RUHR-UNIVERSITÄT BOCHUM

Angular analysis of X → VV systems

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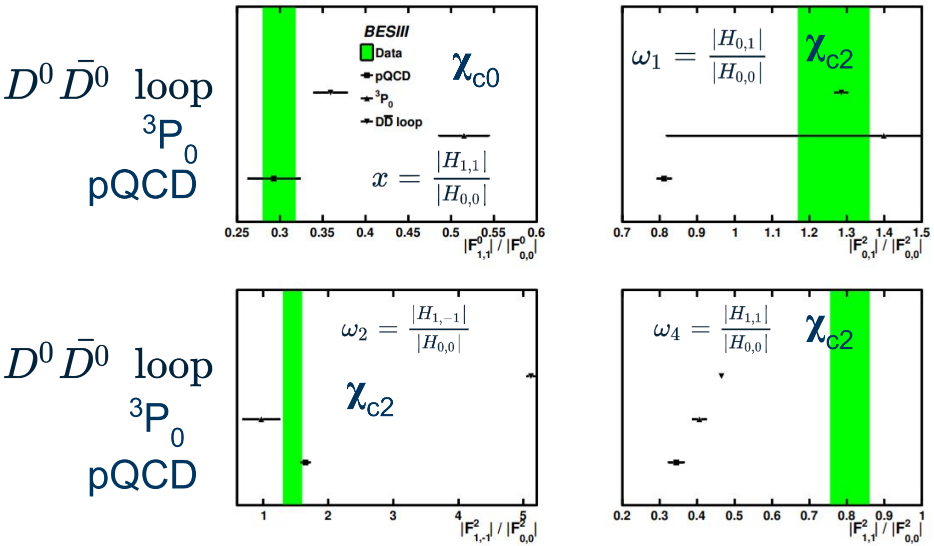
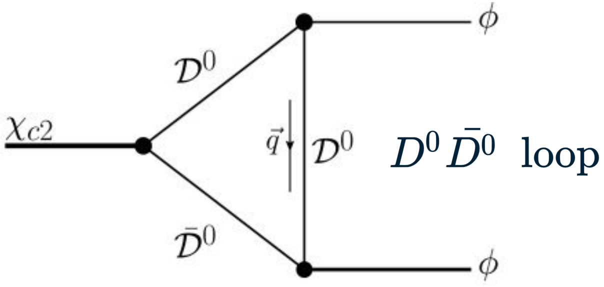
Motivation of $c\bar{c} \to \phi\phi$ angular analysis

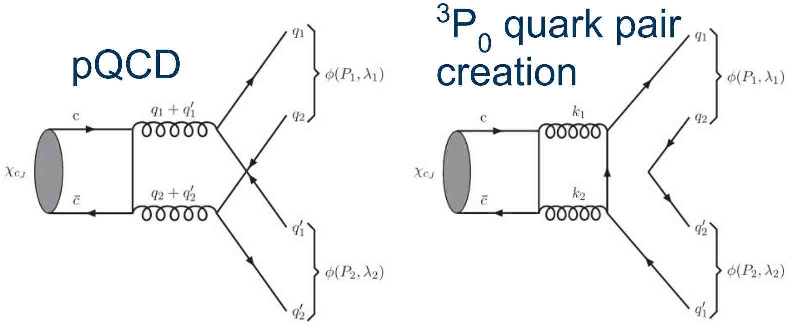
- **Angular analysis of b-hadron → $c\bar{c} \to \phi(\to K^+K^-)\phi(\to K^+K^-)$ provide a powerful tool to study:**
  - Polarization ratios ($|H_{longitudinal}|/|H_{transverse}|$) of χ_{c0} and χ_{c2}
  - Partial wave ratios
  - Discriminate between $c\bar{c} \to \phi\phi$ decay models
  - Get information about inner structure of $c\bar{c}$ states



JHEP 05 (2023) 069



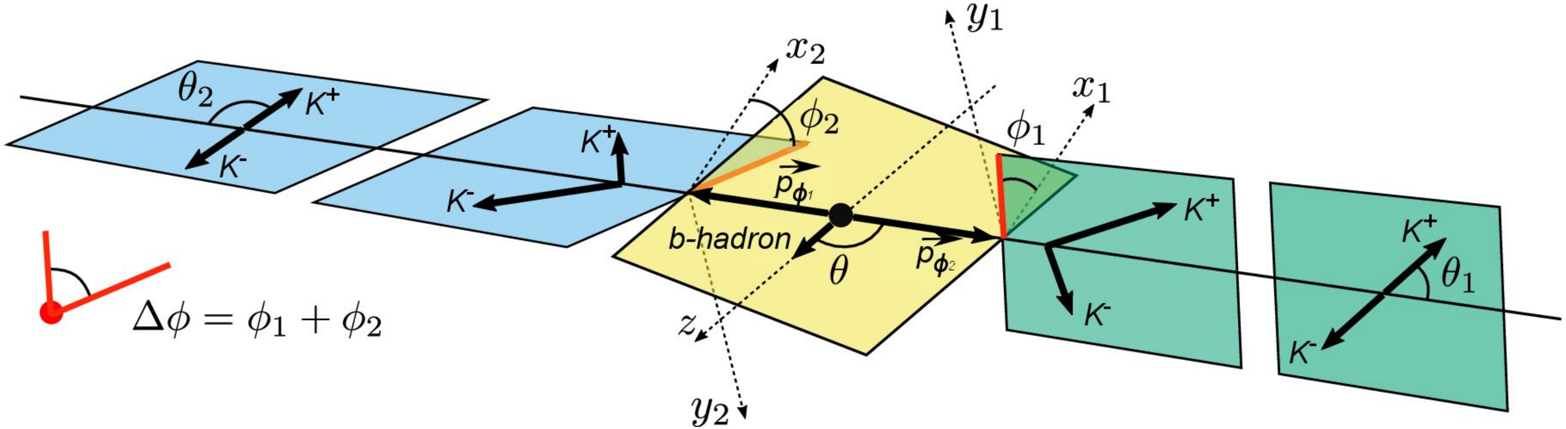
Phys. Rev. D103 (2021) 096006



Phys. Rev. D88 (2013) 034025

Introduction

- **Process considered: b-hadron→$c\bar{c}$→ϕ(→K^+K^-)ϕ(→K^+K^-)**
- **Angular observables:**
  - θ_1, θ_2 - polar angles between kaons and the direction of ϕ mesons
  - $\Delta\phi$ - is a sum of two azimuthal angles between kaons planes and ϕ mesons planes
  - All calculated from the 4-momenta of the kaon candidates

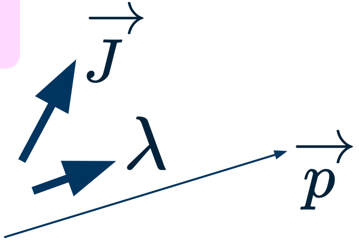


Amplitude model and Helicity matrix

JHEP 07 (2024) 092

- **Decay Amplitude is constructed considering parity conservation and permutation symmetry**

$$A_{J;\nu}(\theta_1,\theta_2,\Delta\phi) = \frac{3}{2}\sum_{\lambda_1,\lambda_2}\delta_{\nu,\lambda_1-\lambda_2} H^J_{\lambda_1,\lambda_2} d^1_{\lambda_1,0}(\theta_1) d^1_{\lambda_2,0}(\theta_2) e^{i\lambda_2\Delta\phi}$$



  - $H^J_{\lambda 1,\lambda 2}$ is a Helicity matrix
  - $d^J_{\lambda,0}(\theta)$ are Wigner d-functions
  - λ_1, λ_2 are helicity states of the ϕ mesons
  - ν is a difference between ϕ mesons helicities, while kaon helicities are 0
- **Helicity matrix contain helicity couplings a,b,c,d**
  - Naturality:

$$\epsilon = P(-1)^J$$

  - Signum:

$$s = (-1)^J$$

$$H = \begin{pmatrix} b & a & c \\ sa & d & \epsilon sa \\ sc & \epsilon a & \epsilon b \end{pmatrix} = \sum_{LS} c_{LS} H_{LS}$$

Symmetry constraints and partial waves

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	n=3,4,5, …
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

All allowed J^P states can be divided into four groups (with special cases) highlighted with different colors

Properties of the analyzed states

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
			2^-	3^-	4^-						3	f-wave
		4^+				2^+	3^+	4^+	5^+	6^+	4	g-wave

- All the states considered in the analysis are spreaded across first five partial waves
- J/ψ and h_c are not accessible since they are prohibited by the symmetry constraints

Properties of the analyzed states

s_1	s_2	S=0	S=1	S=2	L	
1^-	1^-	χ_{c0}: 0^+		χ_{c2}: 2^+	0	s-wave
			η_c: 0^-		1	p-wave
		χ_{c2}: 2^+		χ_{c0}: 0^+, χ_{c1}: 1^+, χ_{c2}: 2^+	2	d-wave
					3	f-wave
				χ_{c2}: 2^+	4	g-wave

- η_c, $\eta_c(2S)$: $J^P=0^-$, special case of Group IV ➞ 1 helicity coupling
- χ_{c0}: $J^P=0^+$, special case of Group I ➞ 2 helicity couplings
- χ_{c1}: $J^P=1^+$, special case of Group II ➞ 1 helicity coupling
- χ_{c2}: $J^P=2^+$, Group I ➞ **4 helicity couplings**, the most complicated case

Angular functions of 3D intensity

- **Using amplitude, intensity can be represented in the simple way as:**

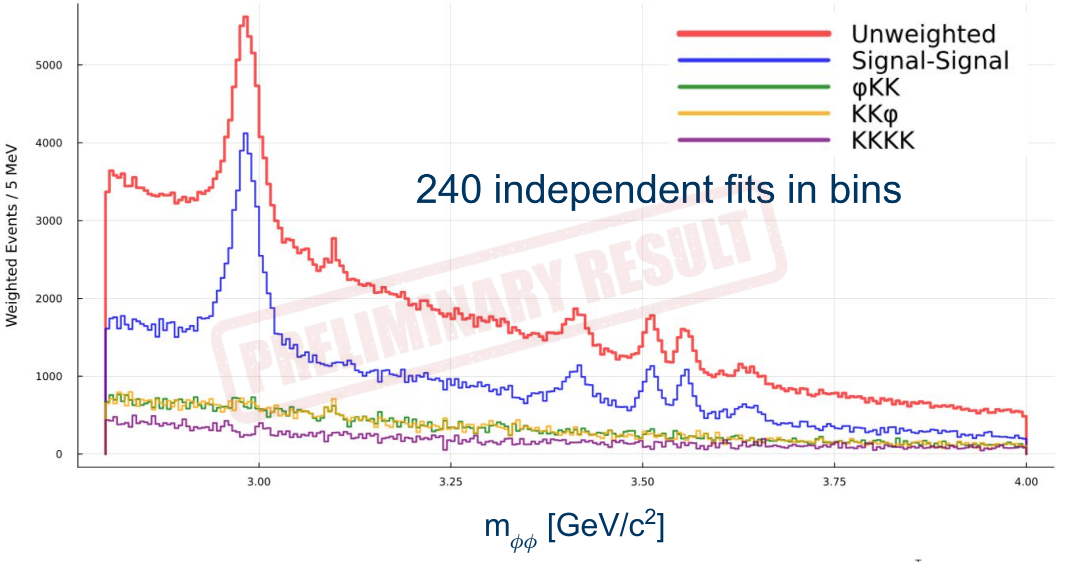
$$I(\theta_1, \theta_2, \Delta\phi) = \sum_{\nu=-2}^{2} |A_\nu(\theta_1, \theta_2, \Delta\phi)|^2 = \sum_{i=1}^{6} c_i f_i$$

i	basis functions, f_i	coeff. $c_i * (4\|a\|^2 + 2\|b\|^2 + 2\|c\|^2 + \|d\|^2)$
1	$9\sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\Delta\phi)/2$	$-2\epsilon\|b\|^2$
2	$\sin(\theta_1)\sin(\theta_2)\cos(\theta_1)\cos(\theta_2)\cos(\Delta\phi)$	$18\epsilon\|a\|^2 s - 9(\epsilon+1)\mathrm{Re}(b^* d)$
3	$9\sin^2(\theta_1)\sin^2(\theta_2)\cos^2(\Delta\phi)/4$	$2\epsilon\|b\|^2 - 8\|a\|^2 + 2\|b\|^2 + 2\|c\|^2 + 4\|d\|^2$
4	$3\sin^2(\theta_1)/2$	$6\|a\|^2 - 6\|d\|^2$
5	$3\sin^2(\theta_2)/2$	$6\|a\|^2 - 6\|d\|^2$
6	1	$9\|d\|^2$

Helicity couplings to fit

Mass dependencies

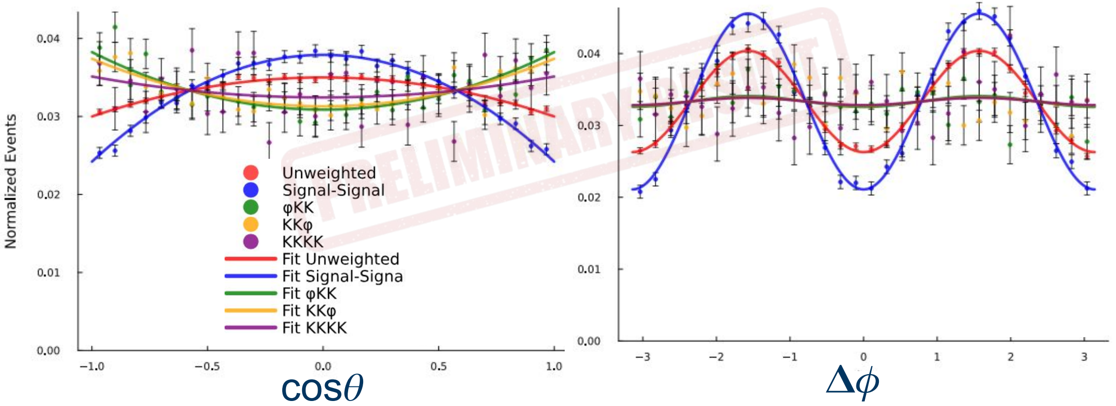
- 2D fits to extract sWeights
- Mass distribution produced based on the fitted yields
  - Less combinatorial background
  - J/ψ peak removed



- Angular distributions are fitted in each mass bin
- Accounting for the sWeights

$$\mathcal{L} = \prod_i \frac{\sum_k (w_k)}{\sum_j (w_j^2)} \, w_i \, \frac{I(\cos\theta_1^i, \cos\theta_2^i, \Delta\phi^i; \vec{c})}{8\pi(c_1 + c_3 + 2c_4 + c_6)}$$

- c_i mass dependencies are extracted (c_6 is fixed to 1 due to normalization)



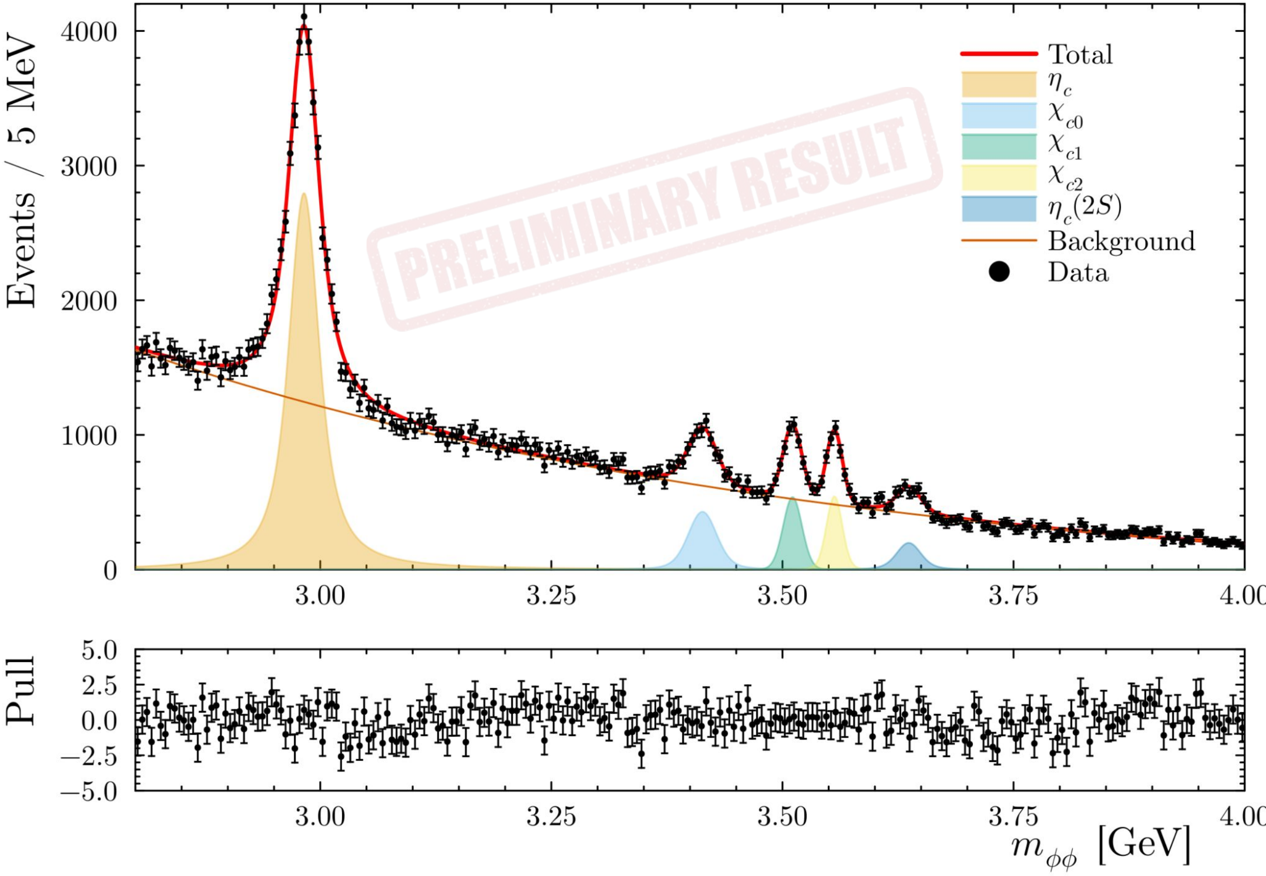
Mass fit

- **Signal**: RBW x Gauss
- **Background**:

$$(1 + \alpha_1 m_{\phi\phi} + \alpha_2 m_{\phi\phi}^2)\exp(-m_{\phi\phi}/\tau)$$

- **Assumption:**
  no interference
  with background

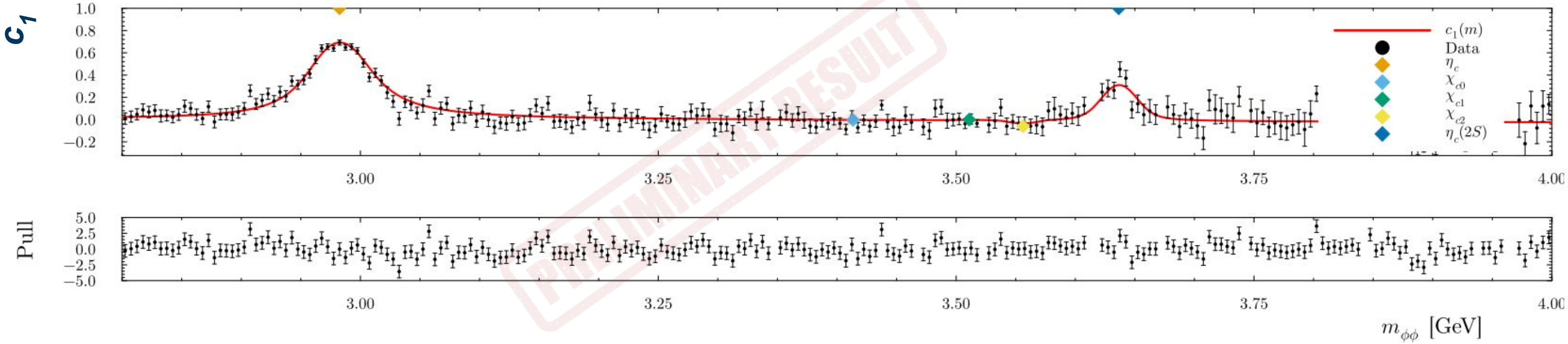
- **Fit quality:**
  - ndf = 222
  - χ^2/ndf = 1.068



Angular fit

- Fractions are fixed from the mass fit
- Bin-to-bin fluctuations due to 3D intensity fits with sWeights
- Fit quality: ndf = 1131, χ^2/ndf = 1.176

- Fit parameters:
  - Contributions of non-minimal partial waves
  - Linear background parameters



Angular fit

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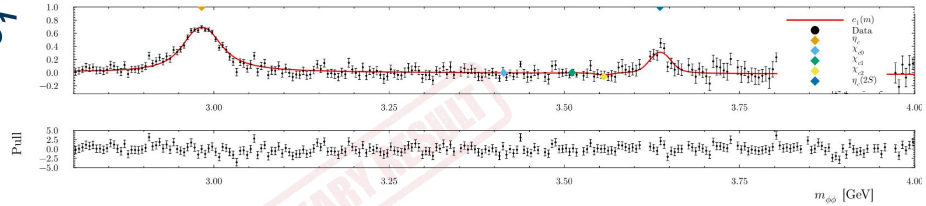
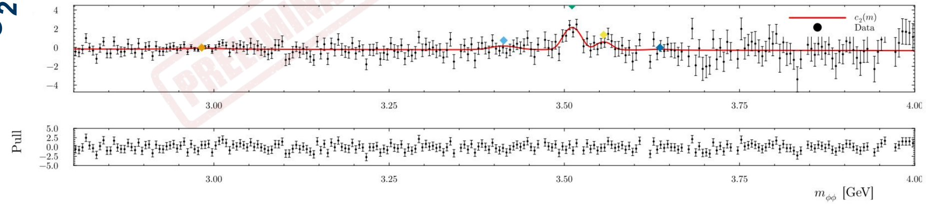
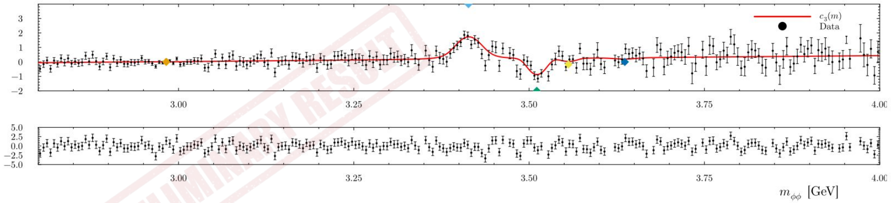
c_1



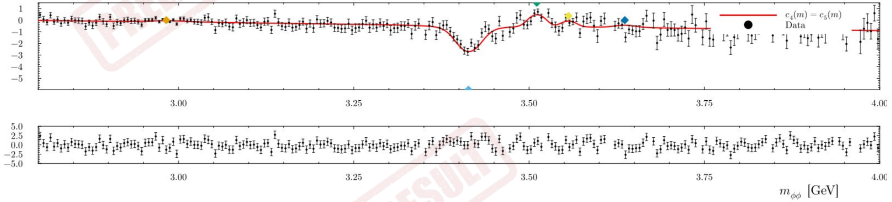
c_2



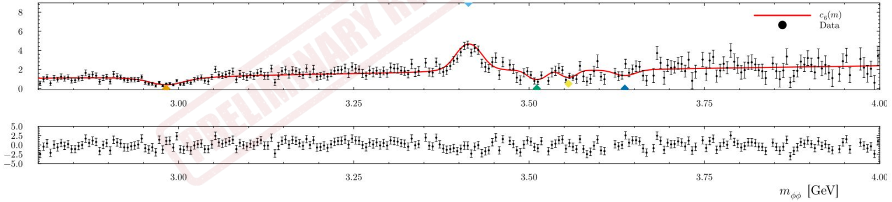
c_3



c_4



c_6



- Fit parameters:
  - Contributions of non-minimal partial waves
  - Linear background parameters

Angular fit

- Fractions are fixed from the mass fit
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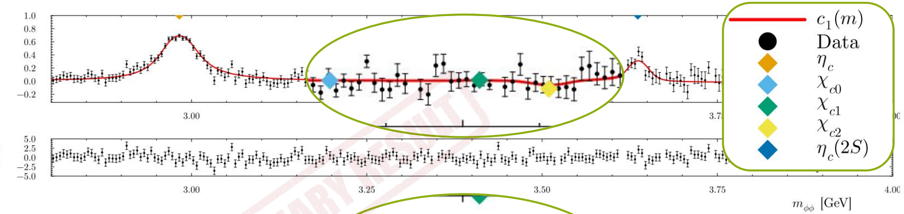
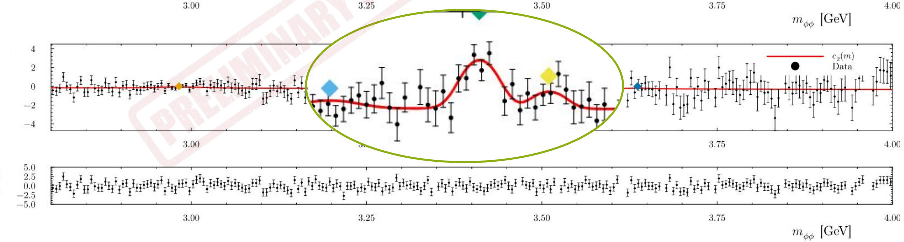
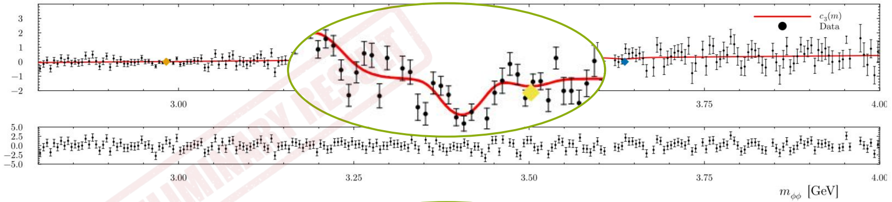
c_1



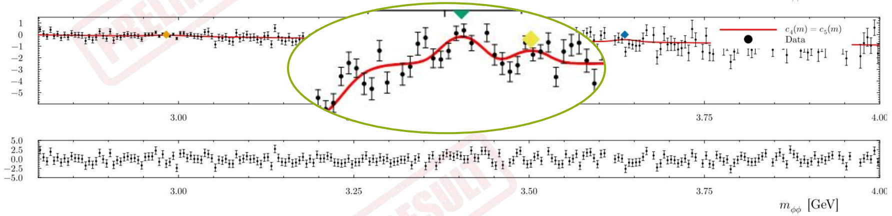
c_2



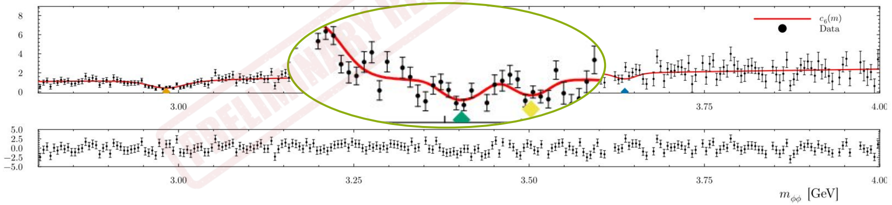
c_3



c_4



c_6



- Fit parameters:
  - Contributions of non-minimal partial waves
  - Linear background parameters

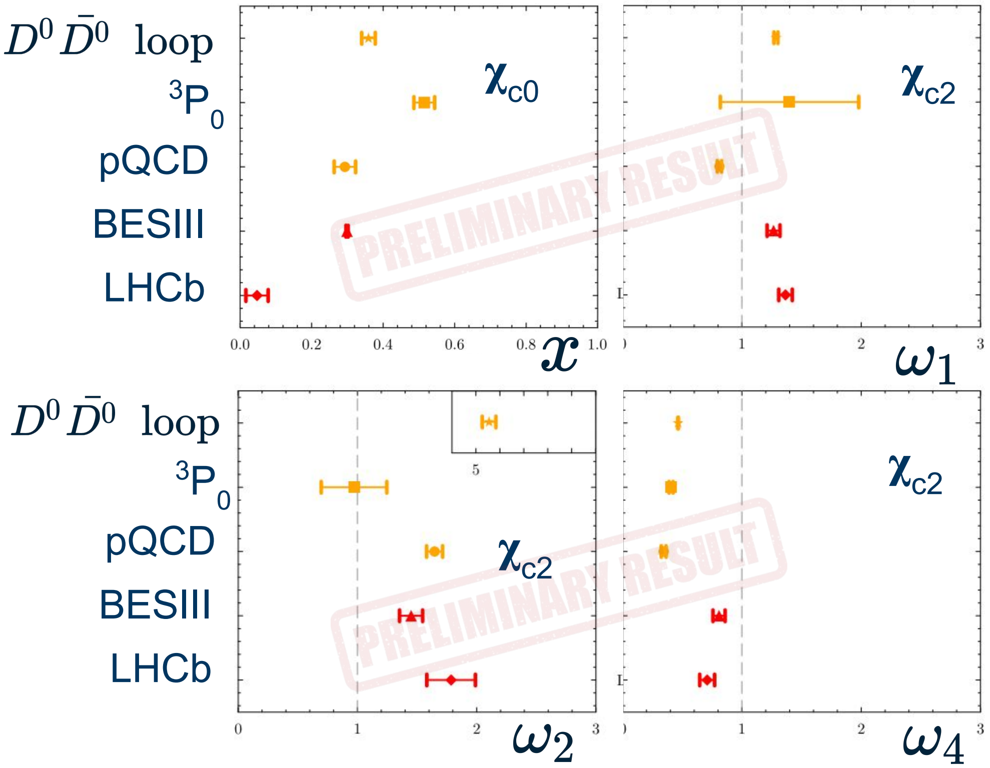
Charmonia polarization ratios

- χ_{c0}: $x = \frac{|H_{1,1}|}{|H_{0,0}|} = \frac{|b|}{|d|} = 0.043 \pm 0.029_{stat}$
- χ_{c2}: $\omega_1 = \frac{|H_{0,1}|}{|H_{0,0}|} = \frac{|a|}{|d|} = 1.37 \pm 0.06_{stat}$

$$\omega_2 = \frac{|H_{1,-1}|}{|H_{0,0}|} = \frac{|c|}{|d|} = 1.79 \pm 0.20_{stat}$$

$$\omega_4 = \frac{|H_{1,1}|}{|H_{0,0}|} = \frac{|b|}{|d|} = 0.71 \pm 0.06_{stat}$$

- Systematic uncertainties are yet to be assessed
- Overall good agreement with BESIII
- Tension with BESIII for χ_{c0} ratio to be understood

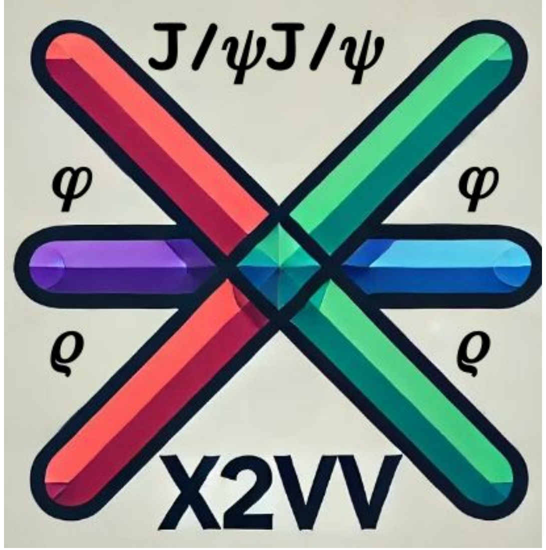


Summary and next steps

- Angular analysis of b-hadron→$c\bar{c}$→$\phi\phi$ decay is performed
- c_i coefficients are used to characterize angular distributions and PW decomposition
- Polarization and PW ratios are computed for χ_{c0} and χ_{c2}
- X2VV analysis framework is developed and used

Next steps:

- Estimate systematic uncertainties for $\phi\phi$ analysis
- Move forward towards publication
- Continue with the investigation of J/ψJ/ψ case
- Later technique to be applied for CEP analyses



Thank you for the attention !

Backup slides

Data selection

- Follow-up on the LHCb-PAPER-2025-058 which is soon going to 2CWC
- Data is kindly provided by Raoul Henderson and Sergey Barsuk

- LHCb Run 2 (~5.9 fb^{-1})
- Online selection (triggers, stripping) through StrippingCcbar2PhiPhiLine

- Offline selections:

Particle	Variable	Denotation	Requirement
Kaons	Track quality	χ^2/ndf	< 3
	Momentum	p, MeV/c	$\in [3000; 200000]$
	Transverse momentum	p_T, MeV/c	All: > 150, at least 3: > 250, at least 2: > 350, at least 1: > 450
	Pseudorapidity	η	$\in [2; 5]$
	Impact parameter	χ^2_IP	> 4.0
	Basic identification	PIDK	> 0
	NN identification	ProbNNK	> 0.2
ϕ	Vertex quality	χ^2/ndf	< 9
	Transverse momentum	p_T, MeV/c	> 800
	Invariant mass	$M_{K^+K^-}$, MeV/c^2	$\in [1002; 1038]$
$\phi\phi$	Vertex quality	χ^2/ndf	< 9
	Invariant mass	$M_{\phi\phi}$, MeV/c^2	$\in [2800; 4000]$
	Pseudorapidity	η	$\in [2; 5]$
	Flight distance quality	χ^2_FD	> 9
	Pseudo-z time of flight	t_z, ps	> 0.3

Background subtraction

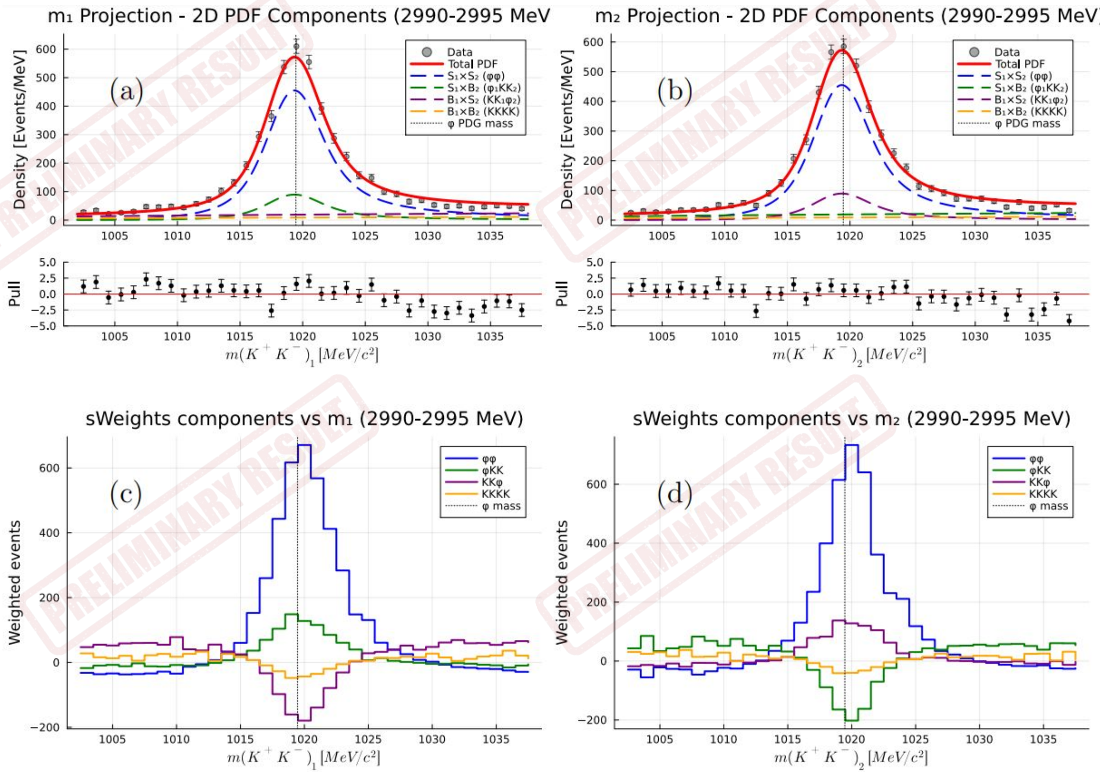
- **2D Model**:

$$F(m_1, m_2) = N_{\phi\phi} S_1(m_1) S_2(m_2)$$
$$+ N_{\phi KK} S_1(m_1) B_2(m_2)$$
$$+ N_{KK\phi} B_1(m_1) S_2(m_2))$$
$$+ N_{KKKK} B_1(m_1) B_2(m_2)$$

- **Signal**: RBW x Gauss
- **Background**: $\sqrt{m - 2m_K}$

- **m($\phi\phi$)**: 240 bins (5 MeV/c^2 width)
- **Fit parameters**:
  - Yields (fractions)
  - ϕ meson mass
  - Resolution σ

- Per event 2D sWeights are calculated



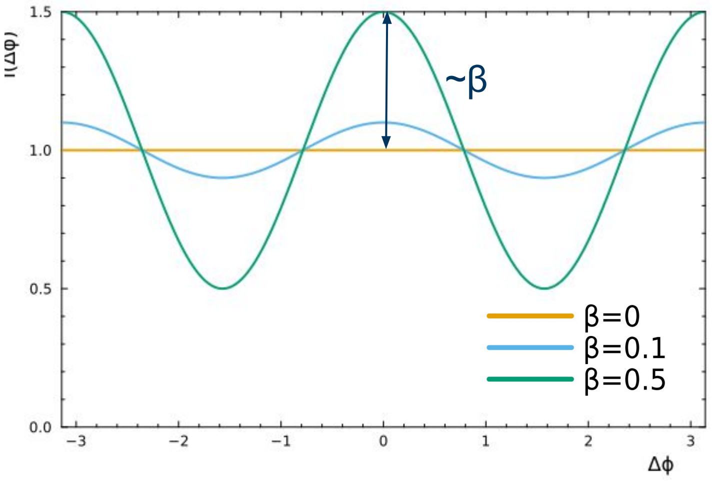
Angular functions

- **Using amplitude, intensity can be calculated in the simple way:**

$$I(\theta_1, \theta_2, \Delta\phi) = \sum_{\nu=-2}^{2} |A_\nu(\theta_1, \theta_2, \Delta\phi)|^2$$

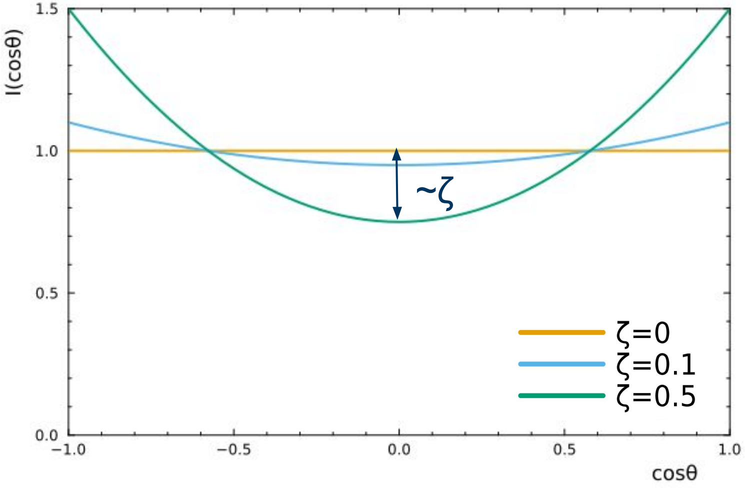
- **Integrating over angles one can get angular distributions of intensity:**
  - For azimuthal angle:

$$I(\Delta\phi) = 1 + \beta\cos(2\Delta\phi)$$



For polar angles:

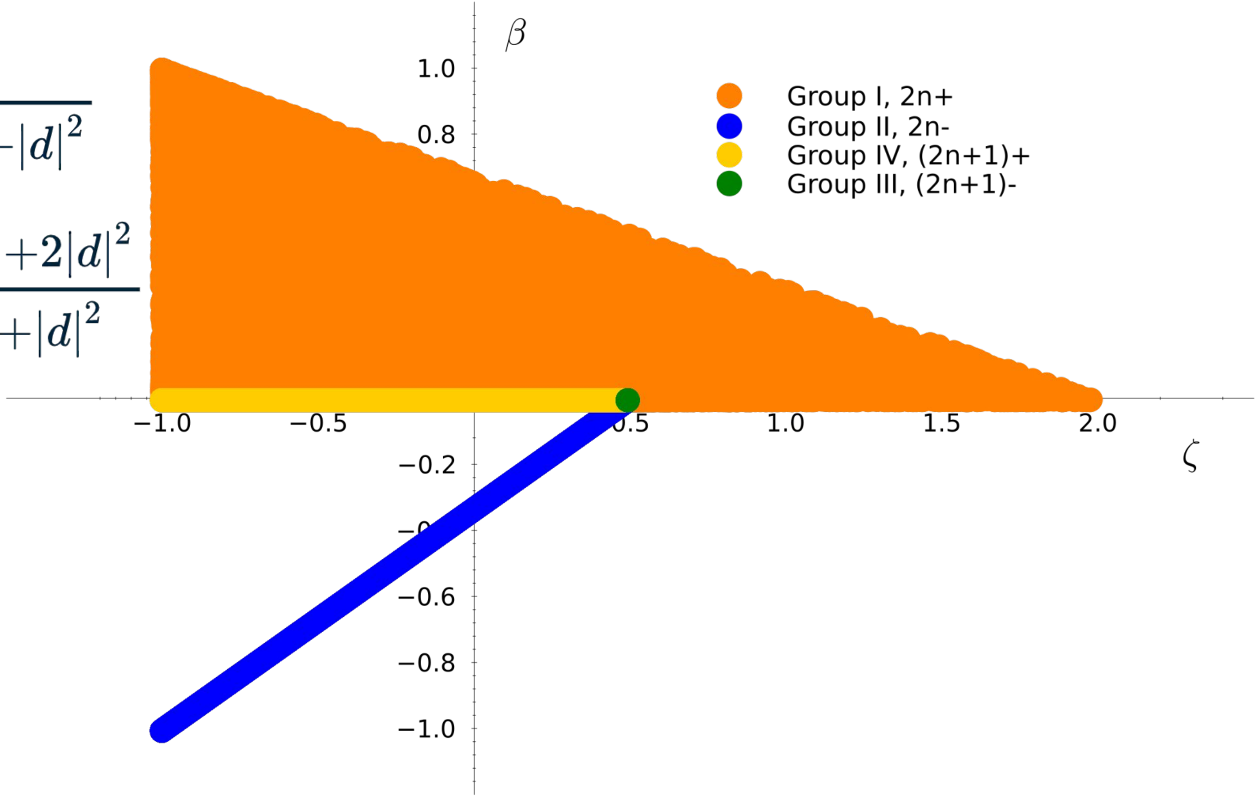
$$I(\cos\theta_i) = 1 + \zeta\frac{3\cos^2\theta_i - 1}{2}, \quad i = 1, 2$$



β and ζ definition

$$\beta = \frac{2\epsilon|b|^2}{4|a|^2+2|b|^2+2|c|^2+|d|^2}$$

$$\zeta = \frac{-2|b|^2-2|c|^2+2|a|^2+2|d|^2}{4|a|^2+2|b|^2+2|c|^2+|d|^2}$$



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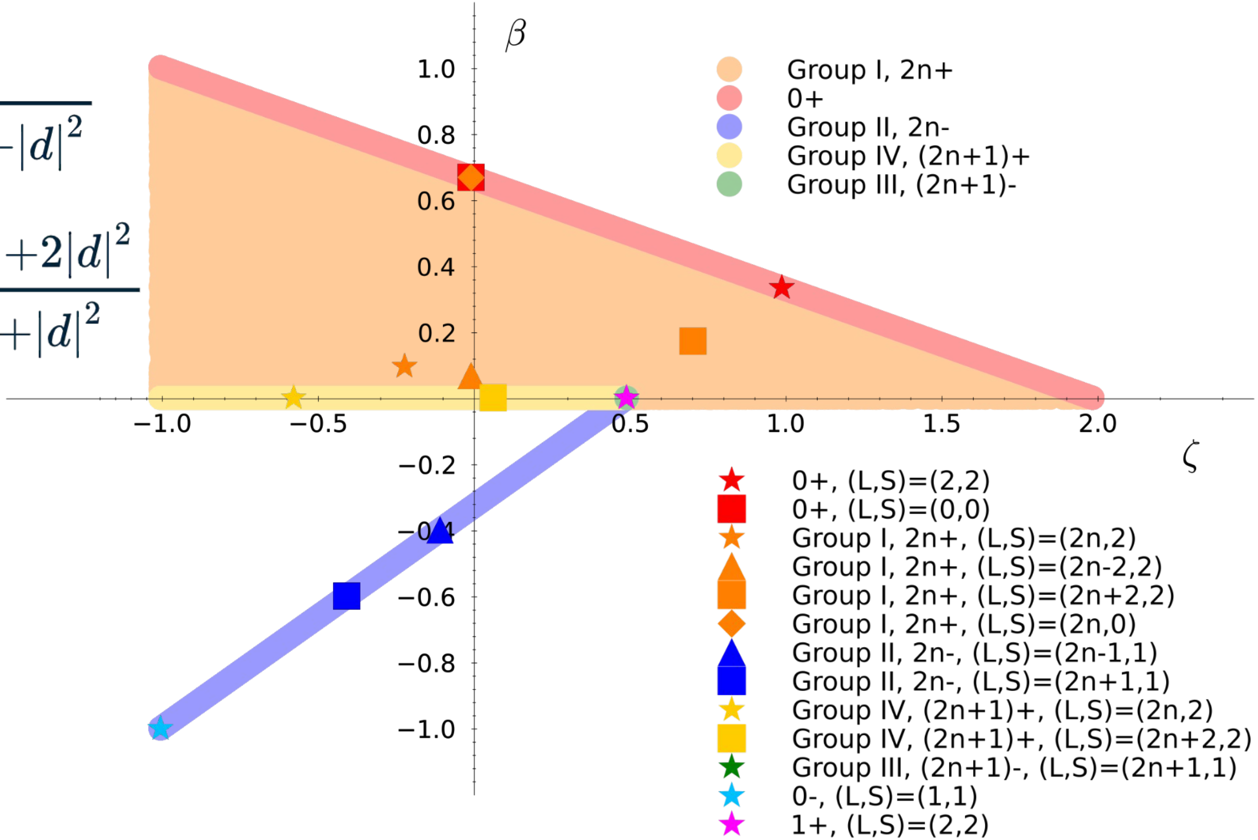

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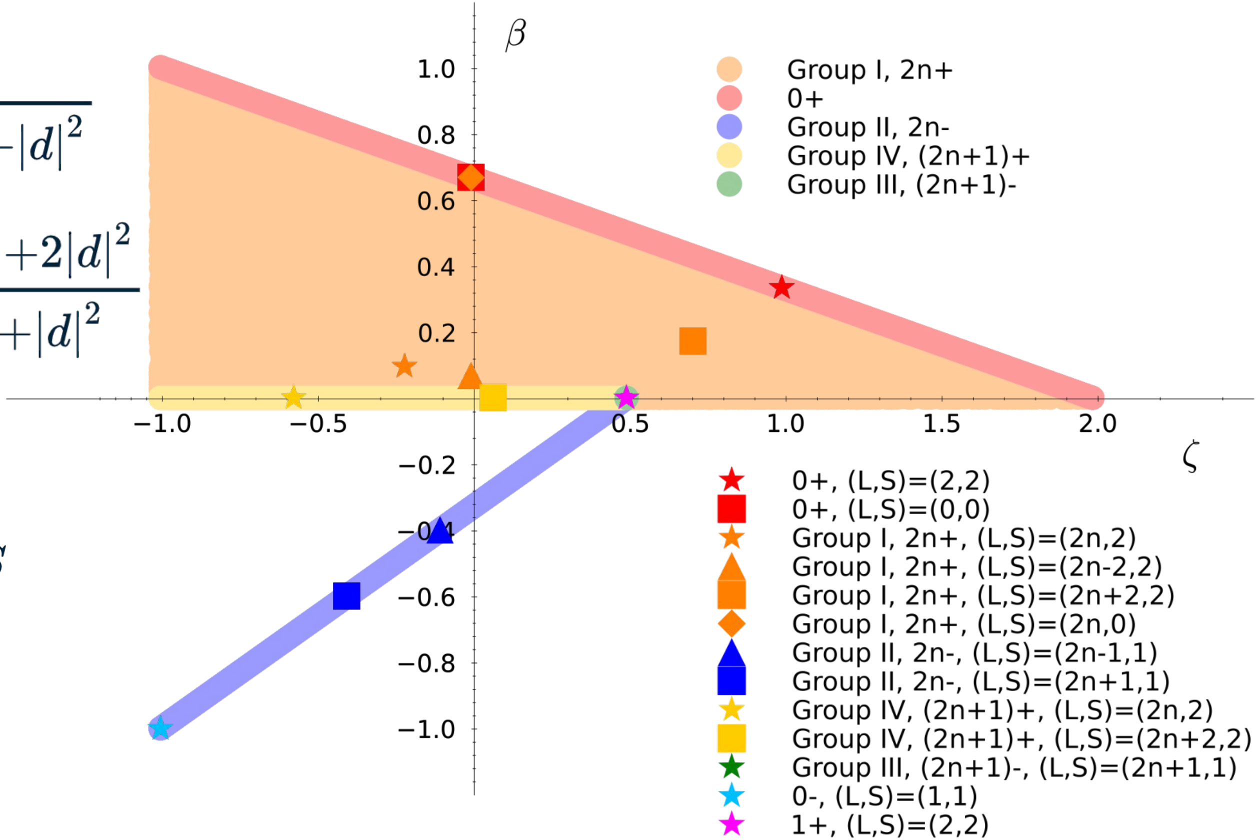
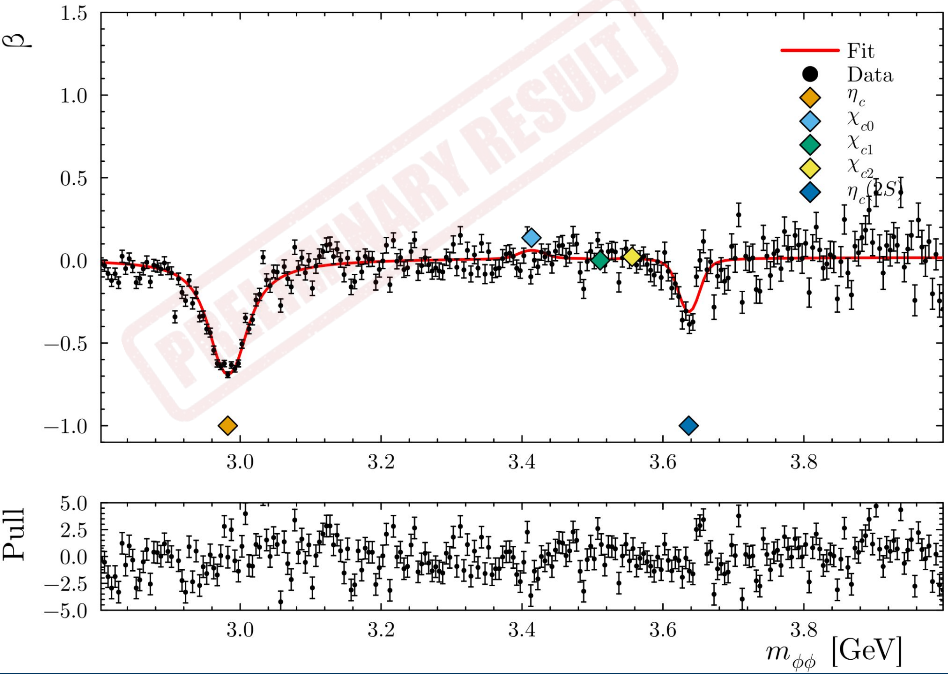
$$H = \sum_{LS} c_{LS} H_{LS}$$

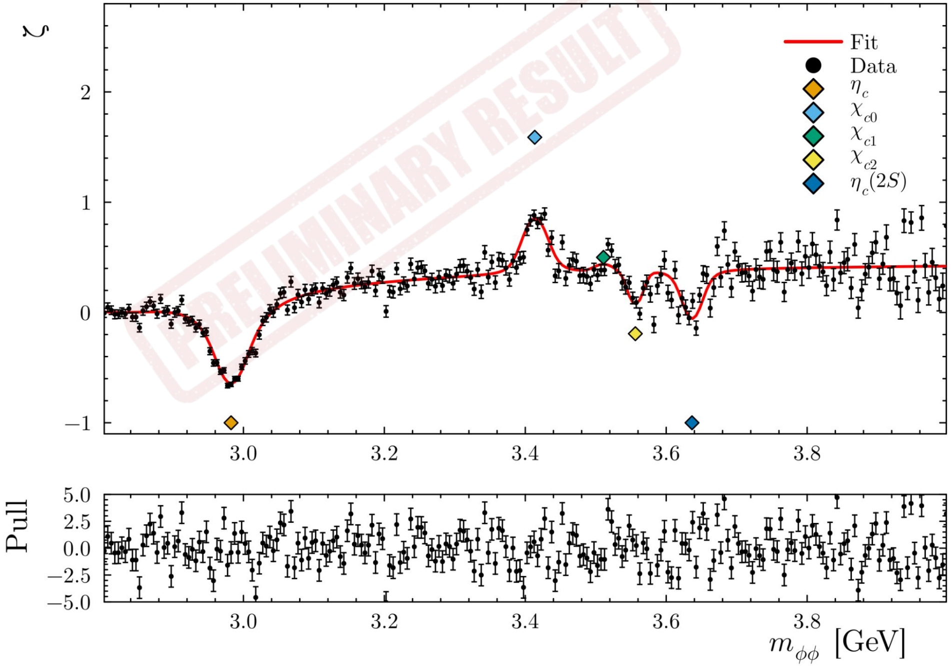


Angular fit

- Fractions are fixed from the mass fit
- Significant bin-to-bin fluctuations
- Fit quality: ndf = 469, χ^2/ndf = 3.192

- Fit parameters:
  - Contributions of non-minimal partial waves
  - Linear background parameters





Angular functions of 3D intensity (possible values of c_i)

- **Using amplitude, intensity can be represented in the simple way:**

$$I(\theta_1, \theta_2, \Delta\phi) = \sum_{\nu=-2}^{2} |A_\nu(\theta_1, \theta_2, \Delta\phi)|^2 = \sum_{i=1}^{6} c_i f_i$$

Case (J^P)	Couplings	ϵ, s	c_1	c_2	c_3	$c_4 (= c_5)$	c_6
Group I ($2n^+$)	a, b, c, d	$1, 1$	$[-1, 0]$	$\left[-\frac{9}{\sqrt{2}}, \frac{9}{\sqrt{2}}\right]$	$[-2, 4]$	$[-6, 1.5]$	$[0, 9]$
Group II ($2n^-$)	a, b ($c, d = 0$)	$-1, 1$	$[0, 1]$	$[-4.5, 0]$	$[-2, 0]$	$[0, 1.5]$	0
Group III ($(2n+1)^-$)	a ($b, c, d = 0$)	$1, -1$	0	-4.5	-2	1.5	0
Group IV ($(2n+1)^+$)	a, c ($b, d = 0$)	$-1, -1$	0	$[0, 4.5]$	$[-2, 1]$	$[0, 1.5]$	0
Special cases							
0^+	b, d ($a, c = 0$)	$1, 1$	$[-1, 0]$	$\left[-\frac{9}{\sqrt{2}}, \frac{9}{\sqrt{2}}\right]$	$[2, 4]$	$[-6, 0]$	$[0, 9]$
0^-	b ($a, c, d = 0$)	$-1, 1$	1	0	0	0	0
1^+	a ($b, c, d = 0$)	$-1, -1$	0	4.5	-2	1.5	0

Systematic uncertainties

- Systematic uncertainties are yet to be estimated during the WG review
- Leading uncertainty is expected to be due to the background model in the angular fit
- Other systematic uncertainties under consideration due to:
  - Interference effects
  - The choice of signal lineshape and resolution function
  - Acceptance correction
  - Binning
  - Real couplings in the angular fit
- Robustness checks:
  - Background subtraction with sWeights
  - Split data by year, magnet polarity, etc.
- Most of these uncertainties are expected to be negligible
- Should be assessed by repeating analysis procedures with different fits, methods, etc.

Partial waves decomposition

- Partial wave decomposition:

$$H = \sum_{LS} c_{LS} H_{LS}$$

- Expression for the PW ratios:

$$\frac{\mathcal{B}(X \to VV_{L_1-wave})}{\mathcal{B}(X \to VV_{L_2-wave})} = \frac{|c_{L_1 S_1}|^2}{|c_{L_2 S_2}|^2}$$

- Measured PW ratios:
  (for the first time)

$$\frac{\mathcal{B}(\chi_{c0} \to \phi\phi_{d-wave})}{\mathcal{B}(\chi_{c0} \to \phi\phi_{s-wave})} = \frac{|c_{22}|^2}{|c_{00}|^2} = 2.7 \pm 0.6 \pm \mathrm{XXX}$$

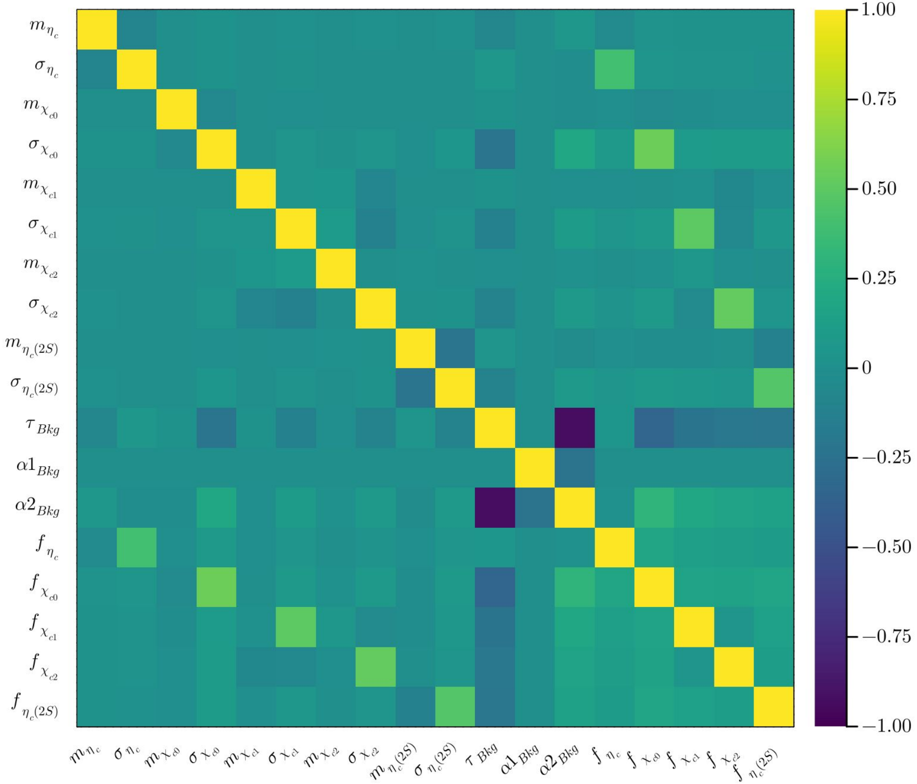
$$\frac{\mathcal{B}(\chi_{c2} \to \phi\phi_{d-wave})}{\mathcal{B}(\chi_{c2} \to \phi\phi_{s-wave})} = \frac{|c_{22}|^2 + |c_{20}|^2}{|c_{02}|^2} = 0.007 \pm 0.003 \pm \mathrm{XXX}$$

$$\frac{\mathcal{B}(\chi_{c2} \to \phi\phi_{g-wave})}{\mathcal{B}(\chi_{c2} \to \phi\phi_{s-wave})} = \frac{|c_{42}|^2}{|c_{02}|^2} = 0.011 \pm 0.002 \pm \mathrm{XXX},$$

- Statistical uncertainties for the χ_{c2} are still under the investigation

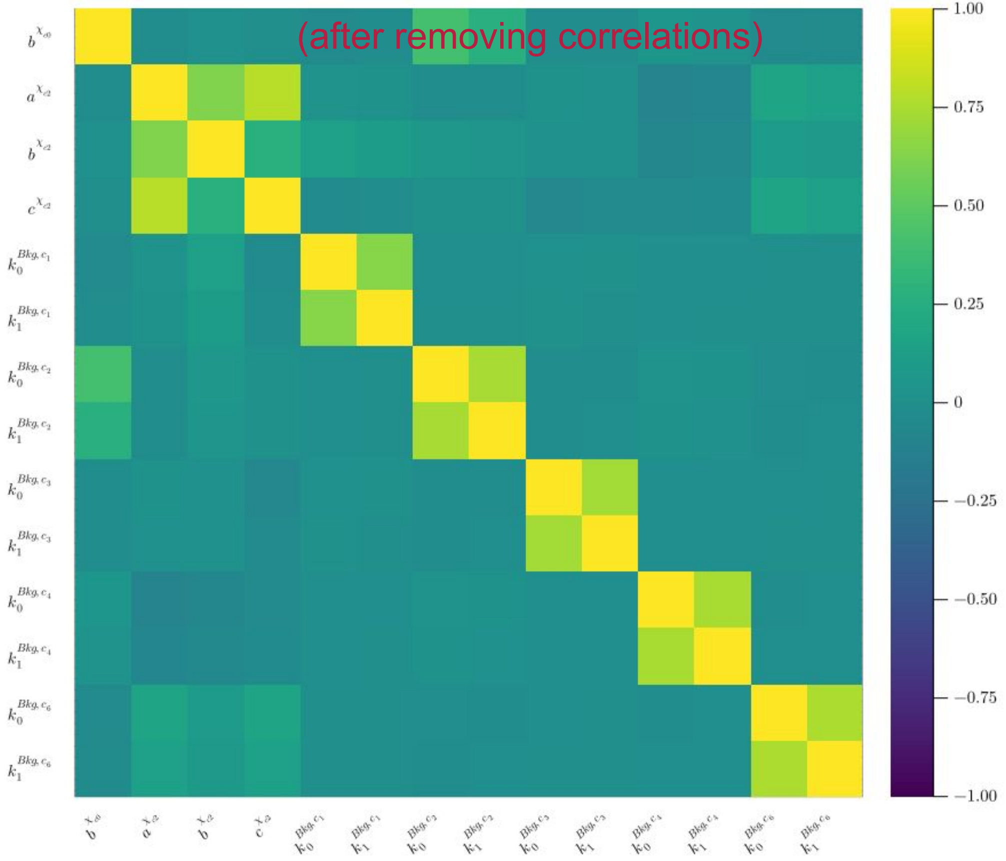
Mass fit parameters

Parameter	Value	Uncertainty
m_{η_c}, GeV/c^2	2.9825	0.0002
σ_{η_c}, GeV/c^2	0.0890	0.0024
$m_{\chi_{c0}}$, GeV/c^2	3.4135	0.0007
$\sigma_{\chi_{c0}}$, GeV/c^2	0.1149	0.0042
$m_{\chi_{c1}}$, GeV/c^2	3.5109	0.0005
$\sigma_{\chi_{c1}}$, GeV/c^2	0.0996	0.0027
$m_{\chi_{c2}}$, GeV/c^2	3.5562	0.0004
$\sigma_{\chi_{c2}}$, GeV/c^2	0.0909	0.0002
$m_{\eta_c(2S)}$, GeV/c^2	3.6365	0.0012
$\sigma_{\eta_c(2S)}$, GeV/c^2	0.1029	0.0021
τ_{bkg}	0.9545	0.0390
$\alpha_{1,\text{bkg}}$	0.0396	0.1701
$\alpha_{2,\text{bkg}}$	-0.0550	0.0364
f_{η_c}	0.1437	0.0015
$f_{\chi_{c0}}$	0.0174	0.0008
$f_{\chi_{c1}}$	0.0128	0.0005
$f_{\chi_{c2}}$	0.0114	0.0004
$f_{\eta_c(2S)}$	0.0070	0.0005



Angular fit parameters (3D Intensity)

Helicity couplings / Background parameters			
η_c (0^-)	b	1	-
χ_{c0} (0^+)	b	-0.043	0.038
	d	fixed	-
χ_{c1} (1^+)	d	1	-
χ_{c2} (2^+)	a	1.446	0.248
	b	0.738	0.279
	c	1.930	0.545
	d	fixed	-
η_c (2S) (0^-)	b	1	-
Background	Linear: $k_0 + k_1(m - 3.5)$		
	k_0^{c1}	-0.013	0.004
	k_1^{c1} (slope)	-0.033	0.011
	k_0^{c2}	-0.257	0.052
	k_1^{c2} (slope)	-0.140	0.113
	k_0^{c3}	0.229	0.024
	k_1^{c3} (slope)	0.409	0.054
	k_0^{c4}	-0.537	0.026
	k_1^{c4} (slope) 24	-0.728	0.057
	k_0^{c6}	1.856	0.030
	k_1^{c6} (slope)	1.085	0.066



Mass dependencies: acceptance corrections

- Corrected log-likelihood:

$$\mathcal{L} = \prod_i \frac{\sum_i (w_i)}{\sum_i (w_i^2)} w_i \; \frac{I(\cos\theta_1, \cos\theta_2, \Delta\phi; \mathbf{c_j})}{I_{\mathcal{A}}(\cos\theta_1, \cos\theta_2, \Delta\phi; \mathbf{c_j})}$$

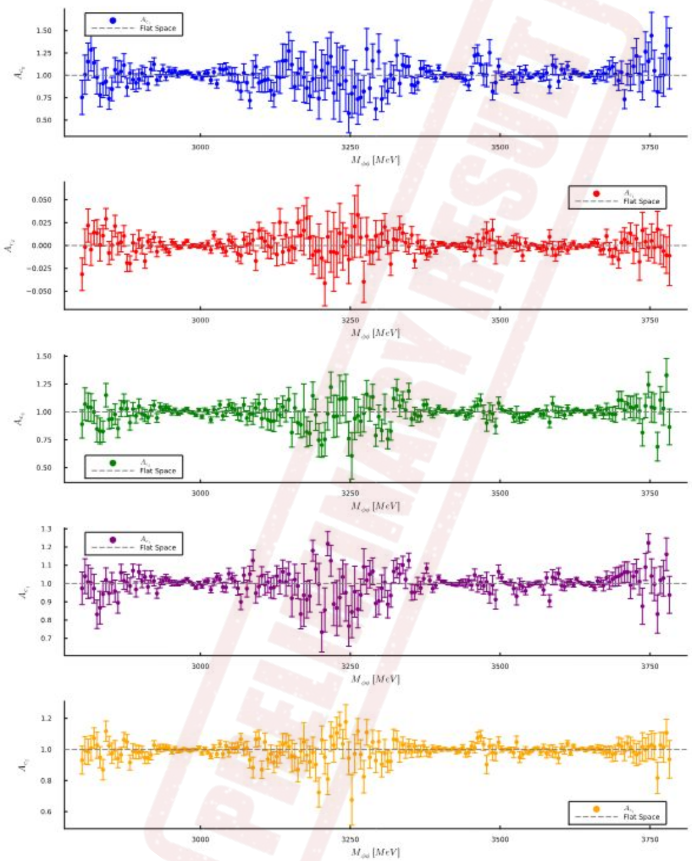
- NLL:

$$-\log\mathcal{L} = -\frac{\sum_i (w_i)}{\sum_i (w_i^2)} \left[\sum_i w_i \log I(\cos\theta_1, \cos\theta_1, \Delta\phi; \mathbf{c_j}) - \log\left(\sum_{\mathbf{j=1}}^{\mathbf{6}} \mathcal{A}_{\mathbf{j}} \mathbf{c_j}\right)\right]$$

- Non-zero acceptance constants are used to correct

$$\mathcal{A}_i = \frac{1}{N} \sum_j f_i^{(j)}$$

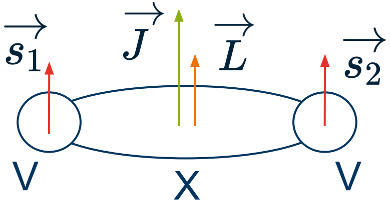
- Calculated based on simulation data



Symmetry constraints and partial waves

s_1	s_2	$S=s_1-s_2$	$S=(s_1-s_2)+1$	$S=(s_1+s_2)$	L
1^-	1^-				

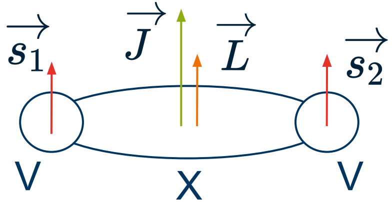
$$\vec{S} = \vec{s_1} + \vec{s_2}$$



Symmetry constraints and partial waves

s_1	s_2	S=0	S=1	S=2	L
1^-	1^-				

$$\vec{S} = \vec{s_1} + \vec{s_2}$$

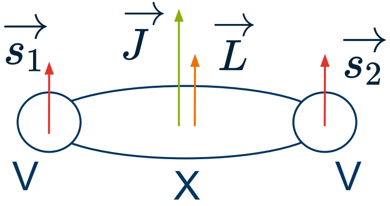


Symmetry constraints and partial waves

s_1	s_2	S=0	S=1	S=2	L
1^-	1^-				

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{S} = \vec{s_1} + \vec{s_2}$$

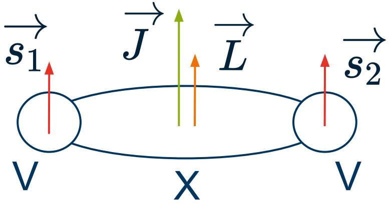


Symmetry constraints and partial waves

s_1	s_2	S=0		S=1				S=2			L	
1^-	1^-	0			1					2	0	s-wave
		1	0	1	2			1	2	3	1	p-wave
		2	1	2	3	0	1	2	3	4	2	d-wave
		3	2	3	4	1	2	3	4	5	3	f-wave

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{S} = \vec{s_1} + \vec{s_2}$$

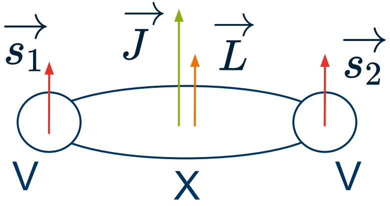


Symmetry constraints and partial waves

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+			1^+					2^+	0	s-wave
		1^-	0^-	1^-	2^-			1^-	2^-	3^-	1	p-wave
		2^+	1^+	2^+	3^+	0^+	1^+	2^+	3^+	4^+	2	d-wave
		3^-	2^-	3^-	4^-	1^-	2^-	3^-	4^-	5^-	3	f-wave

$$P = (-1)^L$$

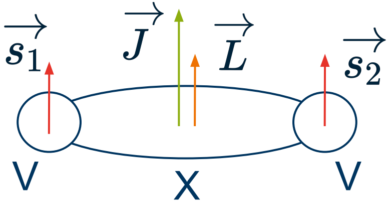
$$\vec{J} = \vec{L} + \vec{S}$$
$$\vec{S} = \vec{s_1} + \vec{s_2}$$



Symmetry constraints and partial waves

s_1	s_2	S=0		S=1				S=2			L	
1^-	1^-	0^+			1^+					2^+	0	s-wave
		1^-	0^-	1^-	2^-			1^-	2^-	3^-	1	p-wave
		2^+	1^+	2^+	3^+	0^+	1^+	2^+	3^+	4^+	2	d-wave
		3^-	2^-	3^-	4^-	1^-	2^-	3^-	4^-	5^-	3	f-wave
		$2n^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	n=2,3,4, …
		$(2n+1)^-$	$2n^-$	$(2n+1)^-$	$(2n+2)^-$	$(2n-1)^-$	$2n^-$	$(2n+1)^-$	$(2n+2)^-$	$(2n+3)^-$	2n+1	

$$\vec{J} = \vec{L} + \vec{S}$$
$$\vec{S} = \vec{s_1} + \vec{s_2}$$



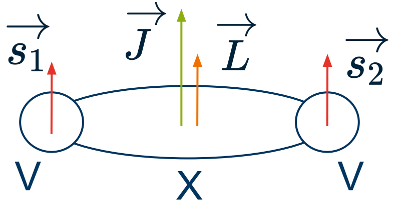
$$P = (-1)^L$$

Symmetry constraints and partial waves

s_1	s_2	S=0		S=1				S=2			L	
1^-	1^-	0^+			1^+					2^+	0	s-wave
		1^-	0^-	1^-	2^-			1^-	2^-	3^-	1	p-wave
		2^+	1^+	2^+	3^+	0^+	1^+	2^+	3^+	4^+	2	d-wave
		3^-	2^-	3^-	4^-	1^-	2^-	3^-	4^-	5^-	3	f-wave
		$2n^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	n=2,3,4,…
		$(2n+1)^-$	$2n^-$	$(2n+1)^-$	$(2n+2)^-$	$(2n-1)^-$	$2n^-$	$(2n+1)^-$	$(2n+2)^-$	$(2n+3)^-$	2n+1	

$\phi\phi$ is a symmetric system, only states of symmetric (even) or antisymmetric (odd) LS are allowed

$$\vec{J} = \vec{L} + \vec{S}$$
$$\vec{S} = \vec{s_1} + \vec{s_2}$$



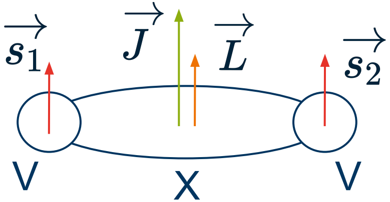
$$P = (-1)^L$$

Symmetry constraints and partial waves

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+			~~1^+~~					2^+	0	s-wave
		~~1^-~~	0^-	1^-	2^-			~~1^-~~	~~2^-~~	~~3^-~~	1	p-wave
		2^+	~~1^+~~	~~2^+~~	~~3^+~~	0^+	1^+	2^+	3^+	4^+	2	d-wave
		~~3^-~~	2^-	3^-	4^-	~~1^-~~	~~2^-~~	~~3^-~~	~~4^-~~	~~5^-~~	3	f-wave
		$2n^+$	~~$(2n-1)^+$~~	~~$2n^+$~~	~~$(2n+1)^+$~~	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	n=2,3,4,
		~~$(2n+1)^-$~~	$2n^-$	$(2n+1)^-$	$(2n+2)^-$	~~$(2n-1)^-$~~	~~$2n^-$~~	~~$(2n+1)^-$~~	~~$(2n+2)^-$~~	~~$(2n+3)^-$~~	2n+1	…

$\phi\phi$ is a symmetric system, only states of symmetric (even) or antisymmetric (odd) LS are allowed

$$\vec{J} = \vec{L} + \vec{S}$$
$$\vec{S} = \vec{s_1} + \vec{s_2}$$



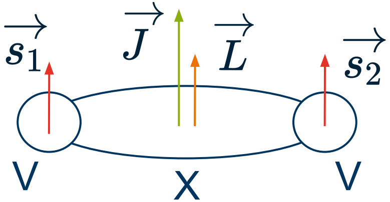
$$P = (-1)^L$$

Symmetry constraints and partial waves

s_1	s_2	S=0		S=1				S=2			L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
			2^-	3^-	4^-						3	f-wave
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	n=2,3,4, …
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	

$\phi\phi$ is a symmetric system, only states of symmetric (even) or antisymmetric (odd) LS are allowed

$$\vec{J} = \vec{L} + \vec{S}$$
$$\vec{S} = \vec{s_1} + \vec{s_2}$$



$$P = (-1)^L$$

Symmetry constraints and partial waves of $c\bar{c} \to \phi\phi$

s_1	s_2	S=0		S=1				S=2			L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	n=3,4,5, …
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

All possible appearances of the certain J^P happens over 5 different waves and divided into 4 groups

Symmetry constraints and partial waves of $c\bar{c} \rightarrow \phi\phi$

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	n=3,4,5,…
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

Group I (even and natural): (L,S)=(2n-2,2),(2n,0),(2n,2),(2n+2,2) → 4 helicity couplings: $H_I = \begin{pmatrix} b & a & c \\ a & d & a \\ c & a & b \end{pmatrix}$

Symmetry constraints and partial waves of $c\bar{c} \rightarrow \phi\phi$

s_1	s_2	S=0		S=1				S=2			L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	n=3,4,5,…
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

Group II (even and unnatural): (L,S)=(2n-1,1),(2n+1,1) → 2 helicity couplings: $H_{II} = \begin{pmatrix} b & a & \\ a & & -a \\ & -a & -b \end{pmatrix}$

Symmetry constraints and partial waves of $c\bar{c} \rightarrow \phi\phi$

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

n=3,4,5, …

Group III (odd and natural): (L,S)=(2n+1,1) → 1 helicity coupling: $H_{III} = \begin{pmatrix} & a & \\ -a & & -a \\ & a & \end{pmatrix}$

Symmetry constraints and partial waves of $c\bar{c} \rightarrow \phi\phi$

s_1	s_2	S=0		S=1				S=2			L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	n=3,4,5, …
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

Group IV (odd and unnatural): (L,S)=(2n,2),(2n+2,2) → 2 helicity couplings: $H_{IV} = \begin{pmatrix} & a & c \\ -a & & a \\ -c & -a & \end{pmatrix}$

Symmetry constraints and partial waves of $c\bar{c} \rightarrow \phi\phi$

s_1	s_2	S=0		S=1				S=2			L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	n=3,4,5, …
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

0^+ is a special case in Group I: (L,S)=~~(-2,2)~~,(0,0),~~(0,2)~~,(2,2) → 2 helicity couplings: a=c=0, $H_{0^+} = \begin{pmatrix} b & & \\ & d & \\ & & b \end{pmatrix}$

Symmetry constraints and partial waves of $c\bar{c} \to \phi\phi$

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	n=3,4,5, …
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

0^- is a special case in Group II: (L,S)=~~(1,1)~~,(1,1) → 1 helicity couplings: a=0, $H_{0^-} = \begin{pmatrix} b & \\ & -b \end{pmatrix}$

Symmetry constraints and partial waves of $c\bar{c} \rightarrow \phi\phi$

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	n=3,4,5, …
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

1^+ is a special case in Group IV: (L,S)=~~(0,2)~~,(2,2) → 1 helicity couplings: c=0, $H_{1^+} = \begin{pmatrix} & a & \\ -a & & a \\ & -a & \end{pmatrix}$

Symmetry constraints and partial waves of $c\bar{c} \to \phi\phi$

s_1	s_2	S=0	S=1			S=2					L	
1^-	1^-	0^+								2^+	0	s-wave
			0^-	1^-	2^-						1	p-wave
		2^+				0^+	1^+	2^+	3^+	4^+	2	d-wave
		$(2n-2)^+$				$(2n-4)^+$	$(2n-3)^+$	$(2n-2)^+$	$(2n-1)^+$	$2n^+$	2n-2	n=3,4,5, …
			$(2n-2)^-$	$(2n-1)^-$	$2n^-$						2n-1	
		$2n^+$				$(2n-2)^+$	$(2n-1)^+$	$2n^+$	$(2n+1)^+$	$(2n+2)^+$	2n	
			$2n^-$	$(2n+1)^-$	$(2n+2)^-$						2n+1	
		$(2n+2)^+$				$2n^+$	$(2n+1)^+$	$(2n+2)^+$	$(2n+3)^+$	$(2n+4)^+$	2n+2	

All other J^Ps can be divided into groups accordingly