Non-Perturbative Hamiltonian Framework for Finite-Volume Energy of N-Body Systems

Kang Yu

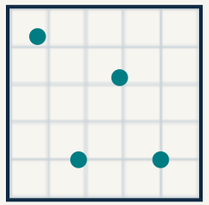
University of Chinese Academy of Sciences (UCAS)

with Jia-Jun Wu, Anthony W. Thomas, Derek B. Leinweber, Guang-Juan Wang and Zhi Yang

Based on **arXiv:2603.07205**

Finite-volume energy and infinite-volume scattering amp.

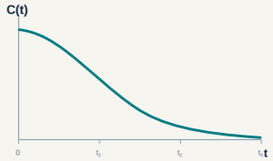
FINITE VOLUME: LATTICE QCD



Finite lattice box

Periodic boundary conditions discretize the momentum and energy and reduce the symmetry from continuous $SO(3)$ to point group.

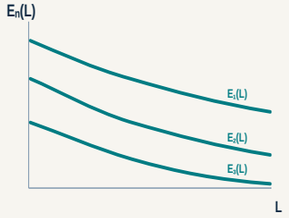
+



Correlation function

$\langle \pi^+\pi^+\pi\pi \rangle$

↓



Finite-volume energy levels

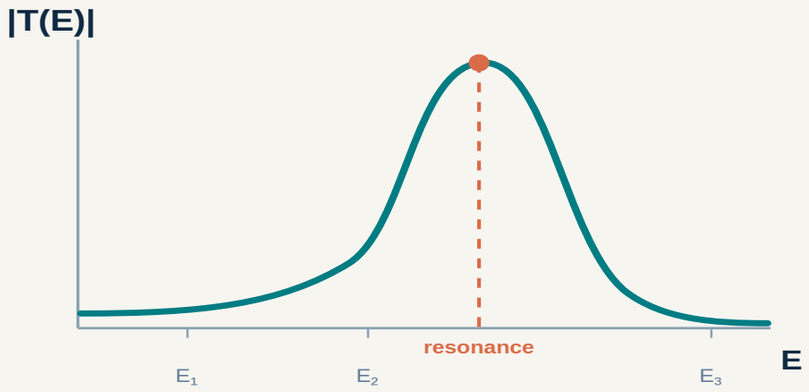
Volume dependence of E encodes dynamic interaction.

FINITE-VOLUME FORMALISM

⇄

relate $E_n(L)$ to $T(E)$

INFINITE VOLUME: HADRON DYNAMICS

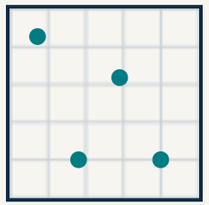


Scattering amp. $T(E)$ at $E_n(L)$

Physics targets: line shapes and resonance poles

Finite-volume energy and infinite-volume scattering amp.

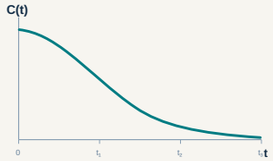
FINITE VOLUME: LATTICE QCD



Finite lattice box

Periodic boundary conditions discretize the momentum and energy and reduce the symmetry from continuous $SO(3)$ to point group.

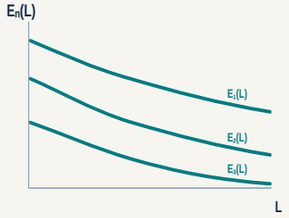
+



Correlation function

$\langle \pi^+\pi^+\pi\pi \rangle$

↓



Finite-volume energy levels

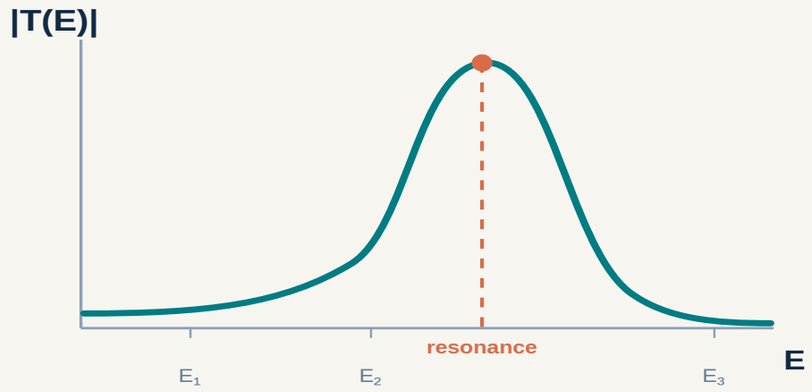
Volume dependence of E encodes dynamic interaction.

FINITE-VOLUME FORMALISM

⇄

relate $E_n(L)$ to $T(E)$

INFINITE VOLUME: HADRON DYNAMICS



Scattering amp. T(E) at $E_n(L)$

Physics targets: line shapes and resonance poles

Lüscher Quantization Condition (for simplest case)

$$\delta(k) + \varphi(q) = n\pi$$

$\delta(k)$: scattering phase shift

k = on-shell momentum at finite-volume energy

$\varphi(q)$: geometric function encoding finite-volume effects

q = kL/2π

Spirit of NPHF

At fixed quantum numbers, lattice spectra extracted from correlation functions are eigenvalues of H^L_{QCD}.

$$C^L(t) = \langle \phi^\dagger(t)\,\phi(0)\rangle \sim \sum_n \left\langle \phi^\dagger(0)\, e^{-H^L_{\mathrm{QCD}}} \,|n\right\rangle \langle n|\,\phi(0)\rangle \sim \sum_n Z_n\, e^{-E_n t}$$

From the perspective of EFT, we don't need to work directly with full QCD Hamiltonian, ...

LATTICE QCD

QCD Hamiltonian under the same boundary conditions.

$$H^L_{\mathrm{QCD}}$$

$$E_n^{\mathrm{LQCD}}(L)$$

$\longleftarrow$

NPHF: FINITE-VOLUME

Construct the Hamiltonian and fit its spectrum to lattice data.

$$H^L_{\mathrm{eff}}(g) = h + V^L_{\mathrm{eff}}(g)$$

$$1 = \frac{1}{L^3}\sum_{\vec{n}} |\vec{n}\rangle\langle\vec{n}|\,, \quad \vec{k}_{\vec{n}} = \frac{2\pi\vec{n}}{L}$$

$$\mathrm{spec}[H^L_{\mathrm{eff}}(g)] \simeq E_n^{\mathrm{LQCD}}(L)$$

$\rightleftarrows$

NPHF: INFINITE-VOLUME

Use the same interaction in the infinite-volume Hilbert space.

$$H^\infty_{\mathrm{eff}}(g) = h + V_{\mathrm{eff}}(g)$$

$$1 = \int d^3\vec{k}\,|\vec{k}\rangle\langle\vec{k}|$$

$$T(E) = V + VG_0(E)T(E)$$

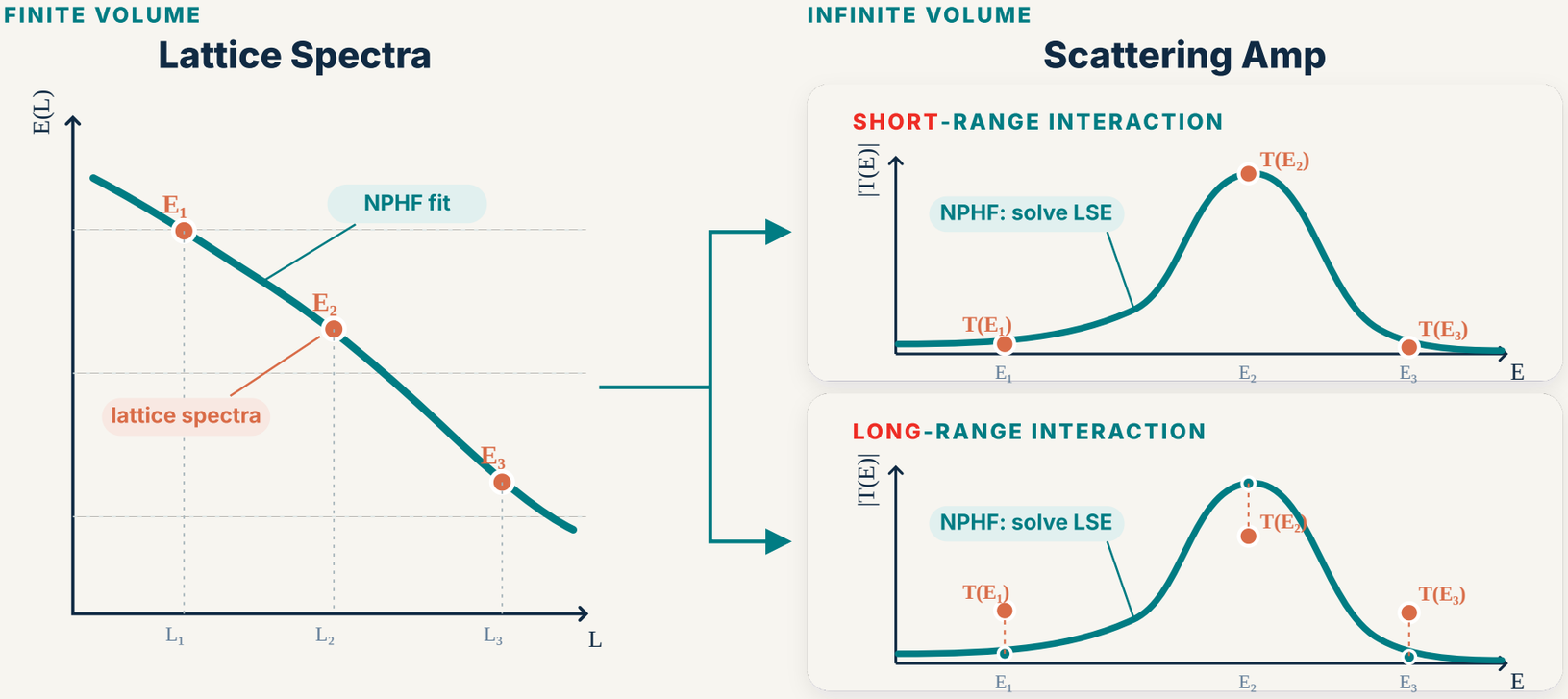
Different Hilbert spaces, **Same** parameters

V_{eff} serves as a bridge connecting finite- and infinite-volume systems.

NPHF and Lüscher formalism for two-body system

For **short-range interaction**, two formalism are proved to be equivalent.

For **long-range interaction**, the discrepancies are observed.



Detailed calculation: J.-J. Wu *et al.*, *Phys. Rev. C* **90**, 055206 (2014); L. Meng and E. Epelbaum, *JHEP* **10**, 051 (2021); K. Yu *et al.*, *JHEP* **04**, 108 (2025).

NPHF PREVIOUS TWO-BODY WORK

Δ(1232) → Nπ
PRD **87**, 094510 (2013)

Λ(1405) as a $\bar{K}$N molecule
PRL **114**, 132002 (2015)

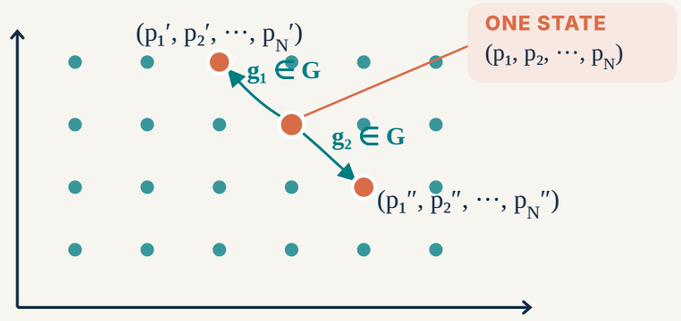
N∗(1535) in lattice QCD
PRL **116**, 082004 (2016)

Roper from lattice constraints
PRD **97**, 094509 (2018)

Moving / elongated volume
PRD **103**, 094518 (2021)

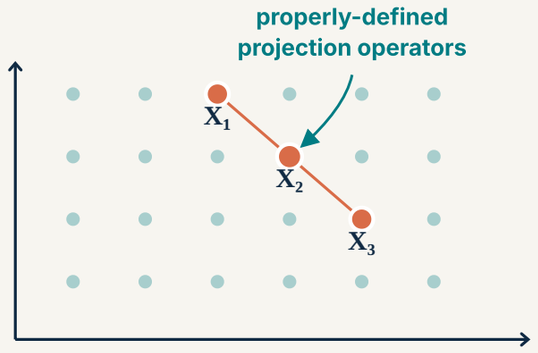
lattice symmetry projection in NPHF

Discrete momentum configurations



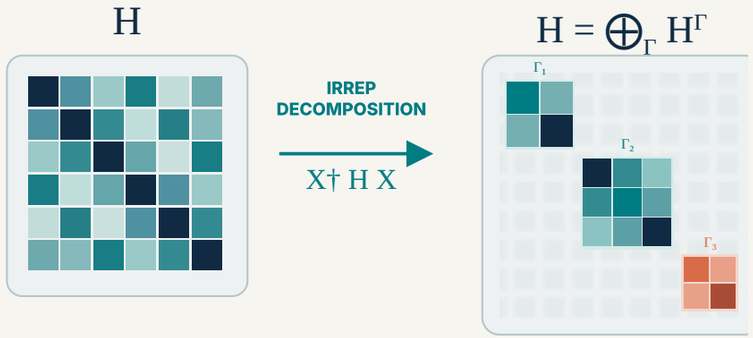
Hilbert space = direct sum of distinct G-orbits.

Construct irrep basis



Given an irrep Γ and a G-orbit,
construct an orthonormal irrep basis: $\sum_i X_i |v_i\rangle$

Block diagonalization by symmetry



Each Hamiltonian block corresponds to an irrep.

$$\hat{P}^{\Gamma}_{\mu r} = \mathcal{N} \sum_{g\in G} \sum_{v} \left(c^{r}\right)_{v} D^{\Gamma *}_{\mu v}(g)\hat{g}$$

normalization
(derived analytically)

linear coefficient

eigenvector associated with the r-th nonzero eigenvalue of the overlap matrix

irrep-matrix-weighted group action

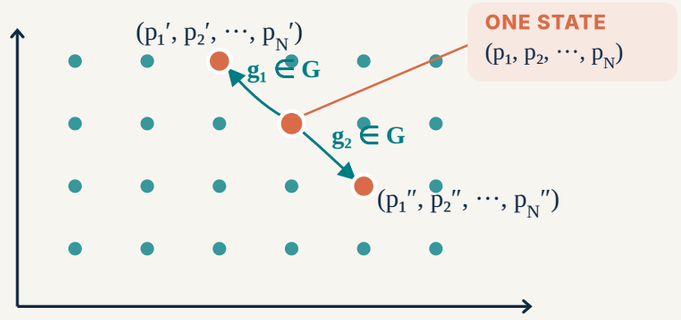
OVERLAP MATRIX I
for the simplest case

$$I_{v'v} = \sum_{g\in G} D^{\Gamma *}_{v'v}(g) \langle v^{\mathrm{ref}} | \hat{g} | v^{\mathrm{ref}} \rangle$$

Encodes all linear information of the G-orbit generated by $|v^{\mathrm{ref}}\rangle$.

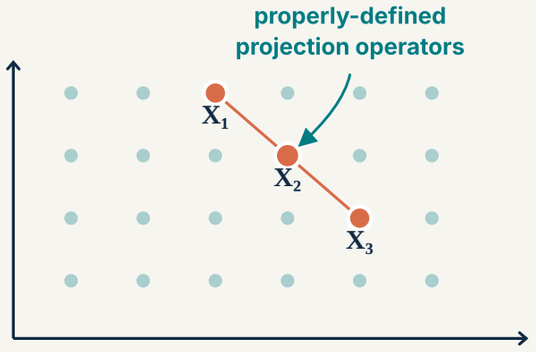
lattice symmetry projection in NPHF

Discrete momentum configurations



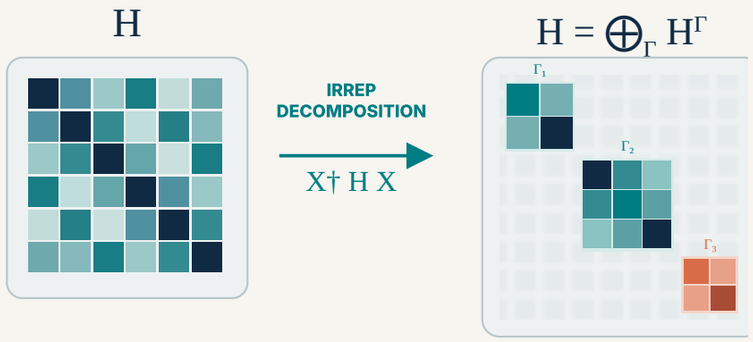
Hilbert space = direct sum of distinct G-orbits.

Construct irrep basis



Given an irrep Γ and a G-orbit,
construct an orthonormal irrep basis: $\sum_i X_i |v_i\rangle$

Block diagonalization by symmetry



Each Hamiltonian block corresponds to an irrep.

$$\hat{P}^{\Gamma}_{\mu r} = \mathcal{N} \sum_{g\in G} \sum_{v} \left(c^{r}\right)_{v} D^{\Gamma *}_{\mu v}(g)\hat{g}$$

normalization
(derived analytically)

linear coefficient

irrep-matrix-weighted group action

eigenvector associated with the r-th nonzero eigenvalue of the overlap matrix

OVERLAP MATRIX I
for the simplest case

$$I_{v'v} = \sum_{g\in G} D^{\Gamma *}_{v'v}(g) \langle v^{\mathrm{ref}} | \hat{g} | v^{\mathrm{ref}} \rangle$$

Encodes all linear information
of the G-orbit generated by $|v^{\mathrm{ref}}\rangle$.

SPIN DEGREES OF FREEDOM

Helicity representation

Under a group action, $\lambda \xrightarrow{g\in G} \pm\lambda$, so helicity is preserved or sign-flipped.

ISOSPIN DEGREES OF FREEDOM

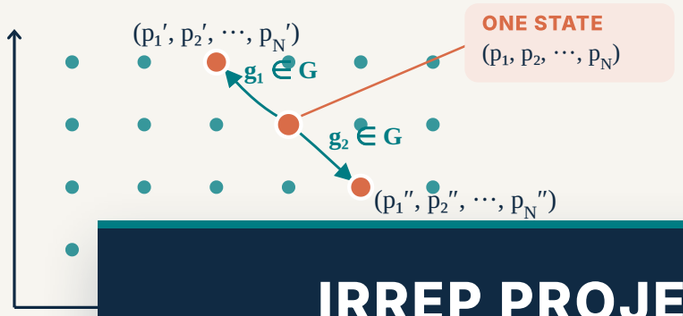
Permutation identification

Include the permutation group S_N and identify physical-equivalent states.

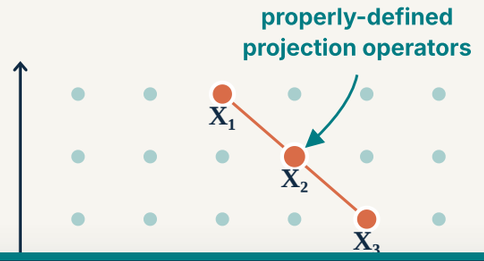
For two particles: $|\vec{p}, -\vec{p}\rangle = \pm|-\vec{p}, \vec{p}\rangle$

lattice symmetry projection in NPHF

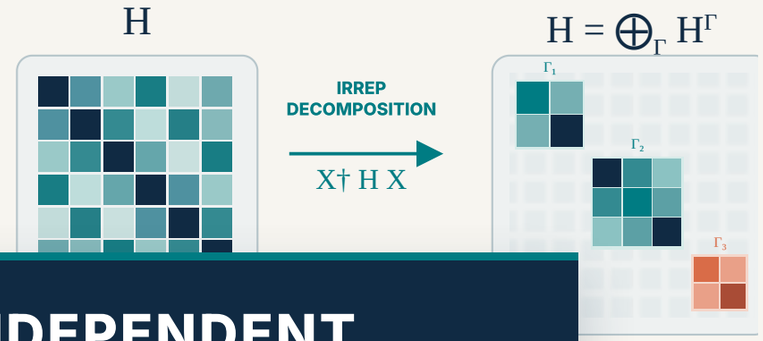
Discrete momentum configurations



Construct irrep basis



Block diagonalization by symmetry



IRREP PROJECTION HERE IS DYNAMICS-INDEPENDENT

Determined solely by kinematics and symmetry

Rest or moving frames · any spin sector · any isospin sector

Hilbert sp … o an irrep.

$$\hat{P}^{\Gamma}_{\mu r} = \mathcal{N} \sum_{g\in G} \sum_{v} \left(c^{r}\right)_{v} D^{\Gamma *}_{\mu v}(g)\hat{g}$$

- normalization (derived analytically)
- linear coefficient: eigenvector associated with the r-th nonzero eigenvalue of the overlap matrix
- irrep-matrix-weighted group action

OVERLAP MATRIX I *for the simplest case*

$$I_{v'v} = \sum_{g\in G} D^{\Gamma *}_{v'v}(g) \langle v^{\mathrm{ref}} | \hat{g} | v^{\mathrm{ref}} \rangle$$

Encodes all linear information of the G-orbit generated by $|v^{\mathrm{ref}}\rangle$.

SPIN DEGREES OF FREEDOM

Helicity representation

Under a group action, $\lambda \xrightarrow{g\in G} \pm\lambda$, so helicity is preserved or sign-flipped.

ISOSPIN DEGREES OF FREEDOM

Permutation identification

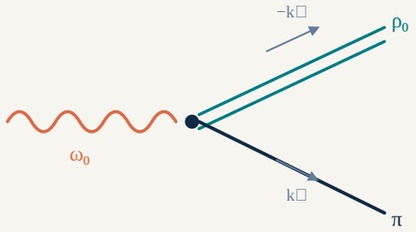
Include the permutation group S_N and identify physical-equivalent states.

For two particles: $|\vec{p}, -\vec{p}\rangle = \pm|-\vec{p}, \vec{p}\rangle$

$\omega \to 3\pi$: a realistic three-body system

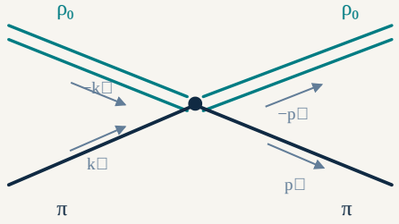
Lattice input: finite-volume spectra for the ω-3π system are already available. Yan *et al.*, *Phys. Rev. Lett.* **133**, 211906 (2024)

Fock states: ω_0 + $\rho_0\pi$ + 3π



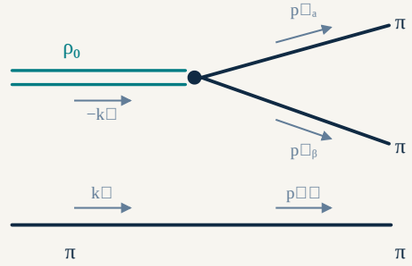
$\boldsymbol{\omega_0 \leftrightarrow \rho_0\pi}$

$\sim g_{\omega\rho\pi}\, k\, f(k, \Lambda_\omega)$



$\boldsymbol{\rho_0\pi \to \rho_0\pi}$

$\sim c_{22}\, p\, k\, f(p, \Lambda_{\rho_0\pi})\, f(k, \Lambda_{\rho_0\pi})$



$\boldsymbol{\rho_0\pi \leftrightarrow 3\pi}$

$\sim \sum_{(a,b,c)} \delta^3(\vec{k} - \vec{p}_c)\, V_{\rho\to\pi\pi}(\vec{p}_a, \vec{p}_b)$

2π-system

	ρ_0	2π
ρ_0	0	$v_{\rho\pi\pi}$
2π	$V^\dagger_{\rho\pi\pi}$	0

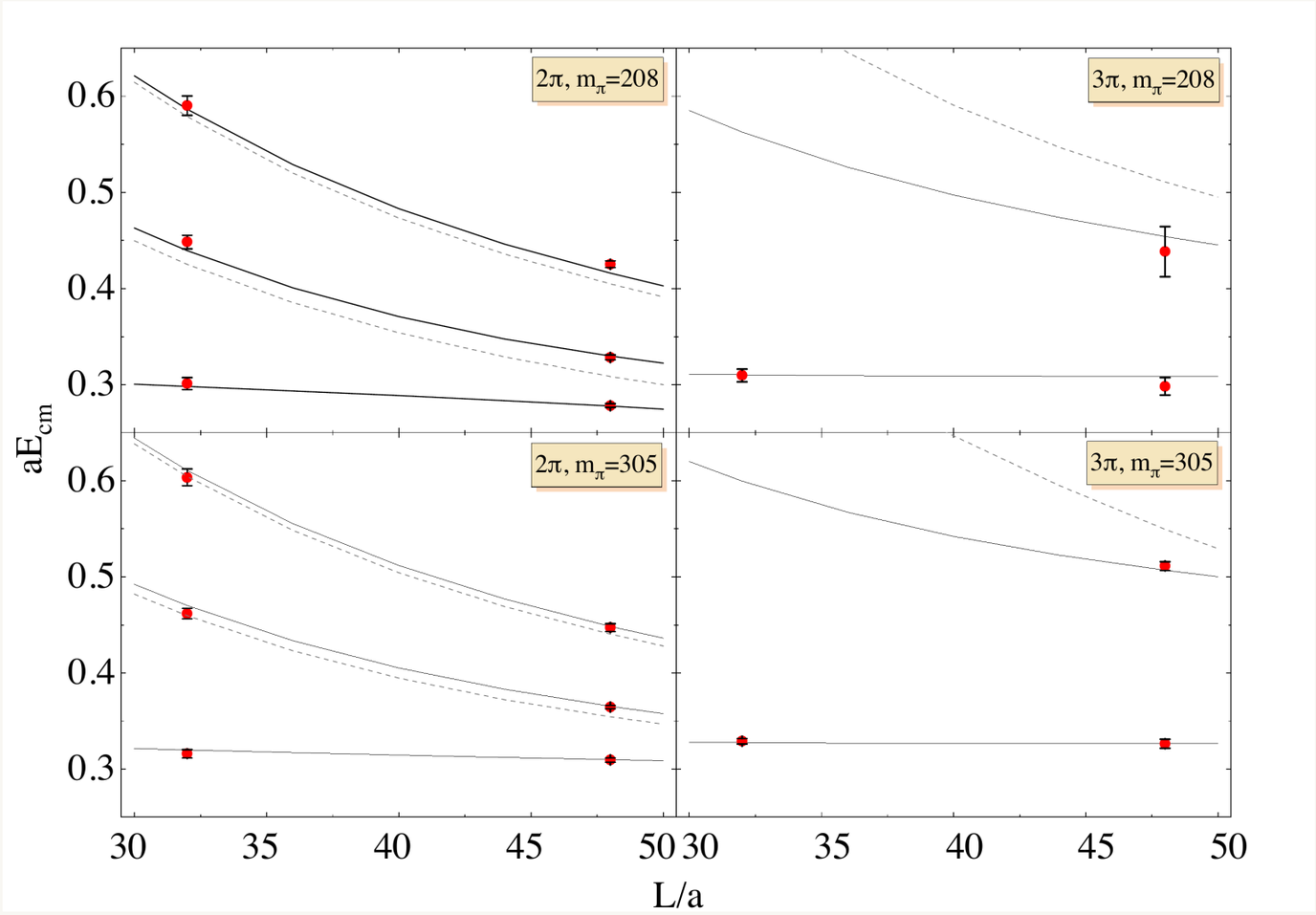
$V_{\rho\to\pi\pi}(\vec{p}_a, \vec{p}_b) \sim g_{\rho\pi\pi}\, p_{\text{cm}}\, f(p_{\text{cm}}, \Lambda_{\pi\pi})$

3π-system

	ω_0	$\rho_0\pi$	3π
ω_0	0	$v_{\omega\rho\pi}$	0
$\rho_0\pi$	$V^\dagger_{\omega\rho\pi}$	$C_{\rho\pi}$	$v_{\rho\pi,3\pi}$
3π	0	$V^\dagger_{\rho\pi,3\pi}$	0

Finite-volume spectrum fit

Simultaneous constraint: isovector 2π and isoscalar 3π spectra at m_π = 208 and 305 MeV. Yan *et al.*, *Phys. Rev. Lett.* **133**, 211906 (2024)



● Lattice spectra — NPHF eigenvalues --- Non-interacting levels

FIT PARAMETERS AND PION-MASS DEPENDENCE

$m_{\rho_0}^{208}$, $m_{\rho_0}^{305}$, $g_{\rho\pi\pi}$, $g_{\omega\rho\pi}$, c_{22}

$m_{\omega_0} = m_{\rho_0} + \delta m$, with δm taken independent of m_π.

FIXED PARAMETERS

Λ_{ω_0} = 1.0 GeV, $\Lambda_{\pi\pi}$ = 1.0 GeV, $\Lambda_{\rho_0\pi}$ = 0.5 GeV, δm

FIT SCHEMES

A	B1	B2	B3
$\delta m = 0$	$c_{22} = 0$ $\delta m = 0$	$c_{22} = 0$ δm = 15 MeV	$c_{22} = 0$ δm = 30 MeV

scheme B2

quantity	fit result
$\hat{\chi}^2$	1.1
$m_{\rho_0}^{208}$	831(13) MeV
$m_{\rho_0}^{305}$	872(10) MeV
$g_{\rho\pi\pi}$	7.69(0.45)
$g_{\omega\rho\pi}$	30.7(3.1)
c_{22}	0 (fixed)

Infinite volume scattering

LSE FOR $\rho_0\pi \to \rho_0\pi$

$$T = V + V\,G_2\,T$$

Scattering kernel $V = C + R + Z$

C
Contact interaction



R
Bare-ω_0 exchange



Z
3π-mediated exchange



Dressed $\rho_0\pi$ propagator

$$G_2(p;E) = \left[E - \omega_{\rho_0}(p) - \omega_\pi(p) - \Sigma_{\pi\pi}(p;E)\right]^{-1}$$

$\rho_0 \to \pi\pi \to \rho_0$ self-energy correction is included.

The 3π **unitary branch cut** renders the integral equation substantially more intricate than in a purely two-body system.

Available numerical strategies include contour deformation, interpolation, and related methods.

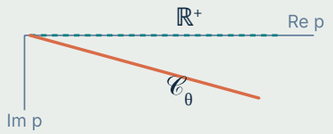
CONTOUR DEFORMATION FOR ANALYTICAL CONTINUATION

POLE CONDITION

$$\det_{\mathbb{R}^+}\left[1 - V(E)\,G_2(E)\right] = 0$$

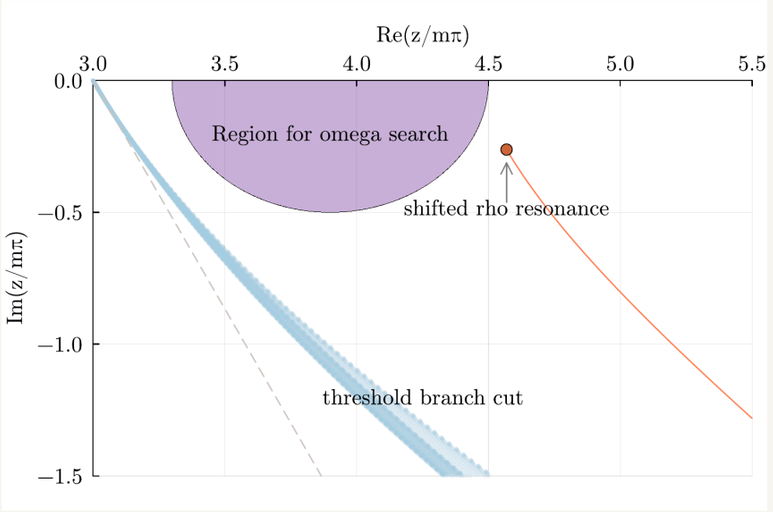
↓ contour deformation

$$\det_{\mathcal{C}_\theta}\left[1 - V(E)\,G_2(E)\right] = 0$$



The self-energy contour in G_2 should be deformed simultaneously.

In the complex energy plane,



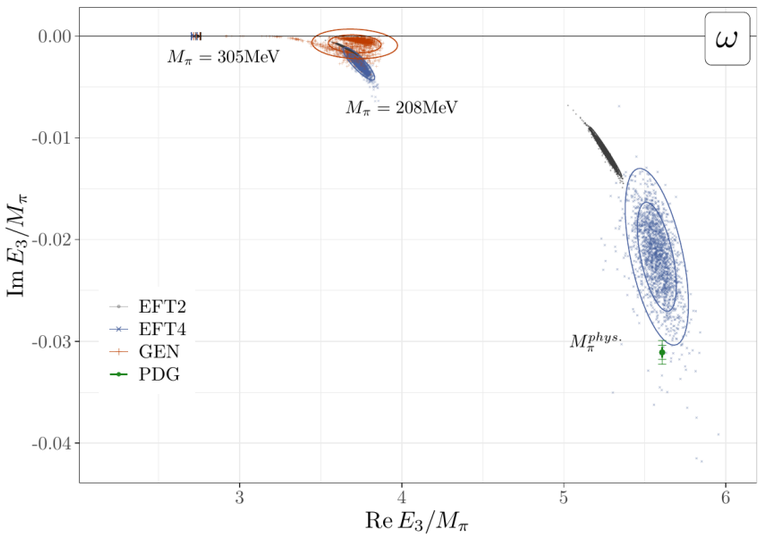
Pole positions

Pole positions across fit schemes

scheme	m_π = 208 **MeV**		m_π = 305 **MeV**	
	z_ρ	z_ω	z_ρ	z_ω
A	742.3(5.3) – *i* 55.0(5.3)	777.9(14.0) – *i* 0.31(0.09)	790.8(4.8) – *i* 28.1(3.3)	823.3(5.8)
B1	744.8(5.5) – *i* 55.3(5.6)	779.0(10.0) – *i* 0.23(0.07)	792.0(4.8) – *i* 27.7(3.3)	829.6(5.1)
B2	743.1(5.2) – *i* 54.9(5.1)	776.7(10.2) – *i* 0.30(0.09)	791.5(5.0) – *i* 28.2(3.4)	827.2(5.4)
B3	743.3(5.3) – *i* 55.1(5.4)	776.3(9.7) – *i* 0.33(0.07)	791.0(4.9) – *i* 28.2(3.5)	826.8(5.3)

All pole positions are in MeV.

Comparison with FVU Yan *et al.*, *Phys. Rev. Lett.* **133**, 211906 (2024)



At the unphysical pion masses, our results are consistent with Finite Volume Unitarity (FVU) result.

No chiral extrapolation is performed in this work.

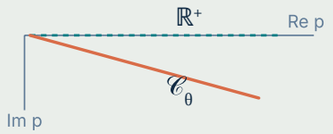
CONTOUR DEFORMATION FOR ANALYTICAL CONTINUATION

POLE CONDITION

$$\det_{\mathbb{R}^+}\left[1 - V(E)\,G_2(E)\right] = 0$$

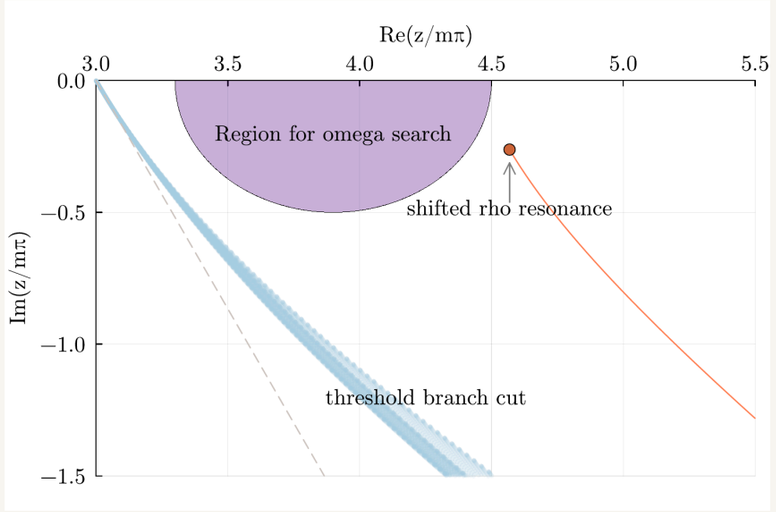
↓ contour deformation

$$\det_{\mathcal{C}_\theta}\left[1 - V(E)\,G_2(E)\right] = 0$$



The self-energy contour in G_2 should be deformed simultaneously.

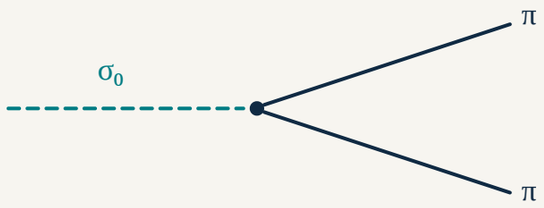
In the complex energy plane,



Beyond three-body: 4π effects on Fin-Vol Energies of σ

FOCK STATES σ_0 + 2π + 4π **A minimal four-body benchmark system**

TWO-PION COUPLING $\sigma \to 2\pi$

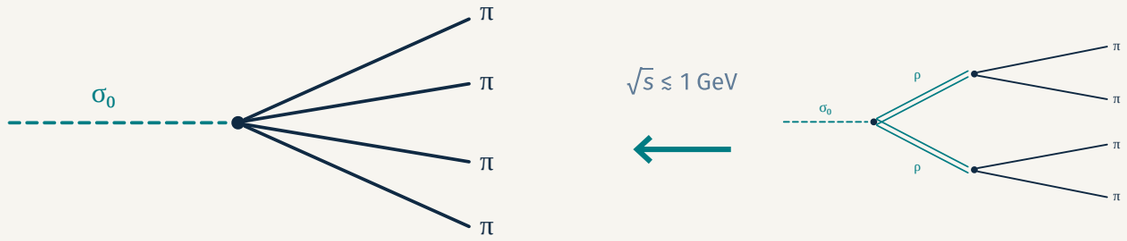


$$\sim g_{\sigma\pi\pi}\left(k^2 + \frac{7}{8}m_\pi^2\right) f(k, \Lambda_{2\pi})$$

Ensures the correct $\pi\pi \to \pi\pi$ S-wave isoscalar Adler zero.

$$k^2 + \frac{7}{8}m_\pi^2 \xrightarrow{\text{on-shell}} s - \frac{1}{2}m_\pi^2$$

FOUR-PION COUPLING $\sigma \to 4\pi$

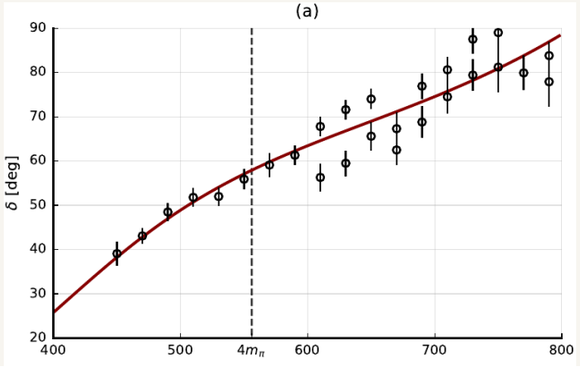


$$\langle \sigma_0 | V | 4\pi, a = 1 \rangle \sim g_{\sigma\to 4\pi}\left[(\vec{k}_1 - \vec{k}_3)\cdot(\vec{k}_2 - \vec{k}_4) + (\vec{k}_1 - \vec{k}_4)\cdot(\vec{k}_3 - \vec{k}_2)\right] \prod_{i=1}^{4} f(k_i, \Lambda_{4\pi})$$

$$\langle \sigma_0 | V | 4\pi, a = 2 \rangle \sim \sqrt{3}\, \langle \sigma_0 | V | 4\pi, a = 1 \rangle \Big|_{\vec{k}_2 \leftrightarrow \vec{k}_3}$$

$I = 0$ 4π furnishes the two-dimensional irrep $[2, 2]$, hence two independent vertices.

$\Lambda_{4\pi} = 300$ MeV such that it is suppressed at $\sqrt{s} \gtrsim 1$ GeV



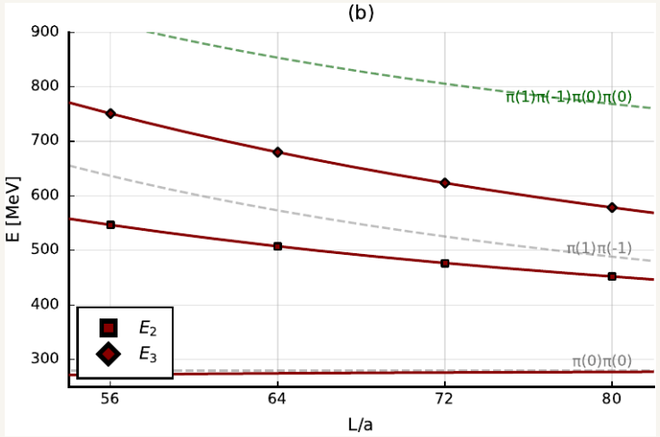
σ–2π FITTED PARAMETERS $g_{\sigma\pi\pi} = 66.9$ $\Lambda_{2\pi} = 346.0$ MeV $m_{\sigma_0} = 798.0$ MeV

Beyond three-body: 4π effects on Fin-Vol Energies of σ

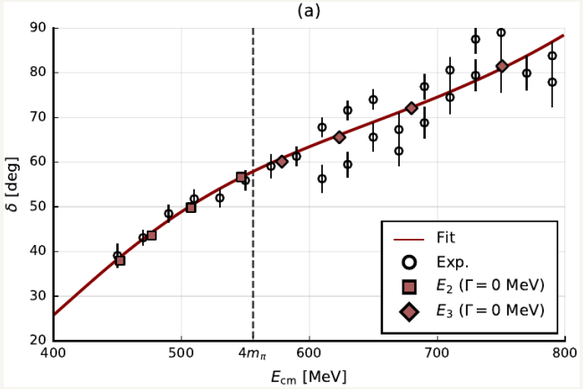
$\Gamma_{\sigma\to 4\pi}$ **determines** $g_{\sigma\to 4\pi}$

$\Gamma_{4\pi} = 0 \Rightarrow g_{\sigma\to 4\pi} = 0$

Finite-volume σ-2π spectrum



Lüscher phase shifts from $E_{2,3}(L)$



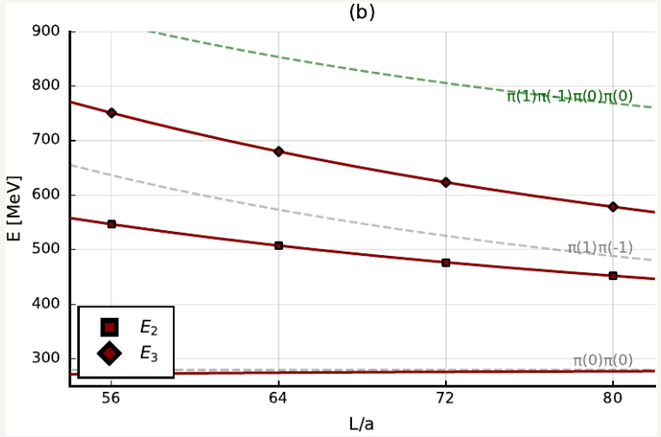
As expected from the equivalence of the two formalisms.

Beyond three-body: 4π effects on Fin-Vol Energies of σ

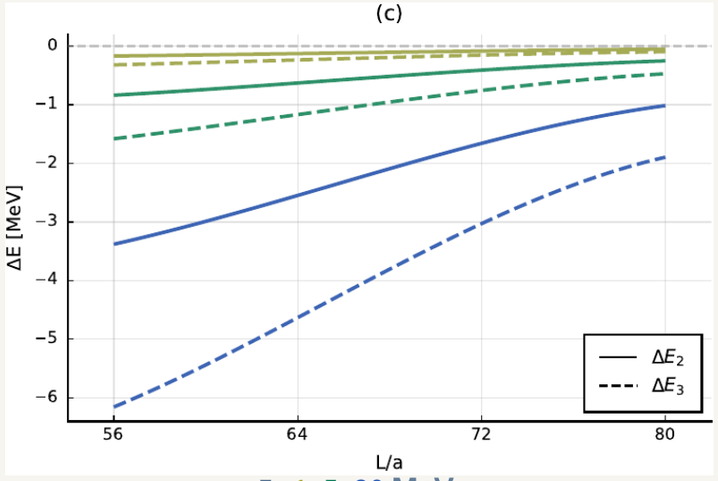
$\Gamma_{\sigma\to 4\pi}$ **determines** $g_{\sigma\to 4\pi}$

$\Gamma_{4\pi}$ = 1, 5, 20MeV ⇒ $g^2_{\sigma\to 4\pi}$ **increases linearly**

Finite-volume σ-2π spectrum

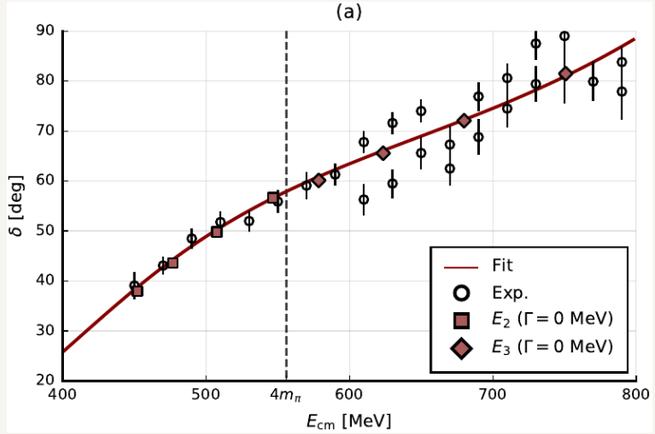


Finite-volume energy shift when switching on $\sigma \to 4\pi$

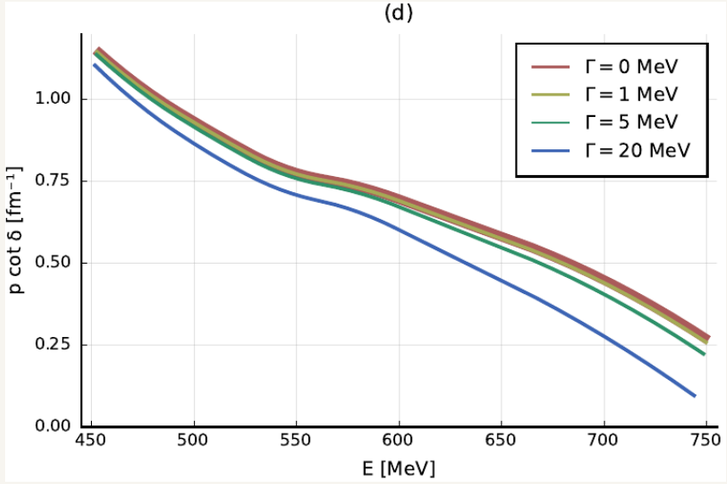


Γ =1, 5, 20 MeV.

Lüscher phase shifts from $E_{2,3}(L)$



$p\cot\delta$ from two-body Lüscher analysis

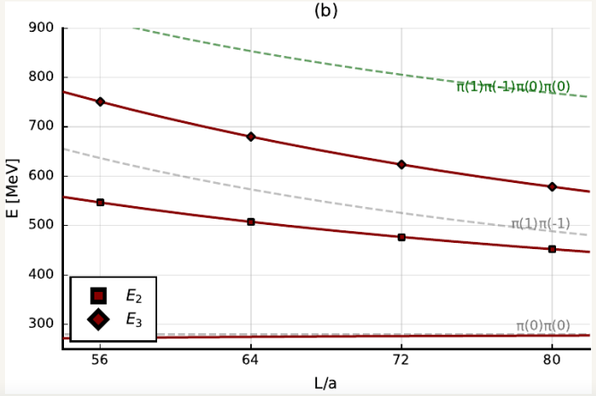


Beyond three-body: 4π effects on Fin-Vol Energies of σ

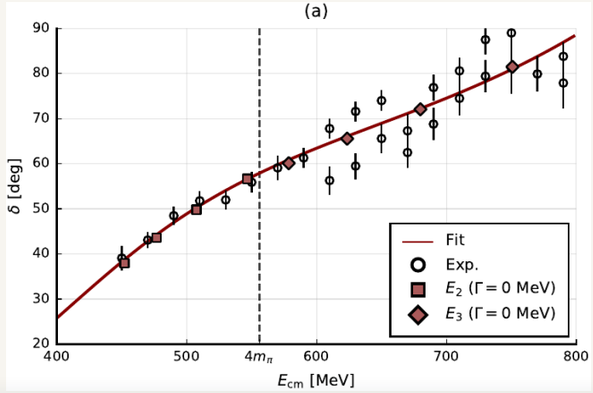
$\Gamma_{\sigma\to 4\pi}$ **determines** $g_{\sigma\to 4\pi}$

$\Gamma_{4\pi}$ = 1, 5, 20MeV ⇒ $g^2_{\sigma\to 4\pi}$ **increases linearly**

Finite-volume σ-2π spectrum

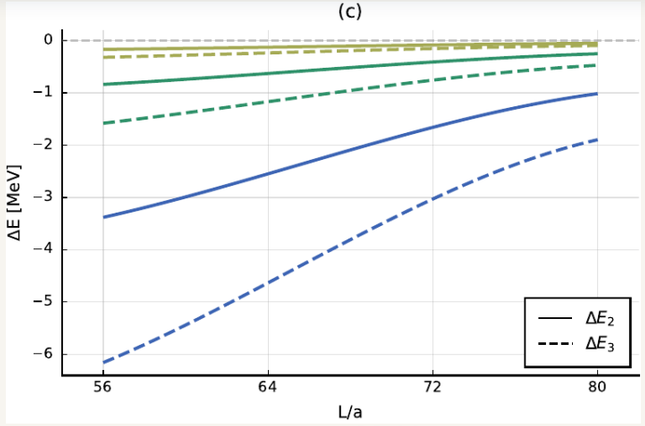


Lüscher phase shifts from $E_{2,3}(L)$



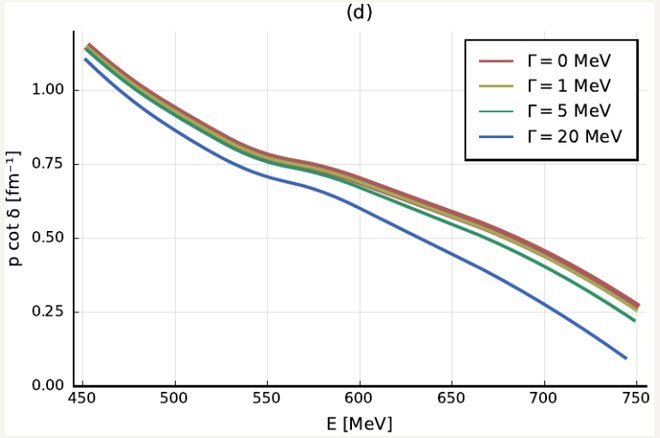
WE CAN INDEED DO A FOUR-BODY FINITE-VOLUME CALCULATION

Finite-volume energy shift when switching on $\sigma \to 4\pi$



Γ =1, 5, 20 MeV.

$p\cot\delta$ from two-body Lüscher analysis



NPHFforFVE.jl: from interactions to finite-volume spectra

$N \leq 4$ · Julia 1.10+

A Julia package that turns a user-defined N-body Hamiltonian into symmetry-projected finite-volume spectra.

YOU PROVIDE

`FockChannel and Quantum Numbers`
particle contents, masses, spin, isospin,...

`Lattice Config Info`
Box size, lattice spacing (only for unit), irreps,...

`User-defined interaction`

→

THE PACKAGE HANDLES

Finite-volume Hamiltonian
Automatically constructs the many-body Hamiltonian from the supplied Fock sectors and interaction matrix elements.

Irrep projection
Projects the Hamiltonian to the selected finite-volume irreducible representation.

→

YOU OBTAIN

Low-lying levels
Ordered energy spectra for every requested irrep.

Fock-channel weights
FockChannel components of each eigenvector.

```
result = compute!(project, my_V, MyParams())
```

Given lattice spectra, the package can also fit the parameters of a user-defined Hamiltonian.

Minimal example: an $I = 1$ $\rho \oplus \pi\pi$ system

Define the system

```
using NPHFforFVE

rho = FockChannel(
    "rho", [1], [:boson],
    [dynamic_mass(:m_rho0)],
    [1//1], [1//1], [-1.0], relativistic,
)

pipi = FockChannel(
    "pipi", [2], [:boson],
    [140.0],                # mπ [MeV]
    [0//1], [1//1], [-1.0], relativistic,
)

project = Project(D000, 1//1, [rho, pipi], [0, 10])

config = add_config!(project, 48, 0.10, ["T1-"], [6])

generate_potential_template(project, "rho_pipi_potential.jl")
```

Provide the interaction

```
# rho_pipi_potential.jl is generated

@params struct YourParameters
    YOUR PARAMETERS
end

# Inside the generated potential file
function my_V(..., chA, chB, ..., params)
if (chA, chB) in ((1,2), (2,1))
    return YOUR_ρ ↔ ππ VERTEX
elseif (chA, chB) == (2,2)
    return YOUR_ππ → ππ INTERACTION
end
end
```

Compute

```
include("rho_pipi_potential.jl")
result = compute!(project, my_V, YourParameters())
```

QUESTIONS & DISCUSSION

Thank you

NPHF VISION

Toward a more direct **connection** between

lattice hadron spectroscopy *(hep-lat)*

hadron physics phenomenology *(hep-ph)*

Details of irrep projection

01 Choose a set of reference states $\{|v_i^{\text{ref}}\rangle\}$ → **02** Generate G-orbits $\mathcal{O}(v_i^{\text{ref}})$ → **04** Projection in each G-orbit → **05** Diagonalize overlap matrix $\mathbf{I}^{\Gamma}$

G-orbit equivalence

As a simple example, consider a two-body system in the CM frame with definite total isospin:

$$|\vec{p},-\vec{p},\lambda,\lambda'\rangle \xrightarrow{\mathbb{P}} |-\vec{p},\vec{p},-\lambda',-\lambda\rangle \underset{\propto}{\text{isospin}} |\vec{p},-\vec{p},-\lambda,-\lambda'\rangle$$

Due to the rearrangement theorem $gG = G \quad \forall\, g \in G$,

$$\mathcal{O}\Big(|\vec{p},-\vec{p},\lambda,\lambda'\rangle\Big) = \mathcal{O}\Big(|-\vec{p},\vec{p},-\lambda',-\lambda\rangle\Big)$$

Analytic form of the overlap matrix

$$I^{\Gamma}_{bv',av} = \sum_{g\in \mathrm{Per}(\{\vec{n}^r\})} \underbrace{e^{-i\varphi^{\eta}(\{\vec{n}^r,\lambda\},g)}}_{\text{phase factor from helicity}} \times \underbrace{D^{\Gamma *}_{v'v}(g)}_{\text{irrep matrix}} \times \underbrace{\sum_{s\in S_N} \delta(s)\,\mathcal{R}^{[\kappa]}_{ba}(s) \prod_{i=1}^{N} \delta_{\vec{n}_{s_i},g\vec{n}_i}\,\delta_{\lambda_{s_i},\epsilon(g)\lambda_i}}_{\text{permutation and statistical factor from isospin symmetry}}$$

nonzero eigenvectors c^r → **independent orthonormal irrep basis**

$\omega - 3\pi$ scattering details

Scattering kernel

$$U(p,k;z) = C(p,k) + R(p,k;z) + Z(p,k;z)$$

C $\rho_0\pi$ contact interaction

$$C(p,k) \propto \frac{\pi c_1}{3}\left[pF(p,\Lambda_{\rho_0\pi})\right]\left[kF(k,\Lambda_{\rho_0\pi})\right]$$

R bare ω_0 s-channel exchange

$$R(p,k;z) \propto \frac{1}{z - m_{\omega_0}}\left(\frac{g_{\omega\rho\pi}}{6\sqrt{2}\,\pi}\right)^2\left[pF(p,\Lambda_{\omega_0})\right]\left[kF(k,\Lambda_{\omega_0})\right]$$

Z connected $\rho_0\pi \to 3\pi \to \rho_0\pi$ exchange kernel

$$Z(p,k;z) = 4\pi^{3/2}\sum_{\sigma,\sigma'} C^{1\sigma}_{1\sigma-\sigma';1\sigma'} C^{1\sigma}_{10;1\sigma} N_{1,\sigma-\sigma'} \int dx\, \frac{P_{1,\sigma-\sigma'}(x)\, v^*_{\sigma'}(-\vec{k},\vec{p}+\vec{k})\, v_\sigma(\vec{p}+\vec{k},-\vec{p})}{z - \omega_\pi(k) - \omega_\pi(p) - \omega_\pi(|\vec{p}+\vec{k}|) + i0^+}$$

$F(k,\Lambda) = \left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^2$; $\mathcal{J}$ collects normalization factors; v_σ is the $\rho_0 \to \pi\pi$ vertex. The Z term generates the 3π unitary branch cut.

Contour-deformed pole condition

$\mathbb{R}^+ \to \mathcal{C}_\theta$ $\arg q_\theta = -\theta$

$$\det_{\mathcal{C}_\theta}\left[\delta_{ij} - U(p_{\theta i}, p_{\theta j}; z)\, q^2_{\theta j} w_j\, G_2(p_{\theta j}; z)\right] = 0$$

Simultaneously deform the self-energy contour in $\Sigma_{\pi\pi}$.

σ–2π–4π: width calibration

The effective $\sigma \to 4\pi$ coupling is calibrated through the spectral-function-averaged partial width.

Spectral-function-averaged $\sigma \to 4\pi$ partial width

$$\Gamma \equiv \Gamma_{\sigma\to\pi^+\pi^-\pi^0\pi^0} = \int_{(4m_\pi)^2}^{\infty} \rho(s)\,\Gamma(s)\,ds$$

Dressed σ spectral function

$$\rho(s) = \frac{\mathrm{Im}\,\Pi_{\pi\pi}(s)}{\left[s - m_{\sigma_0}^2 - \mathrm{Re}\,\Pi_{\pi\pi}(s)\right]^2 + \left[\mathrm{Im}\,\Pi_{\pi\pi}(s)\right]^2}$$

$\Pi_{\pi\pi}(s) = 2m_{\sigma_0}\Sigma'_{\pi\pi}(s)$, **generated by the $\sigma \leftrightarrow 2\pi$ self-energy.**

Energy-dependent partial width

$$\Gamma(s) = \int d\Phi_4 \left|\mathcal{M}_{\sigma\to4\pi}(s)\right|^2$$

$d\Phi_4$ is the four-pion phase-space measure; $\mathcal{M}_{\sigma\to4\pi}$ follows from the effective four-pion vertices.

Coupling calibration

$$\Lambda_{4\pi} = 300\ \mathrm{MeV} \quad + \quad \Gamma_{\mathrm{target}} \quad \Longrightarrow \quad g_{\sigma\to4\pi}$$

At fixed $\Lambda_{4\pi}$, each chosen averaged width determines the coupling $g_{\sigma\to4\pi}$.