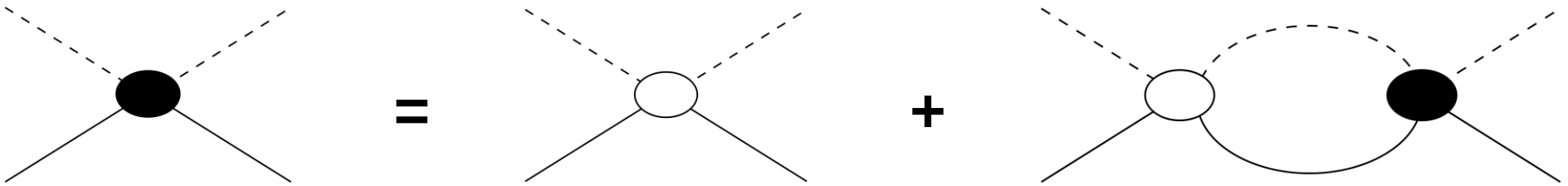
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- The HEFT analysis constrains the model parameters using
  - experimental data
  - Lattice QCD spectra

C. D. Abell et al., Phys. Rev. D 108, 094519 (2023)

J.-j. Wu et al., Phys. Rev. D 97, 094509 (2018)

Contributions from bare states

- HEFT analysis
  - $N^*(1535)/N^*(1650)$: quark-model-like states dressed by meson-baryon interactions
  - $N^*(1440)$: generated mainly by rescattering among coupled meson-baryon channels, with a small quark-model-like component

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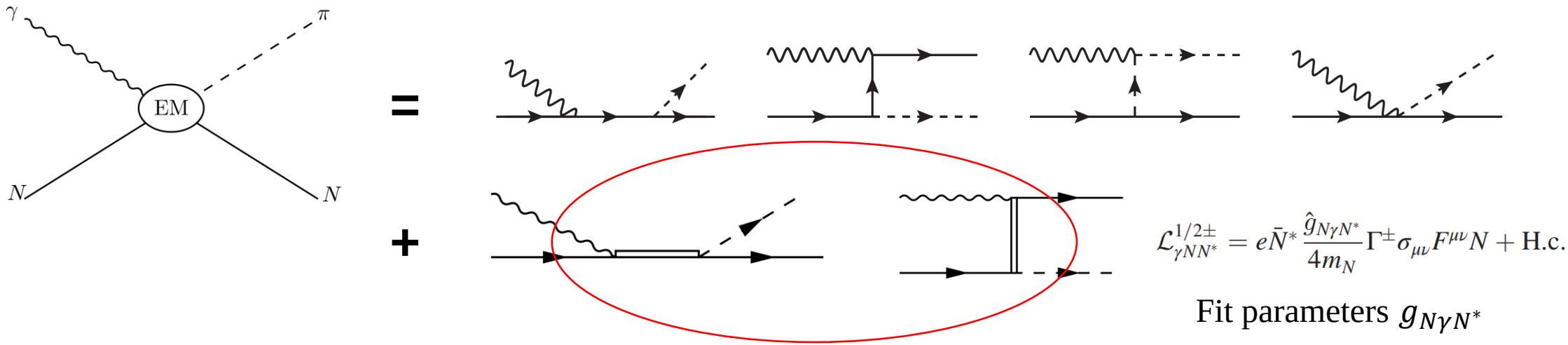
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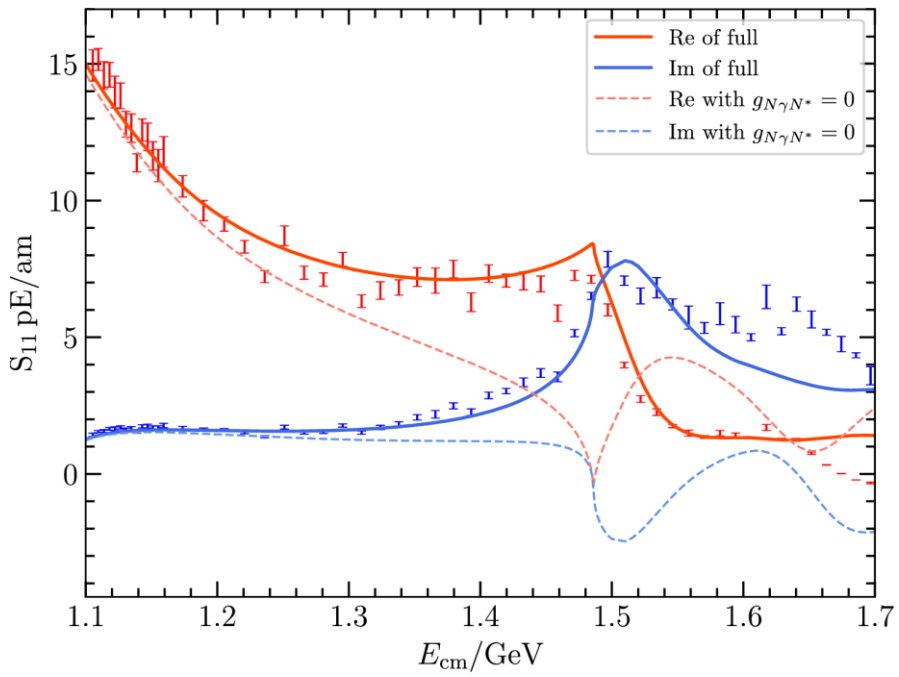
Fit parameters $g_{N\gamma N^*}$

By setting the electromagnetic couplings $g_{N\gamma N^*}$ to zero, we can assess the bare-core contributions to EM transitions

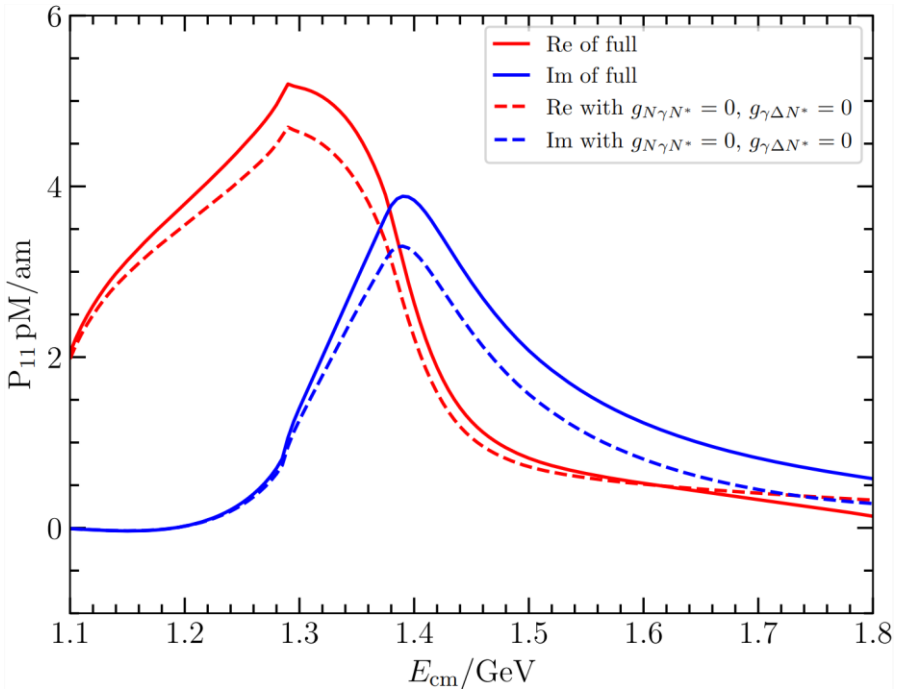
Contributions from bare states

- Fits to the multipole amplitudes E_{0+} and M_{1-}

  - $N^*(1535)$ and $N^*(1650)$: S_{11} E_{0+}



  - $N^*(1440)$: P_{11} M_{1-}



- A bare three-quark core is essential for describing $N^*(1535)$ and $N^*(1650)$, but not $N^*(1440)$, within this framework

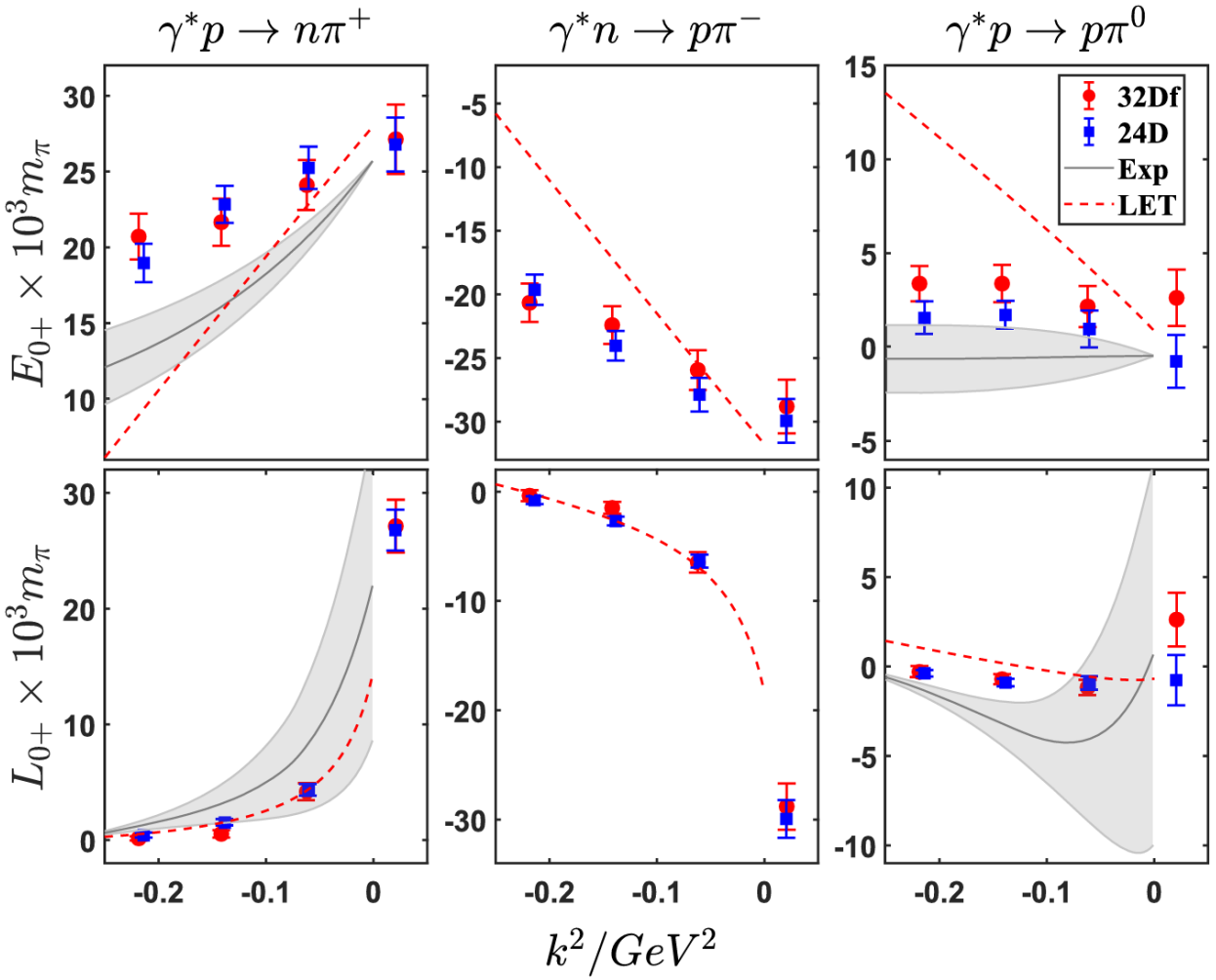
HEFT analysis ⟷ consistent with the pion photoproduction results

III. Extension to pion electroproduction and constraints from lattice QCD results

arXiv:2603.06055v1

First lattice QCD results for multipole amplitudes in $\gamma^* N \to \pi N$

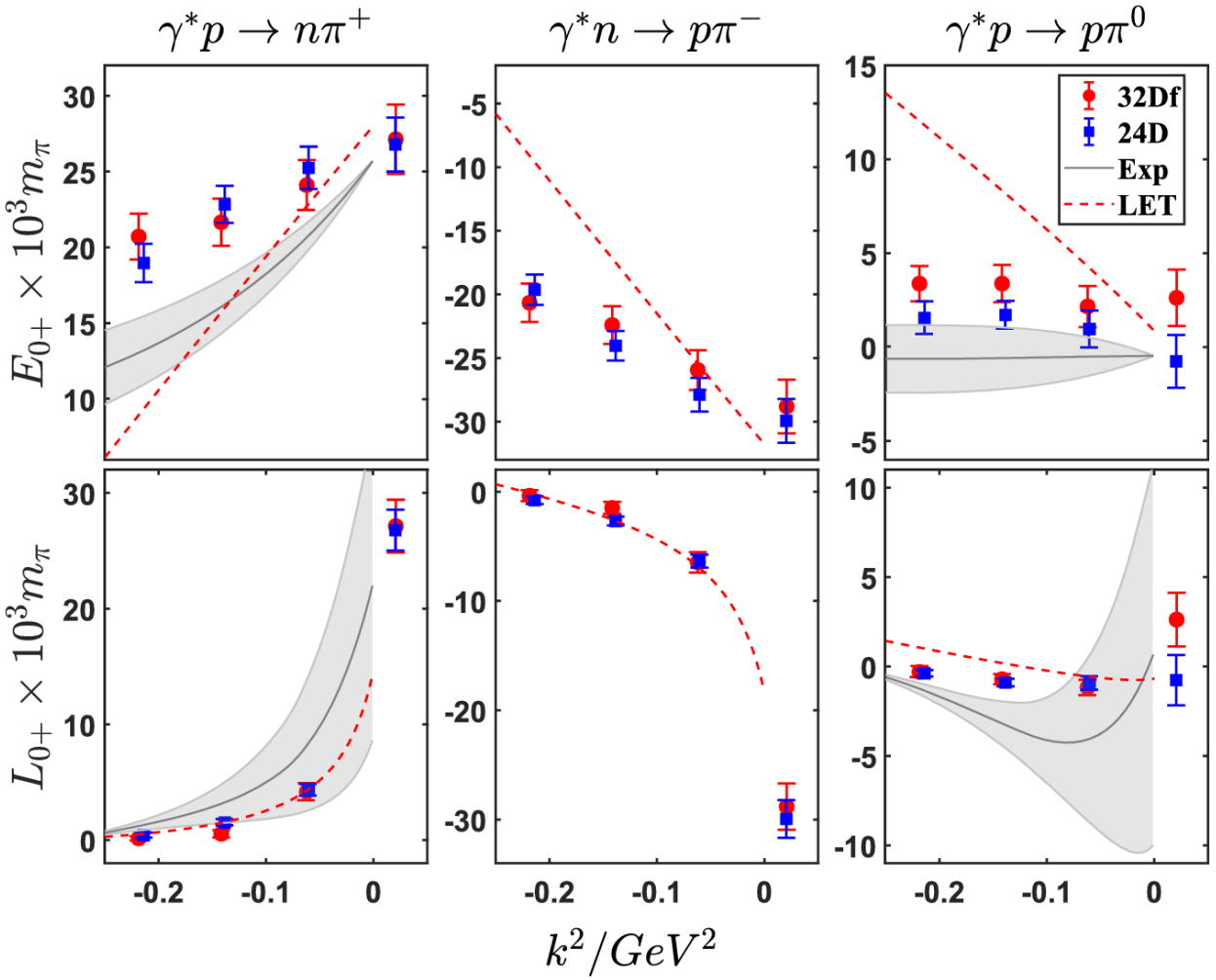
- At the πN threshold and near the physical pion mass
  - The lattice results (data points) are close to those from the Jülich-Bonn-Washington collaboration (solid lines)



Y.-S. Gao, Z.-L. Zhang, X. Feng, L.-C. Jin, C. Liu, and U.-G. Meißner, Lattice QCD Study of Pion Electroproduction and Weak Production from a Nucleon, Phys. Rev. Lett. 134, 171904 (2025)

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The relationship between finite-volume and infinite-volume states is given by

$$|(N\pi)^{I,I_z}\rangle = (2\mu_{N\pi})^{\frac{1}{2}} f_{LL}^{\frac{1}{2}} |(N\pi)^{I,I_z}, \Gamma\rangle_V, \qquad \text{(S 22)}$$

where the Lellouch-Lüscher factor f_{LL} is defined as [57]

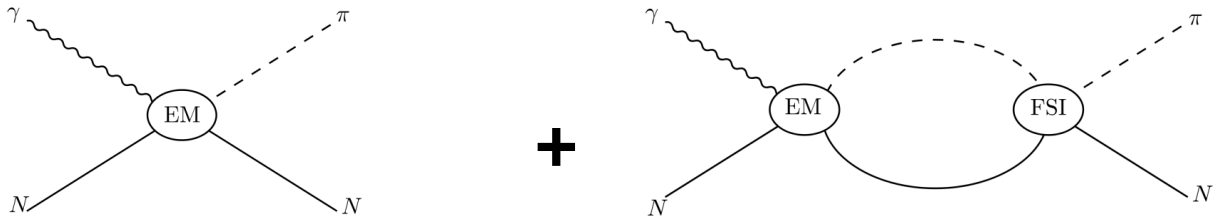
$$f_{LL} = \frac{2\pi}{k^3}\left(q\frac{d\phi^\Gamma}{dq} + k\frac{d\delta}{dk}\right). \qquad \text{(S 23)}$$

Y.-S. Gao et al., Phys. Rev. Lett. 134, 171904 (2025)

Pion electroproduction in infinite volume

- virtual photon

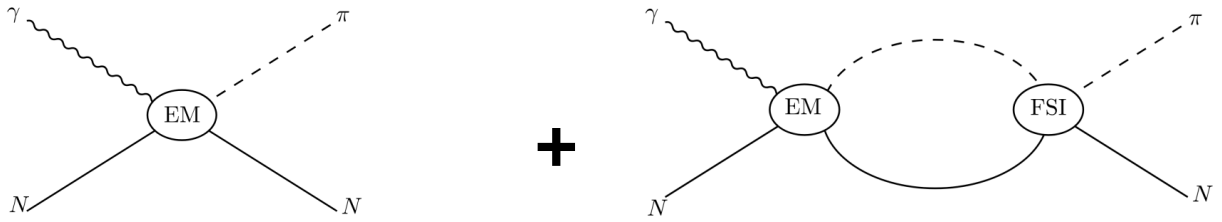
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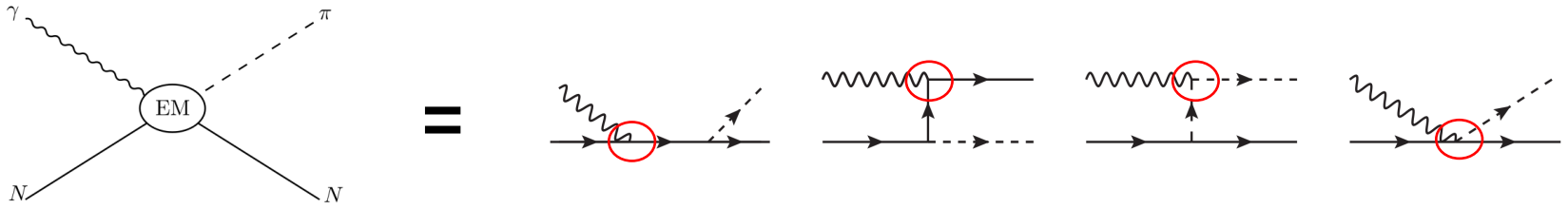
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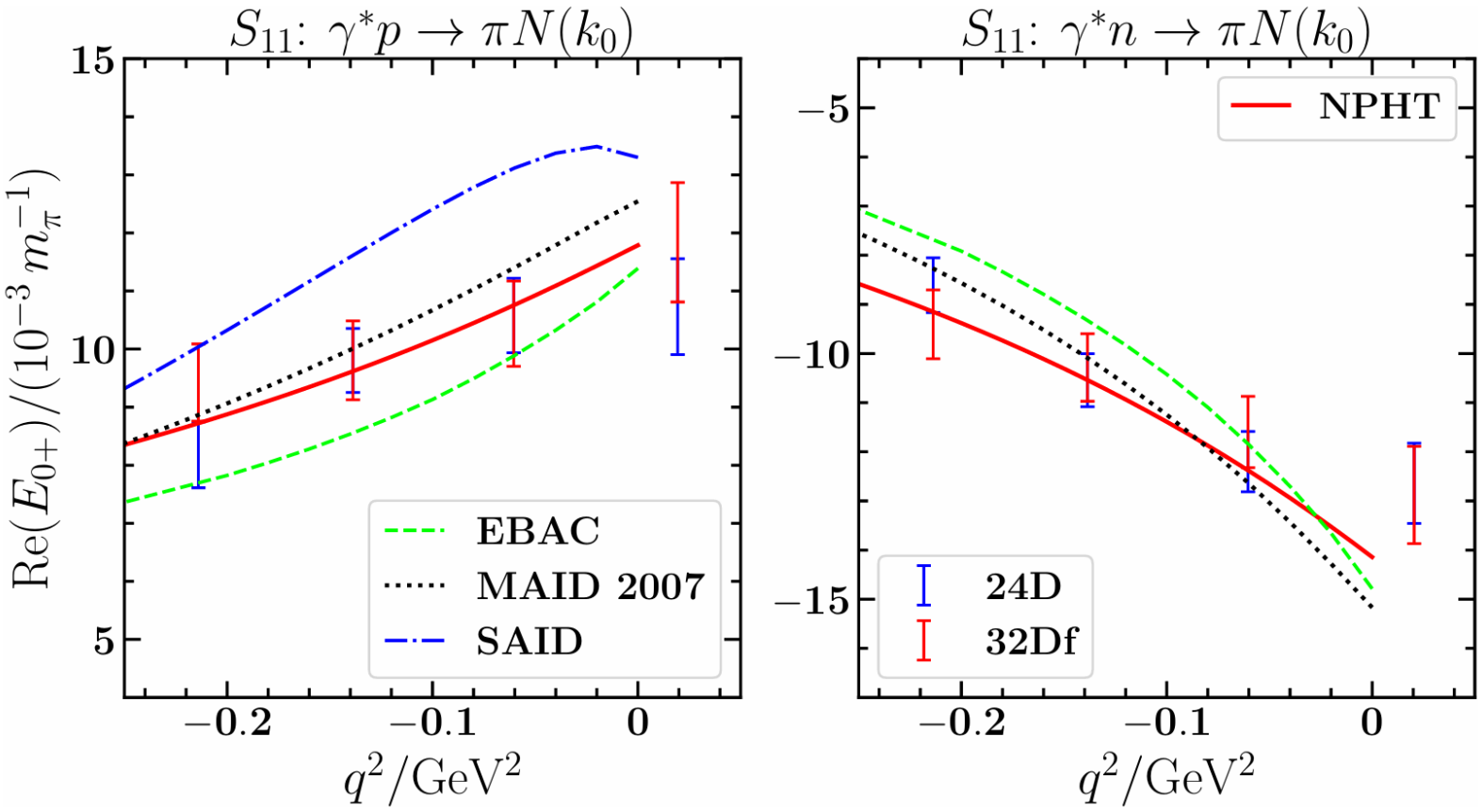
- q^2 dependence: spacelike with $q^2 < 0$



- Include the electromagnetic form factors $F(q^2)$ of the nucleon and pion
- These form factors are well constrained by experimental data

Comparison with the lattice QCD results

- The multipole amplitudes E_{0+} and L_{0+}

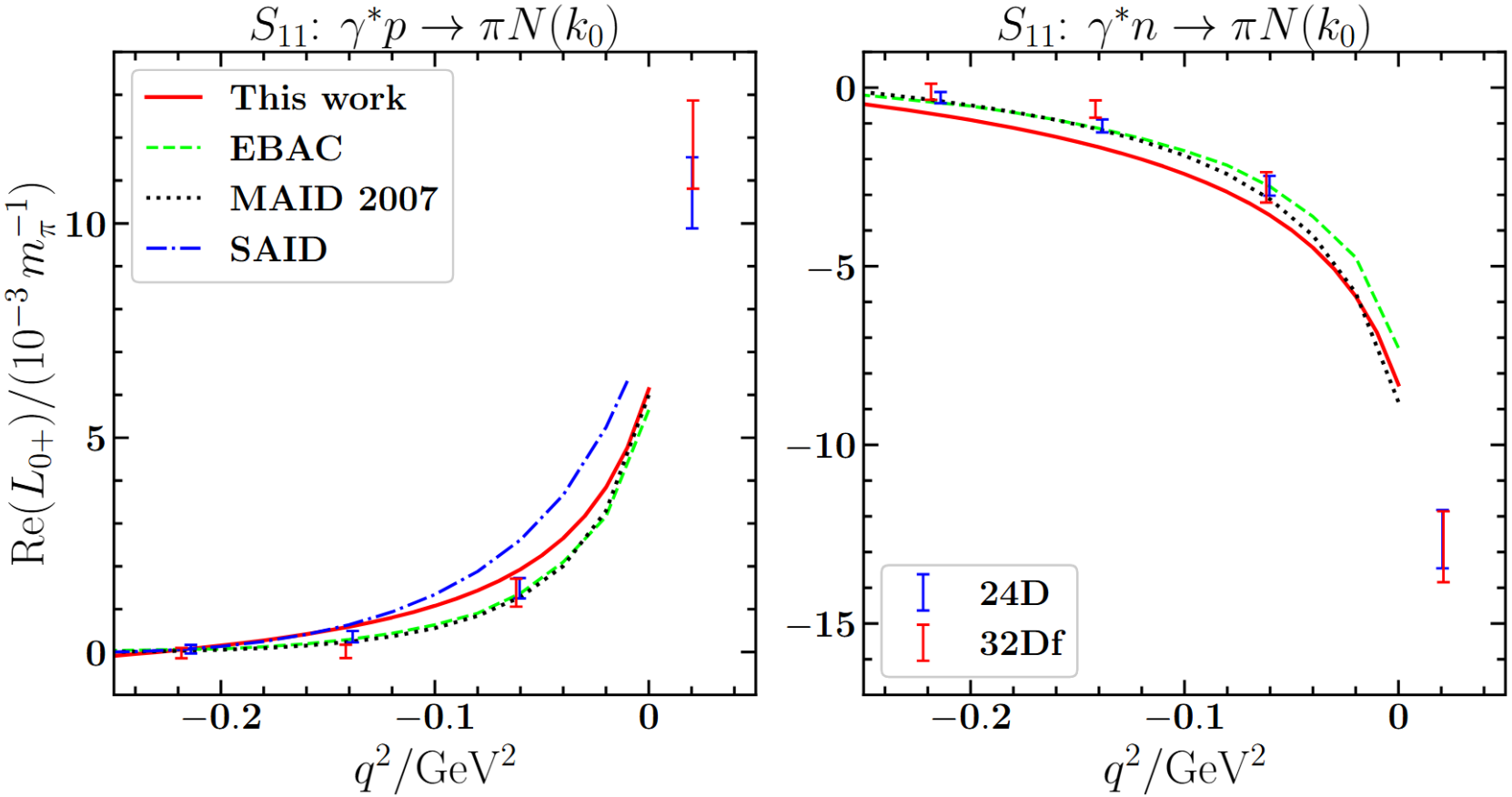


- Our results (red solid lines) describe the lattice data well and are consistent with results from other partial-wave analyses

Y. Zhuge, Z.-W. Liu, D. B. Leinweber, A. W. Thomas, arXiv: 2603.06055

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C. D. Abell et al., Phys. Rev. D 108, 094519 (2023)

Y. Zhuge, Z.-W. Liu, D. B. Leinweber, A. W. Thomas, arXiv: 2603.06055

Pion electroproduction in finite volume

- Infinite volume

$$T^{\lambda_{\gamma^*},\lambda_N}_{\pi N,\gamma^* N}(k,q;E_{\rm cm}) = V^{JLS;\lambda_{\gamma^*},\lambda_N}_{\pi N,\gamma^* N}(k,q) + \sum_\alpha \int {\rm d}k' k'^2 V^{JLS;\lambda_{\gamma^*},\lambda_N}_{\alpha,\gamma^* N}(k',q) \frac{1}{E_{\rm cm} - \omega_\alpha(k') + i\epsilon} T_{\pi N,\alpha}(k,k';E_{\rm cm})$$

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  - The finite-volume amplitude is a sum of transition amplitudes from the initial $\gamma^* N$ state to these basis states

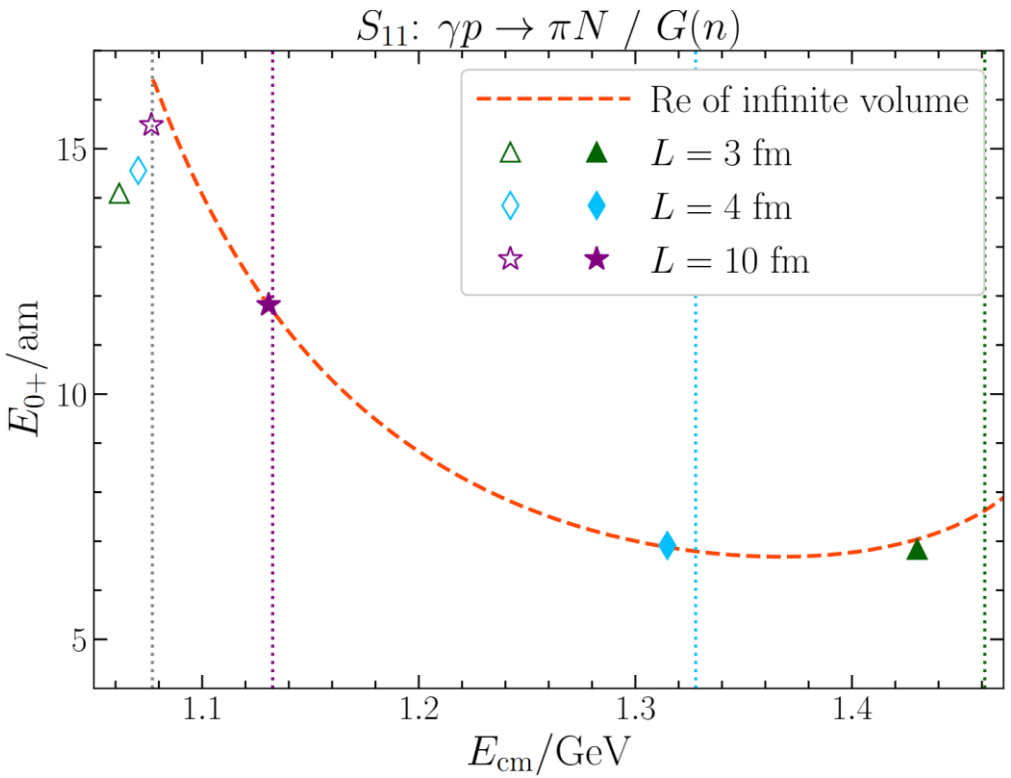
$$E^L_{0+} \sim \sum_i c_i\, G_{N_i,\gamma^* N} + \sum_{\alpha,n_i} c^{(i)}_{n_i}\, V_{\alpha(n_i),\gamma^* N}$$

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Finite-volume effects

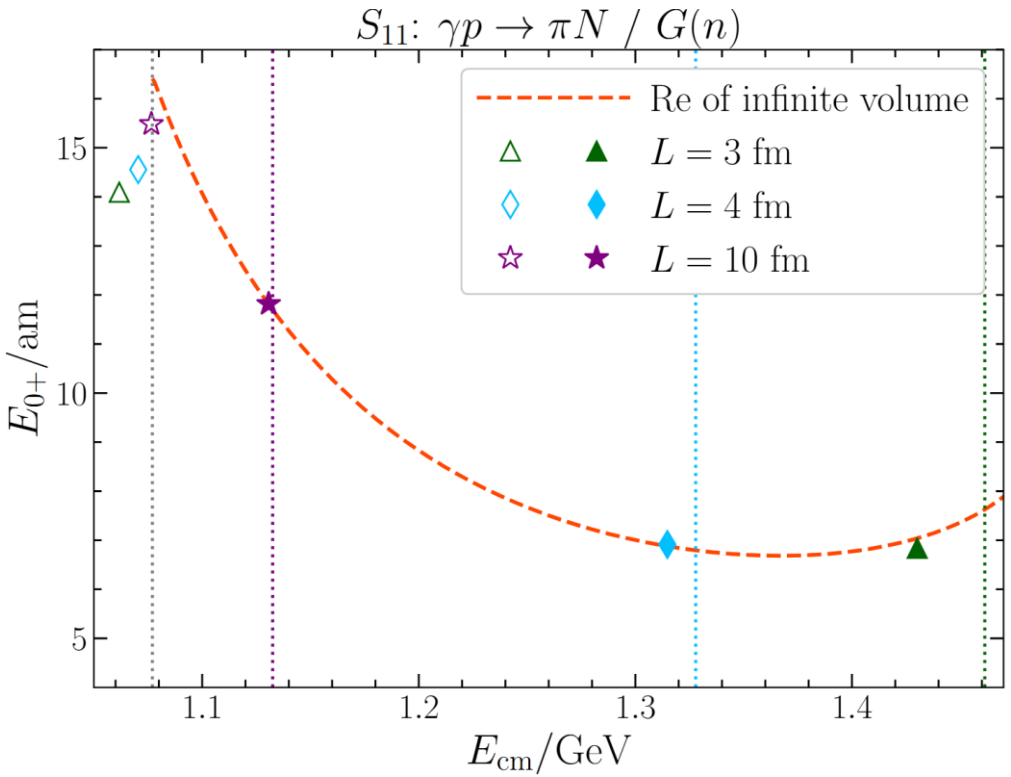
- Finite-volume results at different spatial extents L for $G(0)$ (open) and $G(1)$ (filled)



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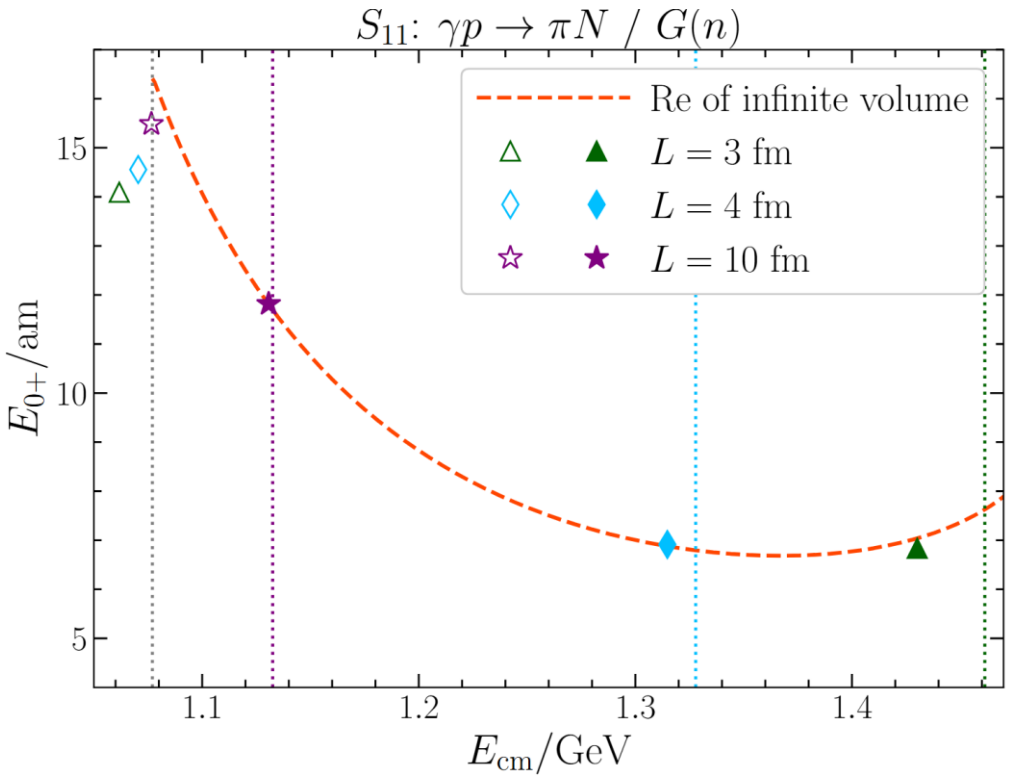


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- As L increases, E_{0+}^{L} approaches the real part of the infinite-volume amplitude E_{0+}
- Finite-volume effects are much smaller for the first excited state $G(1)$

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Lellouch-Lüscher and the complex correction factor

- The Lellouch–Lüscher formula gives the magnitude of the ratio E_{0+}/E_{0+}^{L}

$$F_{\mathrm{LL}} = \left| \frac{E_{0+}|_{E_{G(n)}}}{E_{0+}^{L}|_{E_{G(n)}}} \right| = \sqrt{\frac{2\pi\, C_3(n)}{(kL)^3} \left(\tilde{k} \frac{d\phi}{d\tilde{k}} + k \frac{d\delta}{dk} \right)}$$

Lellouch and M. Lüscher, Weak transition matrix elements from finite volume correlation functions, Commun. Math. Phys. 219, 31 (2001)

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  - F_{sep} depends only on the final-state interactions

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  - E_{0+}/E_{0+}^{L} for the first excited state $G(1)$

L/fm	ReF_{HEFT}	ReF_{sep}	ImF_{HEFT}	ImF_{sep}	$\lvert F_{\mathrm{HEFT}} \rvert$	F_{LL}
3	1.029	1.028	0.376	0.375	1.096	1.095
4	0.995	0.990	0.222	0.221	1.019	1.016
10	0.999	0.999	0.122	0.122	1.007	1.006

- F_{sep} can provide both real and imaginary parts
- F_{sep} agrees with the HEFT results

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Summary

- We extend HEFT to pion photo- and electroproduction
- Our photoproduction analysis supports the HEFT picture of the bare-state contributions to $N^*(1535), N^*(1650)$ and $N^*(1440)$
- Our calculated finite-volume multipole amplitudes describe the recent lattice QCD results
- We propose an extension of the Lellouch–Lüscher formalism that provides access to both the real and imaginary parts of physical amplitudes

Thank you for your attention!

