Pole-Expansion of Two-Particle Imaginary-Time Correlation Function in the Infinite Volume and Its Finite-Volume Correction

Fundamental Problems in Subatomic Physics 2026, Cairns, Sept. 23, 2026

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Previously on our collaboration,

- New method to extract information of near-threshold resonances: Uniformized Mittag-Leffler expansion of Green's function and t matrix,
  W. Yamada and O. Morimatsu,
  Phys. Rev. C 102, 055201 (2020).
- Application of the uniformized Mittag-Leffler expansion to $\Lambda(1405)$,
  W. Yamada and O. Morimatsu,
  Phys. Rev. C 103, 045201 (2021).
- Analytic Map of Three-Channel s Matrix:
  Generalized Uniformization and Mittag-Leffler Expansion,
  W. Yamada, O. Morimatsu and T. Sato,
  Phys. Rev. Lett. 129, 192001 456 (2022).
- Near-threshold spectrum from a uniformized Mittag-Leffler expansion:
  Pole structure of the $z(3900)$,
  W. Yamada, O. Morimatsu, T. Sato and K. Yazaki,
  Phys. Rev. D 105, 014034 (2022)
- Survival probability of unstable states in coupled-channels:
  Nonexponential decay of threshold cusp,
  W. Yamada, O. Morimatsu, T. Sato and K. Yazaki,
  Phys. Rev. D 108, L071502 (2023)
- Pole-expansion of two-hadron imaginary-time correlation function:
  A new analysis method for unstable states in lattice QCD,
  W. Yamada, O. Morimatsu, T. Sato and K. Yazaki,
  Phys. Rev. D 112, 034040 (2025)

In these papers we have proposed and developed the pole expansion (uniformized Mittag-Leffler expansion) of correlation functions and scattering amplitudes as a model-independent method to extract information of hadrons, in particular unstable states near threshold from the results of experiment or lattice QCD simulation.

In this talk we discuss the pole expansion of the two-hadron imaginary-time correlation functions in infinite volume and also finite-volume effects.

Two Key Ideas

- Uniformization
- Mittag-Leffler expansion

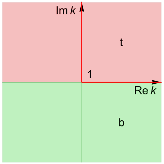
Uniformization

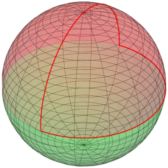
Newton "Scattering Theory of Waves and Particles"

- For the purpose of parametrization, or of the construction of phenomenological models, it is very convenient to introduce another variable, in place of the energy, in terms of which the S matrix is single-valued. Such a procedure is called *uniformization*. It allows us, so to speak, to "open up" the Riemann surface by mapping it on the complex plane.
- The Riemann surface for a three-channel S matrix cannot be topologically mapped onto a plane (or a sphere), but it can be mapped on a torus. This is expressed by saying that its genus is equal to 1.

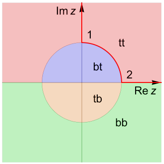
Uniformization of 1 and 2 Channels

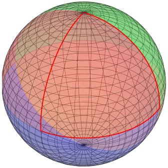
- single-channel
  $(\varepsilon = m + m')$
  trivial
  $$u = k$$
  where
  $$k = \frac{1}{2}(p_0^2 - \boldsymbol{p}^2 - \varepsilon^2)^{1/2}$$





- two-channel
  $(\varepsilon_1 = m_1 + m_1' < \varepsilon_2 = m_2 + m_2')$
  M. Kato (1965)
  $$u = z = \frac{k_1 + k_2}{\Delta}$$
  where
  $$k_i = \frac{1}{2}(p_0^2 - \boldsymbol{p}^2 - \varepsilon_i^2)^{1/2}$$
  $$\Delta = \frac{1}{2}\sqrt{\varepsilon_2^2 - \varepsilon_1^2}$$





Uniformization of 3 Channels

- three-channel
  $(\varepsilon_1 = m_1 + m_1' < \varepsilon_2 = m_2 + m_2 < \varepsilon_3 = m_3 + m_3')$
  W. Yamada, O. Morimatsu and T. Sato (2022)

$$\begin{cases} z_{12} = \dfrac{k_1 + k_2}{\Delta_{12}} \\ z = \dfrac{1}{4K(1/\gamma^2)} \mathrm{sn}^{-1}(\gamma/z_{12}, 1/\gamma^2) \end{cases}$$

where

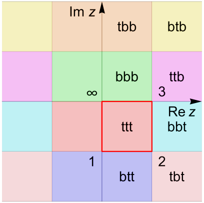
$$k_i = \frac{1}{2}(p_0^2 - \boldsymbol{p}^2 - \varepsilon_i^2)^{1/2}$$

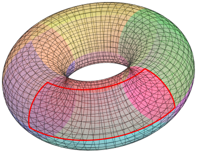
$$\Delta_{ij} = \sqrt{\varepsilon_j^2 - \varepsilon_i^2}$$

$$\gamma = \frac{\Delta_{13} + \Delta_{23}}{\Delta_{12}}$$

and

$$K(t) = \int_0^1 \frac{ds}{\sqrt{(1 - s^2)(1 - t^2 s^2)}}$$

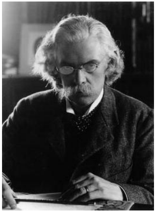




Mittag-Leffler Theorem (From Wikipedia)

- In complex analysis, Mittag-Leffler's theorem concerns the existence of meromorphic functions with prescribed poles.

  Conversely, it can be used to express any meromorphic function as a sum of partial fractions. It is sister to the Weierstrass factorization theorem, which asserts existence of holomorphic functions with prescribed zeros. It is named after Gösta Mittag-Leffler.



Gösta Mittag-Leffler
(16 March 1846 - 7 July 1927)
Swedish mathematician

- Pole expansion of meromorphic functions

  The Mittag-Leffler expansion of a meromorphic function, $\mathcal{A}(z)$, is given by

$$\begin{aligned}\mathcal{A}(z) &= \sum_i \frac{r_i}{z - z_i} + \text{subtraction terms} \\ &= z^n \sum_i \frac{r_i}{(z - z_i)\, z_i^n} + \sum_{k=0}^{n-1} \frac{\mathcal{A}^{(k)}(0)}{k!} z^k\end{aligned}$$

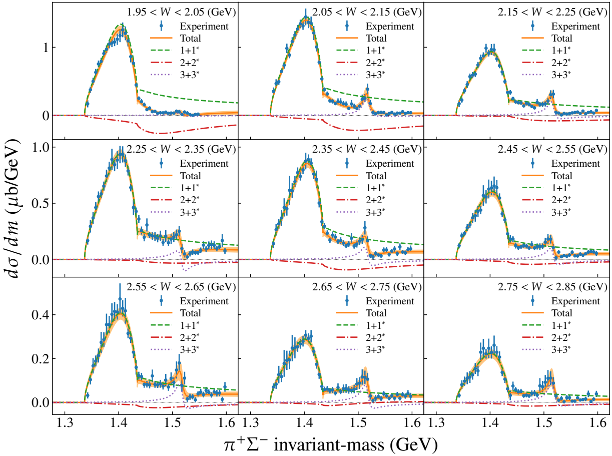
  where $\{z_i\}$ and $\{r_i\}$ are positions and residues of the poles of $\mathcal{A}(z)$.

Pole Expansion of T-matrix (Example of Fii)

Application of the uniformized Mittag-Leffler expansion to $\Lambda(1405)$, W. Yamada and O. Morimatsu, Phys. Rev. C 103, 045201 (2021).

$$T^{ij}(p_0) = \sum_n \left(\frac{t_n^{ij}}{u(p_0) - u_n} - \frac{t_n^{ji*}}{u(p_0) + u_n^*} \right)$$

example of fit by (up to) three pairs of poles



Pole-Expansion of Imaginary-Time Correlation Function

- imaginary-time correlation function

$$C^{ij}(\tau) = \langle 0|\, \mathcal{O}^i(\tau)\mathcal{O}^{j\dagger}(0)\, |0\rangle = \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} e^{ip_4\tau} D_E^{ij}(p_4) = \int_{\varepsilon_{min}}^{\infty} \frac{dp_0}{2\pi i} e^{-p_0\tau} \mathrm{Disc} D^{ij}(p_0)$$

$\mathcal{O}^i$ and $\mathcal{O}^j$: interpolation operators, e.g. $\rho, \pi\pi, D_E^{ij}(p_4) = -D^{ij}(-ip_4)$

$$D^{ij}(u) = \sum_n \left(\frac{r_n^{ij}}{u - u_n} - \frac{r_n^{ji*}}{u + u_n^*} \right) + \text{subtraction terms}$$

u: uniformization variable which makes correlation function single-valued (momentum k in the case of single-channel)

$$C^{ij}(\tau) = \sum_n \left(r_n^{ij} C(\tau, u_n) - r_n^{ji*} C(\tau, -u_n^*) \right)$$

$$C(\tau, u_n) = \int_{\varepsilon_{min}}^{\infty} \frac{dp_0}{2\pi i} e^{-p_0\tau} \mathrm{Disc} \frac{1}{u(p_0) - u_n}$$

We can determine pole positions and residues by fitting the r.h.s. to the lattice results (l.h.s.).

Our Method and Previous Methods

- Our new method

  Imaginary-time correlation function in infinite V $\rightarrow$ mass, width, etc.

$$C^{ij}(\tau) = \sum_n r_n^{ij} C(\tau, u_n)$$

  finite volume effect?

  previous methods (taken from Aoki-san's slides)

- Lüscher's method

  Energy spectrum in finite V $\rightarrow$ phase shift by Lüscher's formula
  $\rightarrow$ mass, width, etc.

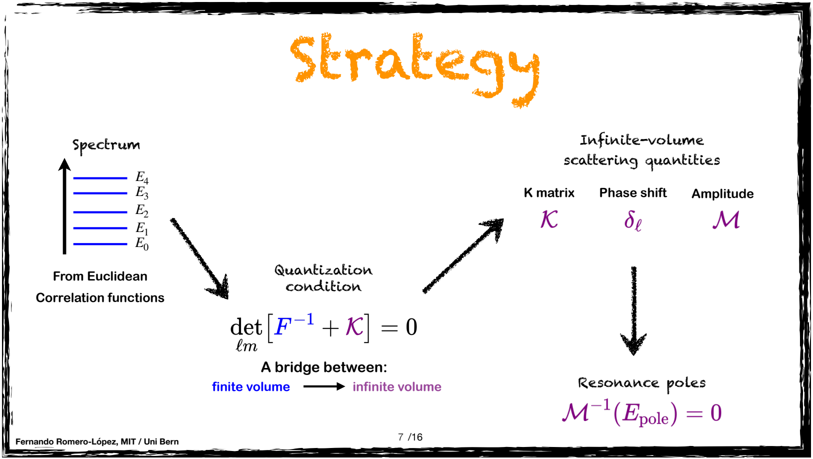
$$\Delta E = 2\sqrt{m^2 + k^2} - 2m$$

- HAL QCD method

  BS wave function $\leftrightarrow$ E-indep. nonlocal "potential"
  $\rightarrow$ phase shifts by solving Schrödinger eq. in infinite V
  $\rightarrow$ mass, width, etc.

$$\left(-\frac{\partial}{\partial t} + \frac{1}{4m}\frac{\partial^2}{\partial t^2} - H_0\right) R(\boldsymbol{r}, t) = \int d\boldsymbol{r}' U(\boldsymbol{r}, \boldsymbol{r}') R(\boldsymbol{r}', t)$$

“Standard” Method

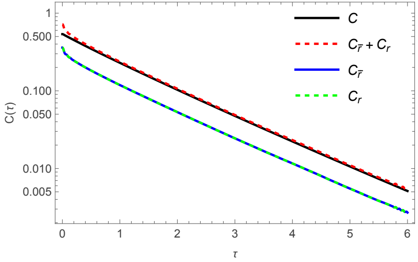


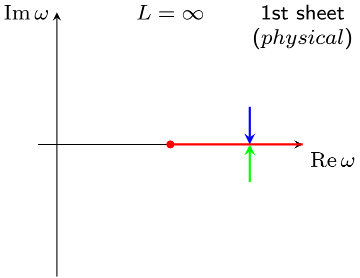
from the talk "Two-pole of the $\Lambda(1405)$ from lattice QCD"
by Fernando Romero-Lopez@J-PARC Hadron 2024

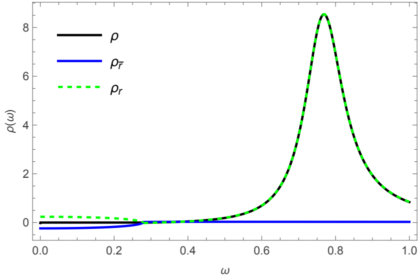
Demonstration with Simple Models

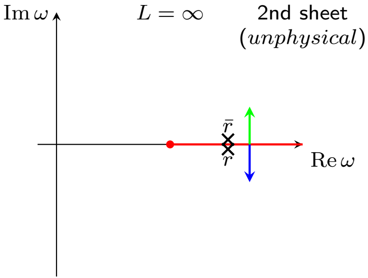
Single Channel

Resonance ρ in $\pi\pi$ Channel (VDM)

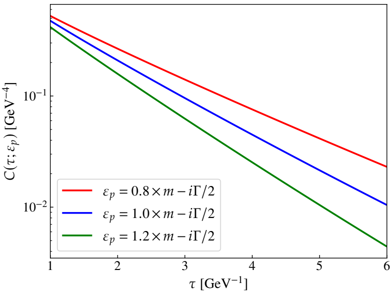


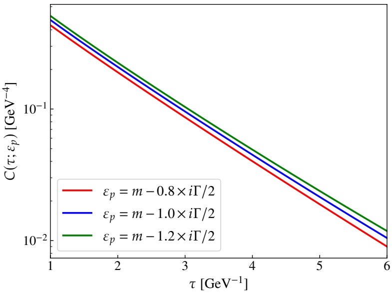




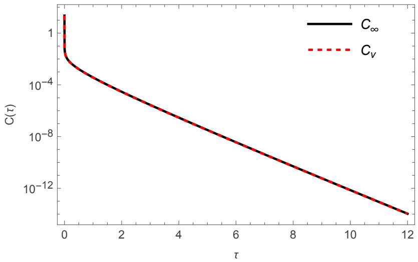


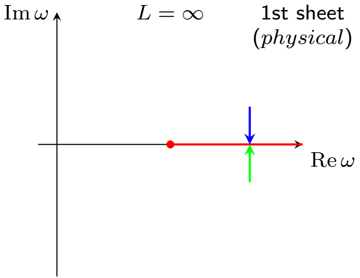
Mass and Width Dependence of C

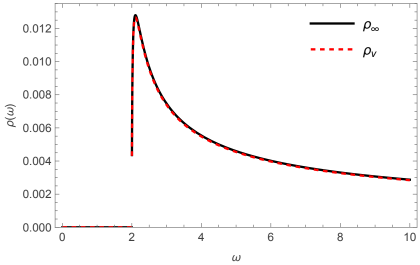


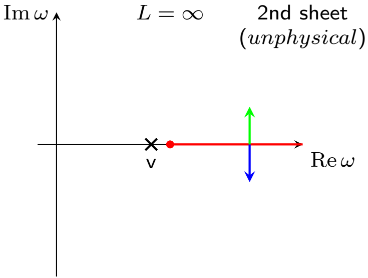


Virtual State in NN Channel (EFT)









Coupled-Channels
$\Lambda(1405)$ in $\bar{K}N$-$\pi\Sigma$ channels

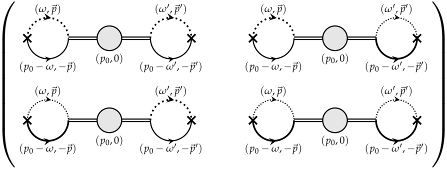
2 ($\bar{K}N$-$\pi\Sigma$) channel ($\Lambda(1405)$)

$$\mathcal{O}^{\bar{K}N}(t) = \frac{1}{\sqrt{2}}\left(K^-(\boldsymbol{q},\tau)p(-\boldsymbol{q},\tau) - \bar{K}^0(\boldsymbol{q},\tau)n(-\boldsymbol{q},\tau)\right)$$

$$\mathcal{O}^{\pi\Sigma}(t) = \frac{1}{\sqrt{3}}\left(\pi^-(\boldsymbol{q}',\tau)\Sigma^+(-\boldsymbol{q}',\tau) + \pi^0(\boldsymbol{q}',\tau)\Sigma^0(-\boldsymbol{q}',\tau) + \pi^+(\boldsymbol{q}',\tau)\Sigma^-(-\boldsymbol{q}',\tau)\right)$$

$$\begin{pmatrix} C^{\bar{K}N\bar{K}N}(\tau) & C^{\bar{K}N\pi\Sigma}(\tau) \\ C^{\pi\Sigma\bar{K}N}(\tau) & C^{\pi\Sigma\pi\Sigma}(\tau) \end{pmatrix}$$
$$= \begin{pmatrix} \langle 0|\, \mathcal{O}^{\bar{K}N}(\tau)\mathcal{O}^{\bar{K}N\dagger}(0)\,|0\rangle & \langle 0|\, \mathcal{O}^{\bar{K}N}(\tau)\mathcal{O}^{\pi\Sigma\dagger}(0)\,|0\rangle \\ \langle 0|\, \mathcal{O}^{\pi\Sigma}(\tau)\mathcal{O}^{\bar{K}N\dagger}(0)\,|0\rangle & \langle 0|\, \mathcal{O}^{\pi\Sigma}(\tau)\mathcal{O}^{\pi\Sigma\dagger}(0)\,|0\rangle \end{pmatrix}$$
$$= \begin{pmatrix} \int_0^\infty \frac{dp_0}{2\pi i} e^{-p_0\tau}\mathrm{Disc}D^{\bar{K}N\bar{K}N}(p_0) & \int_0^\infty \frac{dp_0}{2\pi i} e^{-p_0\tau}\mathrm{Disc}D^{\bar{K}N\pi\Sigma}(p_0) \\ \int_0^\infty \frac{dp_0}{2\pi i} e^{-p_0\tau}\mathrm{Disc}D^{\pi\Sigma\bar{K}N}(p_0) & \int_0^\infty \frac{dp_0}{2\pi i} e^{-p_0\tau}\mathrm{Disc}D^{\pi\Sigma\pi\Sigma}(p_0) \end{pmatrix}$$

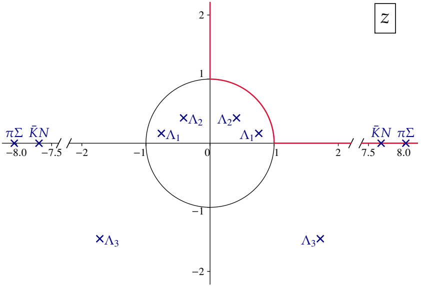
$$\begin{pmatrix} D^{\bar{K}N\bar{K}N}(p_0) & D^{\bar{K}N\pi\Sigma}(p_0) \\ D^{\pi\Sigma\bar{K}N}(p_0) & D^{\pi\Sigma\pi\Sigma}(p_0) \end{pmatrix} =$$



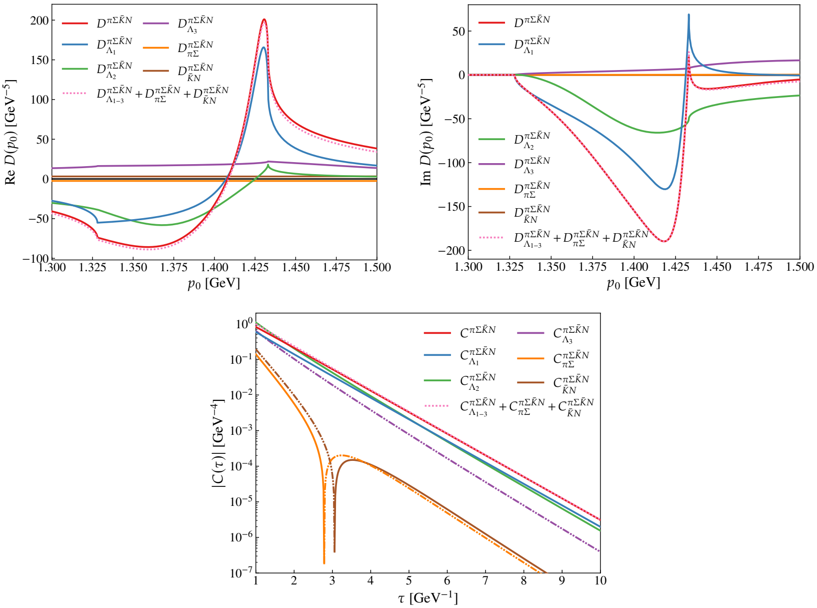
.

Pole positions of $D^{KN\pi\Sigma}$

pole	z	E [GeV]	$r^{KN\pi\Sigma}$ [GeV^{-5}]
Λ_1	$\pm 0.760 + 0.154i$	$1.435 \mp 0.010i$ [bt]	$\pm 43.40 - 3.14i$
Λ_2	$\pm 0.413 + 0.395i$	$1.386 \mp 0.072i$ [bt]	$\mp 10.27 - 24.09i$
Λ_3	$\pm 1.72 - 1.49i$	$1.396 \mp 0.131i$ [bb]	$\mp 18.11 + 17.45i$
$\pi\Sigma$	± 8.03	2.57 [bb]	$\pm 9.35 - 4.34i$
$\bar{K}N$	± 7.68	2.49 [bb]	$\mp 11.64 + 5.16i$



2 ($\bar{K}N$-$\pi\Sigma$) channel ($\Lambda(1405)$)



Finite Volume Effect

Volume Dependence of Energy Spectrum

- Volume dependence of the energy spectrum in massive quantum field theories: I. Stable Particle States,
  M. Lüscher
  Commun. Math. Phys. 104, 177-206 (1986).

$$E(\boldsymbol{p}) = \sqrt{m^2 + \boldsymbol{p}^2} + O(e^{-mL})$$

- Volume dependence of the energy spectrum in massive quantum field theories: II. Scattering States
  M. Lüscher
  Commun. Math. Phys. 105, 153-188 (1986).

$$E(\boldsymbol{p}=0) = 2m - \frac{4\pi a_0}{mL^3}\left\{1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2}\right\} + O(L^{-6})$$

$c_1 = -2.837293, \quad c_2 = 6.375183, \quad a_0$: S-wave scattering length

Two-particle states on a torus and their relation to the scattering matrix
M. Lüscher
Nucl. Phys. B354, 531-578 (1991)

$$k_s(L)\cot\delta(k_s(L)) = \frac{2\pi}{L}\pi^{-3/2}\mathcal{Z}_{00}(1, \hat{k}_s(L)^2)$$

Volume Dependence of Energy Spectrum

- Signatures of unstable particles in finite volume,
  M. Lüscher
  Nucl. Phys. B364, 237-251 (1991)
- Scattering processes and resonances from lattice QCD,
  Briceño, Raúl A., Dudek, Jozef J., Young, Ross D.
  Rev. Mod. Phys. 90, 025001 (2018)
  Resonances in the Lüscher formalism in the narrow-width approximation

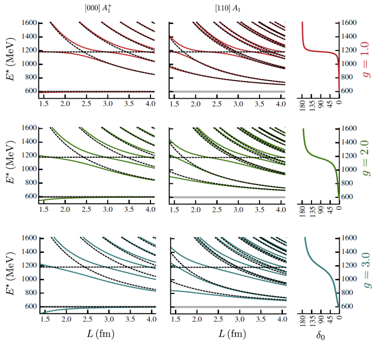
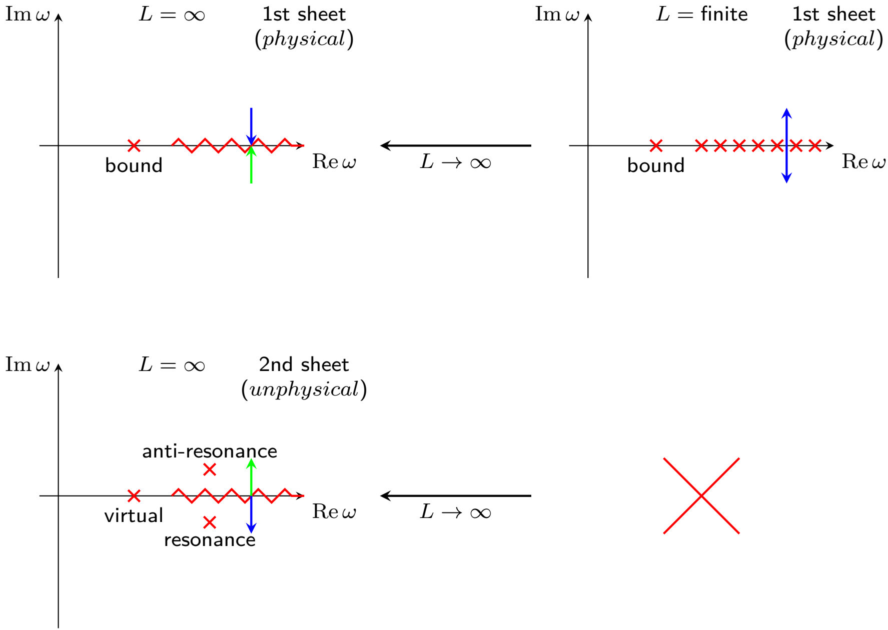
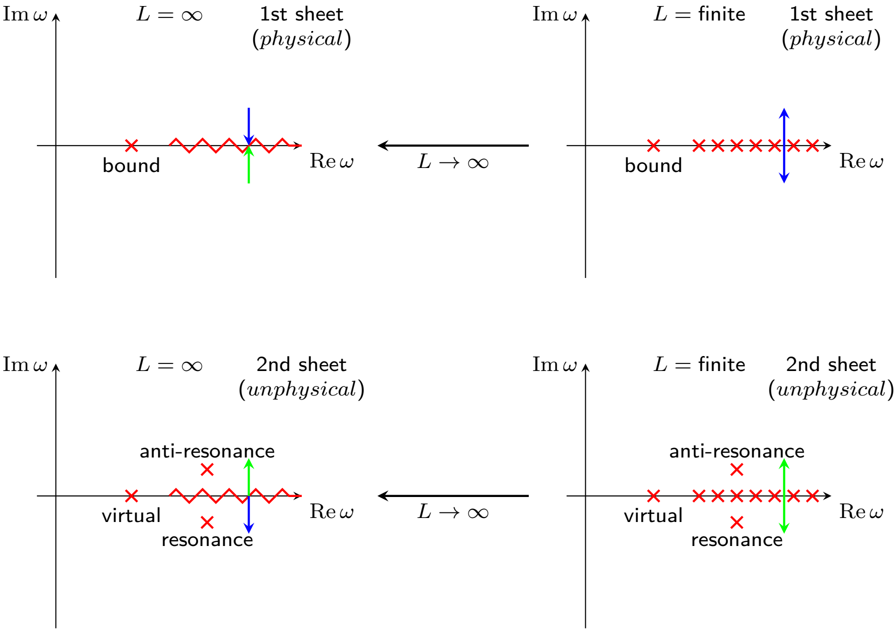


FIG. 5. Finite-volume spectrum in two irreps for a Breit-Wigner resonance with three values of decay coupling. Scattering particles have mass 300 MeV and Breit-Wigner mass is $m = 1182$ MeV. Dashed black curves show noninteracting energy levels, and the gray band at 600 MeV indicates the kinematic threshold. The rightmost panel shows the elastic phase shift in degrees.

Complex-Energy Riemann Sheet (Conventional)



Complex-Energy Riemann Sheet (Our Proposal)



Pole Expansion in Infinite Volume

- imaginary time correlation function in infinite volume

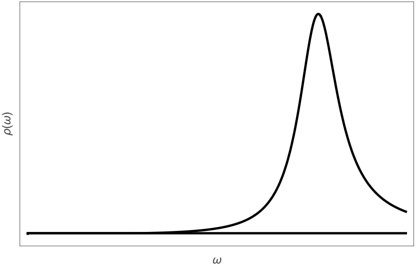
$$C(\tau) = \langle 0| \mathcal{O}(\tau)\mathcal{O}^\dagger(0) |0\rangle = -\int_C \frac{d\omega}{2\pi i} e^{-\omega\tau} D(\omega) = -\int_0^\infty \frac{d\omega}{2\pi i} e^{-\omega\tau} \mathrm{Disc} D(\omega),$$

$\mathcal{O}$ and $\mathcal{O}^\dagger$: interpolation operators, e.g. $\rho, \pi\pi, D_E(p_4) = -D(-ip_4)$

- spectral representation

$$D(\omega) = -\pi \sum_B \frac{\rho_B}{\omega - \omega_B} - \pi \int_{2m}^\infty d\omega' \frac{\rho(\omega')}{\omega - \omega'},$$

$$C(\tau) = \sum_B \rho_B e^{-\omega_B \tau} + \int d\omega' \rho(\omega') e^{-\omega' \tau}.$$



Pole Expansion in Infinite Volume

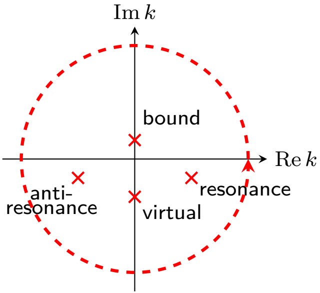
- pole expansion in uniformization variable u

$$D(u) = \sum_n \left(\frac{r_n}{u - u_n} - \frac{r_n^*}{u + u_n^*} \right)$$

$$C(\tau) = \sum_n \left(r_n C(\tau, u_n) - r_n^* C(\tau, -u_n^*) \right)$$

$$C(\tau, u_n) = \int_{\varepsilon_{min}}^{\infty} \frac{dp_0}{2\pi i} e^{-p_0 \tau} \mathrm{Disc} \frac{1}{u(p_0) - u_n}$$

u: uniformization variable (momentum k in the case of single-channel)



Pole Expansion in Finite Volume

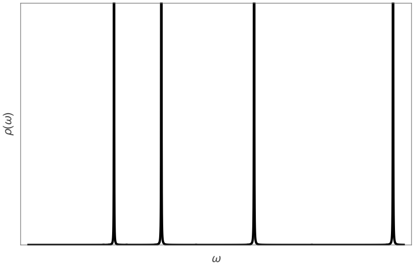
- imaginary time correlation function in finite volume

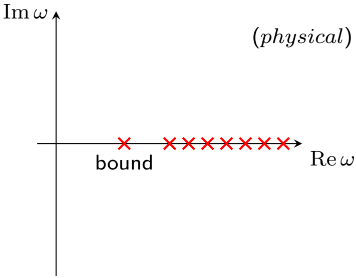
$$C(\tau, L) = \langle 0| \mathcal{O}(\tau)\mathcal{O}^\dagger(0) |0\rangle_L = -\int_C \frac{d\omega}{2\pi i} e^{-\omega\tau} D(\omega, L) = -\int_0^\infty \frac{d\omega}{2\pi i} e^{-\omega\tau} \mathrm{Disc} D(\omega, L)$$

- spectral representation (=pole expansion in energy ω)

$$D(\omega, L) = -\pi \sum_n \frac{\rho(n, L)}{\omega - \omega(n, L)}$$

$$C(\tau, L) = \sum_n \rho(n, L) e^{-\omega(n,L)\tau}$$





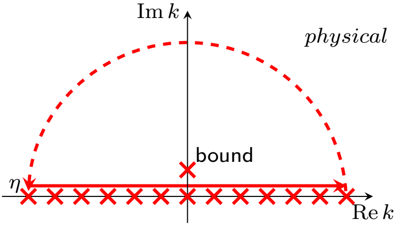
Pole Expansion in Finite Volume

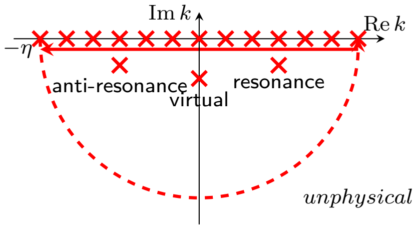
- "pole expansion" in uniformization variable k

$$D(k,L) = \sum_n \left(\frac{r_n(L)}{k - k_n(L)} - \frac{r_n(L)^*}{k + k_n(L)^*} \right) + O(e^{-dL}).$$

where d is the minimum of $\mathrm{Im}\, k$, $\mathrm{Im}\, k_1(L), \cdots$

$$\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \, \frac{D(k'+i\eta, L) - D(k'-i\eta, L)}{k'-k} = O(e^{-dL}) \right)$$





Demonstration with Simple Models (EFT)

$$\mathcal{L} = \psi^\dagger \left(i\frac{\partial}{\partial t} - m + \frac{1}{2m}\boldsymbol{\nabla}^2 \right) \psi - \frac{1}{4} V \psi^\dagger \psi^\dagger \psi \psi$$

$$\mathcal{O}(\tau) = \frac{1}{2L^3} \int d^3x \psi(\tau, \boldsymbol{x}) \psi(\tau, \boldsymbol{x}), \qquad \mathcal{O}^\dagger(0) = \frac{1}{2L^3} \int d^3x \psi^\dagger(0, \boldsymbol{x}) \psi^\dagger(0, \boldsymbol{x})$$

D $\quad G \quad$ $G \quad V \quad G$

- infinite volume

$$D_{1,2}(\omega) = \frac{G_{1,2}}{1 - VG_{1,2}(\omega)}$$

$$G_{1,2}(\omega) = \pm \int^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{\omega - 2m - q^2/m}$$
$$= \pm \frac{M\sqrt{-M\omega}}{4\pi} = -i\frac{Mk}{4\pi}$$

- finite volume

$$D_{1,2}(\omega, L) = \frac{G_{1,2}(\omega, L)}{1 - VG_{1,2}(\omega, L)}$$

$$G_{1,2}(\omega, L) = \pm \frac{1}{L^3} \sum_{\boldsymbol{n}}^{\Lambda} \frac{1}{\omega - 2m - q_{\boldsymbol{n}}^2/m}$$

1 (2) refers to physical (unphysical) sheet

Volume Dependence of Poles

pole energy $\omega(L) = 2m + k(L)^2/m$

$$1 - VG_{1,2}(\omega(L), L) = 0$$

summation formula

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^3}^{\Lambda} \frac{1}{\boldsymbol{n}^2 - \hat{k}^2} = \begin{cases} 2\pi^{1/2}\mathcal{Z}_{00}(1, \hat{k}^2) & (\mathrm{Im}\,\hat{k} = 0) \\ 4\pi\hat{k}^2 \int_0^\infty \frac{d\chi}{\chi^2 - \hat{k}^2} + O(e^{-2\pi L|\mathrm{Im}\hat{k}|}) & (\mathrm{Im}\,\hat{k} \neq 0) \end{cases}$$

- "scattering" state (physical sheet) *known*

$$k(L) = k_s(L) \longrightarrow k_s(L)\cot\delta(k_s(L)) = \frac{2\pi}{L}\pi^{-3/2}\mathcal{Z}_{00}(1, \hat{k}_s(L)^2)$$

- bound state (physical sheet) *known*

$$k(L) = i\kappa_b(L) \longrightarrow \omega_b(L) = \omega_b + O(e^{-\kappa_b L})$$

- virtual state (unphysical sheet) *unknown*

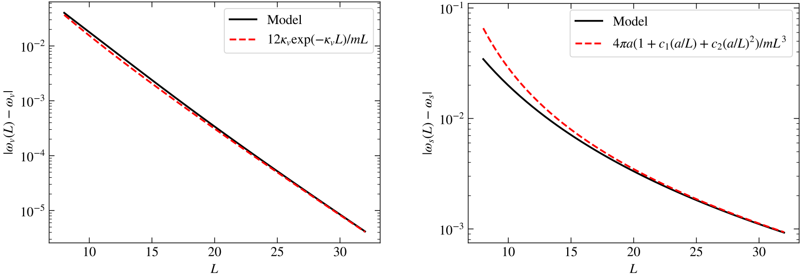
$$k(L) = -i\kappa_v(L) \longrightarrow \omega_v(L) = \omega_v + O(e^{-\kappa_v L})$$

- resonance and anti-resonance (unphysical sheet) *unknown*

$$k(L) = \pm k_r(L) - i\kappa_r(L) \longrightarrow \omega_r(L) = \omega_r + O(e^{-\kappa_r L})$$

Volume Dependence of Poles

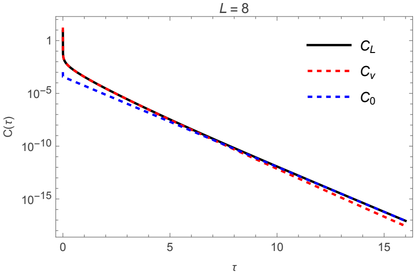
numerical results $\longrightarrow$ exponential volume dependence

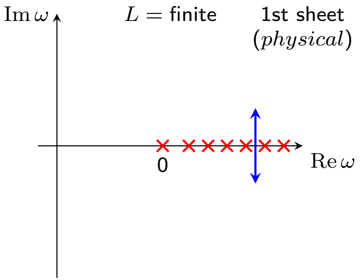


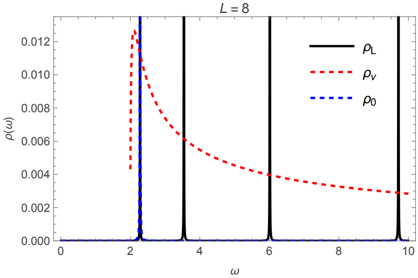
Volume dependece of pole energies for the virtual state (left) and the lowest "scattering" state (right), where the asymptotic formulas for large volume, $12\kappa_v \exp(-\kappa_v L)/mL$, for the virtual pole and the Lüscher formula for the lowest "scattering" state are also shown for comparison.

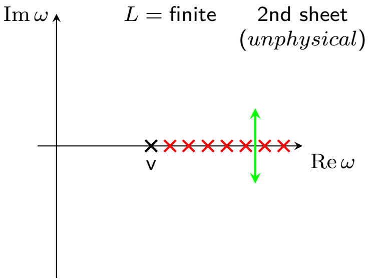
Two Domains of $C(\tau, L)$

virtual state in NN channel (EFT)









Two Domains of $C(\tau, L)$

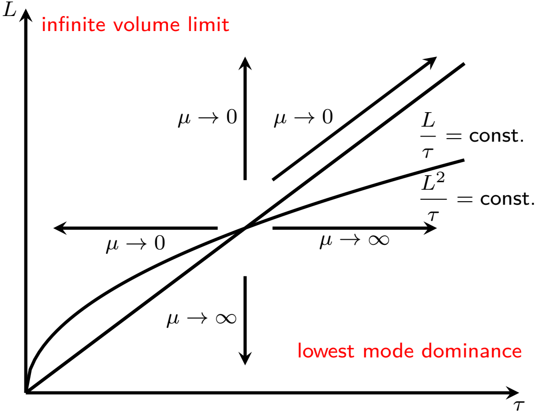
dimensionless parameter $\mu = \dfrac{\tau}{mL^2} \sim \Delta\omega\,\tau \qquad \left(\Delta\omega \sim \dfrac{1}{mL^2}\right)$

- $\mu \gg 1$ lowest-mode-dominance domain

$$C(\tau, L) = \rho(0, L)e^{-\omega(0,L)\tau}\left(1 + O(e^{-4\pi^2\mu})\right) \to \rho(0, L)e^{-\omega(0,L)\tau}$$

- $\mu \ll 1$ infinite-volume-limit domain

$$C(\tau, L) = C(\tau)\left(1 + O(e^{-\frac{1}{4\mu} - 2m\tau})\right) \to C(\tau)$$



Finite Volume Correction, $\Delta C(\omega, L)$ ($\mu \ll 1$)

- general qualitative discussion

$$\Delta C(\tau, L) = -\int_C \frac{d\omega}{2\pi i} e^{-\omega\tau} \Delta D(\omega, L).$$

$$e^{-\omega\tau}\Delta D(\omega, L) \sim e^{-\left(2m+\frac{k^2}{m}\right)\tau} e^{ikL} \sim e^{-\frac{\tau}{m}\left(k-i\frac{mL}{2\tau}\right)^2 - \frac{mL^2}{4\tau} - 2m\tau}$$

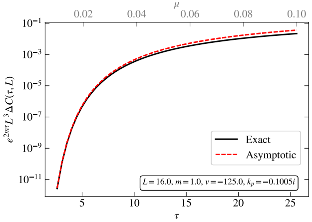
$$\begin{aligned}\Delta C(\tau, L) &= -\int \frac{d\omega}{2\pi i} e^{-\omega\tau} \Delta D(\omega, L) \\ &\sim \int kdke^{-\frac{\tau}{m}\left(k-i\frac{mL}{2\tau}\right)^2 - \frac{mL^2}{4\tau} - 2m\tau} \\ &\sim e^{-\frac{1}{4\mu} - 2m\tau}\end{aligned}$$

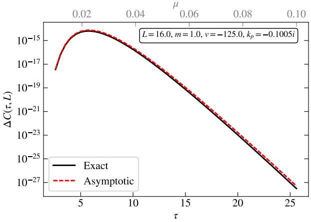
$$C(\tau, L) = C(\tau)\left(1 + O(e^{-\frac{1}{4\mu} - 2m\tau})\right) \to C(\tau).$$

- concrete quantitative results with EFT

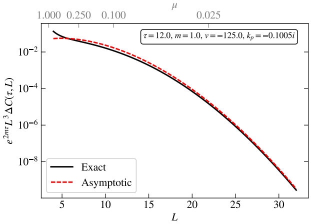
$$\Delta C(\omega, L) \to -\frac{3\kappa_p^2\sqrt{\tau}}{\pi^{3/2}\sqrt{m}L^2} e^{-mL^2/4\tau - 2m\tau} \quad \text{asymptotic form}$$

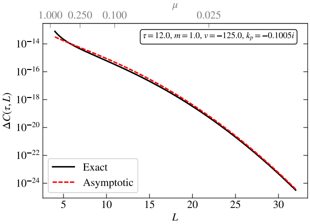
$\Delta C(\omega, L)$ Exact and Asymptotic Form ($\mu \ll 1$)





$e^{2m\tau}L^3\Delta C(\omega, L)$ (left) and $\Delta C(\omega, L)$ (right) vs. τ with L fixed





$e^{2m\tau}L^3\Delta C(\omega, L)$ (left) and $\Delta C(\omega, L)$ (right) vs. L with τ fixed

When $\mu \ll 1$, $\Delta C(\omega, L)$ is small and is well approximated by the asymptotic form.

Summary

- The pole expansion of the imaginary-time correlation function is valid not only for single-channel but also for coupled-channel (two-channel) scatterings.
- The imaginary-time correlation function can be parametrized by relatively few number of pole positions and residues.
- We define the unphysical sheet (the second sheet in the single-channel case) of the correlation function (or the S-matrix) and identify unstable states as poles in the unphysical sheet.
- The finite size effect of unstable states is exponentially small, $\exp(-dL)$, where d is the distance of the unstable state pole the cut of physical scattering states in the complex energy plane.
- The imaginary-time correlation function of a finite volume approaches that of the infinite volume as $\mu \to 0$

  $$C(\tau, L) = C(\tau)\left(1 + O(e^{-\frac{1}{4\mu} - 2m\tau})\right) \to C(\tau),$$

  while it is dominated by the lowest energy mode as $\mu \to \infty$, where $\mu = \tau/(mL^2)$, τ is the imaginary time, L is the size of the volume, and m is the hadrom mass.

  $$C(\tau, L) = \rho(0, L)e^{-\omega(0,L)\tau}\left(1 + O(e^{-4\pi^2\mu})\right) \to \rho(0, L)e^{-\omega(0,L)\tau}.$$

- These results are demonstrated in the single-channel with a simple interaction of EFT. However the results are expected to hold in cases of coupled-channels and/or with more complicated interactions because the loop function plays an essential role which holds true.

Conclusion

- One can extract information of unstable states in the infinite volume such as the mass and the width by fitting the form of the pole expansion in terms of the uniformization variable to the imaginary-time correlation function obtained by lattice simulations in a finite volume in the region, $\mu \ll 1$, where the infinite-volume approximation is valid.
- From this observation we propose the pole expansion of the imaginary-time correlation as a method to extract information of unstable states such as masses and widths from the imaginary-time correlation functions obtained by lattice QCD simulations.
- The next step is obviously to confirm that we can really extract information of unstable states such as masses and widths from the results of actual lattice QCD simulation. We extend the formulation of the survival probability from single channel to coupled-channels (two-channels).