Field Theory Problem set 2

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1. Show that to obtain the canonical equal-time commutators

$$[\phi(\mathbf{x},t),\Pi(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$
$$[\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = [\Pi(\mathbf{x},t),\Pi(\mathbf{y},t)] = 0, \tag{1}$$

where

$$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right) \tag{2}$$
$$\Pi(\mathbf{x},t) = -i\int \frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left(a_{\mathbf{p}}e^{-ip\cdot x} - a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right), \tag{3}$$

it requires the following commutation relations for creation and annihilation operators:

$$[a_{\mathbf{p}},a^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p}},a_{\mathbf{q}}] = [a^\dagger_{\mathbf{p}},a^\dagger_{\mathbf{q}}] = 0. \tag{4}$$

2. Compute the momentum operator for the scalar free theory from the expression

$$\mathbf{P} = -\int d^3\mathbf{x}\Pi(\mathbf{x},t)\nabla\phi(\mathbf{x},t) \tag{5}$$

in terms of creation and annihilation operators.

3. In QM, a propagator $K = \langle q_f|e^{-\frac{i}{\hbar}\hat{H}T}|q_i\rangle$ is

$$K = \int \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}. \tag{6}$$

Consider a free particle theory ($V(q)=0$). One can compute the free particle propagator by discretizing the propagator into N steps with

$$\epsilon = \frac{T}{N}$$

such that the propagator becomes

$$K = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N/2}\int\prod_{n=1}^{N-1}dq_n \exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1}\epsilon\left(\frac{m}{2}\dot{q}_n^2\right)\right] \tag{7}$$

where $\dot{q}_n = \frac{q_{n+1}-q_n}{\epsilon}$. Using the relation

$$\int dq_j \exp\left[\frac{im}{2\hbar\epsilon}\left((q_{j+1}-q_j)^2+\alpha_j(q_j-q_i)^2\right)\right] = \sqrt{\frac{2\pi i\hbar\epsilon}{m(1+\alpha_j)}}\exp\left[\frac{im}{2\hbar\epsilon}\frac{\alpha_j}{1+\alpha_j}(q_{j+1}-q_i)^2\right]$$

and show that

$$K = \sqrt{\frac{m}{2\pi i\hbar T}}\exp\left[\frac{im(q_f-q_i)^2}{2\hbar T}\right],$$

Note that $q_f = q_N$ and $q_i = q_0$ are the final and initial positions, respectively.

4. Show that the generating functional for the scalar field theory

$$Z[J] = \int \mathcal{D}\phi \, \exp\left[-\frac{i}{2}\int d^4x \, \phi(\Box + m^2)\phi + i\int d^4x \, J\phi\right]. \tag{8}$$

becomes

$$Z[J] = Z[0] \exp\left[-\frac{1}{2}\int d^4x \, d^4y \, J(x) D_F(x-y) J(y)\right]. \tag{9}$$

where $D_F(x-y)$ is a Feynman propagator satisfying

$$(\Box + m^2) D_F(x-y) = -i\delta^{(4)}(x-y). \tag{10}$$

5. Consider a phi-4 theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi \, \partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{11}$$

The corresponding generating functional reads

$$Z[J] = \exp\left[-i\frac{\lambda}{4!}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right] Z_0[J] \tag{12}$$

where

$$Z_0[J] = Z_0[0] \exp\left[-\frac{1}{2}\int d^4x \, d^4y \, J(x) D_F(x-y) J(y)\right]. \tag{13}$$

Compute a two-point correlating function of this interacting field theory up to the order of λ^2. Note that

$$\frac{\langle\Omega| T\left\{\hat{\phi}(x_2)\hat{\phi}(x_1)\right\}|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \frac{1}{Z[0]} \left.\frac{(-i)^2\delta^2 Z[J]}{\delta J(x_2)\delta J(x_1)}\right|_{J=0}. \tag{14}$$