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Field Theory part II

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1 Free scalar fields

A bosonic relativistic particle is described by Klein -Gordon Lagrangian density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \quad (1)$$

This scalar field describes spinless scalar particles such as Higgs bosons (complex scalar field to be exact). Note that

$$\partial_\mu \phi \partial^\mu \phi = \partial^\mu \phi \partial_\nu \phi \eta_{\mu\nu} = \left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\vec{\nabla} \phi \right)^2 \quad (2)$$

where we use the Minkowski space metric

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (3)$$

and set $c = \hbar = 1$.

One can show that by applying the Euler-Lagrange equation to the Klein-Gordon Lagrangian (1), one obtains the Klein-Gordon equation as

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (4)$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \quad (5)$$

We can find the general solutions for the scalar field ϕ using the Fourier transform

$$\phi(\vec{x}, t) = \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \tilde{\phi}(\vec{p}, t). \quad (6)$$

Substituting the above expression to the Klein-Gordon equation gives

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{\vec{p}}^2 \right) \tilde{\phi}(\vec{p}, t) = 0 \quad (7)$$

where

$$\omega_{\vec{p}}^2 = \vec{p}^2 + m^2. \quad (8)$$

The equation (7) is nothing but a harmonic oscillator equation. Therefore, we can write the general solution of $\tilde{\phi}$ as

$$\tilde{\phi}(\vec{p}, t) = A(\vec{p}) e^{-i\omega_{\vec{p}} t} + B(\vec{p}) e^{i\omega_{\vec{p}} t}. \quad (9)$$

If we would like to make sure that the scalar field ϕ is real, it requires that

$$A^*(\vec{p}) = B(-\vec{p}). \quad (10)$$

Consequently, (6) becomes

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \left(A(\vec{p})e^{-i\omega_{\vec{p}}t+i\vec{p}\cdot\vec{x}} + A^*(-\vec{p})e^{i\omega_{\vec{p}}t+i\vec{p}\cdot\vec{x}} \right) \quad (11)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left(A(\vec{p})e^{-ip\cdot x} + A^*(\vec{p})e^{ip\cdot x} \right). \quad (12)$$

To obtain the second line, the change of variable $\vec{p} \rightarrow -\vec{p}$ was made. Note that p and x are now 4-vectors which are

$$p^\mu = (\omega_{\vec{p}}, \vec{p}) \quad \text{and} \quad x^\mu = (t, \vec{x}). \quad (13)$$

For later convenience, we re-define new parameter

$$A(\vec{p}) = \frac{a_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}, \quad (14)$$

giving

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}e^{-ip\cdot x} + a_{\vec{p}}^*e^{ip\cdot x} \right). \quad (15)$$

According to the above expression, one finds the conjugate momentum, i.e.

$$\Pi(\vec{x}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} \quad (16)$$

where $\dot{\phi} = \frac{\partial\phi}{\partial t}$, as

$$\Pi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}}e^{-ip\cdot x} - a_{\vec{p}}^*e^{ip\cdot x} \right). \quad (17)$$

2 Canonical Quantization

The canonical quantization refers to the process of promoting classical degrees of freedom to quantum operators together with their commutation relations. In case of particle mechanics where a system is described by the generalized coordinates q_a and their canonical momenta p_a . In quantum mechanics, we can promote them to operators with commutation relations

$$[q_a, q_b] = [p_a, p_b] = 0 \quad \text{and} \quad [q_a, p_b] = i\delta_{ab} \quad (18)$$

where we set $\hbar = 1$.

In field theory, we have the generalized coordinates as fields $\phi_a(\vec{x}, t)$ themselves and their conjugate momenta $\Pi_a(\vec{x}, t)$. These operators obey canonical commutation relations

$$\begin{aligned} [\phi_a(\vec{x}, t), \phi_b(\vec{y}, t)] &= [\Pi_a(\vec{x}, t), \Pi_b(\vec{y}, t)] = 0 \\ [\phi_a(\vec{x}, t), \Pi_b(\vec{y}, t)] &= i\delta^3(\vec{x} - \vec{y})\delta_{a,b}. \end{aligned} \quad (19)$$

The above relations are known as equal-time commutation relations. For the Klein-Gordon theory, when applying the expressions (15) and (17), one obtains

$$[a_{\vec{p}}, a_{\vec{q}}] = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0 \quad (20)$$

and

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \quad (21)$$

This resembles the creation and annihilation operators (or raising/lowering or ladder operators) in quantum harmonic oscillator as $[a, a^\dagger] = 1$. Remember that the expressions for the field operator and its conjugate momentum are now

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^* e^{ip \cdot x} \right). \quad (22)$$

$$\Pi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} e^{-ip \cdot x} - a_{\vec{p}}^\dagger e^{ip \cdot x} \right). \quad (23)$$

2.1 The Hamiltonian

Now, let's compute the Hamiltonian in terms of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$.

$$H = \int d^3\vec{x} \mathcal{H}(\phi, \Pi) = \int d^3\vec{x} \left(\Pi(x) \dot{\phi}(x) - \mathcal{L}(x) \right) \quad (24)$$

$$= \frac{1}{2} \int d^3\vec{x} \left(\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2 \right). \quad (25)$$

Substitute the expressions (22) and (23). It yields

$$\begin{aligned} H = \frac{1}{4} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\omega_{\vec{p}}} \left[(-\omega_{\vec{p}}^2 + \vec{p}^2 + m^2) (a_{\vec{p}} a_{-\vec{p}} + a_{\vec{p}}^\dagger a_{-\vec{p}}^\dagger) e^{-2i\omega_{\vec{p}}t} \right. \\ \left. + (\omega_{\vec{p}}^2 + \vec{p}^2 + m^2) (a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}) \right]. \end{aligned} \quad (26)$$

To obtain the above equation, the integral expression of Dirac delta function

$$\delta^3(\vec{p} - \vec{q}) = \int \frac{d^3\vec{x}}{(2\pi)^3} e^{i(\vec{p}-\vec{q})\cdot\vec{x}} \quad (27)$$

was used. The first term vanishes due to the expression for the frequency $\omega_{\vec{p}}^2 = \vec{p}^2 + m^2$. The equation (26) becomes

$$H = \frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} (a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}). \quad (28)$$

Applying the commutation relation (21), the Hamiltonian takes the form

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} a_{\vec{p}}^\dagger a_{\vec{p}} + E_0 \quad (29)$$

where

$$E_0 = \frac{1}{2} \int d^3\vec{p} \omega_{\vec{p}} \delta^3(0) \quad (30)$$

which refers to the vacuum energy of the theory. Furthermore, this energy is clearly divergent. We will investigate this aspect in the next section.

Furthermore, by explicit calculation, one obtains the commutation relations

$$[H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}} \quad (31)$$

$$[H, a_{\vec{p}}^\dagger] = \omega_{\vec{p}} a_{\vec{p}}^\dagger. \quad (32)$$

2.2 The Vacuum

We define the vacuum state $|0\rangle$ to be the state which is annihilated by all $a_{\vec{p}}$

$$a_{\vec{p}}|0\rangle = 0 \quad \forall \vec{p}. \quad (33)$$

Accordingly

$$H|0\rangle = E_0|0\rangle = \frac{1}{2} \int d^3\vec{p} \omega_{\vec{p}} \delta^3(0)|0\rangle \quad (34)$$

which is clearly infinite. In fact, there are two kinds of infinity which are

- Infra-red divergence from the factor $(2\pi)^3 \delta^3(0)$: This infinity is from the fact that the space where the theory lives is infinite.

$$(2\pi)^3 \delta^3(0) = \lim_{L \rightarrow \infty} \int_{L/2}^{L/2} d^3\vec{x} e^{i\vec{p}\cdot\vec{x}} \Big|_{\vec{p}=0} = V. \quad (35)$$

Therefore, instead of considering the total energy, we rather focus on the energy density

$$\epsilon_0 = \frac{E_0}{V} = \frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}}. \quad (36)$$

Unluckily, ϵ_0 is still divergent.

- UV divergence:

$$\epsilon_0 = \frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} = \frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \sqrt{\vec{p}^2 + m^2} = \frac{1}{(2\pi)^2} \int_0^\infty dp p^2 \sqrt{p^2 + m^2} \quad (37)$$

which goes to infinity as the momentum $p \rightarrow \infty$. This divergence arises because of the breakdown of the theory at high energies. We can dispose of this divergence by arguing that absolute energy cannot be measured (except for the gravitational theory). The only difference in energies (compared to the vacuum energy) matters.

Therefore, we can simply redefine the Hamiltonian by subtracting off this vacuum energy,

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} a_{\vec{p}}^\dagger a_{\vec{p}} \quad (38)$$

giving $H|0\rangle = 0$.

The order of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ is particularly important in quantum theory. The ordering we have used above is called normal ordering. We define normal ordering in such a way that

$$: a_{\vec{p}} a_{\vec{p}}^\dagger := a_{\vec{p}}^\dagger a_{\vec{p}} := a_{\vec{p}}^\dagger a_{\vec{p}}. \quad (39)$$

So, we could write the Hamiltonian (38) as

$$: H := \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} a_{\vec{p}}^\dagger a_{\vec{p}}. \quad (40)$$

2.3 The cosmological constant

In gravity, it turns out that the vacuum energy density ϵ_0 can be observed through the cosmological constant term

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \quad (41)$$

where

$$\Lambda = \epsilon_0. \quad (42)$$

The current observation suggests 70% of the universe is dark energy accelerating the universe whose energy density is around

$$\Lambda \sim (10^{-3} eV)^4 = 10^{-12} (eV)^4. \quad (43)$$

Let's compare it to ϵ_0 from the field theory

$$\epsilon_0 = \frac{1}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \omega_{\vec{p}} \sim \int_0^{\tilde{\Lambda}} dp p^2 \sqrt{p^2 + m^2} \sim \tilde{\Lambda}^4 \quad (44)$$

where $\tilde{\Lambda}$ is a cutoff-scale.

We know that QFT works well at least upto TeV or $10^{12} eV$ which contributes to $\epsilon_0 \sim 10^{48} (eV)^4$. This makes a vast difference between the observed values of vacuum energy density and the theoretical prediction suggested by QFT at least 10^{60} in order. This problem is known as the cosmological constant problem. One of the solutions is to set the vacuum energy to any desired value by renormalization. In exchange, we need to pay a prize by giving up the predictability of the theory.

2.4 Fock space

According to the vacuum state $|0\rangle$ defined in (33) together with the commutation relations (31) and (32), we can construct energy eigenstates by acting the vacuum $|0\rangle$ by $a_{\vec{p}}^\dagger$

$$|\vec{p}\rangle = a_{\vec{p}}^\dagger |0\rangle \quad (45)$$

satisfying

$$H|\vec{p}\rangle = \omega_{\vec{p}} |\vec{p}\rangle. \quad (46)$$

We can create multi-particle states by acting $a_{\vec{p}}^\dagger$ multiple times. For example, an N -particle state can be constructed as

$$a_{\vec{p}_1}^\dagger a_{\vec{p}_2}^\dagger \dots a_{\vec{p}_n}^\dagger |0\rangle = |\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n\rangle. \quad (47)$$

Since all $a_{\vec{p}}^\dagger$'s commute among themselves, the multi-particle states are symmetric under any two particles exchanged. For example,

$$|\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle. \quad (48)$$

The full Hilbert space spanned by acting on the vacuum with all possible combinations of $a_{\vec{p}}^\dagger$'s, i.e.

$$|0\rangle, a_{\vec{p}}^\dagger |0\rangle, a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger |0\rangle, \dots \quad (49)$$

This space is called Fock space. Another useful operator to know is the number operator which counts the number of particles in a multi-particle state. The number operator is defined as

$$N = \int \frac{d^3\vec{p}}{(2\pi)^3} a_{\vec{p}}^\dagger a_{\vec{p}}. \quad (50)$$

This gives

$$N|\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n\rangle = n|\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n\rangle. \quad (51)$$

Moreover, the number operator commutes with the Hamiltonian, i.e.

$$[N, H] = 0. \quad (52)$$

This implies that the number of particles is conserved. Remind that this statement is true only for free theory.

2.5 Relativistic normalization

Although we have constructed the multi-particle states as in (47), their norms are not Lorentz invariant which can be seen by considering

$$\langle\vec{p}|\vec{q}\rangle = \langle 0|a_{\vec{p}}a_{\vec{q}}^\dagger|0\rangle = (2\pi)^3\delta^3(\vec{p} - \vec{q}) \quad (53)$$

where $\langle 0|0\rangle = 1$. This may not be obvious but let's start by investigating the Lorentz invariant quantity

$$\int d^3\vec{p} \frac{1}{2\omega_{\vec{p}}}. \quad (54)$$

Proof. We start from the Lorentz invariant quantities

$$\int d^4p \quad \text{and} \quad p^2 - m^2. \quad (55)$$

Therefore, we can form another Lorentz invariant quantity

$$\begin{aligned} \int d^4p \delta(p^2 - m^2) \Big|_{p_0 > 0} &= \int d^4p \delta(p_0^2 - \vec{p}^2 - m^2) \Big|_{p_0 > 0} \\ &= \int dp_0 d^3\vec{p} \frac{\delta(p_0 - \omega_{\vec{p}})}{(p_0 + \omega_{\vec{p}})} = \int d^3\vec{p} \frac{1}{2\omega_{\vec{p}}}. \end{aligned} \quad (56)$$

Thus, the quantity (54) is Lorentz invariant as claimed. \square

From the result, it is not hard to see that

$$2\omega_{\vec{p}}\delta^3(\vec{p}-\vec{q}) \quad (57)$$

is Lorentz invariant which is because

$$\int d^3\vec{p}\frac{1}{2\omega_{\vec{p}}}2\omega_{\vec{p}}\delta^3(\vec{p}-\vec{q})=1. \quad (58)$$

Consequently, we can define the relativistic norm

$$\langle p|q\rangle=(2\pi)^32\omega_{\vec{p}}\delta^3(\vec{p}-\vec{q}). \quad (59)$$

According to the above definition, we can define a relativistically normalized state as

$$|p\rangle=\sqrt{2\omega_{\vec{p}}}|{\vec{p}}\rangle. \quad (60)$$

2.6 Causality

For causality to hold, we need to make sure that two measurements separated by spacelike distance do not affect each other. Mathematically, it can be written as

$$[O_1(x),O_2(y)]=0\quad\text{if}\quad(x-y)^2<0 \quad (61)$$

where $O(x)$ is a local observable. To confirm this, let's compute

$$\begin{aligned} \Delta(x-y)&:=[\phi(x),\phi(y)] \\ &=\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}\left([a_{\vec{p}},a_{\vec{q}}^\dagger]e^{-ip\cdot x+iq\cdot y}+[a_{\vec{p}}^\dagger,a_{\vec{q}}]e^{ip\cdot x-iq\cdot y}\right) \\ &=\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left(e^{-ip\cdot(x-y)}-e^{-ip\cdot(y-x)}\right). \end{aligned} \quad (62)$$

We know that the measure $\int d^3\vec{p}\frac{1}{2\omega_{\vec{p}}}$ is Lorentz invariant and so is $e^{\pm ip\cdot(x-y)}$. In case $(x-y)$ is spacelike, i.e. $(x-y)^2<0$, we can apply a Lorentz transformation such that $(x^0-y^0)=0$. Thus, the equation (62) becomes

$$\Delta(x-y)\Big|_{(x-y)^2<0}=\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})}-e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right) \quad (63)$$

which is zero when the change of variable $\vec{p}\rightarrow-\vec{p}$ was applied to the second term. This implies that no measurement can affect another measurement outside the light-cone. However, $\Delta(x-y)$ does not vanish for timelike separation. This can be seen by simply choosing the reference frame $(x-y)=(t,\vec{0})$ giving

$$\Delta(x-y)\Big|_{(x-y)^2>0}=\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left(e^{-i\omega_{\vec{p}}t}-e^{i\omega_{\vec{p}}t}\right)\neq 0. \quad (64)$$

2.7 Propagators

One can define a probability amplitude for a particle to propagate between two points in spacetime as

$$D(x - y) \equiv \langle 0 | \phi(x) \phi(y) | 0 \rangle \quad (65)$$

where the particle propagates from y to x . Note that $\phi(x)|0\rangle$ is a position eigenstate. By a mode expansion, one finds

$$D(x - y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip \cdot (x-y)}. \quad (66)$$

Another useful definition of propagator we would like to introduce is the famous Feynman propagator which is

$$D_F(x - y) \equiv \langle 0 | T \phi(x) \phi(y) | 0 \rangle \quad (67)$$

where

$$T \phi(x) \phi(y) = \begin{cases} \phi(x) \phi(y) & \text{if } x^0 \geq y^0 \\ \phi(y) \phi(x) & \text{if } y^0 > x^0. \end{cases} \quad (68)$$

Therefore, This yields

$$D_F(x - y) = \Theta(x^0 - y^0) D(x - y) + \Theta(y^0 - x^0) D(y - x) \quad (69)$$

where $\Theta(x - y)$ is the Heaviside step function. A convenient way to write the Feynman propagator is in the form

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^2 - m^2)} e^{-ip \cdot (x-y)}. \quad (70)$$

with the contour of p^0 illustrated in the figure 1.

Proof. Let's rewrite

$$p^2 - m^2 = (p^0)^2 - (\omega_{\vec{p}})^2 = (p^0 + \omega_{\vec{p}})(p^0 - \omega_{\vec{p}}). \quad (71)$$

It is obvious that there exist 2 poles, i.e. $p^0 = \pm \omega_{\vec{p}}$. This gives

$$D_F(x - y) = \int \frac{d^3\vec{p}}{(2\pi)^4} \int_{-\infty}^{\infty} dp^0 \frac{i}{(p^0 + \omega_{\vec{p}})(p^0 - \omega_{\vec{p}})} e^{-ip \cdot (x-y)}. \quad (72)$$

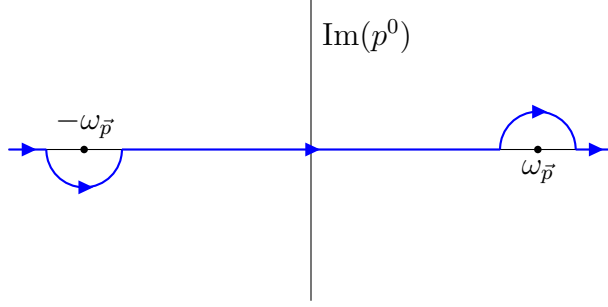


Figure 1: The contour for p^0

In case $x^0 > y^0$, we close the contour in the lower half plane. As $p^0 \rightarrow -i\infty$, $e^{-ip^0(x^0-y^0)}$ vanishes. Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} dp^0 \frac{e^{-ip \cdot (x-y)}}{(p^0 + \omega_{\vec{p}})(p^0 - \omega_{\vec{p}})} &= \oint dp^0 \frac{e^{-ip \cdot (x-y)}}{(p^0 + \omega_{\vec{p}})(p^0 - \omega_{\vec{p}})} \\ &= -2\pi i \frac{e^{-ip \cdot (x-y)}}{2\omega_{\vec{p}}} \end{aligned} \quad (73)$$

To obtain the last line, Cauchy's integral formula was used. Note that the minus sign is due to the fact that the contour is clockwise. Consequently, the Feynman propagator (72) in this case becomes

$$D_F(x-y) \Big|_{x^0 > y^0} = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip \cdot (x-y)} = D(x-y). \quad (74)$$

In case $y^0 > x^0$, we repeat the similar calculation but this time the contour will be closed in the upper-half plane which leads to

$$D_F(x-y) \Big|_{y^0 > x^0} = D(y-x). \quad (75)$$

Accordingly, the equation (69) holds. \square

Instead of specifying the contour in figure 1, it is more convenient to write the Feynman propagator as

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p^2 - m^2 + i\epsilon)} e^{-ip \cdot (x-y)}. \quad (76)$$

In addition, one can view the propagators as Green's functions which can be seen by considering

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}_{\vec{x}}^2 + m^2 \right) D_F(x-y) &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p^2 - m^2)} \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}_{\vec{x}}^2 + m^2 \right) e^{-ip \cdot (x-y)} \\ &= -i \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} = -i \delta^4(x-y). \end{aligned} \quad (77)$$

The Green's function is $iD_F(x-y)$ which is useful to compute inhomogeneous Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + m^2 \right) \phi = J(x). \quad (78)$$

3 Path Integral Formalism

3.1 Path integral in QM

The path integral or functional formalism is an alternative framework to compute observables in QM and QFT apart from the state-operator formalism.

The path integral formulation of QM centers around the transition amplitude. Consider a particle moving from position q_i at time t_i to position q_f at time t_f . The transition amplitude is

$$K(q_f, t_f; q_i, t_i) = \langle q_f | e^{-\frac{i}{\hbar} \hat{H}(t_f - t_i)} | q_i \rangle. \quad (79)$$

This object is also known as the *propagator* or *kernel*.

The quantity

$$|K(q_f, t_f; q_i, t_i)|^2 \quad (80)$$

is related to the probability of finding the particle at (q_f, t_f) given that it started at (q_i, t_i) .

Define the time evolution operator

$$\hat{U}(t_f, t_i) = e^{-\frac{i}{\hbar} \hat{H}(t_f - t_i)}. \quad (81)$$

Then the propagator becomes

$$K(q_f, t_f; q_i, t_i) = \langle q_f | \hat{U}(t_f, t_i) | q_i \rangle. \quad (82)$$

To derive the path integral representation, divide the total time interval

$$T = t_f - t_i \quad (83)$$

into N small intervals of equal size

$$\epsilon = \frac{T}{N} = \delta t. \quad (84)$$

The evolution operator may then be written as

$$\hat{U}(T) = \left(e^{-\frac{i}{\hbar}\epsilon\hat{H}} \right)^N. \quad (85)$$

Hence,

$$K = \langle q_f | \left(e^{-\frac{i}{\hbar}\epsilon\hat{H}} \right)^N | q_i \rangle. \quad (86)$$

Insert the identity operator between each infinitesimal evolution operator:

$$\mathbf{1} = \int dq_n |q_n\rangle \langle q_n|. \quad (87)$$

After inserting $N - 1$ completeness relations, the propagator becomes

$$K = \int \prod_{n=1}^{N-1} dq_n \prod_{n=0}^{N-1} \langle q_{n+1} | e^{-\frac{i}{\hbar}\epsilon\hat{H}} | q_n \rangle, \quad (88)$$

where

$$q_0 = q_i, \quad q_N = q_f. \quad (89)$$

The problem is now reduced to evaluating the short-time matrix element

$$\langle q_{n+1} | e^{-\frac{i}{\hbar}\epsilon\hat{H}} | q_n \rangle. \quad (90)$$

Now, consider the standard Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}). \quad (91)$$

For sufficiently small ϵ , one may expand

$$e^{-\frac{i}{\hbar}\epsilon\hat{H}} \simeq 1 - \frac{i}{\hbar}\epsilon \left(\frac{\hat{p}^2}{2m} + V(\hat{q}) \right) + \mathcal{O}(\epsilon^2). \quad (92)$$

Thus,

$$\langle q_{n+1} | e^{-\frac{i}{\hbar}\epsilon\hat{H}} | q_n \rangle \simeq \langle q_{n+1} | 1 - \frac{i}{\hbar}\epsilon \left(\frac{\hat{p}^2}{2m} + V(\hat{q}) \right) | q_n \rangle \quad (93)$$

Insert the momentum completeness relation

$$\mathbf{1} = \int \frac{dp_n}{2\pi\hbar} |p_n\rangle \langle p_n|. \quad (94)$$

Using

$$\langle q|p\rangle = e^{\frac{i}{\hbar}pq}, \quad (95)$$

we obtain

$$\begin{aligned} \langle q_{n+1}| e^{-\frac{i}{\hbar}\epsilon\hat{H}} |q_n\rangle &\simeq \int \frac{dp_n}{2\pi\hbar} \left(1 - \frac{i}{\hbar}\epsilon \left(\frac{p_n^2}{2m} + V(q_n) \right) \right) e^{\frac{i}{\hbar}p_n(q_{n+1}-q_n)} \\ &\simeq \int \frac{dp_n}{2\pi\hbar} e^{-\frac{i}{\hbar}\epsilon \left(\frac{p_n^2}{2m} + V(q_n) - p\dot{q}_n \right)} \end{aligned} \quad (96)$$

where $\dot{q}_n = \frac{q_{n+1}-q_n}{\epsilon}$ for small ϵ . Applying the Gaussian integral

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}, \quad (97)$$

then the infinitesimal propagator becomes

$$\langle q_{n+1}| e^{-\frac{i}{\hbar}\epsilon\hat{H}} |q_n\rangle \simeq \sqrt{\frac{m}{2\pi i\hbar\epsilon}} e^{i\epsilon \left(\frac{m}{2}\dot{q}_n^2 - V(q_n) \right)} \quad (98)$$

Therefore, the full propagator becomes

$$K = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i\hbar\epsilon} \right)^{N/2} \int \prod_{n=1}^{N-1} dq_n \exp \left[\frac{i}{\hbar} \sum_{n=0}^{N-1} \epsilon \left(\frac{m}{2}\dot{q}_n^2 - V(q_n) \right) \right]. \quad (99)$$

Recognizing the discretized form of the action, define the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q), \quad (100)$$

and the classical action

$$S[x] = \int_{t_i}^{t_f} dt L(q, \dot{q}). \quad (101)$$

Taking the continuum limit yields the Feynman path integral

$$K(q_f, t_f; q_i, t_i) = \int \mathcal{D}q(t) e^{\frac{i}{\hbar}S[q]} \quad (102)$$

where the symbol

$$\int \mathcal{D}q(t) \tag{103}$$

denotes an integration over all possible trajectories connecting the endpoints. Each path contributes a phase

$$e^{\frac{i}{\hbar}S[q]}. \tag{104}$$

Paths with rapidly oscillating phases interfere destructively, while paths near the classical trajectory contribute constructively.

In the classical limit,

$$\hbar \rightarrow 0, \tag{105}$$

the dominant contribution to the path integral comes from stationary points of the action:

$$\delta S = 0. \tag{106}$$

This condition gives the Euler–Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{107}$$

Hence classical mechanics emerges naturally from the path integral formulation.

Example: Free Particle

For a free particle,

$$V(q) = 0, \tag{108}$$

the exact propagator is

$$K(q_f, t_f; q_i, t_i) = \sqrt{\frac{m}{2\pi i \hbar T}} \exp \left[\frac{im(q_f - q_i)^2}{2\hbar T} \right], \tag{109}$$

where

$$T = t_f - t_i. \tag{110}$$

The exponent is precisely the classical action evaluated along the classical trajectory.

Path integral with quadratic action

Let the Lagrangian be quadratic,

$$L(q, \dot{q}) = aq^2 + b\dot{q}^2 + cq\dot{q} + dq + e\dot{q} \tag{111}$$

The action is

$$S[q] = \int_{t_i}^{t_f} dt L(q, \dot{q}). \quad (112)$$

We now decompose the path into a classical part and a fluctuation:

$$q(t) = q_{\text{cl}}(t) + \eta(t), \quad (113)$$

where the fluctuation satisfies fixed endpoint conditions

$$\eta(t_i) = \eta(t_f) = 0. \quad (114)$$

The classical path q_{cl} satisfies the Euler–Lagrange equation. One can show that

$$S[q] = S[q_{\text{cl}}] + S[\eta] \quad (115)$$

where $S[\eta]$ contains only quadratic in η and $\dot{\eta}$ (no linear in η and $\dot{\eta}$). In particular,

$$\begin{aligned} S[q_{\text{cl}}] &= aq_{\text{cl}}^2 + b\dot{q}_{\text{cl}}^2 + cq_{\text{cl}}\dot{q}_{\text{cl}} + dq_{\text{cl}} + e\dot{q}_{\text{cl}} \\ S[\eta] &= a\eta^2 + b\dot{\eta}^2 + c\eta\dot{\eta}. \end{aligned} \quad (116)$$

Accordingly, the propagator reads

$$K = e^{\frac{i}{\hbar}S[q_{\text{cl}}]} \int \mathcal{D}\eta e^{\frac{i}{\hbar}S[\eta]} = e^{\frac{i}{\hbar}S[q_{\text{cl}}]} F(T) \quad (117)$$

Note that the path integral over η depends only on the time interval $T = t_f - t_i$.

Consider the simple harmonic oscillator with Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2. \quad (118)$$

As discussed previously, we decompose the path into

$$q(t) = q_{\text{cl}}(t) + \eta(t), \quad (119)$$

where

$$\eta(t_i) = \eta(t_f) = 0. \quad (120)$$

Hence the propagator factorizes as

$$K(q_f, t_f; q_i, t_i) = e^{\frac{i}{\hbar}S[q_{\text{cl}}]} \int \mathcal{D}\eta e^{\frac{i}{\hbar}S[\eta]}. \quad (121)$$

The fluctuation integral is independent of the endpoints and contributes only an overall normalization factor. Therefore, the main task is to compute the classical action. The classical equation of motion is

$$\ddot{q}_{\text{cl}} + \omega^2 q_{\text{cl}} = 0. \quad (122)$$

Its general solution is

$$q_{\text{cl}}(t) = A \cos(\omega t) + B \sin(\omega t). \quad (123)$$

Imposing the boundary conditions

$$q_{\text{cl}}(0) = q_i, \quad q_{\text{cl}}(T) = q_f, \quad (124)$$

we obtain

$$q_{\text{cl}}(t) = \frac{q_f \sin(\omega t) + q_i \sin[\omega(T - t)]}{\sin(\omega T)}. \quad (125)$$

We now compute the classical action

$$S[q_{\text{cl}}] = \int_0^T dt \left(\frac{m}{2} \dot{q}_{\text{cl}}^2 - \frac{m\omega^2}{2} q_{\text{cl}}^2 \right). \quad (126)$$

Integrating the kinetic term by parts,

$$\int_0^T dt \dot{q}_{\text{cl}}^2 = q_{\text{cl}} \dot{q}_{\text{cl}} \Big|_0^T - \int_0^T dt q_{\text{cl}} \ddot{q}_{\text{cl}}. \quad (127)$$

Using the classical equation

$$\ddot{q}_{\text{cl}} = -\omega^2 q_{\text{cl}}, \quad (128)$$

the bulk terms cancel and the action reduces to a boundary contribution:

$$S[q_{\text{cl}}] = \frac{m}{2} q_{\text{cl}} \dot{q}_{\text{cl}} \Big|_0^T. \quad (129)$$

Differentiating the classical solution,

$$\dot{q}_{\text{cl}}(t) = \frac{\omega q_f \cos(\omega t) - \omega q_i \cos[\omega(T - t)]}{\sin(\omega T)}. \quad (130)$$

Substituting into the action gives

$$S[q_{\text{cl}}] = \frac{m\omega}{2 \sin(\omega T)} [(q_f^2 + q_i^2) \cos(\omega T) - 2q_i q_f]. \quad (131)$$

Therefore, the propagator takes the form

$$K(q_f, T; q_i, 0) = F(T) \exp \left[\frac{i}{\hbar} \frac{m\omega}{2 \sin(\omega T)} \left((q_f^2 + q_i^2) \cos(\omega T) - 2q_i q_f \right) \right], \quad (132)$$

where $F(T)$ is the normalization factor arising from the fluctuation integral. The exact evaluation of the Gaussian fluctuation integral gives

$$F(T) = \sqrt{\frac{m\omega}{2\pi i \hbar \sin(\omega T)}}. \quad (133)$$

Hence the exact propagator for the simple harmonic oscillator is

$$K(q_f, T; q_i, 0) = \sqrt{\frac{m\omega}{2\pi i \hbar \sin(\omega T)}} \exp \left[\frac{i}{\hbar} \frac{m\omega}{2 \sin(\omega T)} \left((q_f^2 + q_i^2) \cos(\omega T) - 2q_i q_f \right) \right]. \quad (134)$$

In the limit $\omega \rightarrow 0$, this reduces to the free-particle propagator

$$K(q_f, T; q_i, 0) = \sqrt{\frac{m}{2\pi i \hbar T}} \exp \left[\frac{im(q_f - q_i)^2}{2\hbar T} \right]. \quad (135)$$

3.2 Path integral in scalar field theory

In quantum field theory, the dynamical variable is no longer a single coordinate $q(t)$, but a field

$$\phi(\mathbf{x}, t). \quad (136)$$

Consequently, the quantum-mechanical path integral generalizes naturally into a functional integral over all field configurations.

In quantum field theory, the analogue of the quantum-mechanical propagator is the vacuum expectation value of time-ordered field operators. For a real scalar field, the two-point correlation function is

$$\langle 0|T \left\{ \hat{\phi}(x_2) \hat{\phi}(x_1) \right\} |0\rangle, \quad (137)$$

In interacting quantum field theory, the true vacuum state differs from the free-theory vacuum. One therefore replaces

$$|0\rangle \rightarrow |\Omega\rangle, \quad (138)$$

where $|\Omega\rangle$ denotes the interacting vacuum state. The physical two-point function is thus

$$\langle \Omega|T \left\{ \hat{\phi}(x_2) \hat{\phi}(x_1) \right\} |\Omega\rangle. \quad (139)$$

The path integral formalism provides a remarkably compact representation of such correlation functions. The vacuum expectation value of any operator $\mathcal{O}[\phi]$ may be written as

$$\frac{\langle \Omega | T \{ \mathcal{O}[\phi] \} | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \frac{\int \mathcal{D}\phi \mathcal{O}[\phi] e^{iS[\phi]}}{\int \mathcal{D}\phi e^{iS[\phi]}}. \quad (140)$$

In particular, the two-point correlation function becomes

$$\frac{\langle \Omega | T \{ \hat{\phi}(x_2) \hat{\phi}(x_1) \} | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \frac{\int \mathcal{D}\phi \phi(x_2) \phi(x_1) e^{iS[\phi]}}{\int \mathcal{D}\phi e^{iS[\phi]}} \quad (141)$$

The denominator serves as a normalization factor and is often denoted by

$$Z = \int \mathcal{D}\phi e^{iS[\phi]}, \quad (142)$$

which is known as the partition function or generating functional. More generally, the n -point correlation function is

$$\frac{\langle \Omega | T \{ \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) \} | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \frac{1}{Z} \int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) e^{iS[\phi]}. \quad (143)$$

Generating functional

A convenient way to compute correlation functions in quantum field theory is through the generating functional. Consider a real scalar field with action

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \quad (144)$$

We introduce an external source $J(x)$ and define the generating functional

$$Z[J] = \int \mathcal{D}\phi \exp \left[iS[\phi] + i \int d^4x J(x) \phi(x) \right] \quad (145)$$

The additional term $\int d^4x J(x) \phi(x)$ is called the source term.

Correlation functions may now be obtained by taking functional derivatives with respect to the source $J(x)$. Since

$$\frac{-i\delta}{\delta J(x)} e^{i \int d^4y J(y) \phi(y)} = \phi(x) e^{i \int d^4y J(y) \phi(y)}, \quad (146)$$

we find

$$\frac{-i\delta Z[J]}{\delta J(x)} = \int \mathcal{D}\phi \phi(x) e^{iS[\phi] + i\int J\phi}. \quad (147)$$

Similarly,

$$\frac{(-i)^2 \delta^2 Z[J]}{\delta J(x_2) \delta J(x_1)} = \int \mathcal{D}\phi \phi(x_2) \phi(x_1) e^{iS[\phi] + i\int J\phi}. \quad (148)$$

Setting $J = 0$, the two-point correlation function becomes

$$\frac{\langle \Omega | T \{ \hat{\phi}(x_2) \hat{\phi}(x_1) \} | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \frac{1}{Z[0]} \left. \frac{(-i)^2 \delta^2 Z[J]}{\delta J(x_2) \delta J(x_1)} \right|_{J=0}. \quad (149)$$

More generally,

$$\frac{\langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \frac{1}{Z[0]} \left. \frac{(-i)^n \delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0}. \quad (150)$$

Consider the free scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \quad (151)$$

Integrating the kinetic term by parts,

$$\begin{aligned} S[\phi] &= \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) \\ &= -\frac{1}{2} \int d^4x \phi(x) (\square + m^2) \phi(x), \end{aligned} \quad (152)$$

where boundary terms have been neglected.

The generating functional becomes

$$Z[J] = \int \mathcal{D}\phi \exp \left[-\frac{i}{2} \int d^4x \phi (\square + m^2) \phi + i \int d^4x J\phi \right]. \quad (153)$$

where $\square = \partial_\mu \partial^\mu$. This is a Gaussian functional integral. To evaluate it, complete the square by shifting

$$\phi(x) = \phi_c(x) + \phi'(x), \quad (154)$$

where ϕ_c satisfies

$$(\square + m^2) \phi_c(x) = J(x). \quad (155)$$

We know that the Feynman propagator $D_F(x - y)$ obeys

$$(\square + m^2)D_F(x - y) = -i\delta^{(4)}(x - y), \quad (156)$$

the classical solution is

$$\phi_c(x) = i \int d^4y D_F(x - y)J(y). \quad (157)$$

Substituting

$$\phi(x) = \phi'(x) + i \int d^4y D_F(x - y)J(y) \quad (158)$$

into the generating functional,

$$Z[J] = \int \mathcal{D}\phi \exp \left[-\frac{i}{2} \int d^4x \phi(\square + m^2)\phi + i \int d^4x J\phi \right], \quad (159)$$

we obtain

$$\begin{aligned} Z[J] = \int \mathcal{D}\phi' \exp & \left[-\frac{i}{2} \int d^4x \left(\phi' + i \int d^4y D_F(x - y)J(y) \right) (\square + m^2) \right. \\ & \times \left(\phi' + i \int d^4z D_F(x - z)J(z) \right) \\ & \left. + i \int d^4x J(x) \left(\phi'(x) + i \int d^4y D_F(x - y)J(y) \right) \right]. \quad (160) \end{aligned}$$

Using (156), one can show that

$$\begin{aligned} Z[J] = \int \mathcal{D}\phi' \exp & \left[-\frac{i}{2} \int d^4x \phi'(\square + m^2)\phi' \right. \\ & \left. - \frac{1}{2} \int d^4x d^4y J(x)D_F(x - y)J(y) \right]. \quad (161) \end{aligned}$$

The second term is independent of ϕ' and may be factored out:

$$\begin{aligned} Z[J] = \exp & \left[-\frac{1}{2} \int d^4x d^4y J(x)D_F(x - y)J(y) \right] \\ & \times \int \mathcal{D}\phi' \exp \left[-\frac{i}{2} \int d^4x \phi'(\square + m^2)\phi' \right]. \quad (162) \end{aligned}$$

The remaining functional integral is precisely $Z[0]$. Therefore,

$$Z[J] = Z[0] \exp \left[-\frac{1}{2} \int d^4x d^4y J(x)D_F(x - y)J(y) \right]. \quad (163)$$

Computation for scalar field propagator

Recall that

$$\langle \Omega | T \{ \phi(x_1) \phi(x_2) \} | \Omega \rangle = \frac{1}{Z[0]} \left. \frac{(-i)^2 \delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0}. \quad (164)$$

We first compute the first functional derivative:

$$\begin{aligned} \frac{\delta Z[J]}{\delta J(x_1)} &= Z[0] \frac{\delta}{\delta J(x_1)} \exp \left[-\frac{1}{2} \int d^4x d^4y J(x) D_F(x-y) J(y) \right] \\ &= -\frac{1}{2} Z[J] \frac{\delta}{\delta J(x_1)} \left(\int d^4x d^4y J(x) D_F(x-y) J(y) \right). \end{aligned} \quad (165)$$

Using

$$\frac{\delta J(x)}{\delta J(x_1)} = \delta^{(4)}(x - x_1), \quad (166)$$

we obtain

$$\frac{\delta Z[J]}{\delta J(x_1)} = -Z[J] \int d^4y D_F(x_1 - y) J(y). \quad (167)$$

Differentiating once more, we obtain

$$\frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} = -\frac{\delta Z[J]}{\delta J(x_2)} \int d^4y D_F(x_1 - y) J(y) - Z[J] D_F(x_1 - x_2). \quad (168)$$

Now set $J = 0$. Since

$$\left. \frac{\delta Z[J]}{\delta J(x)} \right|_{J=0} = 0, \quad (169)$$

the first term vanishes and we are left with

$$\left. \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0} = -Z[0] D_F(x_1 - x_2). \quad (170)$$

Substituting into the definition of the two-point function,

$$\langle \Omega | T \{ \phi(x_1) \phi(x_2) \} | \Omega \rangle = D_F(x_1 - x_2). \quad (171)$$

Including interaction terms

We now generalize the discussion to interacting scalar field theories. Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi), \quad (172)$$

where $V(\phi)$ contains nonlinear interaction terms. A common example is the ϕ^4 theory,

$$V(\phi) = \frac{\lambda}{4!} \phi^4. \quad (173)$$

The generating functional becomes

$$Z[J] = \int \mathcal{D}\phi \exp \left[iS[\phi] + i \int d^4x J(x)\phi(x) \right]. \quad (174)$$

Separating the free and interacting parts of the action,

$$S[\phi] = S_0[\phi] - \int d^4x V(\phi), \quad (175)$$

we may write

$$Z[J] = \int \mathcal{D}\phi \exp \left[iS_0[\phi] + i \int d^4x J\phi \right] \exp \left[-i \int d^4x V(\phi) \right]. \quad (176)$$

The first exponential is precisely the free-theory generating functional. Therefore,

$$Z[J] = \exp \left[-i \int d^4x V \left(\frac{-i\delta}{\delta J(x)} \right) \right] Z_0[J], \quad (177)$$

where

$$Z_0[J] = Z_0[0] \exp \left[-\frac{1}{2} \int d^4x d^4y J(x) D_F(x-y) J(y) \right] \quad (178)$$

is the free generating functional.

Thus, the complete generating functional may be written compactly as

$$Z[J] = \exp \left[-i \int d^4x V \left(\frac{-i\delta}{\delta J(x)} \right) \right] Z_0[J]. \quad (179)$$

This formula is fundamental in perturbative quantum field theory. Correlation functions are obtained by differentiating with respect to $J(x)$, while the interaction terms generate perturbative corrections represented diagrammatically by Feynman diagrams.

For example, in ϕ^4 theory,

$$V(\phi) = \frac{\lambda}{4!} \phi^4, \quad (180)$$

the generating functional becomes

$$Z[J] = \exp \left[-i \frac{\lambda}{4!} \int d^4x \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^4 \right] Z_0[J]. \quad (181)$$

Expanding the exponential in powers of λ generates the perturbation series of the interacting theory.