Lecture on cosmology parts II and III

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Distances in cosmology

- In this lecture, we will learn how to measure distances in the universe.
- According to homogeneity and isotropy of 3D space of the universe, the spacetime of the universe is describe by the FLRW metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1}$$

  where $a(t)$ is a cosmic scale factor and r, θ and ϕ are comoving coordinates.
- The physical length l is related to the comoving length l_c via

$$l = al_c\,. \tag{2}$$

Distances in cosmology

- In GR, the speed of light in vacuum is constant so that light from a source at long distances was emitted long time ago before reaching us today.
- Since the universe is expanding, a scale factor at which the light was emitted from a long-distance source is smaller than the one at present.
- We know that physical wavelength of light is stretched due to the expansion of the universe as

$$\frac{\lambda_0}{\lambda_e} = \frac{a_0 \lambda_c}{a_e \lambda_c} \frac{a_0}{a_e} \equiv 1 + z_e \,, \tag{3}$$

where subscripts $_e$ and $_0$ denote evaluation at emission time and at present . In the above expression λ_c is a comoving wavelength of light.

Distances in cosmology

- Distance to a light source cannot be quantified using only redshift, because relation between redshift and time strongly depends on cosmological models.
- In addition to redshift, information about the distance of the light source is encoded in observed luminosity.
- We will learn that we can also use characteristic length scales in the universe to quantify distances in the universe.

Luminosity distance

- Distances can be implied from luminosity and observed flux of the light sources as follows:

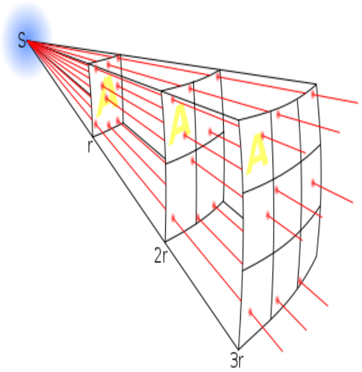

Figure: The flux is the energy per unit time interval per unit area.

Luminosity distance

- The flux measured by observer at distance r from the light source with luminosity L is

$$F = \frac{L}{4\pi r^2}\,, \tag{4}$$

- In the expanding universe the energy of a single photon is redshifted by

$$\frac{E_e}{E_0} = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e} = 1 + z_e\,, \tag{5}$$

  where we have used $e_\gamma \propto 1/\lambda$ and subscript $_\gamma$ denotes photon.

- Physical distances between to adjacent photons at the emitting and detecting time are related by

$$\frac{\delta r_e}{\delta r_0} = \frac{a_e}{a_0} = \frac{1}{1+z_e}\,, \quad \Rightarrow \quad \frac{\delta t_0}{\delta t_e} = 1 + z_e\,, \tag{6}$$

  where we have used $\delta r_e/\delta t_e = \delta r_0/\delta t_0 =$ speed of light.

Luminosity distance

- From the definition of luminosity:

$$L = \frac{E}{\delta t}\,, \quad \Rightarrow \quad L_0 = \frac{L_e}{(1+z_e)^2}\,. \tag{7}$$

- Using the definition of flux:

$$F_0 = \frac{L_0}{4\pi r^2} = \frac{L_e}{4\pi r^2(1+z_e^2)}\,. \tag{8}$$

- Luminosity distance is defined as

$$d_L = (1+z)r\,. \tag{9}$$

Luminosity distance

- In the spatially flat universe, comoving distance propagated by photon is computed from

$$ds^2 = 0 = -dt^2 + a^2(t)dr^2 \,, \tag{10}$$

where we have assumed that light emitted from a source has spherical symmetry.

- Hence, a comoving distance from a light source to observer today can be computed as

$$r = \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e}^{t_0} \frac{da}{Ha^2} = \int_0^{z_e} \frac{dz'}{H(z')} \,, \tag{11}$$

where the Hubble parameter is

$$H \equiv \frac{\dot{a}}{a} \,, \tag{12}$$

Luminosity distance

- From the above equations, the luminosity distance is given by

$$d_L = (1+z)\int_0^z \frac{dz'}{H(z')}\,, \quad \Rightarrow \quad d_L = (1+z)H_0^{-1}\int_0^z \frac{dz'}{E(z')}\,, \tag{13}$$

where we have dropped subscript $_e$ and

$$E(z) \equiv \frac{H(z)}{H_0}\,. \tag{14}$$

- The above equation gives a relation between luminosity and redshift.

Standard candle

- We can use standard candle such as super nova type I to measure luminosity-redshift relation:

$$d_L^O = \sqrt{\frac{L}{4\pi F}} \,. \tag{15}$$

- Comparing d_L from prediction of the model with d_L^O from SN data, we can constrain cosmological parameters.
- The cosmological parameters those satisfy observations correspond to the universe that expand with acceleration at late-time.

Angular diameter distance

- If we know diameter of a sphere, we can determine a distance from us to a sphere by measuring size of a sphere we seen in terms of as angular size as shown in the figure.

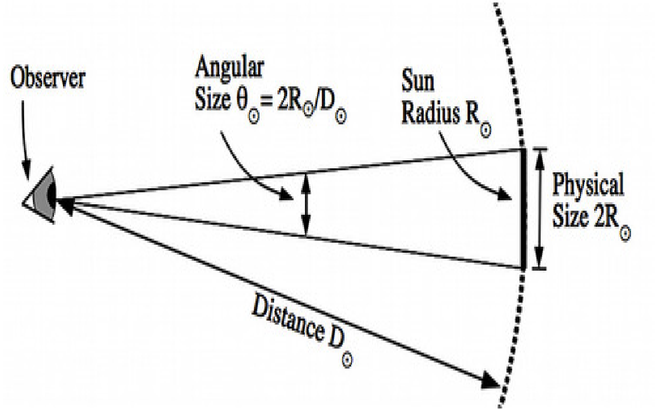

Figure: An angle θ is an angular size of a object.

Angular diameter distance

- The comoving angular diameter distance is defined as

$$d_A \equiv \frac{l}{\theta}\,, \tag{16}$$

  where l is comoving length scales which can be compute model-independently, and θ can be precisely measured.

- The ratio l/θ

$$\frac{l}{\theta} = d_A = r = \int_t^{t_0} \frac{dt'}{a(t')} = \eta_0 - \eta\,, \tag{17}$$

  where η is a conformal time.

- The above equation relates to the cosmological models through

$$\int_t^{t_0} \frac{dt'}{a(t')} = \frac{1}{H_0} \int_0^z \frac{dz'}{E(z')}\,. \tag{18}$$

Standard ruler

- The important length scales in the universe that can be computed theoretically for given models of the universe and their angular size can be observed are the length scales associated with the pattern of the CMB and matter perturbations.

Horizons: particle horizon

- A distance that photon propagates since the big bang until time t is a distance to the particle horizon.

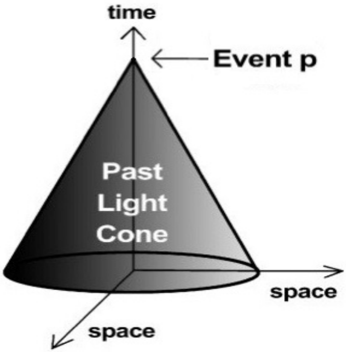

Figure: The particle horizon is the past light cone of the observer at present.

Horizons: particle horizon

- Using the fact that light propagates along the trajectory that satisfies

$$ds^2 = 0 \,, \Rightarrow \frac{dt}{a(t)} = dr \,, \tag{19}$$

where r is a comoving distance.

- A particle horizon can be computed as

$$R_p(t) = a(t) \int_0^t \frac{dt'}{a(t')} \,. \tag{20}$$

Horizons: event horizon

- The farthest distance that photon can propagate from today to the far future is the event horizon.

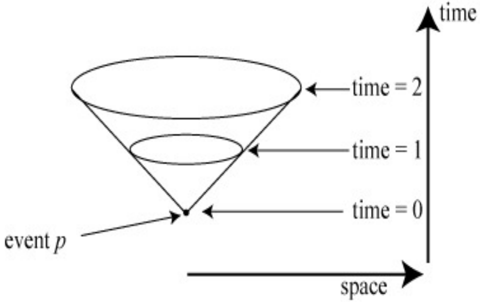

Figure: The event horizon is the future light cone of the observer at present.

Horizons: event horizon

- The event horizon can be computed as

$$R_h(t) = a(t) \int_t^\infty \frac{dt'}{a(t')} \, . \tag{21}$$

- Compute R_h using $a \propto t^p$ where p is constant, and find a range of p that make R_h finite.

Conservation of energy density

- Matter and energy in the universe can be modeled by a fluid which a continuous substance.
- To satisfy homogeneity and isotropy of the background universe, the cosmic fluid cannot flow and deliver energy as well as momentum in the comoving frame of the universe.
- Since both energy and momentum do not flow, the properties of the fluid are characterized by energy density and pressure.
- This type of fluid is a perfect fluid.

Conservation of energy density

- From theory of relativity, the energy of a particle is given by

$$E = \sqrt{p^2 + m^2}\,, \tag{22}$$

where p^2 is a momentum square, m is a rest mass of a particle and speed of light is set to unity.

- For a non-relativistic particle, we have $m^2 \gg p^2$, and therefore

$$E \simeq m\,. \tag{23}$$

- For a relativistic particle, we have $m^2 \ll p^2$, and therefore

$$E \simeq |p|\,. \quad \text{and for a massless particle} \quad E \propto \frac{1}{\lambda}\,, \tag{24}$$

where λ is a wavelength of a particle.

Conservation of energy density

- For a group of non-relativistic particles, the energy density can be computed as

$$\rho = \frac{NE}{V} \simeq \frac{Nm}{V} \propto \frac{1}{V} \propto \frac{1}{a^3 l_c^3} \propto a^{-3}\,, \tag{25}$$

  where N is total number of non-relativistic particle and l_c^3 corresponds to comoving volume.
- For a group of relativistic particles, the energy density can be computed as

$$\rho = \frac{NE}{V} \propto \frac{N}{V\lambda} \propto \frac{1}{a^4 l_c^3 \lambda_c} \propto a^{-4}\,, \tag{26}$$

  where N is total number of relativistic particle.
- A group of non-relativistic particles in the universe is described by a perfect fluid known as matter or dust, while a group of relativistic particles is described by a perfect fluid known as radiation.

Conservation of energy density

- In the following consideration, quantities associated with matter are denoted by subscript $_m$ while those for radiation are denoted by subscript $_r$.
- We now have

$$\rho_m \propto a^{-3}\,, \quad \text{and} \quad \rho_r \propto a^{-3}\,, \tag{27}$$

- In general, the above relations can be derived from the conservation of the energy-momentum tensor:

$$\nabla_\mu T^\mu_\nu = 0\,, \tag{28}$$

where ∇_μ is a covariant derivative.

Conservation of energy density

- In the background universe, the conservation of the energy-momentum tensor yields

$$\dot{\rho} = -3H(\rho + P)\,, \tag{29}$$

where a dot denotes derivative with respect to time, P is a pressure of a fluid and $H \equiv \dot{a}/a$ is the Hubble parameter.

- One of the properties of a fluid is the relation between the energy density and pressure. This relation is the equation of state.
- For a simple perfect fluid, the equation of state can be written in the form

$$P = w\rho\,, \tag{30}$$

where w is an equation of state parameter.

Conservation of energy density

- From $\rho_m \propto a^{-3}$, we get $\dot{\rho}_m = -3H\rho_m$, so that

$$w_m = 0\,. \tag{31}$$

- Use the relation $\rho_r \propto a^{-4}$ to compute w_r.
- Inserting $P = w\rho$ into the Eq. (29), we get

$$\dot{\rho} = -3H(1+w)\rho\,. \tag{32}$$

Conservation of energy density

- For a constant w, Eq. (32) can be integrated as

$$\rho \propto a^{-3(1+w)} \, . \tag{33}$$

- You can check that $w_m = 0$ yields $\rho_m \propto a^{-3}$ and $w_r = 1/3$ yields $\rho_r \propto a^{-4}$.

Cosmological constant

- Let us consider the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{34}$$

where $G_{\mu\nu}$ is the Einstein tensor and G is the Newton's gravitational constant.

- The above equation is the tensor equation. Hence, if we would like to add a constant term into this equation, that term has to be a tensor which the simplest form is

$$g_{\mu\nu}\Lambda\,, \tag{35}$$

where $g_{\mu\nu}$ is a metric tensor and Λ is a constant.

- Inserting the above term into Eq. (34), we get

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\left(T_{\mu\nu} + \underbrace{\frac{\Lambda g_{\mu\nu}}{8\pi G}}_{\equiv T_{\mu\nu\Lambda}}\right)\,. \tag{36}$$

Cosmological constant

- Comparing the energy-momentum tensor of Λ which is in the form

$$T_{\mu\nu\Lambda} = \frac{\Lambda g_{\mu\nu}}{8\pi G}\,, \tag{37}$$

  with that for a perfect fluid in the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & P g_{ij} \end{pmatrix}\,, \tag{38}$$

  we get

$$\rho_\Lambda = -P_\Lambda = -\frac{\Lambda}{8\pi G}\,. \tag{39}$$

- Determine a value of w_Λ from the above equation.

Cosmological constant

- Inserting w_Λ into Eq. (32), we get

$$\dot{\rho}_\Lambda = 0\,, \quad \Rightarrow \quad \rho_\Lambda = \mathrm{constant}\,. \tag{40}$$

- This suggests that the energy associated with the cosmological constant is proportional to 3D spatial volume of spacetime, i.e.,

$$\rho_\Lambda = \mathrm{constant}\,, \quad \Rightarrow \quad E_\Lambda \propto V\,. \tag{41}$$

Basic cosmological models

- Using the FLRW metric for spatially flat spacetime and the perfect fluid model for the energy and matter in the universe, we obtain the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho_T\,, \tag{42}$$

  where ρ_T is the total energy density in the universe.
- Let us first consider the universe containing only a cosmological constant. For this case, the Friedmann equation becomes

$$H^2 = \frac{8\pi G}{3}\rho_\Lambda = \mathrm{constant}\,, \tag{43}$$

  which suggests that the Hubble parameter is constant.
- Compute a scale facter $a(t)$ as a function of time using the above equation

Basic cosmological models

- If there is a cosmological constant only in the universe, one can show that
$$a(t) \propto \mathrm{e}^{Ht} \,. \tag{44}$$
- This model of the universe is the de Sitter model.
- Let us next consider the universe containing only matter.

Basic cosmological models

- If the universe contains only matter, the friedmann equations becomes

$$H^2 = \frac{8\pi G}{3}\rho_m \propto a^{-3}\,. \tag{45}$$

- The solution for the above equation is

$$a \propto t^{2/3}\,, \quad \Rightarrow \quad H = \frac{2}{3t}\,. \tag{46}$$

- This model of the universe is the Einstein – de Sitter model.

Standard model of the universe

- In the ΛCDM model of the universe, the energy and matter relevant to important phenomena in the universe are the follows:
  - $\times$ Photon $\Rightarrow$ radiation
  - $\times$ Massless Neutrino $\Rightarrow$ radiation
  - $\times$ Baryon $\Rightarrow$ matter
  - $\times$ Cold Dark Matter (CDM) $\Rightarrow$ matter
  - $\times$ Cosmological Constant $\Rightarrow$ mysterious form of energy driving an accelerated expansion of the late-time universe

Standard model of the universe

- The cosmological parameter that quantifies the energy fraction of each energy components in the universe is the density parameter defined as

$$\Omega_\alpha = \frac{8\pi G \rho_\alpha}{3H^2}\,, \quad \text{and } quad \Omega_k = -\frac{k}{a^2 H^2}\,, \tag{47}$$

  where index α runs over γ, ν, b, c and Λ for photon, neutrino, baryon, CDM and cosmological constant, respectively. Here, Ω_k quantifies the contribution from the spatial curvature of the universe.

- From the definition of the density parameter, we can write the Friedmann equation with spatial curvature,

$$H^2 = \frac{8\pi G}{3}\rho_T - \frac{k}{a^2}\,, \tag{48}$$

  as

$$1 = \Omega_\gamma + \Omega_\nu + \Omega_b + \Omega_c + \Omega_\Lambda + \Omega_k\,. \tag{49}$$

Dark energy

- According to observation, the present values of the density parameter of each species are

$$\begin{aligned} \Omega_\gamma^0 &\sim 10^{-4}\,, \quad \Omega_\nu^0 \sim 10^{-4}\,, \quad \Omega_k^0 \sim 10^{-2}\,, \\ \Omega_b^0 &\simeq 0.022\,, \quad \Omega_c^0 \simeq 0.3\,, \quad \Omega_\Lambda \simeq 0.68\,. \end{aligned} \tag{50}$$

- Hence, usually we suppose that the universe is spatially flat.

Dark energy

- Observations strongly indicate that the universe is expanding with acceleration at late-time.
- We now discuss how an accelerated expansion of the universe at late-time can be achieved.
- Differentiating the Friedmann equation in Eq. (42) with respect to time, and using the conservation equation in Eq. (32), we get

$$\dot{H} = -\frac{3}{2}\left(1 + w_T\right) H^2 \,, \tag{51}$$

where the equation of state parameter of the total fluid is $w_T = P_T/\rho_T$.

Dark energy

- Hence, we can compute the acceleration equation as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3} H^2 \left(1 + 3w_T\right) . \tag{52}$$

- This suggests that the expansion of the universe will be accelerate if

$$w_T < -\frac{1}{3} . \tag{53}$$

Dark energy

- From $w_T = P_T/\rho_T$ and the fact that the energy density of radiation can be ignored at a late-time, we get

$$w_T = \frac{P_T}{\rho_T} \simeq \frac{w_m \rho_m + w_d \rho_d}{\rho_m + \rho_d} = w_d \Omega_d \,, \qquad (54)$$

  where w_d is the equation of state parameter and ρ_d is the energy density of unknown form of energy called dark energy.

- Inserting the above equation into Eq. (53), we obtain

$$w_d < -\frac{1}{3\Omega_d} \sim -0.5 \,. \qquad (55)$$

- From observation, we have

$$w_d \sim -1 \,. \qquad (56)$$

Dark energy

- The constraint on the previous slide is satisfied well by a cosmological constant.
- Moreover, since a cosmological constant is the simplest model for dark energy, the model of the universe in which an accelerated expansion of the late-time universe is driven by a cosmological constant and dark matter is the cold dark matter is ΛCDM model. The ΛCDM is the standard model of the universe.

Dark matter

- An orbital speed of a point mass with mass m that rotates around a galaxy with mass M is

$$\frac{GMm}{r^2} = \frac{mv^2}{r}, \quad v \propto \frac{1}{\sqrt{r}}, \tag{57}$$

- However, the observed rotation curve is approximately "flat", i.e., the orbital speed is approximately constant over a large range of distances. This seems to indicate that there is a non-visible matter distribution in the galaxy.
- This non-visible matter is dark matter, which could be cold dark matter or hot dark matter, where cold dark matter is non-relativistic matter while hot dark matter is relativistic one.

epochs of the universe

- From observations, the density parameter of dark energy is largest compared with others at present, so that the universe is in the acceleration epoch at a late-time.
- Since the energy density of matter decays faster than that of dark energy, and the energy density of radiation decays fastest, the energy density of the universe will be dominated by matter and radiation respectively when we look back in time.

epochs of the universe

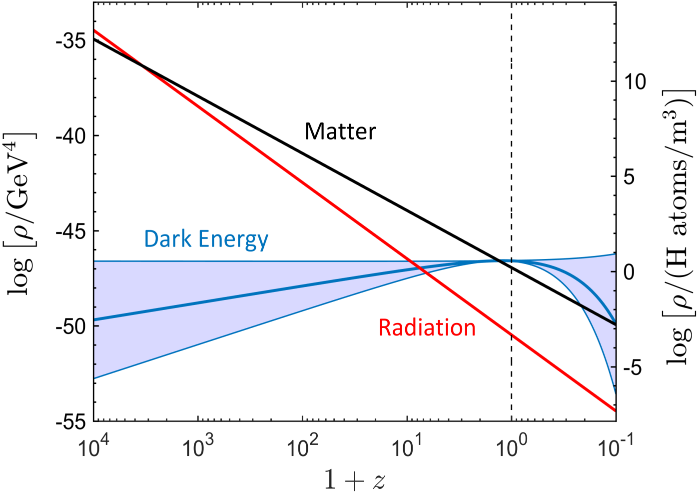

Figure: Plots of the energy density of radiation, matter and dark energy as function of redshift z.

epochs of the universe

In the standard model of the universe, the universe undergoes the following eras after the big bang.

Inflation	$t \sim 10^{-14} - 10^{-43}\,s$; $T \sim 10\,\mathrm{TeV} - 10^{19}$ Gev
Baryon & Lepton synthesis	$t < 10^{-14}\,s$; $T \gtrsim 10^{16}$ GeV
Neutrino decoupling & n_n/n_p freezes out	$t \sim 0.2\,s$; $T \sim 1 - 2$ MeV
electron-positron pairs annihilate	$t \sim 1\,s$; $T \sim 0.5$ MeV
Nucleosynthesis	$t \sim 200 - 300\,s$; $T \sim 0.05$ MeV
Recombination & Last scattering	$t \sim 10^{12} - 10^{13}\,s$
Structure formation	$t \sim 10^{16}\,s$
Acceleration	late time

Cosmic Microwave Background

- Before recombination, photons are coupled to electrons through the Compton scattering and the effects of scattering are transfered to baryons through the Coulomb interaction between (charged) baryons and electrons.
- Since the number density of charged particles is high, photons and baryons are tightly coupled and can be treated as a single fluid known as a photon–baryon fluid.
- Photons and baryons do not uniformly distribute over 3D space because there are deviations from homogeneity and isotropy in the universe.
- Small inhomogeneities and anisotropies are created in the early universe during inflation.

Cosmic Microwave Background

- In the standard notion, such inhomogeneities and anisotropies are seeds of the large-scale structures in the present universe.
- Sizes of these inhomogeneities and anisotropies are small so that they can be treated as perturbations in the homogeneous and isotropic universe (Friedmann universe).
- Due to these inhomogeneities and anisotropies, gradients of gravitational potential are developed.
- The evolution equation for the perturbations in energy density of the photon–baryon fluid takes a form of forced oscillation where a force relevant to the perturbations in photon–baryon fluid is the well of gravitational potential.

Cosmic Microwave Background

- To get a basic picture of the oscillation of the photon-baryon fluid, we consider the evolution equation for the perturbations in photon temperature in the form

$$\Delta_T'' + \frac{r'}{1+r}\Delta_T' + k^2 c_s^2 \Delta_T = \mathcal{F}\,, \tag{58}$$

  where a prime denotes derivative with respect to a conformal time $\eta = \int dt/a$, $\Delta_T \equiv \Delta t/\bar{T}$, $r \equiv 3\rho_b/(4\rho_\gamma)$ and $\mathcal{F}$ is the driving force produced by gravitational potential.

- Here, c_s^2 is the sound speed of the photon-baryon fluid defined as

$$c_s^2 \equiv \frac{1}{3}\frac{1}{1+r}\,. \tag{59}$$

Cosmic Microwave Background

- The solution for this equation can be written in the form

$$
\begin{aligned}
[1+r]^{1/4}\Delta_T &= C_1 \cos kr_s + C_2 \sin kr_s \\
&+ \frac{\sqrt{3}}{k}\int_{\tau_I}^{\tau} d\tau' [1+r(\tau')]^{3/4} \sin[kr_s(\tau) - kr_s(\tau')] F(\tau')\,,
\end{aligned}
\tag{60}
$$

- Here, r_s is the sound horizon defined as

$$
r_s \equiv \int_0^{\tau} c_s d\tau'\,. \tag{61}
$$

- The quantity r_s is the one of important length scales in cosmology, e.g. the cosine term in the above equation has extrema at the wavelength $\lambda = 2nr_s$

Cosmic Microwave Background

- After last scattering, photons freely propagate so that they cannot be described by a perfect fluid, e.g., shears are developed in the photon fluid.

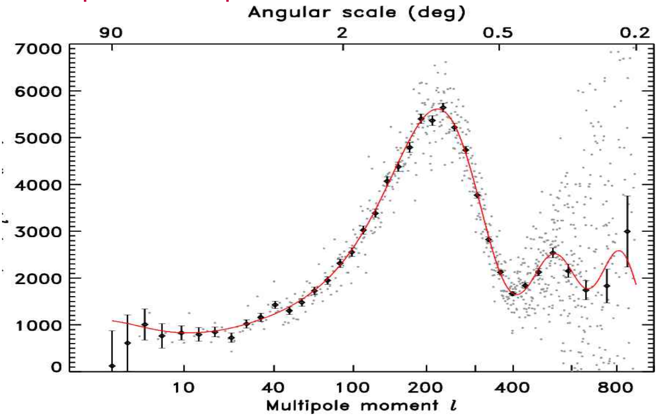

Figure: The angular power spectrum of the Cosmic Microwave Background .

Shortcomings of the hot BB

- According to observations, the universe is nearly homogeneous and isotropic on large scales, e.g., $\Delta_{T\mathrm{CMB}} \sim 10^{-5}$.
- We will learn that the smooth universe cannot be realized by Big Bang (BB) model unless the initial conditions of the universe is fine tuned.

Shortcomings of the hot BB

- We first discuss the horizon problem.
- The horizon scales of the present universe can be computed from the particle horizon which is approximately given by ct_0 where t_0 is the age of the universe and c is a speed of light.
- At the initial time t_i, the scale factor is a_i and therefore the comoving size of this horizon can be written in terms of the quantities at the initial time as

$$\frac{l_i}{a_i} \sim \frac{ct_0}{a_0}\,, \quad \Rightarrow \quad l_i \sim ct_0 \frac{a_i}{a_0}\,. \tag{62}$$

- Compare this scale with the size of a causal region $l_c = ct_i$, we get

$$\frac{l_i}{l_c} = \frac{t_0 a_i}{t_i a_0}\,. \tag{63}$$

Shortcomings of the hot BB

- We suppose that the initial conditions for the universe are specified at the Planck time.
- Hence, $t_i = t_p \sim 10^{-43}$s, $T_i = T_p \sim 10^{32}$K, and then the ratio in Eq. (63) becomes

$$\frac{l_i}{l_c} = \frac{t_0 T_0}{t_p T_p} = \frac{(10^{17})(2.7)}{(10^{-43})(10^{32})} \sim 10^{28}\,, \tag{64}$$

  where we have used $a \propto T$, $T_0 \simeq T_{\rm CMB} \simeq 2.7$K and $t_0 \sim 10^{17}$s.
- We see that at initial time the present horizon scale is 28 of magnitude larger than the size of causal region. This is the horizon problem in the standard hot Big Bang universe.

Shortcomings of the hot BB

- We now consider the flatness problem.
- Let us consider the Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho\,, \qquad (65)$$

where ρ in the above equation is the total energy density.

- This equation can be written as

$$\Omega - 1 = \frac{k}{a^2 H^2}\,. \qquad (66)$$

Shortcomings of the hot BB

- Based on observations, we can suppose for convenience that $|\Omega - 1| \sim \mathcal{O}(1)$ at present, so that at the initial epoch we have

$$|\Omega_i - 1| = |\Omega_0 - 1| \frac{a_0^2 H_0^2}{a_i^2 H_i^2} \,. \tag{67}$$

- If the dominant energy component of the universe has a constant equation of state parameter, one can show that $a \propto t^p$ where p is constant.
- Hence, Eq. (67) can be approximate as

$$|\Omega_i - 1| \sim |\Omega_0 - 1| \frac{a_0^2 t_i^2}{a_i^2 t_0^2} \sim 10^{-56} \,. \tag{68}$$

Shortcomings of the hot BB

- This implies that in order to obtain $|\Omega - 1| \sim \mathcal{O}(1)$ at present, the initial amount of matter must be fine tuned such that the universe is extremely flat at the initial epoch.
- This is the flatness problem.
- To understand how the mentioned problems can be solved, we note that Eqs. (63) and (67) can be approximately written as

$$\frac{l_i}{l_c} = \frac{t_0 a_i}{t_i a_0} \sim \frac{H_i a_i}{H_0 a_0} \sim \frac{\dot{a}_i}{\dot{a}_0}, \tag{69}$$

$$|\Omega_i - 1| = |\Omega_0 - 1| \frac{a_0^2 H_0^2}{a_i^2 H_i^2} = |\Omega_0 - 1| \frac{\dot{a}_0^2}{\dot{a}_i^2}, \tag{70}$$

where we have used $H \sim 1/t$ and $H = \dot{a}/a$.

Solution for the horizon and flatness problems

- From the above approximation, we have

$$\frac{l_i}{l_c} \sim \frac{\dot{a}_i}{\dot{a}_0} \gg 1\,, \tag{71}$$

$$|\Omega_i - 1| = |\Omega_0 - 1|\frac{\dot{a}_0^2}{\dot{a}_i^2} \ll 1\,. \tag{72}$$

- The main problem is that the ratio $\dot{a}_0/\dot{a}_i$ is extremely small.
- This ratio is small because in the BB model the universe expands with deceleration during radiation and matter dominated epochs.

Solution for the horizon and flatness problems

- This suggests that the problems can be solved if there is a mechanism that can enhance $\dot{a}$ before the radiation dominated epoch starts.
- This mechanism is the cosmic inflation in which an expansion of the universe is accelerated before radiation dominated epoch.
- The accelerated expansion of the early universe is known as inflationary epoch.

Graceful exit

- Let us consider the simplest model of inflation in which an accelerated expansion of the early universe is driven by a cosmological constant.
- We suppose that the universe contains only a cosmological constant, so that the Friedmann equation gives

$$H^2 = \frac{8\pi G}{3}\rho_\Lambda = \mathrm{constant}\,, \quad \Rightarrow \quad a \propto \mathrm{e}^{Ht}\,. \tag{73}$$

- From the above equation, we get

$$\ddot{a} = H^2 a\,, \tag{74}$$

which implies that an accelerated expansion of the universe never stops.

Graceful exit

- A realistic inflationary model has to naturally stop after last sufficiently long otherwise the structures in the universe cannot be created.
- The natural end of inflation is known as graceful exit.
- To find a necessary condition for the existence of the graceful exit, we consider a time derivative of the Hubble parameter as

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 \,. \quad \Rightarrow \quad \frac{\ddot{a}}{a} = \dot{H} + H^2 \,. \tag{75}$$

- A graceful exit can be realized if $\ddot{a}$ in the above equation can change from positive (acceleration) to negative (deceleration).

Graceful exit

- Such change can be achieve when $\dot{H}$ is negative, i.e., H decreases with time.
- Moreover, $|\dot{H}|$ has to smaller than H^2 initially and then becomes larger than H^2 at the end of inflation.
- Hence, to get a graceful exit, H has to decrease with time during inflation.

Simple model of inflation

- For the simple model of inflation, an accelerated expansion of the universe is driven by a scalar field known as inflaton.
- The action of scalar field in gravitational field is

$$S = \int d^4x\sqrt{-g}\left\{\frac{M_p^2}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\}, \quad (76)$$

where $M_p \equiv 1/\sqrt{8\pi G}$ is the reduced Planck mass, g is the determinant of the metric tensor $g_{\mu\nu}$, R is the Ricci scalar and $V(\phi)$ is the potential of scalar field ϕ.

Simple model of inflation

- Varying the action (76) with respect to the field ϕ, we get

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) - \frac{dV}{d\phi} = 0\,. \tag{77}$$

- Variation of the action (76) with respect to the metric tensor $g^{\mu\nu}$ yields

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{78}$$

- The energy-momentum tensor of a scalar field is

$$T^\mu_\nu = \partial_\nu\phi\partial^\mu\phi + \delta^\mu_\nu\left(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)\right)\,. \tag{79}$$

Simple model of inflation

- In the homogeneous and isotropic universe, the scalar field has to homogeneous, i.e., does not depend on spatial coordinates, so that

$$-T^0_0 = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V\,, \quad \text{and}\, T^i_i = P_\phi = \frac{1}{2}\dot{\phi}^2 - V\,. \quad (80)$$

- Hence,

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V} \quad (81)$$

- To drive an accelerated expansion of the universe, $w_\phi \sim -1$, so that $\dot{\phi}^2 \ll V$ (slowly rolling) is required during inflation.

Simple model of inflation

- In the homogeneous and isotropic universe, EOM of a scalar field is
$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0\,, \tag{82}$$
- The Friedmann equation is
$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right)\,. \tag{83}$$
- The conditions on the inflaton potential for slow-roll inflation will be analyzed.

Slow-roll Inflation

- In the slow-roll approximation:

$$\dot{\phi}^2 \ll V(\phi)\,, \quad |\ddot{\phi}| \ll 3H|\dot{\phi}|\,. \tag{84}$$

- Based on these slow-roll approximation, we get

$$3H\dot{\phi} \simeq -V_\phi\,, \quad H^2 \simeq \frac{8\pi G}{3}V(\phi)\,. \tag{85}$$

- The slow-roll evolution during inflation is quantified by slow-roll parameters.

Slow-roll Inflation

- The conditions for the graceful exit, $|\dot{H}| < H^2$, can be characterized by

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \tag{86}$$

where $\epsilon < 1$ during inflation.

- Inflation must last long enough to resolve the horizon and flatness problems, so that

$$\epsilon_1 \equiv \frac{\dot{\epsilon}}{H\epsilon}\,. \tag{87}$$

- The inflation ends when $\epsilon \sim 1$.

Slow-roll Inflation

- The parameter ϵ is written in the terms of the field as

$$\epsilon = 4\pi G \frac{\dot{\phi}^2}{H^2} \, . \tag{88}$$

- The slow-roll conditions for the inflaton field yield

$$\epsilon = 4\pi G \frac{V_\phi^2}{9H^4} = \frac{1}{16\pi G} \frac{V_\phi^2}{V^2} \, , \tag{89}$$

and

$$\epsilon_1 = 2 \underbrace{\frac{\ddot{\phi}}{H\dot{\phi}}}_{\equiv \eta} + 2\epsilon \tag{90}$$

Slow-roll Inflation

- Differentiating $3H\dot{\phi} = V_\phi$ with respect to time gives

$$\eta_1 = \frac{1}{8\pi G}\frac{V_{\phi\phi}}{V} < 1\,. \qquad (91)$$

- Using $H = d\ln a/dt$, the EOM of a scalar field becomes

$$\frac{d\phi}{d\ln a} = -\frac{V_\phi}{3H^2}\,. \qquad (92)$$

- Further using Eq. (85), we get

$$\frac{d\ln a}{d\phi} \simeq -\frac{8\pi G V}{V_\phi}\,. \qquad (93)$$

- This equation can be integrated as

$$a(t) \simeq a_i \exp\left(-8\pi G\int_{\phi_i}^{\phi}\frac{V}{V_\phi}d\phi\right)\,. \qquad (94)$$

Slow-roll Inflation

- Hence, during the slow-roll evolution, the number of e-foldings can be computed as

$$N = \ln\left(\frac{a}{a_i}\right) \simeq 8\pi G \int_{\phi}^{\phi_i} \frac{V}{V_{\phi'}} d\phi' \,. \tag{95}$$

- The total number of e-folding depends on the field value at the beginning ϕ_i and the end ϕ_f of inflation as

$$N \simeq 8\pi G \int_{\phi_f}^{\phi_i} \frac{V}{V_{\phi}} \mathrm{d}\phi \,. \tag{96}$$

- According to observation, the horizon and flatness problems can be solved if the inflation has to last longer than $N > 60$.