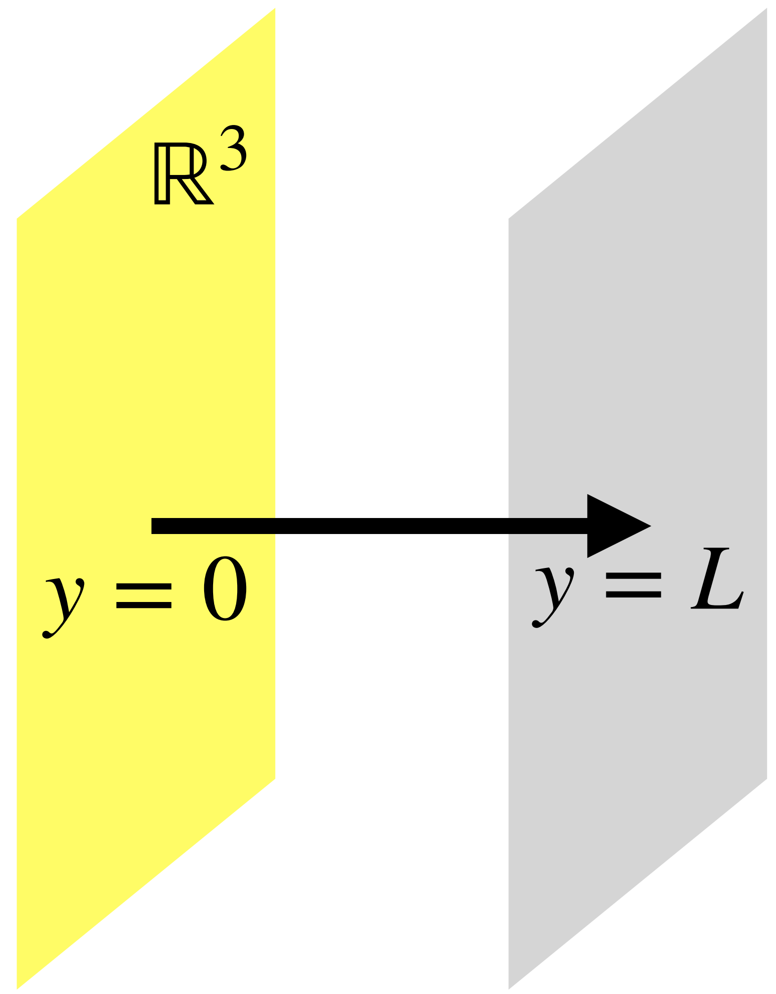
Extra Dimensions and Neutrino Physics

Hiroshi Isono
(礒野 裕)

High Energy Physics Research Unit
Chulalongkorn University, Bangkok, Thailand

Neutrino & extra dimensions



- A model of Dirac neutrino masses from 5D fermions (Auttakit's lecture)
  suppression of neutrino masses through the size of the extra dimension
  contains an infinite tower of sterile neutrinos
  --> their masses may give a hint for the extra dimension.

- Technically:
  -- Kaluza-Klein expansion of 5D spinors to obtain 4D spinors
  -- 5D spinors must contain both left- and right-handed spinors symmetrically, while 4D Standard Model not.
  -- Left-Right asymmetry is realised through boundary conditions in 5D.

- Goals of this lecture:
  >> Introduction to extra dimension and Kaluza-Klein expansion
  >> Introduction to 4D and 5D spinors, in particular left-right asymmetry through extra dimension

  Convey a manner of arguments in theoretical high energy physics with field theory

Extra Dimensions

Brief history

- 1919 Kaluza: 5D spacetime: Minkowski + 1D space
  Decompose 5D metric into 4D metric and 4D vector
  --> 4D gravity + 4D Maxwell : 1st generation of unification of forces

  1926 Klein: Compactification: Make the extra direction sufficiently small

- 1970's and later: string theory
  string's oscillation modes --> a set of particles containing massless graviton, gauge fields etc.
  Consistent only in a spacetime with extra directions: 4D+6D
  4D Minkowski with 6D tiny, compact geometry
  Tiny 6D geometry's topology --> a theory with matter fields in 4D at lower energies.

- 1990's and later: Large Extra Dimensions (LED)
  4D Minkowski spacetime is embedded as a "brane" in a larger ("bulk") spacetime
  Standard Model is confined in the brane. Gravitational force propagates in the bulk.
  The size of the extra dimensions can reduce the bulk gravitational coupling to small 4D one.

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Force unification oriented

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- 1990's and later: Large Extra Dimensions (LED)

  Energy hierarchy resolution oriented

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Plan

Spinors in 4D and 5D

Kaluza-Klein expansion

Find a neutrino mass

Spinors in 4D and 5D

Pre-introductory high energy field theory

- Action and equation of motion of a scalar field (Klein-Gordon)

equation of motion

$$S[\phi] = \int dt d\vec{x} \Big(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 \Big) \xrightarrow{\text{variation in } \phi} \Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \vec{x}^2} - m^2 \Big) \phi(t, \vec{x}) = 0$$

In field theory: spacetime derivative $\partial/\partial x^\mu$ --> replace it by 4 momentum p_μ

--> EoM gives relativistic dispersion relation $E^2 = \vec{p}^2 + m^2$

--> m = particle mass

- Keep in mind: in field-theory actions,

Non-derivative quadratic term $\varphi \cdot \varphi$ is called the mass term

The 2 or 1 derivative term gives E and p: called the kinetic term

Pre-introductory high energy field theory

- High energy physics: based on special relativistic field theory
  basic guiding principle: Lorentz invariance of an action (= spacetime integral of a Lagrangian)

- To make Lorentz invariant actions, better to make a catalogue of convenient field transformations

  Examples of field transformation $\varphi \mapsto \varphi'$ under Lorentz transformation $x' = \Lambda x$:

  scalar field --> $\varphi'(\Lambda x) = \varphi(x)$: spin 0 particles (Higgs)

  vector field --> $\varphi'^{\mu}(\Lambda x) = \Lambda^{\mu}{}_{\nu}\varphi^{\nu}(x)$: spin 1 particles (photon, weak bosons W, Z)

  spinor field: the topic in what follows! : spin 1/2 particles (quarks, leptons)

Spinors in 3D

Start with 3D rotation of spin 1/2 states (size 2 vectors) in quantum mechanics

Small rotation $\vec{x} \mapsto (I+\omega)\vec{x}$ with 3x3 matrix ω

Rotation: $(I+\omega)(I+\omega^T) = I \quad \to \quad \omega^T = -\omega \quad \to$ three parameters

ω_{12} : rotation around axis 3

ω_{23} : rotation around axis 1

ω_{31} : rotation around axis 2

ω_{ij} : rotation in (ij)-plane

Angular momenta as generators of 3D rotation: $\dfrac{\sigma_i}{2}$ (Pauli matrix) rotation around axis i

For small rotation $x_i \mapsto x_i + \omega_{ij} x_j$, a state of spin 1/2 particle changes as $\psi \mapsto \left(I + i\omega_{23}\dfrac{\sigma_1}{2} + i\omega_{31}\dfrac{\sigma_2}{2} + i\omega_{12}\dfrac{\sigma_3}{2} \right)\psi$

$$= \left(I + \frac{1}{2}\sum_{i,j=1}^{3} \frac{1}{4}\omega_{ij}[\sigma_i, \sigma_j] \right)\psi$$

$$I + i\omega_{23}\frac{\sigma_1}{2} + i\omega_{31}\frac{\sigma_2}{2} + i\omega_{12}\frac{\sigma_3}{2} = I + \omega_{23}[\frac{\sigma_2}{2}, \frac{\sigma_3}{2}] + \omega_{31}[\frac{\sigma_3}{2}, \frac{\sigma_1}{2}] + \omega_{12}[\frac{\sigma_1}{2}, \frac{\sigma_2}{2}]$$

$$= I + \frac{1}{2}\sum_{i,j=1}^{3} \omega_{ij}[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}] \quad = I + \frac{1}{2}\sum_{i,j=1}^{3} \frac{1}{4}\omega_{ij}[\sigma_i, \sigma_j]$$

Spinors in 4D and 5D

3D small rotation

$$x_i \mapsto x_i + \omega_{ij} x_j \qquad \omega_{ij} = -\omega_{ji}$$

$$\psi \mapsto \left(I + \frac{1}{2} \sum_{i,j=1}^{3} \frac{1}{4} \omega_{ij} [\sigma_i, \sigma_j] \right) \psi$$

$\Downarrow$

4D small Lorentz transformation

$$x^\mu \mapsto x^\mu + \omega^\mu{}_\nu x^\nu \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$\psi \mapsto \left(I - \frac{1}{2} \sum_{\mu,\nu=0}^{3} \frac{1}{4} \omega^{\mu\nu} [\gamma_\mu, \gamma_\nu] \right) \psi$$

$\Downarrow$

5D small Lorentz transformation

$$x^M \mapsto x^M + \omega^M{}_N x^N \qquad \omega_{MN} = -\omega_{NM}$$

$$\psi \mapsto \left(I - \frac{1}{2} \sum_{M,N=0}^{4} \frac{1}{4} \omega^{MN} [\gamma_M, \gamma_N] \right) \psi$$

Dirac matrices

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$$

$$\{\gamma_M, \gamma_N\} = -2\eta_{MN}$$

Lorentz metric: mostly plus

{a,b}:=ab+ba

Convenient choice

Dirac matrices

4D Dirac matrices	$\gamma_0, \gamma_1, \gamma_2, \gamma_3$	4x4 matrices	$\gamma^\mu = \begin{pmatrix} O_2 & \sigma^\mu \\ \bar{\sigma}^\mu & O_2 \end{pmatrix}$	$\sigma^0 = I_{2\times 2}$, $\bar{\sigma}^0 = I_{2\times 2}$, $\bar{\sigma}^i = -\sigma^i$
5D Dirac matrices	$\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4$	4x4 matrices	the same γ^μ &	$\gamma^4 = i \begin{pmatrix} I_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & -I_{2\times 2} \end{pmatrix}$

Weyl spinors

<u>4D Dirac spinors:</u> $\psi \mapsto \left(I+\frac{1}{2}\omega^{\mu\nu}s_{\mu\nu}\right)\psi$ $\quad s_{\mu\nu} := -\frac{1}{4}[\gamma_\mu, \gamma_\nu]$ $\quad \gamma^\mu = \begin{pmatrix} O_2 & \sigma^\mu \\ \bar{\sigma}^\mu & O_2 \end{pmatrix}$

However, $[\gamma_\mu, \gamma_\nu]$ is 2x2 block diagonal $\Longrightarrow$ Upper and lower 2 components transform independently

$$s_{\mu\nu} = \begin{pmatrix} s^{\mathrm{L}}_{\mu\nu} & O_{2\times2} \\ O_{2\times2} & s^{\mathrm{R}}_{\mu\nu} \end{pmatrix} \Longrightarrow \psi = \begin{pmatrix} \psi^{\mathrm{L}} \\ \psi^{\mathrm{R}} \end{pmatrix} \Longrightarrow \begin{array}{l} \psi^{\mathrm{L}} \mapsto \psi^{\mathrm{L}} + \frac{1}{2}\omega^{\mu\nu}s^{\mathrm{L}}_{\mu\nu}\psi^{\mathrm{L}} \\ \psi^{\mathrm{R}} \mapsto \psi^{\mathrm{R}} + \frac{1}{2}\omega^{\mu\nu}s^{\mathrm{R}}_{\mu\nu}\psi^{\mathrm{R}} \end{array}$$

Left-handed Weyl spinors

Right-handed Weyl spinors

$s^{\mathrm{L}}_{\mu\nu} := -\frac{1}{4}[\sigma_\mu, \bar{\sigma}_\nu]$

$s^{\mathrm{R}}_{\mu\nu} := -\frac{1}{4}[\bar{\sigma}_\mu, \sigma_\nu]$

No mixing between L and R

<u>5D spinors:</u> $\Psi \mapsto \left(I+\frac{1}{2}\Omega^{MN}s_{MN}\right)\Psi, \quad s_{MN} := -\frac{1}{4}[\gamma_M, \gamma_N]$

Big difference from 4D: In 5D, the upper and lower 2 components are mixed (NO Weyl spinors).

$$\psi^{\mathrm{L}} \mapsto \psi^{\mathrm{L}} \,\&\, \psi^{\mathrm{R}}$$
$$\psi^{\mathrm{R}} \mapsto \psi^{\mathrm{R}} \,\&\, \psi^{\mathrm{L}}$$

Relation between 4D and 5D spinors

We will obtain 4D spinors from 5D spinors through KK expansion.

How can we relate 4D and 5D spinors?

Good news: The two have the same size. So, just identify them!

$$\begin{array}{cc} \text{5D} & \text{4D} \\ \Psi = & \begin{pmatrix} \psi^{\mathrm{L}} \\ \psi^{\mathrm{R}} \end{pmatrix} \end{array}$$

Consistency check: Under 4D Lorentz transformation, Ψ must behave as if they were 4D spinors.

4D Lorentz transformation = special case of 5D one:

$x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$ and $x^4 \mapsto x^4$ = $x^M \to x^M + \Omega^M{}_N x^N$ with $\Omega_{M4} = 0,\ \Omega_{\mu\nu} = \omega_{\mu\nu} \Longrightarrow \Lambda^M{}_N = \begin{pmatrix} \Lambda^\mu{}_\nu & 0 \\ 0 & 1 \end{pmatrix}$

$$\Psi \mapsto \left(I + \frac{1}{2}\Omega^{MN} s_{MN}\right)\Psi \quad \xRightarrow{\Omega_{M4} = 0,\ \Omega_{\mu\nu} = \omega_{\mu\nu}} \quad \Psi \mapsto \left(I + \frac{1}{2}\omega^{\mu\nu} s_{\mu\nu}\right)\Psi$$

5D spinor transformation

4D spinor transformation

Lorentz invariant combination

Under 4D Lorentz transformations, $\Psi \mapsto \left(I + \frac{1}{2}\omega^{\mu\nu}s_{\mu\nu}\right)\Psi \quad \Longrightarrow$

$$\Psi^{\mathrm{L}} \mapsto \left(I + \frac{1}{2}\omega^{\mu\nu}s^{\mathrm{L}}_{\mu\nu}\right)\Psi^{\mathrm{L}}$$
$$\Psi^{\mathrm{R}} \mapsto \left(I + \frac{1}{2}\omega^{\mu\nu}s^{\mathrm{R}}_{\mu\nu}\right)\Psi^{\mathrm{R}}$$

$$s_{\mu\nu} := -\frac{1}{4}[\gamma_\mu, \gamma_\nu] \qquad s_{\mu\nu} = \begin{pmatrix} s^{\mathrm{L}}_{\mu\nu} & O_{2\times 2} \\ O_{2\times 2} & s^{\mathrm{R}}_{\mu\nu} \end{pmatrix} \qquad \Psi = \begin{pmatrix} \Psi^{\mathrm{L}} \\ \Psi^{\mathrm{R}} \end{pmatrix}$$

$$s^{\mathrm{L}}_{\mu\nu} = -\frac{1}{4}[\sigma_\mu, \bar{\sigma}_\nu] \qquad s^{\mathrm{R}}_{\mu\nu} = -\frac{1}{4}[\bar{\sigma}_\mu, \sigma_\nu]$$

Their Hermitian conjugation flip L and R as well as the overall sign $(s^{\mathrm{L}}_{\mu\nu})^\dagger = -\,s^{\mathrm{R}}_{\mu\nu}, \quad (s^{\mathrm{R}}_{\mu\nu})^\dagger = -\,s^{\mathrm{L}}_{\mu\nu}$

$$\begin{aligned} \Psi^{\mathrm{L}} &\mapsto \left(I + \frac{1}{2}\omega^{\mu\nu}s^{\mathrm{L}}_{\mu\nu}\right)\Psi^{\mathrm{L}} \\ \Psi^{\mathrm{R}} &\mapsto \left(I + \frac{1}{2}\omega^{\mu\nu}s^{\mathrm{R}}_{\mu\nu}\right)\Psi^{\mathrm{R}} \end{aligned} \quad \overset{\dagger}{\Longrightarrow} \quad \begin{aligned} \Psi^{\mathrm{L}\dagger} &\mapsto \Psi^{\mathrm{L}\dagger}\left(I - \frac{1}{2}\omega^{\mu\nu}s^{\mathrm{R}}_{\mu\nu}\right) \\ \Psi^{\mathrm{R}\dagger} &\mapsto \Psi^{\mathrm{R}\dagger}\left(I - \frac{1}{2}\omega^{\mu\nu}s^{\mathrm{L}}_{\mu\nu}\right) \end{aligned}$$

$\sigma_\mu, \bar{\sigma}_\nu$: Hermitian

$\Psi^{\mathrm{R}\dagger}\Psi^{\mathrm{L}} + \Psi^{\mathrm{L}\dagger}\Psi^{\mathrm{R}}$ is 4D Lorentz invariant

Lesson: (L dagger).R + (R dagger).L is 4D Lorentz invariant

quadratic, nonderivative, **LR mixed** Lorentz invariant combination --> Dirac mass term

cf. quadratic, nonderivative, **one-chirality (LL, RR)** Lorentz invariant combination --> Majorana mass term

Dirac mass terms through bulk-brane interaction

We have shown: under 4D Lorentz transformations, R-part of a 5D spinor transforms just as a 4D R-spinors

$$\Psi^{\mathrm{R}} \mapsto \left(I+\frac{1}{2}\omega^{\mu\nu}s^{\mathrm{R}}_{\mu\nu}\right)\Psi^{\mathrm{R}} \qquad \Psi^{\mathrm{R}\dagger} \mapsto \Psi^{\mathrm{R}\dagger}\left(I-\frac{1}{2}\omega^{\mu\nu}s^{\mathrm{L}}_{\mu\nu}\right)$$

Original motivation: make a mass term for Standard Model (=4D) left-handed neutrinos!

$$\nu^{\mathrm{L}} \mapsto \left(I+\frac{1}{2}\omega^{\mu\nu}s^{\mathrm{L}}_{\mu\nu}\right)\nu^{\mathrm{L}} \qquad \nu^{\mathrm{L}\dagger} \mapsto \nu^{\mathrm{L}\dagger}\left(I-\frac{1}{2}\omega^{\mu\nu}s^{\mathrm{R}}_{\mu\nu}\right)$$

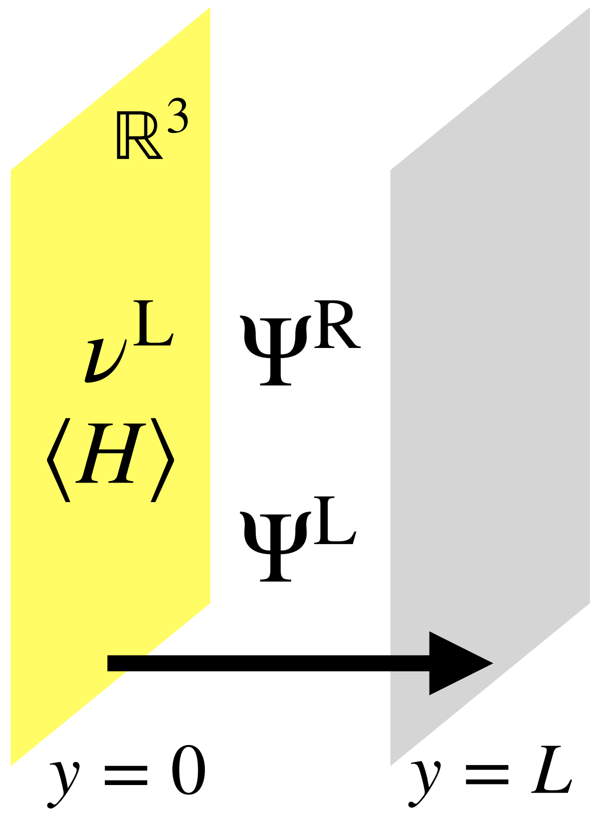
We can form a Dirac mass term with them!

bulk-brane interaction $$S_{b\partial} = g\langle H\rangle \int dt d^3x \, [\Psi^{\mathrm{R}\dagger}(y=0)\nu^{\mathrm{L}} + \nu^{\mathrm{L}\dagger}\Psi^{\mathrm{R}}(y=0)]$$

brane (4D)

from bulk

bulk-brane interaction gives mass scale gv of neutrino.



Remark 1: It's not an exact mass. We need the Kaluza-Klein expansion (shown later).

Remark 2: The bulk-brane coupling can be rewritten in an SU(2) invariant form.
(the same form as the Yukawa coupling for the up quark in SM)

Planck scale suppression of bulk-brane coupling

We have just made $$g v \int dt d^3x \, [\Psi^{\mathrm{R}\dagger}(y=0)\nu^{\mathrm{L}} + \nu^{\mathrm{L}\dagger}\Psi^{\mathrm{R}}(y=0)]$$

Q: What's the dimension of g? Use kinetic terms:

$\int d^4x \, |\partial_\mu H|^2$ 4=2+2[H] --> [v]=[H]=1

$\int d^4x dy \, i\bar{\Psi}\gamma^M \partial_M \Psi$ 5=1+2[Ψ] --> [Ψ]=2

$\int d^4x \, i\bar{\psi}\gamma^\mu \partial_\mu \psi$ 4=1+2[Ψ] --> [Ψ]=3/2

4=[g]+[v]+[Ψ]+[ν] --> [g] = −1/2

g has dimension --> what determines its scale? (large/small compared with what scale?)

It's a theory in 5D. So the only fundamental scale is 5D Planck scale M_*.

$\Longrightarrow$ Natural to think that the dimension is determined by M_*

$\Longrightarrow$ $g = \dfrac{y}{M_*^{1/2}}$ suppression by the 5D Planck mass $M_{\mathrm{Pl}}^2 = M_*^3 L$ $M_{\mathrm{Pl}} \simeq 10^{19}$ GeV

Current bound: L < about 30 micron $\Longrightarrow$ $M_* > 10^{10}$ GeV

Kaluza-Klein expansion of 5D spinors

Kaluza-Klein expansion

- A 4D field is denoted by $\varphi(t, x_1, x_2, x_3)$

  Field in the presence of extra dimensions $\Phi(t, x_1, x_2, x_3, y_1, y_2, \cdots, y_n)$

  extra dimensions of finite-size --> boundary condition --> mode expansion (recall wave & oscillation course)

$$\Phi(t, x_1, x_2, x_3, y_1, y_2, \cdots, y_n) = \sum_n \underbrace{\varphi_n(t, x_1, x_2, x_3)}_{\text{4D fields}} \underbrace{f_n(y_1, y_2, \cdots, y_n)}_{\text{mode functions}}$$

Consider n=1

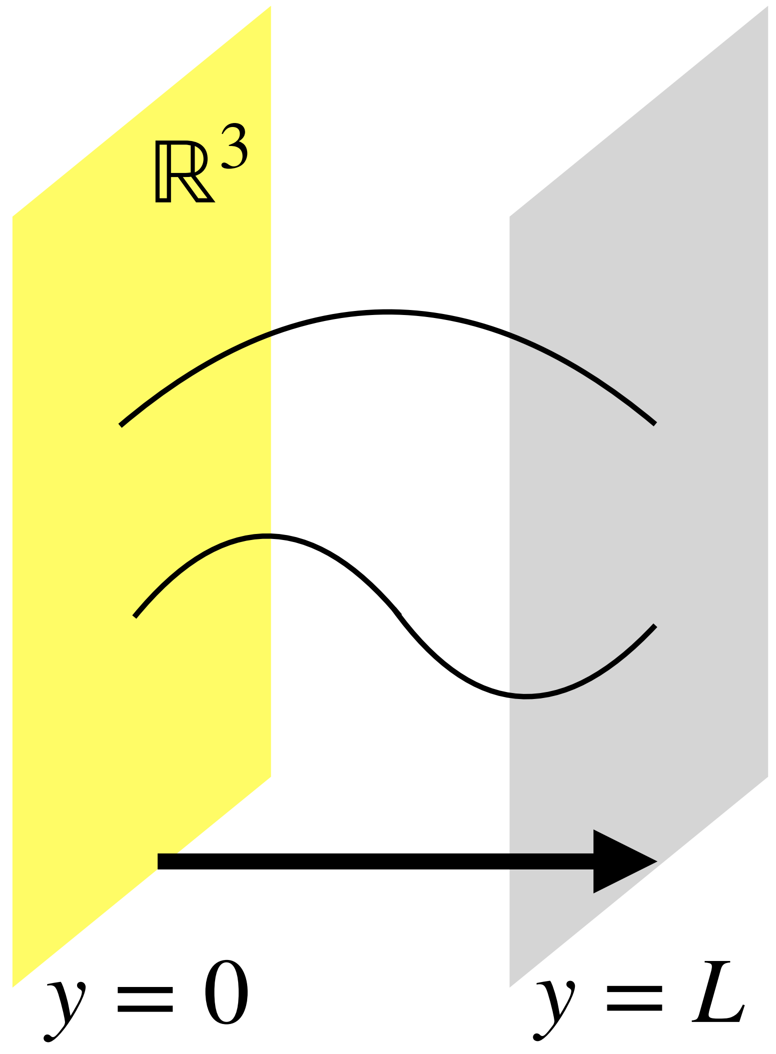
$$\Phi(t, x_1, x_2, x_3, y) = \sum_n \underbrace{\varphi_n(t, x_1, x_2, x_3)}_{\text{amplitude as 4D fields}} \underbrace{f_n(y)}_{\text{mode functions = single-frequency waves}}$$

Kaluza-Klein expansion

<u>Consider n=1</u>

$$\Phi(t, x_1, x_2, x_3, y) = \sum_n \varphi_n(t, x_1, x_2, x_3)\, f_n(y)$$

amplitude as 4D fields — mode functions (single-frequency waves)



periodic boundary condition f(y)=f(y+L) $\quad f_n(y) = \exp\left(i\frac{2\pi n}{L}y\right), \quad n \in \mathbb{Z}$

Dirichlet boundary condition f(y)=f(y+L)=0 $\quad f_n(y) = \sin\left(\frac{\pi n}{L}y\right), \quad n = 1,2,\cdots$

Neumann boundary condition f'(y)=f'(y+L)=0 $\quad f_n(y) = \cos\left(\frac{\pi n}{L}y\right), \quad n = 0,1,2,\cdots$

A higher dim. field produces an infinite tower of 4dim fields with label n through mode expansion: Kaluza-Klein modes

Next question: what are their masses?

Kaluza-Klein masses

KK expansion $\Phi(t, \vec{x}, y) = \sum_n \varphi_n(t, \vec{x})\, f_n(y)$

To obtain masses, write down equation of motion.

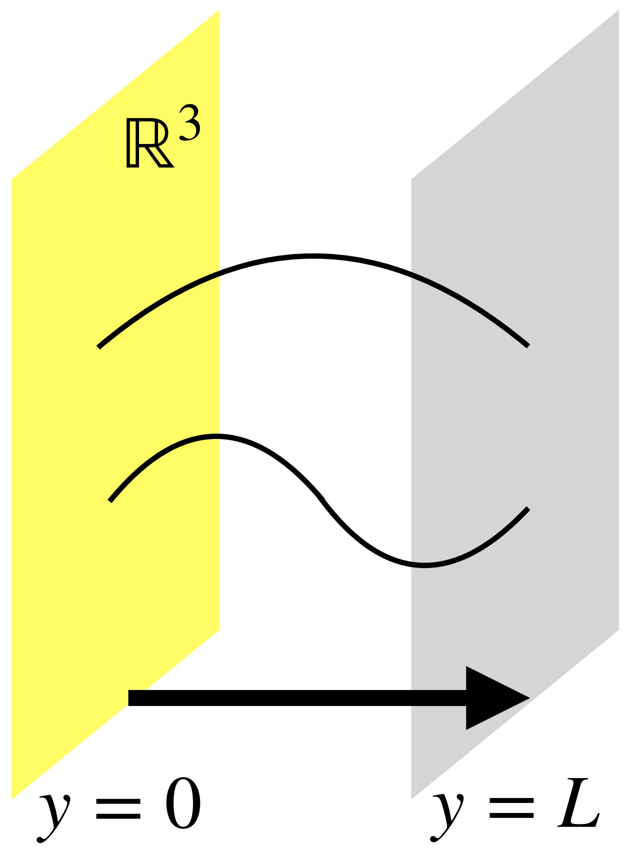
setup: Higher dimensional equation of motion

example: massless 5D Klein-Gordon $(\partial_t^2 - \vec{\nabla}_x^2 - \partial_y^2)\Phi(t, \vec{x}, y) = 0$

Substituting the KK expansion into the bulk EoM,
we obtain EoMs of the 4D fields φ_n and then read off their masses: Kaluza-Klein masses

If you ignore $\vec{x}$, it's nothing but a string between two boundaries
in undergrad course on the wave & oscillation.
You can just regard $\vec{x}$ as indices on top of the mode label n.

Kaluza-Klein masses



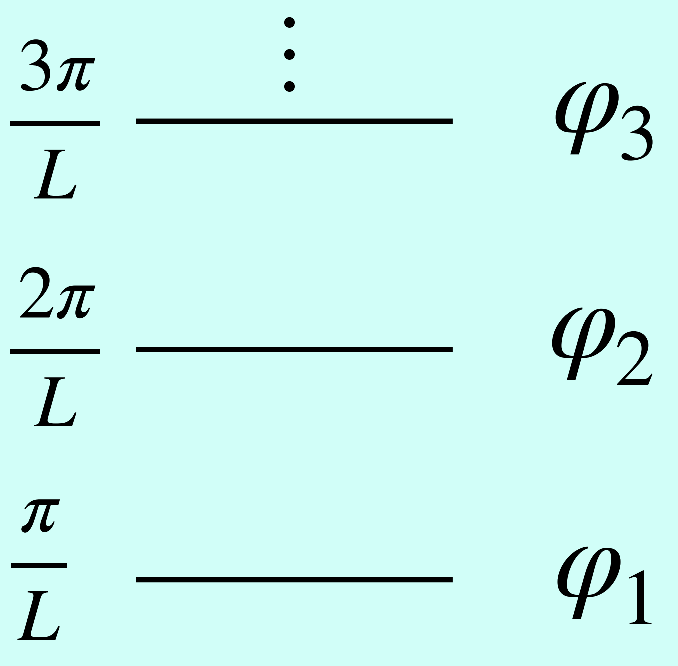
$$\Phi(t, \vec{x}, y) = \sum_n \varphi_n(t, \vec{x})\, f_n(y) \qquad + \qquad (\partial_t^2 - \vec{\nabla}_x^2 - \partial_y^2)\Phi(t, \vec{x}, y) = 0$$

Dirichlet b.c. $\quad f_n(y) = \sin\left(\frac{\pi n}{L} y\right) \quad \to \quad \left(\partial_t^2 - \vec{\nabla}_x^2 + \left(\frac{\pi}{L} n\right)^2\right) \varphi_n(t, \vec{x}) = 0$

5dim field Φ = an infinite tower of φ_n of mass $\pi |n| / L$ for $n > 0$

No zero mode. Only massive modes.

If L is too small, all modes will not be observed.

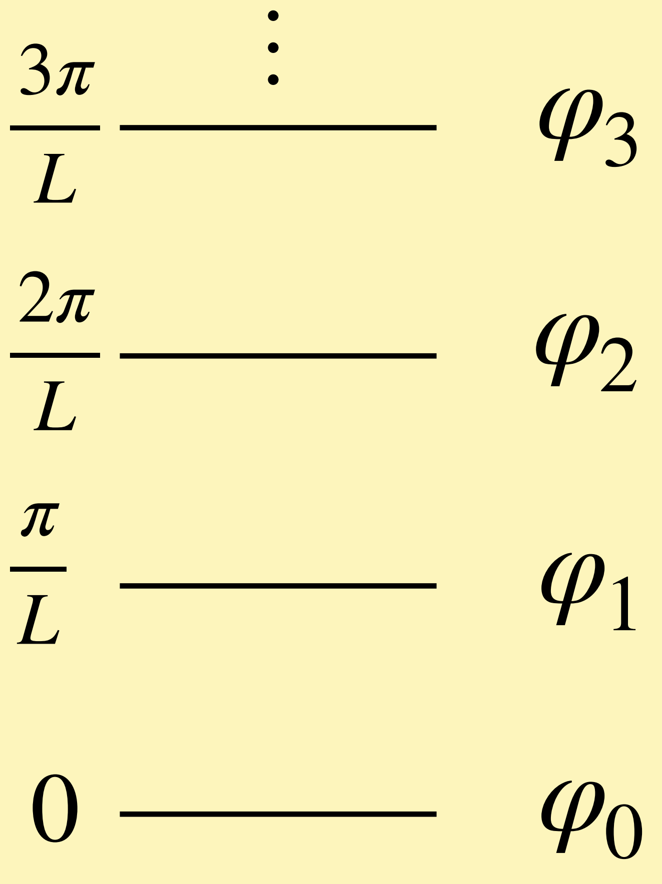


Neumann b.c. $\quad f_n(y) = \cos\left(\frac{\pi n}{L} y\right) \quad \to \quad \left(\partial_t^2 - \vec{\nabla}_x^2 + \left(\frac{\pi}{L} n\right)^2\right) \varphi_n(t, \vec{x}) = 0$

5dim field Φ = an infinite tower of φ_n of mass $\pi |n| / L$ for $n \geq 0$

n=0: massless, |n|>0: massive

If L is sufficiently small, only zero mode will survive at low energies



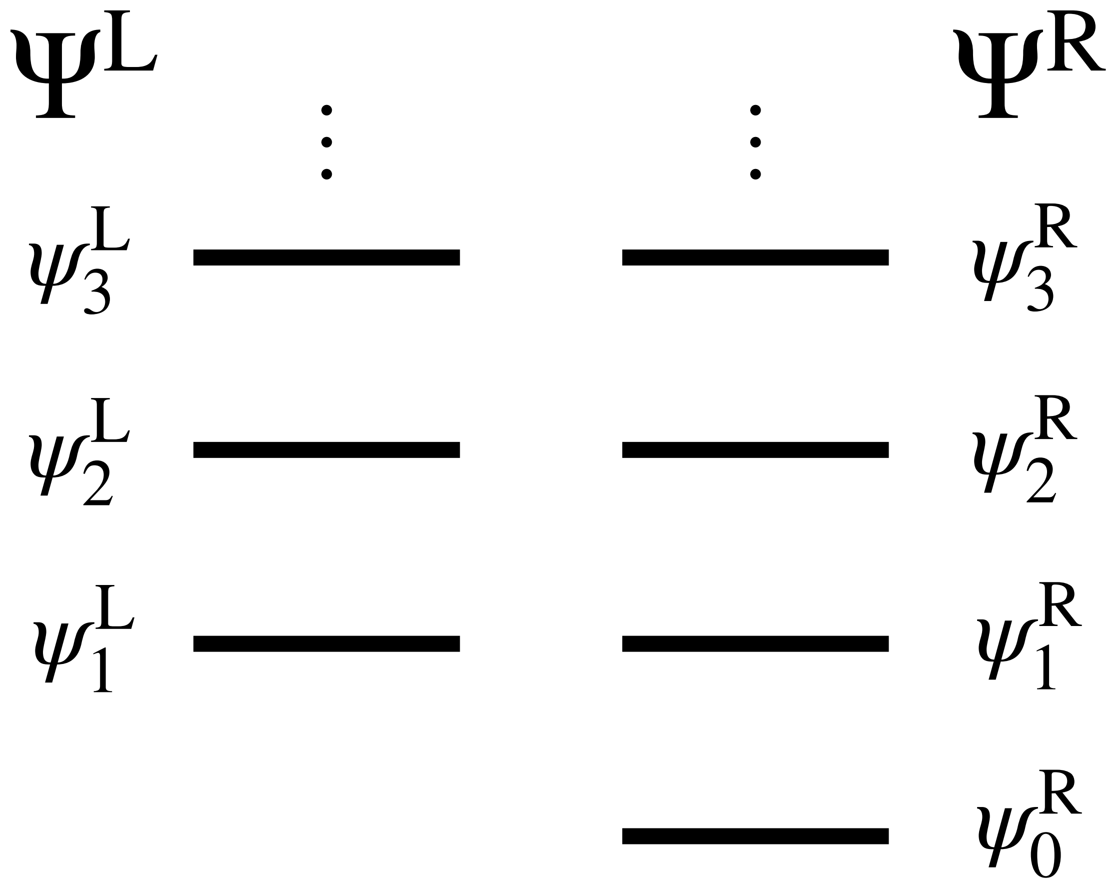
Kaluza-Klein expansion of 5D spinor

KK expansion $\Psi = \begin{pmatrix} \Psi^{\mathrm{L}} \\ \Psi^{\mathrm{R}} \end{pmatrix}$ $\quad \Psi^{\mathrm{L}}(t, \vec{x}, y) = \sum_n \psi_n^{\mathrm{L}}(t, \vec{x}) f_n^{\mathrm{L}}(y)$ $\quad \Psi^{\mathrm{R}}(t, \vec{x}, y) = \sum_n \psi_n^{\mathrm{R}}(t, \vec{x}) f_n^{\mathrm{R}}(y)$

Under 4D transformations, Ψ^{L} and Ψ^{R} transform as 4D L and R spinors:

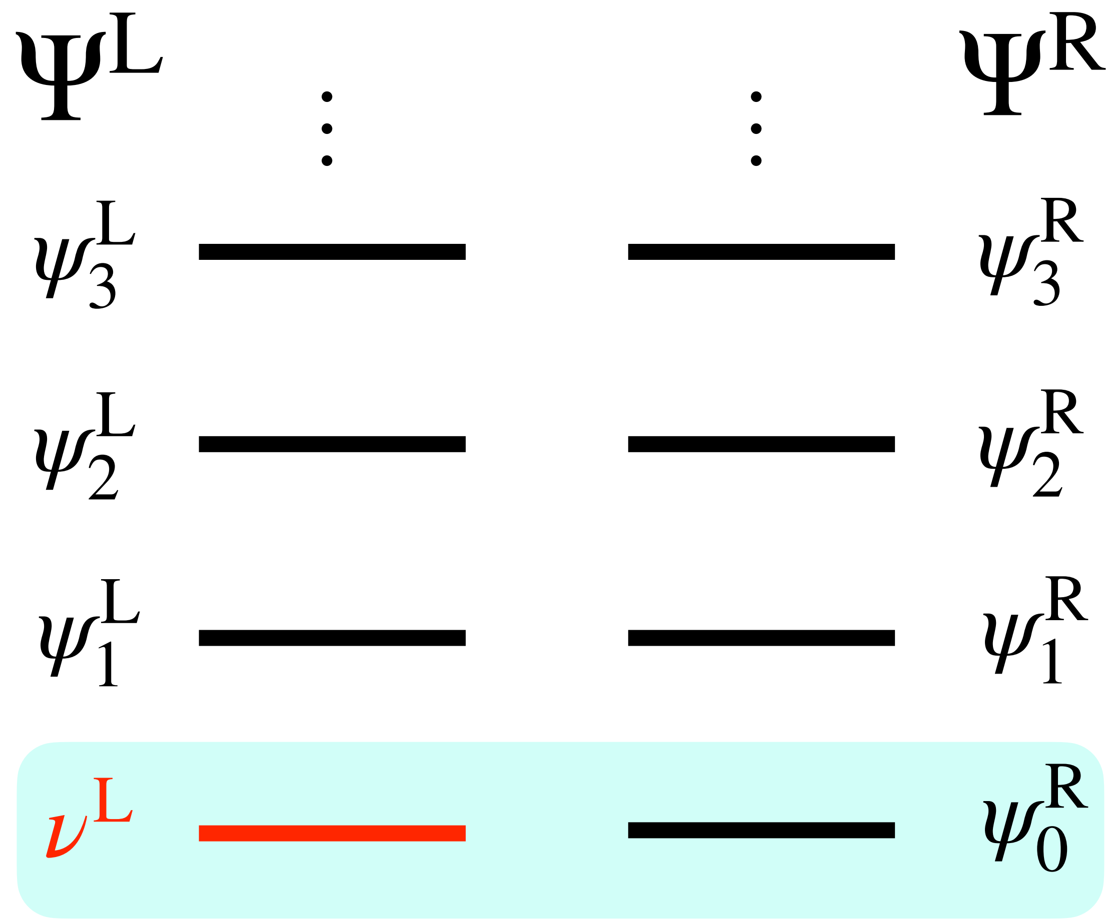
So, 4D KK fields ψ_n^{L} and ψ_n^{R} also transform as 4D L and R spinors, as desired.

Scenario: without bulk-brane coupling



If there is a L-zero mode on top of neutrino,
it may have been already detected
--> let's suppress it as a minimum setup.

with bulk-brane coupling



Dirac mass

bulk-brane interaction

Total action of the 5D model

What is the total action of the system?

Dirac action of the bulk fermion $\int d^4xdy\ i\bar{\Psi}\gamma^M\partial_M\Psi$ Lorentz invariant kinetic term

$\Downarrow$

equation of motion: $i\gamma^M\partial_M\Psi = 0 \iff$
$$i\bar{\sigma}^\mu\partial_\mu\Psi^{\mathrm{L}} + \partial_y\Psi^{\mathrm{R}} = 0$$
$$i\sigma^\mu\partial_\mu\Psi^{\mathrm{R}} - \partial_y\Psi^{\mathrm{L}} = 0$$

Left-Right mixed

$$\gamma^\mu = \begin{pmatrix} O_{2\times2} & \sigma^\mu \\ \bar{\sigma}^\mu & O_{2\times2} \end{pmatrix}$$

$$\gamma^4 = i\begin{pmatrix} I_{2\times2} & O_{2\times2} \\ O_{2\times2} & -I_{2\times2} \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \Psi^{\mathrm{L}} \\ \Psi^{\mathrm{R}} \end{pmatrix}$$

$$\bar{\Psi} = (\Psi^{\mathrm{R}\dagger}, \Psi^{\mathrm{L}\dagger})$$

$+$

bulk-boundary interaction $g\langle H\rangle \int dtd^3x\,[\Psi^{\mathrm{R}\dagger}(y=0)\nu^{\mathrm{L}} + \nu^{\mathrm{L}\dagger}\Psi^{\mathrm{R}}(y=0)]$

1. We first make KK expansion of the bulk part,
2. Add the bulk-brane interaction.

LR asymmetry in boundary conditions

Impose Dirichlet b.c. on L: $\Psi^{\mathrm{L}}(y=0) = \Psi^{\mathrm{L}}(y=L) = 0$

$f_n^{\mathrm{L}}(y) = \sin(n\pi y/L), \quad n = 1, 2, \cdots$ No zero mode

This <u>automatically fixes</u> the boundary condition of the R-part through the equation of motion

Ψ^{L}

$\vdots$

ψ_3^{L}

ψ_2^{L}

ψ_1^{L}

LR asymmetry in boundary conditions

Impose Dirichlet b.c. on L: $\Psi^{\mathrm{L}}(y=0) = \Psi^{\mathrm{L}}(y=L) = 0$

$f_n^{\mathrm{L}}(y) = \sin(n\pi y/L), \quad n = 1,2,\cdots$ No zero mode

This automatically fixes the boundary condition of the R-part through the equation of motion

Dirac equation of 5D spinor

$$i\bar{\sigma}^\mu \partial_\mu \Psi^{\mathrm{L}}(t,\vec{x},y) + \partial_y \Psi^{\mathrm{R}}(t,\vec{x},y) = 0$$ Left-Right mixed

$\Downarrow$

At the boundaries y=0,L, this term vanishes since it has no y-derivative. $\Longrightarrow$ 0 $\Longrightarrow$

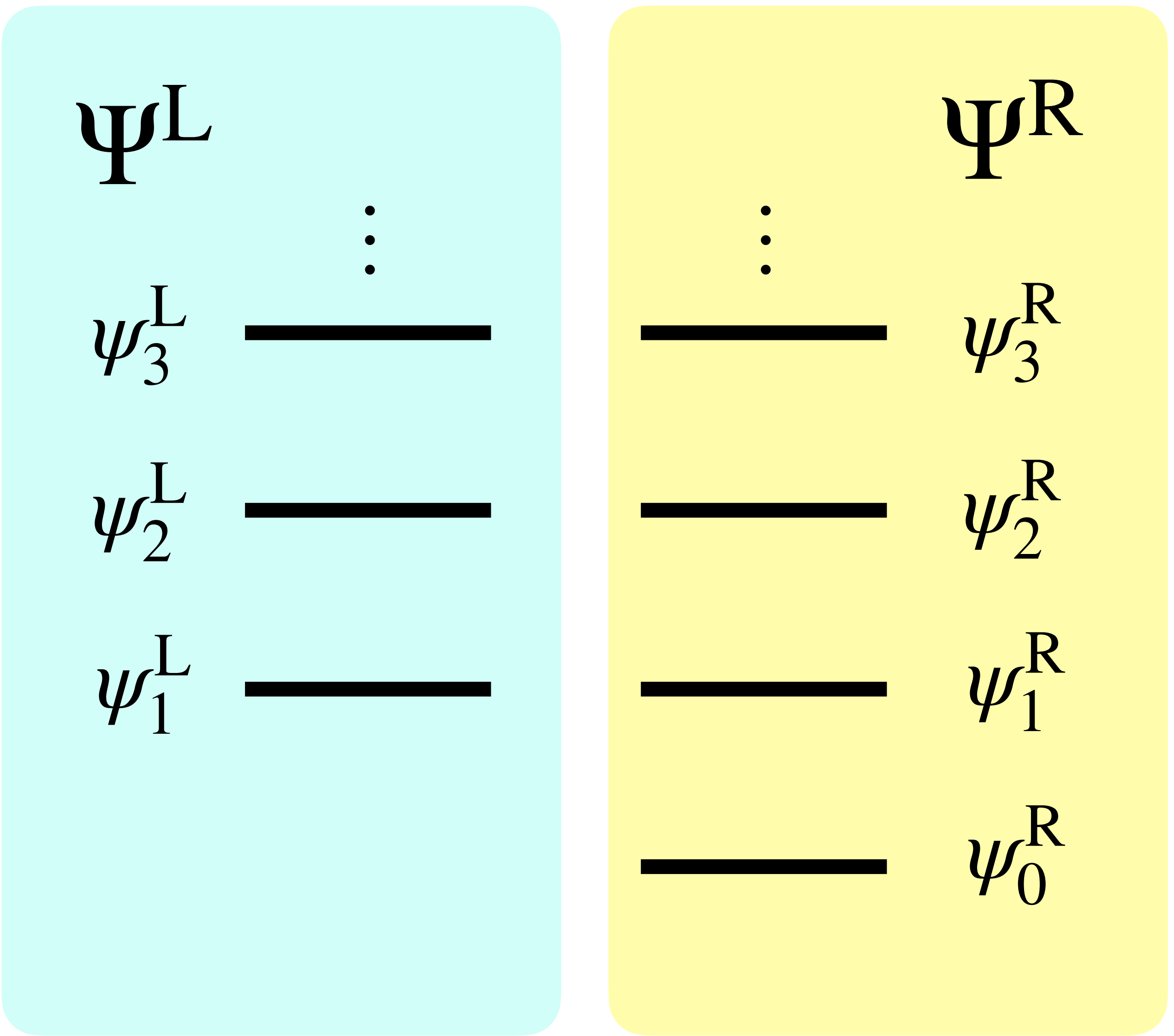
$\partial_y \Psi^{\mathrm{R}}|_{y=0,L} = 0$

Neumann boundary condition

$f_n^{\mathrm{R}}(y) = \cos(n\pi y/L), \quad n = 0,1,2,\cdots$

Zero mode exists!

Geometrical realisation of Left-Right asymmetry



Kaluza-Klein mass spectrum

5D Dirac equations + mode expansions --> 4D Dirac/Weyl equations

How can we read off the mass of spinor fields?

4D Dirac equations $\quad i\sigma^\mu\partial_\mu\psi^{\mathrm{R}} - m\psi^{\mathrm{L}} = i\bar{\sigma}^\mu\partial_\mu\psi^{\mathrm{L}} - m\psi^{\mathrm{R}} = 0 \quad \Longrightarrow \quad$ mass = |m|

4D Weyl equations $\quad i\sigma^\mu\partial_\mu\psi^{\mathrm{R}} = i\bar{\sigma}^\mu\partial_\mu\psi^{\mathrm{L}} = 0 \quad \Longrightarrow \quad$ mass = 0

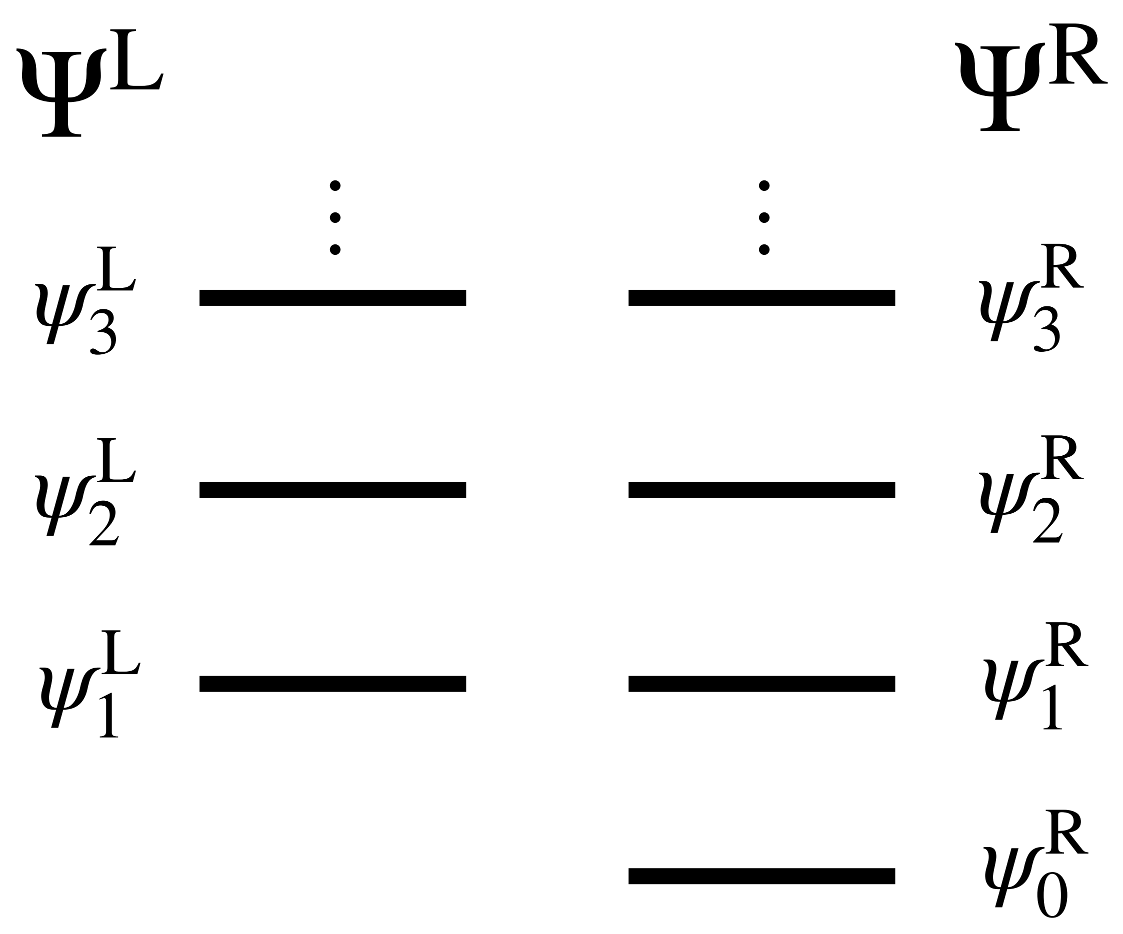
$$i\sigma^\mu\partial_\mu\Psi^{\mathrm{R}}(t,\vec{x},y) - \partial_y\Psi^{\mathrm{L}}(t,\vec{x},y) = 0 \quad \Longrightarrow \quad i\sigma^\mu\partial_\mu\psi_n^{\mathrm{R}} - (n\pi/L)\psi_n^{\mathrm{L}} = 0$$

$$i\bar{\sigma}^\mu\partial_\mu\Psi^{\mathrm{L}}(t,\vec{x},y) + \partial_y\Psi^{\mathrm{R}}(t,\vec{x},y) = 0 \quad \Longrightarrow \quad i\sigma^\mu\partial_\mu\psi_n^{\mathrm{L}} + (n\pi/L)\psi_n^{\mathrm{R}} = 0$$

$\Longrightarrow$ mass spectrum $\quad m_n = \dfrac{\pi}{L}n$

$$\Psi^{\mathrm{L}}(t,\vec{x},y) = \sum_n \psi_n^{\mathrm{L}}(t,\vec{x})\ \sin(\pi n y/L)$$

$$\Psi^{\mathrm{R}}(t,\vec{x},y) = \sum_n \psi_n^{\mathrm{R}}(t,\vec{x})\ \cos(\pi n y/L)$$



Dirac equation and mass

4D Dirac equations	$i\sigma^\mu \partial_\mu \psi^{\mathrm{R}} - m\psi^{\mathrm{L}} = i\bar{\sigma}^\mu \partial_\mu \psi^{\mathrm{L}} - m\psi^{\mathrm{R}} = 0$	$\Longrightarrow$	mass = |m|
4D Weyl equations	$i\sigma^\mu \partial_\mu \psi^{\mathrm{R}} = i\bar{\sigma}^\mu \partial_\mu \psi^{\mathrm{L}} = 0$	$\Longrightarrow$	mass = 0

WHY?

Dirac equation: $i\sigma^\mu \partial_\mu \psi^{\mathrm{R}} - m\psi^{\mathrm{L}} = i\bar{\sigma}^\mu \partial_\mu \psi^{\mathrm{L}} - m\psi^{\mathrm{R}} = 0 \leftrightarrow i\gamma^\mu \partial_\mu \psi - m\psi = 0$ $\quad \gamma^\mu = \begin{pmatrix} O_{2\times 2} & \sigma^\mu \\ \bar{\sigma}^\mu & O_{2\times 2} \end{pmatrix}$

$$m^2\psi = m(m\psi) = m(i\gamma^\mu \partial_\mu \psi) = i\gamma^\mu \partial_\mu (m\psi) = (i\gamma^\mu \partial_\mu)(i\gamma^\nu \partial_\nu)\psi = -\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu \psi = \partial_\mu \partial^\mu \psi$$

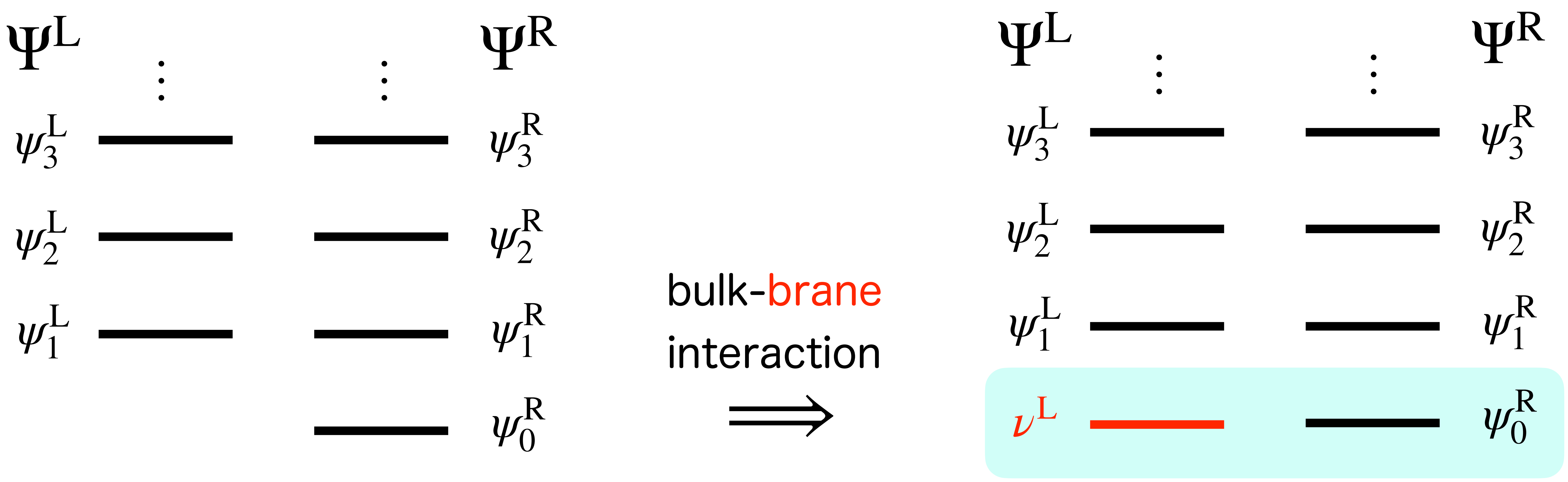
$$\Downarrow$$

$$(\partial_\mu \partial^\mu - m^2)\psi = 0$$ Klein-Gordon equation

$$\Downarrow$$

mass = |m|

With bulk-brane interaction



Dirac mass of SM neutrino

If 1/L is larger than SM energy scale, the higher modes are not observed directly.

bulk-brane coupling | Dirac mass

$$\frac{y\langle H\rangle}{M_*^{1/2}}[\Psi^{\rm R\dagger}(y=0)\nu^{\rm L}+\nu^{\rm L\dagger}\Psi^{\rm R}(y=0)] \supset \frac{y\langle H\rangle f^{\rm R}(0)}{M_*^{1/2}}[\psi_0^{\rm R\dagger}\nu^{\rm L}+\nu^{\rm L\dagger}\psi_0^{\rm R}]$$

$$\Psi^{\rm R}(t,\vec{x},y)=\sum_n \psi_n^{\rm R}(t,\vec{x})f_n^{\rm R}(y) \qquad f^{\rm R}(0)=\frac{1}{\sqrt{L}}$$

Remark 1: There are also mixings of the neutrino with higher modes. So, we need to diagonalise the mass terms to read off the exact mass. But, KK spectrum structure and Planck scale suppression are still kept.

Remark 2: We don't have a direct coupling between bulk fermion and SM weak boson Thus, KK fermions behave as sterile neutrinos.

Conclusion

- Extra dimensional setup can offer natural suppression of interactions thourgh higher dim. Planck scale.
- 4D and 5D spinors look almost the same.
  5D spinors necessarily mix L and R handed spinors (left-right symmetric) as opposed to 4D spinors.
  Once they are put on a finite interval and make the KK expansion,
  we can suppress the L part and keep only R part.
  This realises the violation of the left-right symmetry in 4D.
- We can apply this to the Dirac mass term generation of the Standard Model neutrino
  through a bulk-brane coupling between the R zero mode of a bulk 5D fermion and the L-handed SM neutrino.
  Higher KK mode fermions behave as sterile neutrinos.
- The bulk-brane coupling has a negative mass dimension, which allows the Planck mass suppression of the coupling and achieves a small neutrino mass.
- If masses of the higher KK modes are detected by neutrino experiments (direct mass, oscillation), it would give a hint for the extra dimensions.

Thank you very much!

Backup

Gravitational forces in 4D and bulk

- Consider a (4+n)D bulk with a 4D spacetime embedded
  gravitational force propagates in the bulk
  Question: How does the bulk gravitational force feel in the (3+1)D Minkowski spacetime brane?

- 1st task: find gravitational constant in general dimensions
  Gauss law as in electrostatics (<-- in both cases, potentials satisfy Laplace eq. with a source in space)
  Coulomb force = (electric charge) x (electric field), electric field satisfies Gauss law
  Gravitational force = (mass) x (gravitational field), gravitational field satisfies Gauss law

$$F = qE, \quad 4\pi r^2 E = \varepsilon_0 Q \qquad\qquad F = qE, \quad r^{2+n} E \sim \varepsilon_{4+n} Q$$

$$F = mg, \quad 4\pi r^2 g = -G_{\mathrm{N}} M \qquad\qquad F = mg, \quad r^{2+n} g \sim -G_{4+n} M$$

$$[G_N] = \mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2} \qquad\qquad [G_{4+n}] = \mathrm{M}^{-1}\mathrm{L}^{3+n}\mathrm{T}^{-2}$$

Gravitational constant and Planck mass

- Use energy scales to classify physical phenomena
- Planck mass = [mass] quantity made of the fundamental constants $c, \hbar, G$

$$M_P = \sqrt{\frac{c\hbar}{G_\mathrm{N}}} \qquad M_{4+n} = \left(\frac{c^{1-n}\hbar^{1+n}}{G_{4+n}}\right)^{\frac{1}{2+n}}$$

- use the natural unit: $c = \hbar = 1$: [mass] = [energy] = [1/length] = [1/time]

$$M_P = G_\mathrm{N}^{-\frac{1}{2}} \qquad M_{4+n} = G_{4+n}^{-\frac{1}{2+n}}$$

- The Planck mass gives the maximum energy (mass) scale beyond which quantum gravity will be needed.
- 4D Planck mass is much^3 higher than the energy scale of Standard Model (electroweak scale) O(100) GeV
- 4D Minkowski brane in the bulk: $G_\mathrm{N} = \dfrac{G_{4+n}}{V_n}$ $\qquad M_\mathrm{Pl}^2 = M_{4+n}^{2+n} V_n$

An idea: explain the huge M_Pl in terms of smaller M_{4+n} and the volume V_n

Gravitational forces in 4D and bulk

- Next task: How does the bulk gravitational force feel in (3+1)D brane?

  R: size of each direction in the extra dimensions

  Region of scale r >> R: extra dimensions look too small --> effectively 4D --> 4D gravity

  Region of scale r < R: extra dimensions get visible --> bulk --> (4+n)D gravity

  --> at r=R, the two gravitational forces may be comparable

$$\frac{G_{\mathrm{N}}}{R^2} = \frac{G_{4+n}}{R^{2+n}} \quad \Longrightarrow \quad G_{\mathrm{N}} = \frac{G_{4+n}}{R^n} = \frac{G_{4+n}}{V_n}$$

- The volume of the extra directions reduces the bulk gravitational constant

  --> the bulk gravitational constant as a fundamental scale of the system

  --> 4D gravitational constant emerges as an effective one, suppressed by the volume of the extra dimensions

Gravitational forces in 4D and bulk

sloppy notation: $1 = i = e = \pi$

- A simple way to derive this: look at the Einstein-Hilbert action

$$\frac{1}{G_{4+n}}\int d^4x d^n y\sqrt{-G}R(G)$$

decompose the metric: $G_{MN} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & h_{ij}(y) \end{pmatrix}$

curvature: $R(G) \sim R(g) + R(h)$

$M, N = 0,1,2,\cdots,3+n$
$\mu, \nu = 0,1,2,3$
$i, j = 1,2,\cdots,n$

The y-integral gives an integral over 4D Minkowski

$$V_n = \int d^n y\sqrt{h}$$

$$\frac{1}{G_{4+n}}\int d^4x d^n y\sqrt{-G}R(G) \supset \frac{1}{G_{4+n}}\int d^n y\sqrt{h}\int d^4x\sqrt{-g}R(g) \sim \frac{V_n}{G_{4+n}}\int d^4x\sqrt{-g}R(g) = \frac{1}{G_{\mathrm{N}}}\int d^4x\sqrt{-g}$$

This is the effective 4D action we want

$$\frac{V_n}{G_{4+n}}\int d^4x\sqrt{-g}R(g) = \frac{1}{G_{\mathrm{N}}}\int d^4x\sqrt{-g}R(g) \implies \frac{V_n}{G_{4+n}} = \frac{1}{G_{\mathrm{N}}} \implies G_{\mathrm{N}} = \frac{G_{4+n}}{V_n}$$

In Planck masses

$$M_{\mathrm{Pl}}^2 = M_{4+n}^{2+n}V_n$$