Tutorial Cosmology I

Exercise

1. Consider a surface in four-dimensional Euclidean space with some radius a, with line element

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{x}^2 + \mathrm{d}z^2, \quad z^2 + \mathbf{x}^2 = a^2,$$

and a hyperbolic surface in four-dimensional pseudo-Euclidean space, with line element

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{x}^2 - \mathrm{d}z^2, \quad z^2 - \mathbf{x}^2 = a^2.$$

Show that, by rescaling i.e. $\mathbf{x} \to a\mathbf{x}$, the 3-dimensional line element is

$$\mathrm{d}s^2 = a^2 \left[\mathrm{d}\mathbf{x}^2 + K \frac{(\mathbf{x} \cdot \mathrm{d}\mathbf{x})^2}{1 - K\mathbf{x}^2} \right],$$

where

$$K = \begin{cases} +1 & \text{spherical,} \\ -1 & \text{hyperbolic,} \\ 0 & \text{Euclidean.} \end{cases}$$

Further show that

$$\mathrm{d}s^2 = a^2 \left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \right].$$

2. From the definition of the conformal time, we can define the conformal Hubble parameter as

$$\mathcal{H} = \frac{a'}{a} = \frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}\eta}.$$

Show that the Friedmann equation becomes:

$$\mathcal{H}^2 = \frac{8\pi G}{3} \rho a^2 + \frac{\Lambda c^2 a^2}{3} - Kc^2,$$

and that the acceleration equation becomes:

$$\frac{a''}{a} = \frac{4\pi G}{3} \left(\rho - \frac{3P}{c^2} \right) a^2 + \frac{2\Lambda c^2 a^2}{3} - Kc^2 .$$

3. If the random field is statistically homogeneous and statistically isotropic, show that the 2-point correlation function has the following property:

$$\xi(\mathbf{x}_1, \mathbf{x}_2) = \xi(\mathbf{x}_1 - \mathbf{x}_2) = \xi(r_{12}) ,$$

where $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$. That is, the 2-point correlation function depends only on the distance between the two points.

4. Show that for a Gaussian random field, δ, and an integer $n \geq 1$,

$$\begin{aligned} \langle \delta^{2n} \rangle &\equiv \frac{\int_{-\infty}^{\infty} \mathrm{d}\delta \; \delta^{2n} \exp\left(-\frac{1}{2}a\delta^2\right)}{\int_{-\infty}^{\infty} \mathrm{d}\delta \; \exp\left(-\frac{1}{2}a\delta^2\right)} \\ &= \frac{1}{a^n}(2n-1)(2n-3)\ldots 5 \cdot 3 \cdot 1 \\ &= (2n-1)(2n-3)\ldots 5 \cdot 3 \cdot 1 \; \langle \delta^2 \rangle^n, \end{aligned}$$

and

$$\langle \delta^{2n-1} \rangle \equiv \frac{\int_{-\infty}^{\infty} \mathrm{d}\delta \ \delta^{2n-1} \exp\left(-\frac{1}{2}a\delta^2\right)}{\int_{-\infty}^{\infty} \mathrm{d}\delta \ \exp\left(-\frac{1}{2}a\delta^2\right)} = 0.$$

Hence, justify the validity of the Wick's theorem.

5. Write down the diagrammatic connecting parts for 4-point connected correlation functions. Then verify with the equation in (4.42).

6. The top-hat window function in 3-dimensional space is given by

$$W_R(\mathbf{x}) = \begin{cases} \dfrac{3}{4\pi R^3}, & \text{if } x \leq R, \\ 0, & \text{otherwise .} \end{cases}$$

Show that the Fourier transform of the top-hat window function is given by

$$W_R(\mathbf{k}) = 3j_1(kR) = 3\left[\frac{\sin(kR)}{(kR)^3} - \frac{\cos(kR)}{(kR)^2}\right].$$

$j_\nu(kR)$ is the spherical Bessel function of the first kind.

7. From the expansion of the angular correlation function in terms of Legendre polynomial,

$$C(\theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta),$$

and the expansion of the temperature anisotropy in terms of spherical harmonics,

$$\Theta(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi).$$

(a) Show that

$$C_\ell^{\text{th}} = \langle a_{\ell m} a_{\ell m}^* \rangle = |a_{\ell m}|^2.$$

(b) Prove that

$$C_\ell^{\text{obs}} = \frac{1}{2\ell+1} \sum_m a_{\ell m} a_{\ell -m}$$

You may need the spherical harmonic addition theorem

$$P_\ell(\cos\theta) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n}) Y_{\ell m}^*(\hat{n}'),$$

where $\cos\theta = \hat{n} \cdot \hat{n}'$.