General Relativity and Black Holes

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The Institute for Fundamental study,
Naresuan University

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Contents

1 Energy momentum tensor

1.1 Individual particle

A 4-momentum P^μ can be used to provide a complete description of energy and momentum of a particle, so that the dynamics of the particle can be described. From SR, the 4-momentum P^μ can be written in terms of 4-velocity V^μas

$$P^\mu = mV^\mu, \tag{1}$$

where m is a rest mass of the particle. The 4-velocity can be defined as

$$V^\mu = \frac{dx^\mu}{d\tau}, \tag{2}$$

where τ is the proper time and the 4-velocity obeys the normalization

$$V_\mu V^\mu = -1. \tag{3}$$

The 4-velocity can be thought as a tangent vector along the timelike worldline. Note that, the rest frame of the particle, the 4-velocity can be written as $V^\mu = (1, 0, 0, 0)$.

1.2 System of particles

Rather than specify the individual 4-momentum of all particles, we instead describe the system by a "fluid". To specify the fluid, one may need to know the the macroscopic quantities such as density, pressure, entropy, viscosity and so no. Therefore, the single 4-momentum of the fluid is not sufficient to describe the fluid. We can go further to describe the fluid by using the symmetric (2,0) tensor called "Energy Momentum Tensor" (EMT), $T^{\mu\nu}$. A general definition of EMT is the flux of 4-momentum, P^μ across a surface of constant x^ν. In order to explore the physical meaning of each components of energy momentum tensor , let us consider the infinitesimal element of the fluid in its rest frame with a volume V.

- T^{00}: flux of P^0 (energy) in x^0 (time) $\rightarrow$ "energy density".
- $T^{0i} = T^{i0}$: flux of P^i (momentum) in x^0 (time) $\rightarrow$ "momentum density".
- $T^{ij}(i = j)$: flux of P^i (momentum) in x^j $\rightarrow$ This represents the transfer momentum of element in i direction into j direction corresponding to the force per unit volume in i direction acting on the plane with $x^j =$ constant. Therefore, for $i = j$, this corresponds to the "pressure."

- $T^{ij}(i \neq j)$: As the same strategy, for $i \neq j$, this corresponds to the "shear" due to viscosity of the fluid.

1.3 Simple and useful example (dust)

Dust is a system of particles that are at rest with respect to each other. The 4-velocity of the fluid is the same for all particles. The number-flux 4-vector can be defined as

$$N^\mu = nV^\mu, \tag{4}$$

where n is the number density of particles measured in their rest frame. Suppose that the particles have the same mass m, the energy density at the rest frame can be written as

$$\rho_0 = mn, \tag{5}$$

In the rest frame, one can write N^μ and P^μ as $N^\mu = (n, 0, 0, 0)$ and $p^\mu = (m, 0, 0, 0)$, so that the ENT of dust can be written as

$$T^{\mu\nu}_{(\mathrm{dust})} = P^\mu N^\nu = mnU^\mu U^\nu = \rho_0 U^\mu U^\nu. \tag{6}$$

1.4 Conservation of EMT (dust)

To see clearly how EMT conserves, let us consider Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu}$. As a result, the conservation equation can be written as

$$\partial_\mu T^{\mu\nu} = 0. \tag{7}$$

From exercise, the EMT in moving frame can be written as $T^{00} = \rho$, $T^{0i} = \rho v^i$, $T^{ij} = \rho v^i v^j$, where $\rho = \rho_0\gamma^2$ and $\gamma = (1 - v^2)^{-1/2}$.

- zero component:

$$\begin{aligned} \partial_\mu T^{\mu 0} &= \partial_0 T^{00} + \partial_i T^{i0}, \\ &= \partial_t \rho + \partial_i(\rho v^i), \\ &= \partial_t \rho + \vec{\nabla}(\rho\vec{v}) = 0. \end{aligned} \tag{8}$$

  This corresponds to the conservation of energy/mass. To see more clearly, let us consider the familiar one, which is the moving charge with charge density ρ. The total charge and the current density can be written, respectively, as

$$Q = \int \rho dV, \tag{9}$$

$$\vec{J} = \rho\vec{v}. \tag{10}$$

During time δt, the change of the charges enclosed by the surface A can be written as

$$\delta Q_1 = \frac{\partial Q}{\partial t}\delta t = \int \left(\frac{\partial \rho}{\partial t} dV\right) \delta t. \tag{11}$$

The charges escaping through the surface can be written as

$$\delta Q_2 = \left(\oint \vec{J} \cdot d\vec{a}\right) \delta t = \left(\int \vec{\nabla} \cdot \vec{J} dV\right) \delta t, \tag{12}$$

where we have used the divergence theorem. The charges must be conserved, $\delta Q_1 + \delta Q_2 = 0$. As a result, we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} &= 0, \\ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) &= 0. \end{aligned} \tag{13}$$

Compared to our case, it implies that the equation belongs to the conservation of mass/energy.

- i component:

$$\begin{aligned} \partial_\mu T^{\mu i} &= \partial_0 T^{0i} + \partial_j T^{ji}, \\ &= \partial_t(\rho v^i) + \partial_j(\rho v^j v^i) = 0. \end{aligned} \tag{14}$$

  This equation is also in the same form as the previous one, so it corresponds to the conservation equation as well.

As a result, one finds that $\partial_\mu T^{\mu\nu} = 0$ is the conservation equation of EMT in flat Minkowski. One can generalize this equation to one in the curved spacetime by replacing ∂_μ by ∇_μ

$$\nabla_\mu T^{\mu\nu} = 0. \tag{15}$$

2 Newtonian limit

One of the important conditions to construct the Einstein equation is that the equation must be reduced to the Newtonian theory. Such the conditions are as follows: 1) A particle must move slowly, comparable to the speed of light. 2) the gravitational field should be weak. This condition allows us to use the perturbation method to perform calculations, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \ll 1$. Another condition is that the gravitational field must be static.

This condition is imposed since it provides an easy way to compare with the Newtonian theory.

In Newtonian theory for the central force, the gravitational field can be written as

$$\vec{E}_g = -\vec{\nabla}\Phi, \quad \Phi = -\frac{GM}{r}, \tag{16}$$

where Φ is the gravitational potential.

Now let us consider the equation in Einstein's theory. As we know, the equation that explains how a particle moves due to the curvature of the spacetime is the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{17}$$

where $\Gamma^\mu_{\nu\rho}$ are components of affine connection. From the first condition, "moving slowly", one can use the approximation as follows

$$t \approx \tau, \quad \frac{dx^0}{d\tau} \approx 1, \quad \frac{dx^i}{d\tau} \ll 1 \approx 0. \tag{18}$$

Applying this condition to the geodesic equation, one has

$$\begin{aligned} \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} &= 0, \\ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} &= 0, \\ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{00} &= 0. \end{aligned} \tag{19}$$

Let us consider the connection by applying the second condition, "weak field limit," and also the static one, $\partial_0 g_{\mu\nu} = 0$, one obtains

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma}\right), \tag{20}$$

$$\begin{aligned} \Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\nu}\left(\partial_0 g_{\nu0} + \partial_\sigma g_{\nu0} - \partial_\nu g_{00}\right), \\ \Gamma^\mu_{00} &= -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}, \\ \Gamma^i_{00} &= -\frac{1}{2}\eta^{ij}\partial_j h_{00}. \end{aligned} \tag{21}$$

Substituting this connection into Eq. (19), the geodesic equation in the

Newtonian limit can be written as

$$
\begin{aligned}
\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{00} &= 0, \\
\frac{d^2x^i}{dt^2} - \frac{1}{2}\delta^{ij}\partial_j h_{00} &= 0, \\
\frac{d^2x^i}{dt^2} - \frac{1}{2}\partial_i h_{00} &= 0, \\
\vec{E}_g &= \frac{1}{2}\vec{\nabla} h_{00}. \qquad (22)
\end{aligned}
$$

By comparing the gravitational field in Eq. (22) from Einstein theory and one in Eq. (16) from Newtonian theory, one obtains

$$
h_{00} = -2\Phi = \frac{2GM}{r} \;\rightarrow\; g_{00} = -\left(1 - \frac{2GM}{r}\right). \qquad (23)
$$

3 Einstein equation

In order to construct the equation for describing the relation between matter/energy and spacetime curvature, one has to impose two requirements such that

- The equation must be reduced to Newtonian theory.
- The curvature part must contain the metric tensor $g_{\mu\nu}$ and its derivatives such as $\Gamma^\rho_{\mu\nu}, R^\rho{}_{\mu\sigma\nu}, R_{\mu\nu}$ and R while the matter/enegy should be proportional to the EMT $T_{\mu\nu}$.

From the first requirement, the important equation in Newtonian theory is the Poisson equation $\nabla^2\Phi = 4\pi G\rho$. As we discussed before, the component of the metric tensor is proportional to the gravitational potential $g_{00} \propto \Phi$. Therefore, in order to reduce the master equation to the Poisson equation, the curvature part must be proportional to the second derivative of the metric. These quantities are $R^\rho{}_{\mu\sigma\nu}, R_{\mu\nu}$ and R.

3.1 Vacuum equation

For the vacuum equation, the matter/energy part vanishes, and then the curvature part will vanish

$$
f(R^\rho{}_{\mu\sigma\nu}, R_{\mu\nu}, R) = 0. \qquad (24)
$$

One may first guess for this equation, such that

$$R^{\rho}{}_{\mu\sigma\nu} = 0\,(?). \tag{25}$$

However, one found that this may not be possible since this equation provides the flat spacetime near the massive source. So that we can make a further guess by

$$R_{\mu\nu} = 0\,(?). \tag{26}$$

This is a good choice since the Ricci tensor has ten dof. like the metric tensor, hoping that ten dof. of the metric transfer to ten dof. of the Ricci tensor through their second derivatives, making from source nearby.

3.2 Equation with source

Now, let us consider the equation with a source. By adding the EMT, the equation may be written as

$$R_{\mu\nu} = kT_{\mu\nu}\,(?), \tag{27}$$

where k is the proportional constant. This is still be the good choice since the index symmetry also satisfies. However, as we discussed before, the EMT obeys the conservation equation $\nabla_\mu T^{\mu\nu} = 0$ while the Ricci tensor does not satisfy in general. As a result, one may find other quantities to satisfy this condition as well as maintain the mentioned requirements. Fortunately, from the Bianchi indentity, $\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = 0$, it serve us the conservation quantity as follows

$$\nabla^\mu G_{\mu\nu} = 0, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{28}$$

where $G_{\mu\nu}$ is called Einstein tensor. Note that the derivation of $\nabla^\mu G_{\mu\nu} = 0$. As a result, the equation can be constructed as

$$G_{\mu\nu} = kT_{\mu\nu}. \tag{29}$$

Next task for this construction is that we have to find the proportional constant k as well as check whether this equation satisfies the vacuum equation or not. To perform this evaluation, let us take the trace of the above equation as follows

$$\begin{aligned} R - \frac{1}{2}(4)R &= kT, \\ R &= -kT \end{aligned} \tag{30}$$

Substituting R from this equation into Eq. (29), one obtains

$$\begin{aligned} R_{\mu\nu} + \frac{1}{2}kTg_{\mu\nu} &= kT_{\mu\nu}, \\ R_{\mu\nu} &= k\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right). \end{aligned} \tag{31}$$

From this equation, one can see that the vacuum equation is still satisfied where the source is eliminated, $T_{\mu\nu} = T = 0$.

Now, we will find the proportional constant by taking the Newtonian limit into Eq. (31). As a result, the EMT can be written as

$$T^{\mu\nu} = \rho \begin{pmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & v^1v^1 & v^1v^2 & v^1v^3 \\ v^2 & v^2v^1 & v^2v^2 & v^2v^3 \\ v^3 & v^3v^1 & v^3v^2 & v^3v^3 \end{pmatrix} \sim \rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

Then we have

$$\begin{aligned} T^{\mu\nu} - \frac{1}{2}Tg^{\mu\nu} &= \frac{1}{2}\rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\rho\delta^{\mu\nu}, \\ T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} &= \frac{1}{2}\rho\delta_{\mu\nu}. \end{aligned} \tag{33}$$

Now let us consider the left-hand side of Eq. (31),

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\nu}\Gamma^\rho{}_{\lambda\rho} - \Gamma^\lambda{}_{\mu\rho}\Gamma^\rho{}_{\lambda\nu}. \tag{34}$$

By using the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and then keeping only first order perturbations, one obtains

$$R_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\rho\partial_\mu h_{\nu\sigma} + \partial_\rho\partial_\nu h_{\mu\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu} - \partial_\nu\partial_\mu h_{\rho\sigma}\right). \tag{35}$$

Exercise: Show that by using the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and then keeping only first order perturbations, the Ricci tensor can be written as the above equation.

Imposing the static condition $\partial_0 h_{\mu\nu} = 0$, the component R_{00} can be written as

$$R_{00} = -\frac{1}{2}\eta^{ij}\partial_i\partial_j h_{00} = -\frac{1}{2}\nabla^2 h_{00} = \nabla^2\Phi, \tag{36}$$

where we have used Eq. (23). Substituting results into Eq. (31), one obtains

$$\nabla^2\Phi = \frac{1}{2}k\rho. \tag{37}$$

By comparing this equation to the Poisson equation $\nabla^2\Phi = 4\pi G\rho$, the constant k can be written as

$$k = 8\pi G. \tag{38}$$

Finally, the Einstein equation is completely constructed as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{39}$$

4 Schwarzschild geometry

From the Einstein equation (39), if we expand the Einstein tensor in terms of the derivative of the metric, it is found that there is a complicated non-linear differential equation. Thus, it is not easy (or impossible) to solve it analytically. In order to simplify the equation, one has to impose some conditions or assumptions on the problem. Systematically, we can impose some symmetries on the system we want to study. Since the simple astronomical objects are approximately spherical, it is worthwhile to impose the spherical symmetry in our system. Moreover, for simplicity, we can impose the condition such that the system is also static. Therefore, we are now at the point to begin to solve the Einstein equation, and our task is to find the solution of the Einstein equation with the static spherically symmetric solution in empty space.

4.1 Schwarzschild metric

Before we go to derive the solution, let us clarify what static spacetime is and how it is different from stationary spacetime. It is clear and makes sense that static spacetime means that the metric does not depend on the timelike coordinate x^0. However, what is the difference with the stationary spacetime? One more condition for a static spacetime is that the line element is invariant under the transformation $x^0 \to -x^0$. For example, a spinning spherical object is stationary but not static. In terms of the line element, the term such that $dx^0 dx^i$ will disappear for static spacetime. Note that this term can be eliminated by using the proper coordinate transformation.

The most general form of the line element for static spherically symmetric spacetime can be written as

$$ds^2 = -A(r)c^2dt^2 + B(r)dr^2 + C(r)r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{40}$$

The coordinate r is only a radial parameter, not an actual radial distance. Therefore, it may be replaced by any function of r. By introducing the new radial parameter as $\tilde{r} = r\sqrt{C(r)}$ and then $d\tilde{r} = (1 + C'r/(2C))\sqrt{C}dr$, the line element becomes

$$ds^2 = -A(\tilde{r})c^2dt^2 + B(\tilde{r})d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{41}$$

Removing the tilde and then rewriting the function A and B as e^{2A} and e^{2B}, one obtains

$$ds^2 = e^{\alpha(r)}c^2dt^2 - e^{\beta(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{42}$$

The Schwarzschild metric can be written as

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{43}$$

The contravariant or the inverse version of the metric can be obtained by using $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$ and then be in the form

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{-2\beta(r)} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}. \tag{44}$$

It is found that there are only two independent functions to identify the metric. This is a result of imposing the spherical symmetry. Now we are in a position to solve the Einstein equation in vacuum,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{45}$$

Taking the trace of this equation, one obtains $R = 0$. Thus, the simpler equation we have to solve is that

$$R_{\mu\nu} = 0, \tag{46}$$

where $R_{\mu\nu}$ can be derived from the metric through the affine connection, defined in equation (20). The Ricci tensor can be written as

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\nu}\Gamma^\rho{}_{\lambda\rho} - \Gamma^\lambda{}_{\mu\rho}\Gamma^\rho{}_{\lambda\nu}. \tag{47}$$

Substituting the metric and its inverse into this equation, the nontrivial equations can be written as

$$\begin{aligned}
R_{00} &= -e^{2\alpha-2\beta}\left(\alpha'^2 - \beta'\alpha' + \alpha'' + \frac{2\alpha'}{r}\right) = 0, && (48)\\
R_{11} &= \alpha'^2 - \beta'\alpha' + \alpha'' - \frac{2\beta'}{r} = 0, && (49)\\
R_{22} &= -e^{-2\beta}\left(e^{2\beta} - r\alpha' + r\beta' - 1\right) = 0, && (50)\\
R_{33} &= -e^{-2\beta}\sin^2\theta\left(e^{2\beta} - r\alpha' + r\beta' - 1\right) = \sin^2\theta R_{22} = 0. && (51)
\end{aligned}$$

There are three independent equations since equations (50) and (51) give the same result. By using (48) and (49), one finds that

$$2\frac{\alpha'}{r} + 2\frac{\beta'}{r} = 0 \;\Rightarrow\; \alpha + \beta = \text{ const.} \tag{52}$$

The constant can be found from the fact that the metric at $r \to \infty$ must approach the Minkowski metric. It turns out that the constant must be zero, and then one obtains

$$\beta = -\alpha. \tag{53}$$

Substituting the result back into equation (50), one gets

$$(1+2r)e^{2\alpha} = 1 \;\Rightarrow\; \frac{d}{dr}\left(re^{2\alpha}\right) = 1 \;\Rightarrow\; e^{2\alpha} = 1 + \frac{C}{r}, \tag{54}$$

where C is an integration constant. From weak field limit,

$$g_{00} = e^{2\alpha} = 1 + \frac{2\Phi}{c^2} = 1 - \frac{2GM}{c^2 r} = 1 - \frac{2\mu}{r} \;\Rightarrow\; C = -2\mu = -\frac{GM}{c^2}. \tag{55}$$

Finally, one obtains the Schwarzschild line element as

$$ds^2 = \left(1 - \frac{2\mu}{r}\right)c^2dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{56}$$

From this metric, there are a few issues to clarify. First, it is about the singularity. From the Schwarzschild metric, one can see that the metric components are infinite at $r = 0$ and $r = 2\mu$. This is a signal that may lead to the singularity. However, the metric components are coordinate-dependent. The singularity may disappear when we use the appropriate coordinate. If we can find the coordinate transformation to eliminate the singularity, it is said that this singularity is a ”coordinate singularity”. Therefore, the coordinate singularity does not tell us about the breakdown of the spacetime geometry. For example, for the polar coordinate in a plane, $ds^2 = dr^2 + r^2 d\theta^2$, the component $g^{\theta\theta} = r^{-2}$ is infinite at $r = 0$. However, there is nothing in the plane at $r = 0$. Also, we know that there is a coordinate transformation such that the metric becomes Euclidian which is well-defined at the origin.

In order to figure out the real or intrinsic spacetime singularity, one has to find the quantities that are coordinate independent. Since spacetime curvature is measured by the Riemannian curvature tensor, the consequent scalar quantities of the Riemannian curvature tensor can be used to find the intrinsic singularity of the spacetime. However, there are many scalar quantities, and it is not guaranteed that the spacetime does not have the singularity if the scalar quantities are well-defined. For example, we know that the Ricci scalar is a consequent result of the Riemannian curvature tensor and is well-defined since it is always zero. It does not mean that spacetime has no singularity. However, if we can find the scalar quantity that has a singularity, it is said that there is a singularity of spacetime. For example, for the Riemannian curvature tensor square,

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48\mu^2}{r^6}, \tag{57}$$

it is infinite at $r = 0$. Therefore, the spacetime has the real or intrinsic singularity at $r = 0$. For $r = 2\mu$, it is a coordinate singularity, and we will see later that there exists the coordinate transformation in which the singularity can be eliminated. The radius $r = 2\mu = 2GM/c^2$ is called the Schwarzschild radius. There are many interesting points at this radius, and we will consider them in detail later.

Some people doubt why we have mass M in our solution since at the beginning we solved the equation in the vacuum or empty space, which is no matter in spacetime. The answer is that we are considering the curved spacetime, which is the empty spacetime. M is just a parameter that has mass dimension to make the spacetime curves. We can imagine that we put mass M in the flat spacetime to make it curve and then freeze or fix it, as well as take the mass out and then study this geometry of spacetime. It is important to note that our solution is valid only in the region without any mass. For example, the sun has mass $M = 1.99 \times 10^{30}$ kg. This leads to the Schwarzschild radius about $r_s = 2.95$ km while the radius of the sun is about 6.96×10^5 km. So that our solution is valid in the region outside the sun and outside its Schwarzschild radius. It is important to note that GR and Newtonian theory share the same character in which, outside the object mass M, the gravitational field or the spacetime curvature do not depend on the distribution of the mass but depend only on the total mass M.

4.2 Birkhoff's theorem

Besides the static condition of the metric, we can generalize the static spherical symmetry to be the spherical symmetry. Therefore, the metric is now dependent on the x^0 coordinate. However, the solution for empty spacetime will reduce to the Schwarzschild solution, which is exactly the same as the static case. Let us start with the most general line element for the time-dependent and spherically symmetric metric,

$$ds^2 = -A(r,t)dt^2 + B(r,t)dtdr + C(r,t)dr^2 + D(r,t)r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{58}$$

Here, we omit c^2 in the time coordinate for convenience. However, we can obtain it back by redefining the timelike coordinate in the final result. One can play in the same way as done in the static case by introducing the new radial parameter such that $\tilde{r} = r\sqrt{D(r)}$ and $d\tilde{r} = (1 + D'r/(2D))\sqrt{D}dr$, and then redefining the functions and omitting the tilde. We can eliminate function $D(r,t)$ and then the line element becomes

$$ds^2 = -A(r,t)dt^2 + B(r,t)dtdr + C(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{59}$$

Let $F(r,t)$ be any well-defined function. Then the change of this function can be written as

$$dF = \dot{F}dt + F'dr, \tag{60}$$

where prime and dot denote the derivative with respect to coordinates r and t, respectively. Furthermore, the change can be written in the exact or perfect differential by defining $\dot{F} = f(r,t)A(r,t)$ and $F' = -f(r,t)B(r,t)/2$. The change becomes

$$dF = f(Adt - Bdr), \tag{61}$$

where $f(r,t)$ is a function satisfy the relation

$$\frac{\partial \dot{F}}{\partial r} - \frac{\partial F'}{\partial t} = 0 = \frac{\partial (fA)}{\partial r} + \frac{\partial (fB/2)}{\partial t} \Rightarrow \frac{\partial f}{\partial t} = -\frac{2}{B}\left(f(A' + \dot{B}/2) + f'A\right). \tag{62}$$

In other words, f plays the role of integrating factor to make the change the perfect differential. From equation (61), one finds that

$$dF^2 = f^2\left(A^2dt^2 - ABdrdt + \frac{1}{4}B^2dr^2\right) \Rightarrow Adt^2 - Bdrdt = \frac{dF^2}{f^2A} - \frac{B^2}{4A}dr^2. \tag{63}$$

Substituting this result into the line element (59), it becomes

$$ds^2 = -\frac{1}{f^2 A} dF^2 + \left(C + \frac{B^2}{4A}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{64}$$

Finally, redefining the time parameter as $c\, d\tilde{t} = dF$ and also the functions $e^{-2\tilde{\alpha}(r,\tilde{t})} = f^2 A$ as well as $e^{2\tilde{\beta}(r,\tilde{t})} = C + \frac{B^2}{4A}$ and then omitting the tilde, the line element becomes

$$ds^2 = -e^{2\alpha(r,t)} c^2 dt^2 + e^{2\beta(r,t)} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{65}$$

Again, the result of the spherical symmetry yields only two independent functions left in the metric. However, the functions are now time-dependent. In order to find the solution of the Einstein equation, substituting this metric into the definition of the affine connection (20) and Ricci tensor (47), the nonzero component of the Einstein equation becomes

$$\begin{aligned}
R_{00} &= -e^{2(\alpha-\beta)}\left(\alpha'^2 - \beta'\alpha' + \alpha'' + \frac{2\alpha'}{r}\right) + \frac{1}{c^2}\left(\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}\right) = 0, && (66)\\
R_{11} &= \alpha'^2 - \beta'\alpha' + \alpha'' - \frac{2\beta'}{r} - \frac{e^{-2(\alpha-\beta)}}{c^2}\left(\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}\right) = 0, && (67)\\
R_{01} &= -2\frac{\dot{\beta}}{r} = 0, && (68)\\
R_{33} &= -e^{-2\beta}\sin^2\theta\left(-r\alpha' + e^{2\beta} + r\beta' - 1\right) = \sin^2\theta R_{22} = 0. && (69)
\end{aligned}$$

Besides the static case, the last three terms of the components $(0,0)$ and $(1,1)$ are the additional terms. There is one more equation, which is the component $(0,1)$. The components $(2,2)$ and $(3,3)$ are still the same. Note again that the functions α and β are now time-dependent. Using (68), it is found that the function β does not depend on time, $\beta = \beta(r)$. Combining the equations (66) and (67), one has

$$2\frac{\alpha'}{r} + 2\frac{\beta'}{r} = 0 \;\Rightarrow\; \alpha(r,t) + \beta(r) = \eta(t), \tag{70}$$

where η is a function depending only on the time. Substituting the result into equation (69), one obtains

$$(1-2r)e^{-2\beta} = 1 \;\Rightarrow\; \frac{d}{dr}\left(re^{-2\beta}\right) = 1 \;\Rightarrow\; e^{-2\beta} = 1 + \frac{C}{r} = 1 - \frac{2\mu}{r}. \tag{71}$$

The last equality comes from the fact that it must be reduced to the static case. Consequently, the function α becomes $e^{2\alpha(r,t)} = e^{2\eta(t)} e^{-2\beta(r,t)} = e^{2\eta(t)}\left(1 - \frac{2\mu}{r}\right)$. Substituting all results into the line element, one obtains

$$ds^2 = -e^{2\eta(t)}\left(1 - \frac{2\mu}{r}\right) c^2 dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{72}$$

Finally, one can redefine the time coordinate such that $d\tilde{t} = e^{\eta(t)}dt$ and then omit the tilde. This leads to the metric

$$ds^2 = -\left(1 - \frac{2\mu}{r}\right)c^2dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (73)$$

which is exactly the same as the Schwarzschild metric or the solution of the Einstein equation in the static case. Now we are at the point to state that "any spherically symmetric solution of the Einstein field equation in the empty spacetime is necessarily static". Commonly, this statement refers to the Birkhoff's theorem. From this theorem, there is an interesting result. It is found that the radially pulsating star has exactly the same gravitational field. Note also that our solution is valid only in the region outside the mass distribution. In other words, there is no gravitational radiation from a radially pulsating star. We will see in detail later (after midterm examination) that the star must oscillate in the terms of quadrupole moment in order to generate the gravitational radiation.

4.3 Orbit motion of particles

In this section, we will qualitatively consider how the results of the general relativity are different from those of Newtonian theory. We specialize in the differences in behavior of the motion of the particle by using the effective potential. We will first review the Newtonian theory and then derive the effective potential from general relativity in order to compare the results with those from Newtonian theory.

4.3.1 Newtonian theory

It is interesting to consider the motion of a particle mass m moving in a plane with Newtonian potential energy $U(r) = -GMm/r$. In polar coordinates, the Lagrangian of this particle can be written as

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GMm}{r}. \quad (74)$$

Using the Euler-Lagrange equation, the equation of motion of the particle can be written as

$$m\ddot{r} - mr\dot{\phi}^2 + \frac{GMm}{r^2} = 0, \quad (75)$$

$$mr^2\dot{\phi} = 0 \Rightarrow mr^2\dot{\phi} = L_\phi = \text{constant}. \quad (76)$$

It is obvious that the Lagrangian is independent of ϕ. This leads to the conserved quantity. In this system, the conserved quantity is angular momentum, L_ϕ, of the particle, implying from equation (76). Moreover, there is another conserved quantity, which is the total energy of the particle, which can be written as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{GMm}{r}. \tag{77}$$

Instead of using the equation (75), we will use the energy constraint equation (77) to analyze the motion of the particle. In other words, we can analyze the orbit motion of the particle by using the effective potential. Rewriting equation (77) in terms of the effective potential, one obtains

$$\frac{1}{2}\dot{r}^2 + V_{eff} = \bar{E} = \frac{E}{m}, \tag{78}$$

$$V_{eff} = \frac{1}{2}\frac{l^2}{r^2} - \frac{GM}{r}, \tag{79}$$

where $l = L_\phi/m$. The minimum of the effective potential can be obtained by differentiating the effective potential with respect to r and then setting to zero in order to solve for r. By doing that, we have

$$r_{min} = \frac{l^2}{GM}, \tag{80}$$

$$V_{eff(min)} = -\frac{(GM)^2}{2l^2} = -\frac{1}{2r_{min}}. \tag{81}$$

The key concept to analyze and identify the behavior of the particle is the value of the total energy $\bar{E}$. One more thing is that one has to sketch the shape of the effective potential, and it is shown in the figure 1. We plot it in different three values of the parameter $l = 1/2r_{min}^{-1}, r_{min}^{-1}, r_{min}^{-1}$. One can see that the potential has a local extremum. This can be interpreted as meaning that there exists a bound orbit of the motion. If the local extremum is a local minimum (maximum), the orbit motion will be stable (not stable). If total energy is in the well of the potential (in figure 1 we set the total energy to be $\bar{E} = 0.7V_{eff(min)}$, the particle will move in the potential well (it will hit the potential and then turn back). Therefore, one can see that, with the energy $V_{eff(min)} < \bar{E} < 0$, the particle will move in an elliptic orbit. If the total energy $\bar{E} = V_{eff(min)}$, the orbit motion will be a circular orbit. In the case of $\bar{E} > 0$, the motion will not be bounded. The particle has too much kinetic energy. It will hit the potential and then turn back to $r \to \infty$. The trajectory of the particle will look like the parabolic or hyperbolic motion.

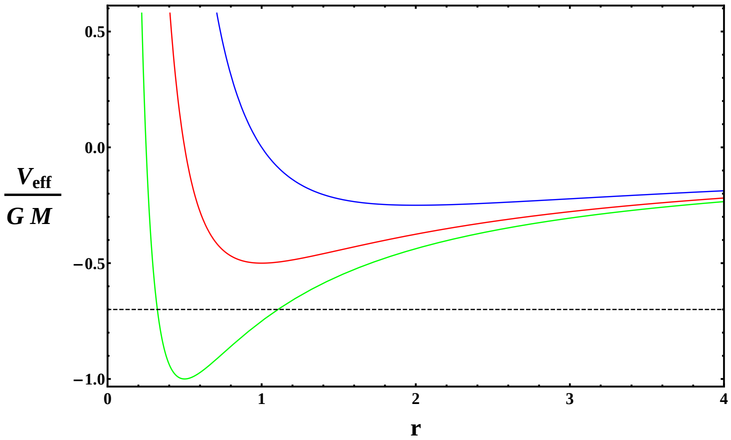

Figure 1: The effective potential with various angular momentum per mass ω. The green, red and blue lines represent the plot with $l = r_{min}^{-1}/2, r_{min}^{-1}, r_{min}^{-1}$ respectively. Dashed line corresponds to the total energy per mass $\bar{E} = 0.7V_{eff(min)}$.

Before we close this subsection, consider the motion of the massless particle in Newtonian theory. In Newtonian theory, the massless particle will not influence with the gravitational force. Therefore, the potential terms in the effective potential will be omitted and take the form

$$V_{eff} = \frac{1}{2}\frac{l^2}{r^2}. \tag{82}$$

The effective potential for a massless particle can be sketched in figure 2. We will see that there are no local extremum points in this effective potential. It implies that the trajectory of the particle is not bound to be the orbit. In fact, the trajectory is a strength line. The particle will hit the potential and then turn back to $r \to \infty$.

4.3.2 General relativity

The goal of this subsection is to find the effective potential of a particle by using general relativity. As we mentioned before, the trajectory or the motion of a particle in general relativity can be obtained by using the geodesic equation. There are a few forms of the geodesic equation. Here we will use the form, which can be analogous to classical mechanics. The geodesic equation can be obtained by using Lagrangian formalism, where the Lagrangian can be written as

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\left(1 - \frac{2\mu}{r}\right)c^2\dot{t}^2 + \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2). \tag{83}$$

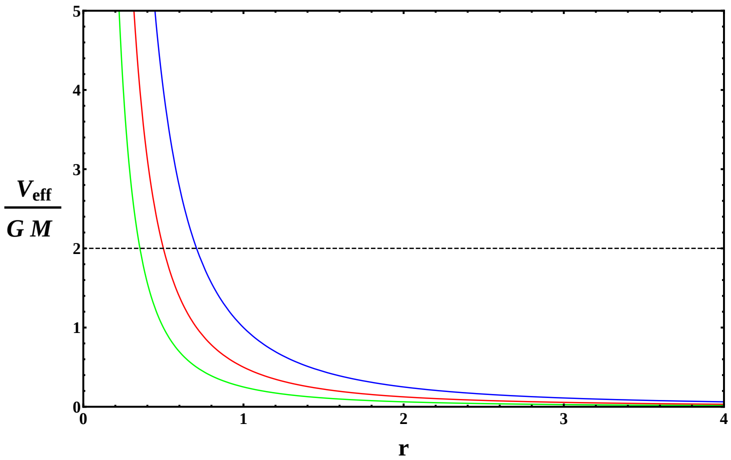

Figure 2: The effective potential for a massless particle in Newtonian theory with various angular momentum per mass ω. The green, red, and blue lines represent the plot with $l = r_{min}^{-1}/2, r_{min}^{-1}, r_{min}^{-1}$ respectively. Dashed line corresponds to the total energy per mass $\bar{E} = 2V_{eff(min)}$.

From this Lagrangian, we can see that it is independent of t and ϕ. This implies that there are two conserved quantities in this system. By varying this Lagrangian with respect to t, r, θ and ϕ, we obtain the equation of motion respectively

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k, \quad (84)$$

$$\left(1 - \frac{2\mu}{r}\right)^{-1}\ddot{r} - \left(1 - \frac{2\mu}{r}\right)^{-2}\frac{2\mu\dot{t}^2}{r^2} + \frac{c^2\mu\dot{t}^2}{r^2} - r(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = 0, \quad (85)$$

$$\ddot{\theta} + \frac{\dot{\theta}\dot{r}}{r} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0, \quad (86)$$

$$r^2\sin^2\theta\dot{\phi} = l \quad (87)$$

There is another constraint equation that the geodesic equations must satisfy, and this equation can be expressed as

$$-\epsilon = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\left(1 - \frac{2\mu}{r}\right)c^2\dot{t}^2 + \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2), \quad (88)$$

where $\epsilon = 0$ for null geodesic and $\epsilon = c^2$ for time-like geodesic. In order to compare the result with one from Newtonian theory, we will consider a particle moving in a plane. Due to the spherical symmetry, one can choose the plane in which the calculation is simplified. This corresponds to the plane

$\theta = \pi/2$. This also automatically satisfies the equation (86). Generally, one can choose any planes they want. However, one can rotate the axis to satisfy the plane $\theta = \pi/2$ due to the spherical symmetry. Therefore, the equations become

$$\begin{aligned}
\left(1 - \frac{2\mu}{r}\right)\dot{t} &= k, &(89)\\
\left(1 - \frac{2\mu}{r}\right)^{-1}\ddot{r} - \left(1 - \frac{2\mu}{r}\right)^{-2}\frac{2\mu\dot{t}^2}{r^2} + \frac{c^2\mu\dot{t}^2}{r^2} - \dot{\phi}^2 &= 0, &(90)\\
r^2\dot{\phi} &= l &(91)\\
\left(1 - \frac{2\mu}{r}\right)c^2\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 &= \epsilon. &(92)
\end{aligned}$$

The constant k is not obvious to find the physical meaning. In order to see the physical meaning of the constant k, let us consider the total energy per unit mass $\bar{E}$ of a particle observed by the rest observer at infinity with the four velocity $V^\mu = (1, 0, 0, 0)$, which can be expressed as

$$\bar{E} = P_\mu V^\mu = p_0 = g_{00}\dot{t} = c^2\left(1 - \frac{2\mu}{r}\right)\dot{t} = c^2 k, \tag{93}$$

where $P^\mu = \dot{x}^\mu$ is a four momentum of a particle. Note that the components of the four-velocity $V^\mu = dx^\mu/d\tau$ are defined in the coordinates (t, r, θ, ϕ). Therefore, the constant k reads

$$k = \frac{\bar{E}}{c^2} = \frac{E}{mc^2}. \tag{94}$$

One can see that the constant k is just the dimensionless version of the total energy of the particle, which is scaled by its rest mass multiplied by c^2. In order to obtain the effective potential, one can substitute the constant l from equation (91) and k from equation (89) into equation (92). This leads to the equation

$$\begin{aligned}
\frac{1}{2}\dot{r}^2 + \left(1 - \frac{2\mu}{r}\right)\frac{l^2}{2r^2} + \left(1 - \frac{2\mu}{r}\right)\frac{\epsilon}{2} &= \frac{c^2k^2}{2}, &(95)\\
\frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} - \frac{\epsilon\mu}{r} - \frac{\mu l^2}{r^3} &= \frac{c^2k^2 - \epsilon}{2}. &(96)
\end{aligned}$$

Therefore, the effective potential and effective total energy per unit mass take the form

$$\begin{aligned}
V_{eff} &= \frac{l^2}{2r^2} - \frac{\epsilon\mu}{r} - \frac{\mu l^2}{r^3}, &(97)\\
E_{eff} &= \frac{c^2k^2 - \epsilon}{2} = c^2\left(\frac{E^2}{m^2c^4} - \frac{\epsilon}{c^2}\right). &(98)
\end{aligned}$$

From the effective potential, the last term is the additional term to the effective potential in Newtonian theory. This term is proportional to r^{-3} and has a negative sign. Therefore, this term will dominate at short distances. This leads to one of the ways to distinguish general relativity and Newtonian theory, and this is the reason why we try to test general relativity with the precession of Mercury since it is the nearest planet to the sun. Before we analyze the effective potential in detail, let us consider the effect of the total energy per unit mass.

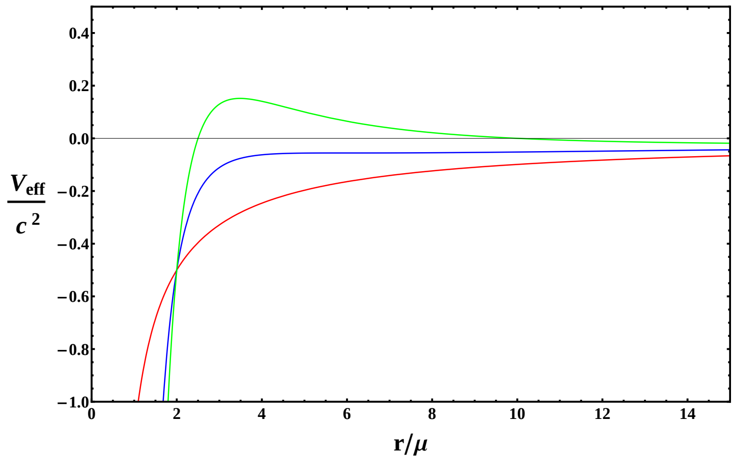

Figure 3: The effective potential for a massless particle in Newtonian theory with various angular momentum per mass ω. The green, blue, and red lines represent the plot with $l = 5c\mu, 2\sqrt{3}c\mu, c\mu/2$ respectively.

First, let us consider the massive particle by setting $\epsilon = c^2$. In order to characterize the motion of the particle, firstly we have to sketch the effective potential, and the plots are shown in figure 3. The plots are done by using three different values of $l = 5c\mu, 2\sqrt{3}c\mu, c\mu/2$. The minimum of the potential is not clear to see in the figure. However, one can find it by using

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow GMr^2 - \omega r + \frac{3GM\omega^2}{c^2} = 0. \tag{99}$$

Solving this equation for r, one gets

$$r = r_\pm = \frac{l^2}{2GM}\left(1 \pm \sqrt{1 - 12\left(\frac{GM}{lc}\right)^2}\right). \tag{100}$$

In order to clearly understand the behavior of the motion of the particle, we can divide the consideration into two parts, high l and low l. From figure

3, the green line represents the plot at high angular momentum and the red line represents the the plot at low momentum. The blue line represents the specific value of l, which we will discuss this point in detail later.

For high angular momentum, the extremum points can be approximated as

$$
\begin{aligned}
r_+ &= \frac{l^2}{2GM}(1+1) = \frac{l^2}{GM}, & (101)\\
r_- &= \frac{l^2}{2GM}\left(1-\left(1-\frac{1}{2}12\left(\frac{GM}{lc}\right)^2\right)\right) = \frac{l^2}{2GM}\left(\frac{GM}{lc}\right)^2 = 3\frac{GM}{c^2} = 3\mu & (102)
\end{aligned}
$$

At high l, there is a circular orbit at $r = 3\mu$, but it is not stable. If there are small perturbations of the particle, it will move inward to the center or move outward to infinity. However, there also exists the stable point in which the particle will move in a circular or elliptic orbit.

For low angular momentum, the extremum point will be imaginary. This implies that there are no extremum points. Actually, the effective potential will be proportional to $-1/r$, which has no local extremum point. This is also shown by the red line in the figure 3. The particle with low angular momentum will move inward to the center.

The interesting situation occurs when we reduce the angular momentum from a high to a low value. From equation (100), we will see that r_+ and r_- will come closer together until $l = \sqrt{12}\frac{GM}{c} = 2\sqrt{3}c\mu$, and then they are in the same value as

$$
r_+ = r_- = \frac{l^2}{2GM} = 12\left(\frac{GM}{c}\right)^2\frac{1}{2GM} = 6\frac{GM}{c^2} = 6\mu. \tag{103}
$$

Therefore, at $l = 2\sqrt{3}c\mu$ is a minimum value of the angular momentum of the particle in which the particle can have an orbital motion. The plot with this specific value is represented by the blue line in the figure 3. Moreover, the radius $r = 6\mu$ is a minimum radius for a stable circular orbit. The unstable circular orbit of the particle will be in the region $3\mu < r < 6\mu$.

For the massless particle, the effective potential becomes

$$
V_{eff} = \frac{l^2}{2r^2} - \frac{\mu l^2}{r^3}. \tag{104}
$$

The extremum of the effective potential can be obtained by

$$
\frac{dV_{eff}}{dr} = 0 \Rightarrow -\frac{1}{r^3} + \frac{3\mu}{r^4} = 0 \Rightarrow r = 3\mu. \tag{105}
$$

This is a maximum point, which implies that it is not stable. The plot of this effective potential is shown in figure 4. We can see that there is a circular orbit of a photon at $r = 3\mu$, even though it is not stable, while it is impossible for the Newtonian theory. Moreover, we also found that the trajectory of the light can be a curved line in general relativity, while it is always a straight line in Newtonian theory. This difference leads us to test gravity theory of Einstein by the experiments or observations of light trajectory bending.

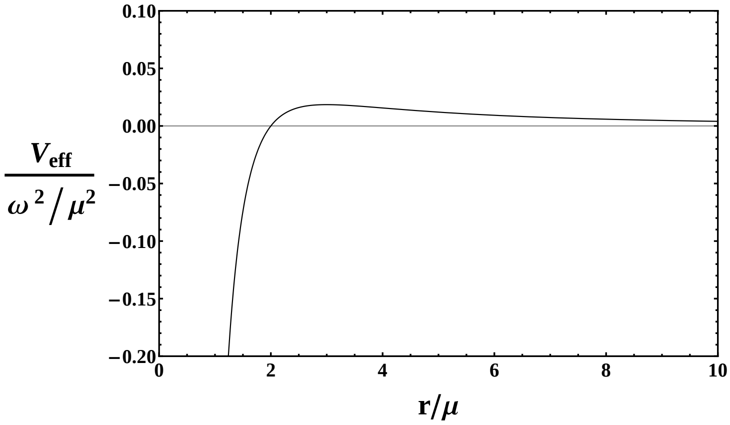

Figure 4: The effective potential for a massless particle in General relativity. There always exists the maximum point at $r = 3\mu$.

4.4 Radial motion of particles

To see the effect of particle motion at the Schwarzschild radius or event horizon, let us consider the radial motion of the particle by setting $\dot{\phi} = 0 \Rightarrow l = 0$. Therefore, the equation (96) becomes

$$\frac{1}{2}\dot{r}^2 - \frac{\epsilon\mu}{r} = \frac{c^2k^2 - \epsilon}{2}. \tag{106}$$

Considering the case of a massive particle, setting $\epsilon = c^2$, one obtains

$$\frac{1}{2}\dot{r}^2 - \frac{c^2\mu}{r} = \frac{c^2k^2 - c^2}{2} \Rightarrow \dot{r}^2 + c^2\left(1 - \frac{2\mu}{r}\right) = c^2k^2. \tag{107}$$

In order to relate the time coordinate t to r, one has

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = \dot{t}\frac{dr}{dt} = k\left(1 - \frac{2\mu}{r}\right)^{-1}\frac{dr}{dt}. \tag{108}$$

Therefore equation (107) becomes

$$k^2\left(1-\frac{2\mu}{r}\right)^{-2}\left(\frac{dr}{dt}\right)^2+c^2\left(1-\frac{2\mu}{r}\right)=c^2k^2 \quad\Rightarrow\quad \left(\frac{dr}{dt}\right)^2=\frac{\left(1-\frac{2\mu}{r}\right)^2c^2}{k^2}\left(k^2-\left(1-\frac{2\mu}{r}\right)\right),$$
$$\Rightarrow\quad \frac{dr}{dt}=\pm\frac{\left(1-\frac{2\mu}{r}\right)c}{k}\sqrt{k^2-\left(1-\frac{2\mu}{r}\right)}. \tag{109}$$

Note that the minus/plus sign refers to the incoming/outgoing particle. Considering the particle falls inward from the rest at $r=R$, observed by an observer at rest at infinity. This corresponds to the boundary condition such that

$$\frac{dr}{dt}=0,\ r=R>2\mu\Rightarrow k^2=\left(1-\frac{2\mu}{R}\right). \tag{110}$$

Substituting the constant k into equation (109), one obtains

$$\begin{aligned}\frac{dr}{dt} &= -\frac{\left(1-\frac{2\mu}{r}\right)c}{\left(1-\frac{2\mu}{R}\right)^{1/2}}\sqrt{2\mu\left(\frac{1}{r}-\frac{1}{R}\right)},\\ &= -\frac{(2\mu)^{1/2}c}{(R-2\mu)^{1/2}}\frac{(r-2\mu)(R-r)^{1/2}}{r^{3/2}},\end{aligned} \tag{111}$$

and then

$$\frac{dt}{dr} = -\frac{(R-2\mu)^{1/2}}{(2\mu)^{1/2}c}\frac{r^{3/2}}{(r-2\mu)(R-r)^{1/2}}=-A\frac{r^{3/2}}{(r-2\mu)(R-r)^{1/2}} \tag{112}$$

Finally, we have the integrand

$$t = -A\int_R^r\frac{\bar{r}^{3/2}}{(\bar{r}-2\mu)(R-\bar{r})^{1/2}}d\bar{r}. \tag{113}$$

We can see that the integrand diverges at $r=2\mu$. In order to see how the time coordinate r behaves, let us consider the integrand near the horizon by introducing $\bar{r}=2\mu+\varepsilon\Rightarrow d\bar{r}=d\varepsilon$ where ε is small. Thus, the integrand becomes

$$\begin{aligned}t &= -A\int_{R-2\mu}^{r-2\mu}\frac{(2\mu+\varepsilon)^{3/2}}{\varepsilon(R-(2\mu+\varepsilon))^{1/2}}d\varepsilon,\\ &\sim -A\frac{(2\mu)^{3/2}}{(R-2\mu)^{1/2}}\int_{R-2\mu}^{r-2\mu}\frac{d\varepsilon}{\varepsilon},\\ &\sim -\frac{(R-2\mu)^{1/2}}{(2\mu)^{1/2}c}\frac{(2\mu)^{3/2}}{(R-2\mu)^{1/2}}\ln\frac{r-2\mu}{R-2\mu},\\ &\sim -\frac{2\mu}{c}\ln\frac{r-2\mu}{R-2\mu},\end{aligned} \tag{114}$$

Thus, we have

$$r - 2\mu \;\; = \;\; (R - 2\mu)e^{-\frac{c}{2\mu}t} \tag{115}$$

Therefore, it is found that the rest observer at infinity will see the particle approaching $r \to 2\mu$ at time $t \to \infty$. It is interesting to consider the radial collapsing star. The observer will see that the star collapses more and more slowly, and never collapses to a point. This phenomenon is called "frozen star". Next, we will see how the particle clock evolves. In other words, we will see the evolution of the proper time. By using the chain rule of time parameter, one has

$$\begin{aligned}\frac{d\tau}{dr} &= \frac{d\tau}{dt}\frac{dt}{dr} = \frac{1}{\dot{t}}\frac{dt}{dr} = \frac{(r-2\mu)}{kr}\frac{dt}{dr}, \\ &= -A\frac{(r-2\mu)}{kr}\frac{r^{3/2}}{(r-2\mu)(R-r)^{1/2}} = -\frac{(R-2\mu)^{1/2}}{(2\mu)^{1/2}c}\frac{R^{1/21}}{(R-2\mu)^{1/2}}\frac{r^{1/2}}{(R-r)^{1/2}}, \\ &= -\frac{R^{1/2}}{(2\mu)^{1/2}c}\frac{r^{1/2}}{(R-r)^{1/2}}.\end{aligned} \tag{116}$$

Therefore, the integral form of the proper time is that

$$\tau \;\; = \;\; -\frac{R^{1/2}}{(2\mu)^{1/2}c}\int_R^r \frac{\bar{r}^{1/2}}{(R-\bar{r})^{1/2}}d\bar{r}. \tag{117}$$

By introducing new variable such that $u = R\sin^2\theta$, the integrand becomes

$$\int_R^r \frac{\bar{r}^{1/2}}{(R-\bar{r})^{1/2}}d\bar{r} = -R\left(\sin^{-1}\left(\sqrt{\frac{R-r}{R}}\right) + \sqrt{\frac{(R-r)r}{R^2}}\right), \tag{118}$$

and then the proper time can be written as

$$\tau \;\; = \;\; \frac{R^{3/2}}{(2\mu)^{1/2}c}\left(\sin^{-1}\left(\sqrt{\frac{R-r}{R}}\right) + \sqrt{\frac{(R-r)r}{R^2}}\right). \tag{119}$$

One can see that the proper time is finite during a particle crossing the horizon or until it reaches the origin. Therefore, it is surprising that even if we fall passing the horizon, the observer never sees us pass it.

For a massless particle, one can set $\epsilon = 0$. The equation (106) becomes

$$\dot{r}^2 = c^2k^2 = \dot{t}^2\left(\frac{dr}{dt}\right)^2 = k^2\left(1 - \frac{2\mu}{r}\right)^{-2}\left(\frac{dr}{dt}\right)^2 \Rightarrow \frac{c\,dt}{dr} = \pm\left(1 - \frac{2\mu}{r}\right)^{-1} \tag{120}$$

Note that the plus/minus sign corresponds to the outgoing/incoming massless particle. Therefore, the coordinate ct can be written as

$$\begin{aligned} c\,dt &= \pm \int \left(1 - \frac{2\mu}{r}\right)^{-1} dr = \pm \int \frac{r}{r - 2\mu} dr \\ &= \pm \int \left(1 + \frac{2\mu}{r - 2\mu}\right) dr \\ &= \pm \left(r + 2\mu \ln \left|\frac{r}{2\mu} - 1\right|\right) + \text{constant.} \end{aligned} \tag{121}$$

From equation (120), the slope of the light cone in $r - ct$ diagram becomes $\pm\infty$. This implies that the signal (emission of photon) sent by the particle at $r = 2\mu$ will reach the rest observer at far distance by $t \to \infty$. In other words, the rest observer far away never received any signal from the emitter at the horizon, even though the signal is a photon. This is the reason why it is usually called a "black hole". We will study black holes in detail in the next section.

4.5 Exercise

1. From metric

$$g_{\mu\nu} = \begin{pmatrix} e^{2A(r,t)} & 0 & 0 & 0 \\ 0 & -e^{2B(r,t)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix},$$

where $A(r,t)$ and $B(r,t)$ are arbitrary function, find non-zero components of Christoffel symbol $\Gamma^{\rho}{}_{\mu\nu}$ and Ricci tensor $R_{\mu\nu}$.

2. Line element for the Schwarzschild metric can be written as

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

By introducing new radial coordinate as

$$\tilde{r} = \frac{1}{2}\left(r - \mu + \sqrt{r^2 - 2\mu r}\right),$$

showing that the line element in spatial Euclidean coordinates can be written as

$$ds^2 = \left(1 - \frac{\mu}{2\tilde{r}}\right)^2 \left(1 + \frac{\mu}{2\tilde{r}}\right)^{-2} dt^2 + \left(1 + \frac{\mu}{2\tilde{r}}\right)^4 (dx^2 + dy^2 + dz^2).$$

3. By using the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0, \tag{122}$$

where $g_{\mu\nu}$ is the Schwarzschild metric, find the equation of motion of the particle.

4. From the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

find Reimannian curvature tensor $R_{\mu\nu\rho\sigma}$ and its inverse $R^{\mu\nu\rho\sigma}$ and then find $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$.

5. Show that

$$\int_R^r \frac{\bar{r}^{1/2}}{(R - \bar{r})^{1/2}} d\bar{r} = -R\left(\sin^{-1}\left(\sqrt{\frac{R - r}{R}}\right) + \sqrt{\frac{(R - r)r}{R^2}}\right). \tag{123}$$

5 Schwarzschild black hole

In the previous section, we studied the properties of Schwarzschild geometry by considering the trajectory of particles. In the study, we learned that there are difficulties in studying the motion of the particle at the Schwarzschild radius $r = 2\mu$. We also know that, by the spirit of general relativity, physics does not depend on the coordinates we use. Therefore, it seems that the Schwarzschild coordinate is not the proper to use to describe physics in all regions of the Schwarzschild spacetime. In this chapter, we will study the physics of Schwarzschild geometry by using the coordinate transformation to find proper coordinates. Trajectories of particles are discussed by using a spacetime diagram in the $r - ct$ plane.

5.1 Schwarzschild coordinates

A spacetime diagram is very useful to analyze the behavior of the particle motion. In fact, if we know the structure of the lightcone for every point in the diagram, we will find the possible directions of the particle trajectory. In order to find the structure of the lightcone, let us recall the equation for null geodesics in which θ and ϕ are constant. From the results in the previous section, we obtain

$$\frac{c\,dt}{dr} = \pm\left(1 - \frac{2\mu}{r}\right)^{-1}, \tag{124}$$

where plus and minus signs refer to outgoing and incoming photons, respectively. This equation provides us with the slope of the lightcone. At large r, the slope becomes ± 1 corresponding lightcone with angle 45° as shown in figure 5. This also provides us with a flat Minkowski spacetime at large r as we expect. At r approaches 2μ, the slope of the incoming ray tends to be ∞ and of the outgoing ray tends to be $-\infty$. This means that the lightcone becomes more and more close up as shown in figure 5. This behavior of the lightcone implies that massive particles or even light cannot escape from the point $r = 2\mu$ to $r > \mu$. Since there are no particles going out from inside the radius $r = 2\mu$, this object is usually called ”black hole”. The radius $r = 2\mu$ is called the Schwarzschild radius and is also referred to as the radius of a black hole. To go further into the radius of the black hole, let us consider the relation of ct and r by recalling equation (121),

$$c\,t = \pm\left(r + 2\mu\ln\left|\frac{r}{2\mu} - 1\right|\right) + \text{constant}. \tag{125}$$

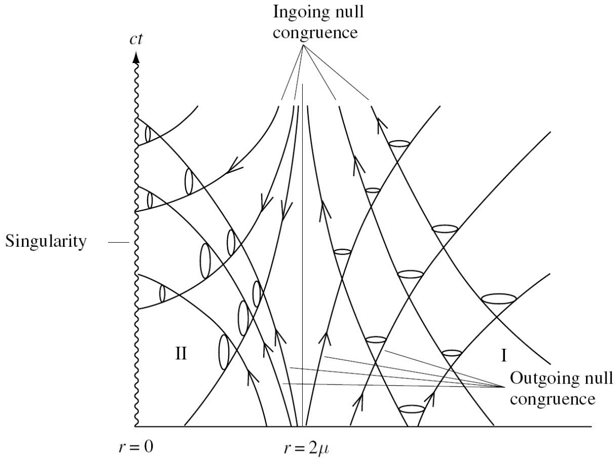

Figure 5: Spacetime diagram in $r - ct$ plane with Schwarzschild coordinates [1]. The future lightcone is flipped from the increasing time to the decreasing radius when the the lightcone crosses the Schwarzschild radius.

At $r \lesssim 2\mu$, the slope of the incoming ray becomes $-\infty$, and ct starts from $+\infty$ and then becomes decreasing as r decreases, as shown in figure 5. For the outgoing ray, its slope starts from $+\infty$ then becomes decreasing while ct starts from $-\infty$ and then becomes increasing as r decreases. This provides the lightcone structure in which the future lightcone is flipped from the increasing time to the decreasing radius when the the lightcone crosses the Schwarzschild radius. This structure of the lightcone controls the motion of the particle to the origin only. This behavior of the lightcone can be seen from the Schwarzschild line element,

$$ds^2 = -\left(1 - \frac{2\mu}{r}\right) c^2 dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{126}$$

We can see that the coordinates ct and r will change their character together. The coordinate r changes from a spacelike coordinate to a timelike coordinate while the coordinate ct changes from a timelike coordinate to a spacelike coordinate. Therefore, in the world outside the black hole, time will unavoidably increase while, in the world inside the black hole, r will unavoidably decrease.

It is important to note that this result is experienced by the observer outside and far away from the black hole. As we have discussed before, from the particle's point of view, there is nothing at the Schwarzschild radius since its clock still works well, and the proper time has no singularity. The massive particle motion can be obtained by using the lightcone structure since it has

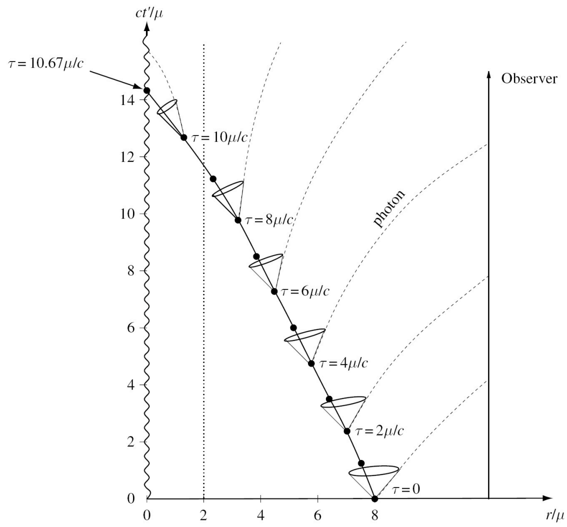

Figure 6: Trajectory of the massive particle in a spacetime diagram with Schwarzschild coordinates [1]. The periodic signal sent from the particle is received by the rest observers at far away. The periods of the receiving signal are longer as particles go closer to the event horizon.

to move only in the future lightcone of an event as seen in figure 6. From this figure, if the particle sends a periodic light-signal during its fall into the black hole to the rest observer far away from the black hole, the period of receiving the signal of the observer will be longer and longer as the particle gets closer and nearer the black hole. Until the particle approaches the black hole radius, the observer will receive the signal at $t = \infty$. From this analysis, we see that one can use only the qualitative analysis, and then the results of the motion of both massive and massless particles can be obtained without complicated calculations. Therefore, it is common to study the properties of the black hole and particle motion by using the lightcone structure in the spacetime diagram. In the next section, we will find the lightcone structure in the new coordinate system.

5.2 Eddington-Finkelstein coordinates

From the previous section, the Schwarzschild coordinates are blown up at $r = 2\mu$. In order to alleviate this behavior, let us consider the trajectory of the incoming light ray

$$ct = -r - 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant}. \tag{127}$$

We will see that the singular term comes from the second term. In order to take away this term, one can introduce the new coordinate such that

$$c\,\bar{t} = c\,t + 2\mu \ln\left|\frac{r}{2\mu} - 1\right| = -r + \text{constant}. \tag{128}$$

The coordinates with $\bar{t}$ in this form are usually called "advanced Eddington-Finkelstein coordinates". This transformation provides us with the relation

$$c\,dt = c\,d\bar{t} - 2\mu\left(\frac{r}{2\mu} - 1\right)^{-1}\frac{dr}{2\mu} = c\,d\bar{t} - \left(\frac{2\mu}{r-2\mu}\right)dr. \tag{129}$$

This leads to the relation

$$c^2\,dt^2 = c^2\,d\bar{t}^2 - \left(\frac{4\mu c}{r-2\mu}\right)d\bar{t}dr + \left(\frac{4\mu^2}{(r-2\mu)^2}\right)dr^2, \tag{130}$$

$$\left(1-\frac{2\mu}{r}\right)c^2dt^2 = \left(1-\frac{2\mu}{r}\right)c^2d\bar{t}^2 - \left(\frac{4\mu c}{r}\right)d\bar{t}dr + \left(\frac{4\mu^2}{r^2}\right)\left(1-\frac{2\mu}{r}\right)^{-1}dr^2 \tag{131}$$

Substituting this result into the Schwarzschild line element in equation (126), one obtains

$$\begin{aligned} ds^2 &= -\left(1-\frac{2\mu}{r}\right)c^2d\bar{t}^2 + \left(\frac{4\mu c}{r}\right)d\bar{t}dr + \left(1-\frac{2\mu}{r}\right)^{-1}\left(1-\frac{4\mu^2}{r^2}\right)dr^2, \\ &= -\left(1-\frac{2\mu}{r}\right)c^2d\bar{t}^2 + \left(\frac{4\mu c}{r}\right)d\bar{t}dr + \left(1-\frac{2\mu}{r}\right)^{-1}\left(1-\frac{2\mu}{r}\right)\left(1+\frac{2\mu}{r}\right)dr^2, \\ &= -\left(1-\frac{2\mu}{r}\right)c^2d\bar{t}^2 + \left(\frac{4\mu c}{r}\right)d\bar{t}dr + \left(1+\frac{2\mu}{r}\right)dr^2 \end{aligned} \tag{132}$$

From these coordinates, we will see that the metric no longer has the singularity at $r = 2\mu$. This is also proof that it is the coordinate singularity at $r = 2\mu$, not a real or intrinsic singularity. Note that it seems like the coordinate transformation above is not well-defined at $r = 2\mu$. However, it is normal. Physics is still the same, does not depend on coordinates. We will see the result soon after the lightcone structure is investigated. The significant difference of these two coordinates is that, for the advanced Eddington-Fenkelstein coordinates, the range of radial coordinate r can be applied as $0 < r < \infty$ while for the Schwarzschild coordinates, r can be applied as $2\mu < r < \infty$.

Now we are ready to find the lightcone structure in the advanced Eddington-Fenkelstein coordinates. Let us consider the null geodesics by using the line

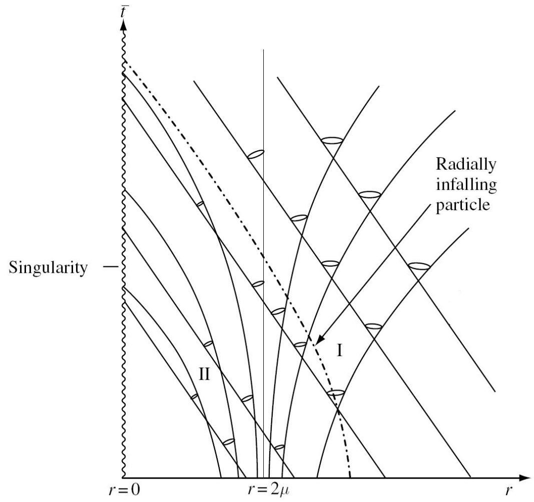

Figure 7: Spacetime diagram in $r - ct$ plane with advance Eddington-Fenkelstein coordinates [1]. The future lightcone is flipped from the increasing time to the decreasing radius when the the lightcone crosses the Schwarzschild radius. The incoming light ray does not blow up at the Schwarzschild radius.

element in equation (132),

$$\begin{aligned}
\left(1-\frac{2\mu}{r}\right)c^2d\bar{t}^2-\left(\frac{4\mu c}{r}\right)d\bar{t}dr-\left(1+\frac{2\mu}{r}\right)dr^2 &= 0,\\
c^2d\bar{t}^2-\left(\frac{4\mu c}{r-2\mu}\right)d\bar{t}dr-\left(\frac{r+2\mu}{r-2\mu}\right)dr^2 &= 0,\\
(cd\bar{t}+dr)\left(c\,d\bar{t}-\frac{r+2\mu}{r-2\mu}dr\right) &= 0,\\
\left(c\frac{d\bar{t}}{dr}+1\right)\left(c\,\frac{d\bar{t}}{dr}-\frac{r+2\mu}{r-2\mu}\right) &= 0. \qquad (133)
\end{aligned}$$

The first parenthesis refers to the equation for the incoming ray, which has the its slope and the solution as follows

$$c\frac{d\bar{t}}{dr} = -1, \qquad (134)$$

$$c\,\bar{t} = -r+\text{constant}. \qquad (135)$$

The slope is always equal to -1, and the solution is sketched in figure 7. This solution corresponds to the incoming ray as we expect from equation

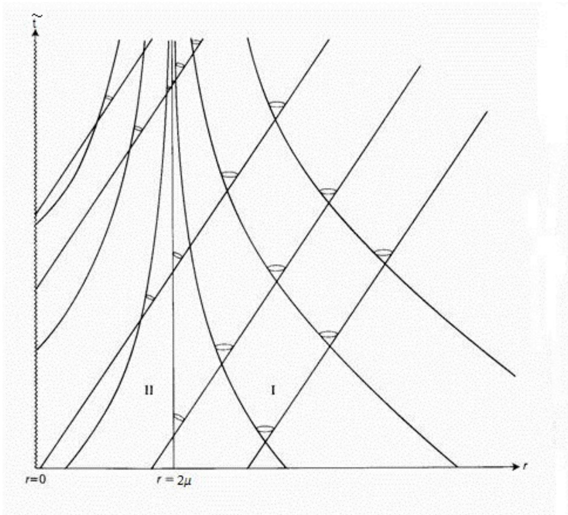

Figure 8: Spacetime diagram in $r - ct$ plane with retarded Eddington-Fenkelstein coordinates [3]. The future lightcone is flipped from the increasing time to the decreasing radius when the the lightcone crosses the Schwarzschild radius. The outgoing light ray does not blow up at the Schwarzschild radius. These coordinates lead to another region of the solution in Schwarzschild geometry.

(128). For the outgoing ray, its slope takes the form

$$c\,\frac{d\bar{t}}{dr} = \frac{r+2\mu}{r-2\mu}. \tag{136}$$

At large r, the lightcone is still the same, resulting in the Schwarzschild coordinates, since it approaches the Minkowski spacetime. The solution of the above equation can be obtained as follows

$$\begin{aligned} c\,\bar{t} &= \int \frac{r+2\mu}{r-2\mu} dr, \\ &= \int \left(1 + \frac{4\mu}{r-2\mu}\right) dr, \\ &= r + 4\mu \ln\left|\frac{r}{2\mu} - 1\right| + \text{constant}. \end{aligned} \tag{137}$$

We can see that the outgoing ray does not significantly change from the Schwarzschild case. The slope still changes sign, and its absolute value still goes to infinity. The solution also still blows up at $r = 2\mu$. Thus, the sketching of the outgoing light ray is still qualitatively the same as in the

Schwarzschild case, as seen in the figure 7. The result of the particle motion is also the same since the lightcone outside the black hole still has the future lightcone in the direction of increasing time, and the lightcone inside the black hole still has the future lightcone in the direction of decreasing radial r.

Let us introduce new coordinates by using an analogous way as we have done in the advanced Eddington-Fenkelstein case. Now we can try to fix the problem in the outgoing solution instead of the incoming solution as follows

$$c\,\tilde{t} = c\,t - 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| = r + \text{constant}. \tag{138}$$

The coordinates with this $\tilde{t}$ are usually called "retarded Eddington-Fenkelstein" coordinates. We can play in the same way as we have done in the advanced case, and then the line element becomes

$$ds^2 = -\left(1 - \frac{2\mu}{r}\right) c^2 d\tilde{t}^2 - \left(\frac{4\mu c}{r}\right) d\tilde{t}dr + \left(1 + \frac{2\mu}{r}\right) dr^2 \tag{139}$$

Note that the mathematical difference occurs only in the change of sign of the second term in equation (131). This leads to the changing sign between equations (132) and (139). The metric in these coordinates is also still non-singular in the range $0 < r < \infty$, similarly to the advanced case. The slope and solution of the outgoing ray can be written in the form

$$c\frac{d\tilde{t}}{dr} = 1, \tag{140}$$

$$c\,\tilde{t} = r + \text{constant}, \tag{141}$$

while, in the case of incoming ray, they take the form

$$c\,\frac{d\tilde{t}}{dr} = -\frac{r + 2\mu}{r - 2\mu}, \tag{142}$$

$$c\,\tilde{t} = -r - 4\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant}. \tag{143}$$

The lightcone structure can be sketched in the figure 8. At large r, the solution is still the same. However, at $2\mu \lesssim r$, physics is very different. It is found that particles always go out from the Schwarzschild radius and cannot move past this radius. In fact, these coordinates provide us the new solution that is hidden when the Schwarzschild coordinates are used. Since this solution provides only outgoing particles, which is in contrast with the black hole solution, it is named as "white hole" solution. This solution can be treated as the time-reversed black hole solution. Even though the time reversal invariance is an important symmetry in nature, the white hole seems like that it does not to exist in nature while black holes do. Theoretical researches of white holes still works, and most of the investigations suggest that the white hole solution is not stable. This issue is interesting.

5.3 Kruskal-Szekeres coordinates

From Eddington-Finkelstein coordinates, it is found that there are two regions of the solutions, one corresponding to the Schwarzschild black hole and the other corresponding to a white hole. These two regions are described by two spacetime diagrams. Therefore, it is worthwhile to find the new coordinates that can combine these two regions in one diagram. This is the main purpose of this section, and it corresponds to the Kruskal-Szekeres coordinates. Other motivations come from the fact that, in Eddington-Finkelstein coordinates, one of the coordinates in the spacetime diagram is still singular at $r = 2\mu$ and the lightcone structure is not easy to sketch in the spacetime diagram. In any conformally flat coordinates, the lightcone structure is still the same as the lightcone in Minkowski spacetime. Therefore, it is easy the sketch the lightcone since the angle of the lightcone is fixed to 45° in all regions of the diagram. Therefore, let us begin by introducing the coordinates which are conformally flat as follows

$$\tilde{r} = r + 2\mu \ln\left|\frac{r}{2\mu} - 1\right|, \tag{144}$$

$$d\tilde{r} = \left(1 - \frac{2\mu}{r}\right)^{-1} dr \Rightarrow dr^2 = \left(1 - \frac{2\mu}{r}\right)^{2} d\tilde{r}^2. \tag{145}$$

Substituting into the Schwarzschild line element in equation (126), one obtains

$$ds^2 = \left(1 - \frac{2\mu}{r}\right)(-c^2dt^2 + d\tilde{r}^2). \tag{146}$$

Conveniently, we omit the contribution from the solid angle by fixing $\theta = \pi/2$ and $\phi =$ constant. If we include it, the lightcone structure will not change. Each point in the spacetime diagram will represent the two spheres of θ and ϕ. Note that r is no longer the coordinate. It plays the role of the function of the coordinate $r(\tilde{r})$. We see that the coordinate $\tilde{r}$ still has a singularity at $r = 2\mu$. This implicitly leads to the singularity of the metric at $r = 2\mu$. In order to see how the singularity emerges in the metric, one can investigate the metric at $r \sim 2\mu$. This leads to the relation

$$\tilde{r} \simeq 2\mu \ln\left|\frac{r}{2\mu} - 1\right| \Rightarrow \frac{r}{2\mu} \simeq 1 \pm e^{\frac{\tilde{r}}{2\mu}} \Rightarrow \left(1 - \frac{2\mu}{r}\right) \simeq \pm e^{\frac{\tilde{r}}{2\mu}}, \quad (147)$$

where the upper sign denotes the region $r > 2\mu$ and the lower sign denotes the region $r < 2\mu$. Thus the line element in equation (146) at $r \sim 2\mu$ can be approximated as

$$ds^2 \simeq \pm e^{\frac{\tilde{r}}{2\mu}}(-c^2dt^2 + d\tilde{r}^2). \quad (148)$$

From this line element, one can see that the metric blows up at $r = 2\mu$ since the coordinate $\tilde{r}$ will be infinity. In order to find the new coordinates to get rid of this factor, one can firstly transform the null coordinates such that

$$\begin{aligned} p = ct + \tilde{r},\ q = ct - \tilde{r} \qquad &\Rightarrow\ d\tilde{r} = \frac{1}{2}(dp - dq),\ c\,dt = \frac{1}{2}(dp + dq), \quad (149)\\ &\Rightarrow\ -c^2dt^2 + d\tilde{r}^2 = -dp\,dq. \end{aligned}$$

Substituting the results of the coordinate transformation into the approximated line element above, one obtains

$$ds^2 \simeq \mp e^{\frac{\tilde{r}}{2\mu}}dp\,dq = \mp e^{\frac{p-q}{4\mu}}dp\,dq. \quad (150)$$

Now we see that, in order to get rid of the divergent factor, one can find the new coordinates that satisfy the relation

$$d\tilde{p} \propto e^{\frac{p}{4\mu}}dp, \text{ and } d\tilde{q} \propto e^{-\frac{q}{4\mu}}dq. \quad (151)$$

Therefore, one of the simple choices is that

$$\tilde{p} = e^{\frac{p}{4\mu}}, \text{ and } \tilde{q} = \mp e^{\frac{-q}{4\mu}}. \quad (152)$$

This leads to the relation

$$\begin{aligned} dp\,dq &= \pm 16\mu^2 e^{-\frac{p-q}{4\mu}}d\tilde{p}\,d\tilde{q} = \pm 16\mu^2 e^{-\frac{\tilde{r}}{2\mu}}d\tilde{p}\,d\tilde{q}, \\ &= \pm 16\mu^2 e^{-\frac{r}{2\mu} - \ln\left|\frac{r}{2\mu}-1\right|}d\tilde{p}\,d\tilde{q}, \\ &= \pm 16\mu^2 e^{-\frac{r}{2\mu}}\left(\left|\frac{r}{2\mu} - 1\right|\right)^{-1}d\tilde{p}\,d\tilde{q}, \\ &= 16\mu^2 e^{-\frac{r}{2\mu}}\left(\frac{2\mu}{r - 2\mu}\right)d\tilde{p}\,d\tilde{q}. \quad (153) \end{aligned}$$

Substituting this results back into the line element (146) with using the relation in equation (150), one obtains

$$\begin{aligned} ds^2 &= -\left(\frac{r-2\mu}{r}\right)16\mu^2 e^{-\frac{r}{2\mu}}\left(\frac{2\mu}{r-2\mu}\right)d\tilde{p}\,d\tilde{q}, \\ &= -32\mu^3\frac{e^{-\frac{r}{2\mu}}}{r}d\tilde{p}\,d\tilde{q}. \end{aligned} \tag{154}$$

Finally, let us transform the coordinates to the one which is in the conformally flat form. This can be achieved by introducing new coordinates such that

$$\tilde{p} = v+u, \tilde{q} = v-u \Rightarrow d\tilde{p}\,d\tilde{q} = -du^2+dv^2, \tag{155}$$

Substituting the result back into equation (154), the line element becomes

$$ds^2 = 32\mu^3\frac{e^{-\frac{r}{2\mu}}}{r}(-dv^2+du^2). \tag{156}$$

Now r is a function of u and v. We can see that there are no singularities of the metric at $r = 2\mu$ as well as it satisfies the conformally flat form. The real singularity still explicitly appears in the metric. In order to sketch the spacetime diagram, one has to find the relation between Schwarzschild and Kruskal-Szekeres coordinates. Firstly, let us consider the simple relation

$$v^2-u^2 = \tilde{p}\,\tilde{q} = \mp e^{\frac{p-q}{4\mu}} = \mp e^{\frac{\tilde{r}}{2\mu}} = \mp e^{\frac{r}{2\mu}}\left(\left|\frac{r}{2\mu}-1\right|\right) = -e^{\frac{r}{2\mu}}\left(\frac{r}{2\mu}-1\right) \tag{157}$$

This relation provides us the lines at the event horizon such that $v = \pm u$. These lines are sketched as a crossing sign, which is shown in the figure 9. These crossing lines separate the spacetime diagram into four regions: left (I'), right (I), upper (II) and lower (II') regions. For any constant r, it is a hyperbolic locus. For $r > 2\mu$, it corresponds to a hyperbolic locus in left and right regions of the diagram, and, for $r < 2\mu$, it corresponds to a hyperbolic locus in upper and lower regions of the diagram which contain the locus of the singularity.

To see more information about the diagram, let us consider the relation for constant time. Firstly, we can find the relation of v in terms of r and t

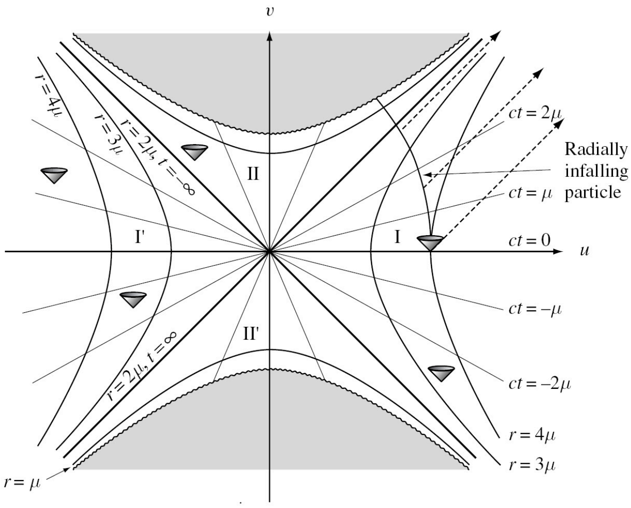

Figure 9: Spacetime diagram in Kruskal-Szekeres coordinates [1]

which can be written as

$$\begin{aligned} v &= \frac{1}{2}(\tilde{p}+\tilde{q}) = \frac{1}{2}\left(e^{\frac{p}{4\mu}} \mp e^{\frac{p}{4\mu}}\right) = \frac{1}{2}\left(e^{\frac{ct+\tilde{r}}{4\mu}} \mp e^{-\frac{ct-\tilde{r}}{4\mu}}\right), \\ &= e^{\frac{\tilde{r}}{4\mu}}\left(e^{\frac{ct}{4\mu}} \mp e^{-\frac{ct}{4\mu}}\right) = e^{\frac{r}{4\mu}}\left(\frac{r}{2\mu}-1\right)^{1/2}\frac{1}{2}\left(e^{\frac{ct}{4\mu}} \mp e^{-\frac{ct}{4\mu}}\right), \\ &= e^{\frac{r}{4\mu}}\left(\frac{r}{2\mu}-1\right)^{1/2}\sinh\left(\frac{ct}{4\mu}\right), \quad \text{for}\, r>2\mu, \qquad (158) \\ &= e^{\frac{r}{4\mu}}\left(1-\frac{r}{2\mu}\right)^{1/2}\cosh\left(\frac{ct}{4\mu}\right), \quad \text{for}\, r<2\mu. \qquad (159) \end{aligned}$$

We can obtain the relation of u in terms of r and t in the same manner. The results can be expressed as

$$\begin{aligned} u &= e^{\frac{r}{4\mu}}\left(\frac{r}{2\mu}-1\right)^{1/2}\cosh\left(\frac{ct}{4\mu}\right), \quad \text{for } r>2\mu, \qquad (160) \\ &= e^{\frac{r}{4\mu}}\left(1-\frac{r}{2\mu}\right)^{1/2}\sinh\left(\frac{ct}{4\mu}\right), \quad \text{for } r<2\mu, \qquad (161) \end{aligned}$$

Now, the relations of u and v in terms of t can be written as

$$\tanh\left(\frac{ct}{4\mu}\right) = \frac{v}{u}, \quad \text{for } r>2\mu, \qquad (162)$$

$$\tanh\left(\frac{ct}{4\mu}\right) = \frac{u}{v}, \quad \text{for } r<2\mu. \qquad (163)$$

From these relation, we found that the axis $t = 0$ corresponds to the axis $v = 0$ for $r > 2\mu$ and corresponds to the axis $u = 0$ for $r < 2\mu$. For $r > 2\mu$ in the right region, any value of t corresponds the strength line. The more value of t, the more slope of the strength line until the slope becomes 1 corresponding to $t = \infty$ as shown in figure 9. The lightcone structure is still the same everywhere in the diagram and have the future lightcone in the direction of increasing v. This leads to the fact that the upper region correspond to the interior of the black hole while the lower region corresponds to the interior of white hole. We also see from the lightcone at the line $r = 2\mu$ in the upper region that the particle never escaped from black hole while, in the lower region, particle never moved into the white hole. The interesting results of this diagram are that there are two regions which have an asymptotically flat Minkowski spacetime in the left and right regions, or in regions I and I'. Region I completely describes our spacetime outside the black hole. This leads to the fact that there exists another world in the region I' which cannot be influenced by our world. We can see that the particles at the origin of the diagram are restricted to the region II, which is the black hole. Thus, it is impossible to move from region I to region I' or vice versa from region I' to region I. The join between these two regions is called "wormhole," which we will discuss in detail in the next sections.

5.4 Wormhole and Einstein-Rosen bridge

As we have mentioned in the previous section, there is a join between two worlds, which is the origin in Kruskal-Szekeres spacetime diagram. Remembering that each point in the spacetime diagram represents a 2-sphere of the solid angle. Thus, there are others view point to consider this joint. We have learned that it is convenient to consider a two-dimensional diagram. Now, we can choose to consider a diagram in which $v = 0$ and $\theta = \pi/2$. This corresponds to the line element

$$ds^2 = 32\mu^3 \frac{e^{-\frac{r}{2\mu}}}{r} du^2 + r^2 d\phi^2. \tag{164}$$

By using the relation of u^2 and r in equation (157) , one obtains

$$ds^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \tag{165}$$

It is convenient to consider this surface by embedding it into three-dimensional Euclidean space. Since we have the coordinate ϕ and r, which look similar to the polar cylindrical coordinates, it is convenient to consider this three-dimensional Euclidean space in polar cylindrical coordinates, which can be written as

$$ds^2 = dz^2 + d\rho^2 + \rho^2 d\psi^2. \tag{166}$$

In order to embed our two-sphere into this three-dimensional Euclidean space, we have to find a constraint equation to satisfy the line element (165). This can be achieved by introducing the coordinates such that $\rho = \rho(r)$, $z = z(r)$. Therefore, the line element in equation (166) becomes

$$ds^2 = \left(\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2\right) dr^2 + \rho^2 d\psi^2. \tag{167}$$

Then, by setting the coordinate such that $\psi = \phi$, $\rho = r$, one obtains

$$ds^2 = \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 d\phi^2. \tag{168}$$

Compared to equation (165), the constraint equation in differential form can be written as

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} \Rightarrow \left(\frac{dz}{dr}\right)^2 = \frac{r}{r-2\mu} - 1 = \frac{2\mu}{r-2\mu}. \tag{169}$$

This leads to the constraint equation,

$$z = (2\mu)^{1/2}\int (r-2\mu)^{-1/2} dr = (8\mu)^{1/2}(r-2\mu)^{1/2} + \text{constant}. \tag{170}$$

From this constraint equation, the generality of the surface structure is not lost by setting the constant to be zero, and we can see that this is the locus of a parabolic line. At $z = 0$ corresponds to $r = 2\mu$ which is the closet point to the z-axis and recognizing that for $v = 0$ corresponds to $r \geq 2\mu$. To obtain the surface, one can turn the parabolic locus around z-axis, and then the surface can be shown in figure (10). From this figure, the two worlds can be connected together through the throat of the surface at $z = 0$ corresponding to the origin of the Kruskal-Szekeres diagram. If we return to consider the Kruskal-Szekeres diagram, we cannot stay at the origin point and are forced to move to region II where $v > 0$. Since, in region II, the coordinate r becomes a timelike coordinate, the surface which depends on r will become to be dynamics. At $v = v_0 =$ constant where $0 < v_0 < 1$, the metric is still in the same form, and then the surface also has the same form. However, from

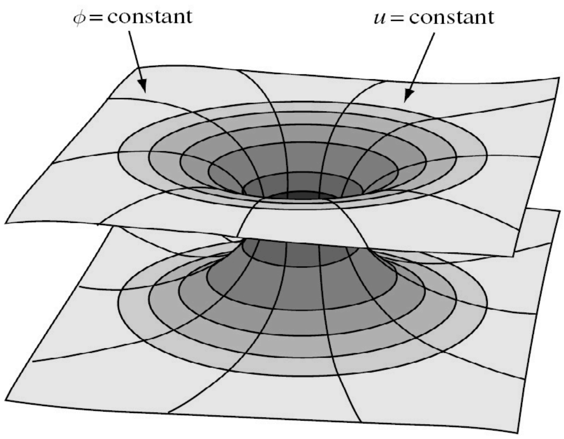

Figure 10: Wormhole or Einstein-Rosen bridge [1]. It is another viewpoint of the join between our world and the other, which corresponds to the line $v = 0$ in Kruskal-Szekeres diagram.

the Kruskal-Szekeres diagram, when we scan $-\infty < u < \infty$, the minimum of r is in the range $0 < r < 2\mu$. Therefore, the throat of the surface becomes more and more narrow as v increases until $v = 1$, corresponding to $r = 0$, the throat is pinched off. The cross-section of the surface with various v is shown in figure 11. From the point of view of the Einstein-Rosen bridge, it may be

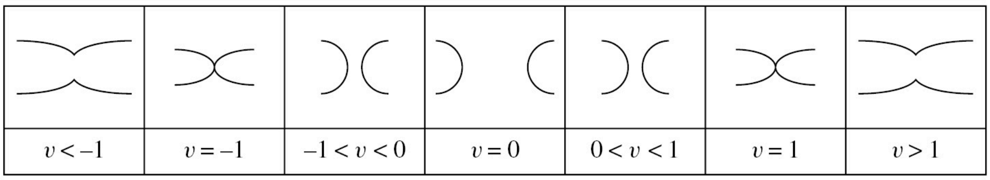

Figure 11: Cross section of Wormhole or Einstein-Rosen bridge with various values of v [1]

possible to travel from one world to the other before the throat of the bridge is pinched off. However, the information from the Kruskal-Szekeres is still valid and provides us that the throat is pinched off too quickly for any timelike particle can cross it from one world to the other. It is important to note that this is only the solution of the Einstein equation in empty spacetime. It may be possible to construct the spacetime geometry by introducing exotic matter in which the wormhole will not pinch off too quickly. This is also the basic study of the time traveling which is commonly mentioned in pop-science movies. The research area of this subject is still active.

5.5 Conformal diagram

From the Kruskal-Szekeres diagram, we have seen that it is a powerful tool to characterize the properties of the spacetime as well as the behavior of the particle motion. In this section, we will study one of the useful diagrams, namely the "conformal diagram" or "Carter-Penrose diagram", or just "Penrose diagram". The general properties of a conformal diagram are somewhat the generalized version of Kruskal-Szekeres diagram. It is supposed to preserve the lightcone angle and include all regions of the spacetime solution as well as compact all regions into a finite diagram. The third one is the additional properties of the Kruskal-Szekeres diagram.

It is a good starting point to study the simple spacetime such as Minkowski metric in spherical coordinates,

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2. \tag{171}$$

It is well-known that the lightcone in this spacetime automatically preserves the angle to 45°. The range of the coordinates θ and ϕ is already finite, while the range of the coordinates r and t is not finite and takes the region as $0 \leq r < \infty$ and $-\infty < t < \infty$. Our task is just to make them be in a finite region of the diagram. However, by doing this, it usually destroys the preserving angle of the lightcone. The technique is to introduce the null coordinates first and then make them finite. Let us introduce the null coordinate such that

$$p = ct + r, \ \ q = ct - r. \tag{172}$$

Thus the ranges of the coordinates can be expressed as $-\infty < p < \infty$, $-\infty < q < \infty$ and $q \leq p$. Using these coordinates, Minkowski line element in equation (171) becomes

$$ds^2 = -dp\,dq + \frac{1}{4}(p-q)^2d\Omega^2. \tag{173}$$

Now, we are at a point to make the finite coordinates. It is very useful to use the function arctangent since it can map $\pm\infty$ to $\pm\pi/2$. Thus, the new coordinates can be written as

$$\tilde{p} = \tan^{-1} p, \ \ \tilde{q} = \tan^{-1} p. \tag{174}$$

These lead to the relations,

$$p - q = \tan \tilde{p} - \tan \tilde{q} = \frac{\sin \tilde{p} \cos \tilde{q} - \sin \tilde{q} \cos \tilde{p}}{\cos \tilde{p} \cos \tilde{q}} \;\Rightarrow\; (p-q)^2 = \frac{\sin^2(\tilde{p}-\tilde{q})}{\cos^2 \tilde{p}\ \cos^2 \tilde{q}}, \tag{175}$$

and

$$dp = \frac{d\tilde{p}}{\cos^2 \tilde{p}},\; dq = \frac{d\tilde{q}}{\cos^2 \tilde{q}} \;\Rightarrow\; dp\, dq = \frac{d\tilde{p}\, d\tilde{q}}{\cos^2 \tilde{p}\ \cos^2 \tilde{q}}. \tag{176}$$

By using these relations, the line element in equation (173) becomes

$$ds^2 = \frac{1}{4 \cos^2 \tilde{p}\ \cos^2 \tilde{q}} \left(-4 d\tilde{p}\, d\tilde{q} + \sin^2(\tilde{p} - \tilde{q}) d\Omega^2 \right). \tag{177}$$

Note that the finite coordinates have regions as $-\pi/2 < \tilde{p} < \pi/2$, $-\pi/2 < \tilde{q} < \pi/2$ and $\tilde{q} \leq \tilde{p}$. Let us return to the coordinates which preserve the lightcone angle such that

$$v = \tilde{p} + \tilde{q},\; u = \tilde{p} - \tilde{q} \;\Rightarrow d\tilde{p}\, d\tilde{q} = \frac{1}{4} \left(dv^2 - du^2 \right), \tag{178}$$

and then the line element becomes

$$ds^2 = (\cos u + \cos v)^{-2} \left(-dv^2 + du^2 + \sin^2 u\, d\Omega^2 \right), \tag{179}$$

where we have used a relation

$$2 \cos \tilde{p}\ \cos \tilde{q} = 2 \cos \left(\frac{1}{2}(u+v) \right) \cos \left(\frac{1}{2}(u-v) \right) = \cos u + \cos v. \tag{180}$$

These coordinate transformations lead to the ranges of the new coordinates such that

$$0 < u < \pi \;\text{ and }\; |v| + u < \pi. \tag{181}$$

The line element in equation (179) looks similar to the coordinates in $\mathbf{R} \times S^3$ spacetime, which can be sketched in cylindrical coordinates as shown in figure 12. The rang of the coordinates of $\mathbf{R} \times S^3$ can be written as $-\infty < v < \infty$ and $0 < u < \pi$. Note that each circle in the surface represents a three-sphere. The shaded region in the cylinder surface represents a part of our coordinate region in equation (181) as shown in figure 12. Therefore, the conventional diagram of this spacetime is just the unrolled version of this shaded region. It is convenient to consider half of the part and the boundary of the unrolled diagram since it already contains all information of the spacetime as shown in figure 13. Now, let us consider the relations of the new coordinates and

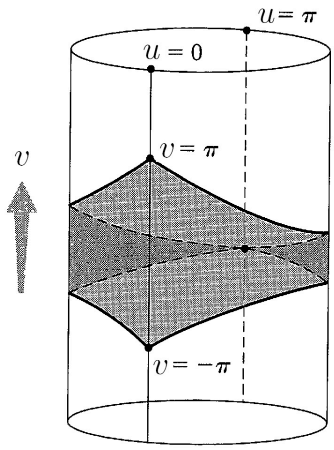

Figure 12: $\mathbf{R} \times S^3$ spacetime embedding in cylindrical coordinates [2]

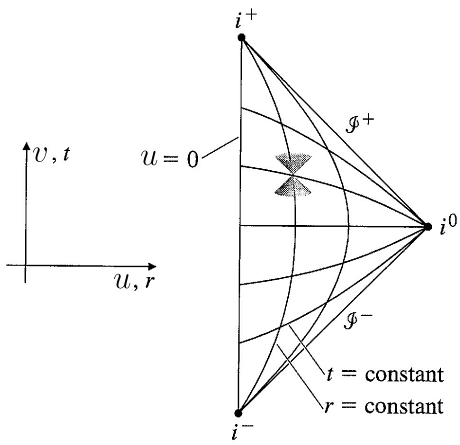

Figure 13: Conformal diagram for Minkowski spacetime [2].

the Schwarzschild coordinates, which can be written as

$$v = \tilde{p} + \tilde{q} = \tan^{-1} p + \tan^{-1} q = \tan^{-1}(ct + r) + \tan^{-1}(ct - r), \quad (182)$$
$$u = \tilde{p} - \tilde{q} = \tan^{-1} p - \tan^{-1} q = \tan^{-1}(ct + r) - \tan^{-1}(ct - r). \quad (183)$$

At $r = 0$ with any ct, this corresponds to line $u = \tan^{-1}(ct) - \tan^{-1}(ct) = 0$. There are the end points of this line corresponding to the point $r = 0$, $ct = -\infty$, i^- called "past timelike infinity" and the point $r = 0$, $ct = \infty$, i^+ called "future timelike infinity". At $r = \infty$ and $ct = 0$, this point corresponds to $u = \pi$ and $v = 0$, i^0, called "spatial infinity". Therefore, the vertical-curved lines correspond to the line with $r =$ constant. There are two other interesting lines: $u + v = \pi$ and $u - v = \pi$ called past and future null infinity, corresponding to $ct + r = \infty$ and $ct - r = \infty$, respectively. Finally, the horizontal-curved lines correspond to the line with $ct =$ constant.

Now we are at the point to consider the conformal diagram for Schwarzschild spacetime. Let us recall the line element in null coordinates as

$$ds^2 = -32\mu^3 \frac{e^{-\frac{r}{2\mu}}}{r} d\tilde{p}\, d\tilde{q} + r^2 d\Omega^2, \quad (184)$$

where

$$\tilde{p} \;=\; e^{\frac{ct+\tilde{r}}{4\mu}}, \text{ with range } 0 < \tilde{p} < \infty, \quad (185)$$
$$\tilde{q} \;=\; \mp e^{\frac{-(ct-\tilde{r})}{4\mu}}, \text{ with range } -\infty < \tilde{q} < \infty. \quad (186)$$

Therefore, we can make null coordinates finite by using similar coordinates as we have done in Minkowski case. These new coordinates can be written as

$$\tilde{\tilde{p}} \;=\; \tan^{-1} \tilde{p}, \text{ with range } 0 < \tilde{\tilde{p}} < \frac{\pi}{2}, \quad (187)$$
$$\tilde{\tilde{q}} \;=\; \tan^{-1} \tilde{q}, \text{ with range } -\frac{\pi}{2} < \tilde{\tilde{q}} < \frac{\pi}{2}. \quad (188)$$

By using these transformations, the line element becomes

$$ds^2 = -32\mu^3 \frac{e^{-\frac{r}{2\mu}}}{r} \sec^2 \tilde{\tilde{p}} \sec^2 \tilde{\tilde{q}}\; d\tilde{\tilde{p}}\, d\tilde{\tilde{q}} + r^2 d\Omega^2, \quad (189)$$

In order to obtain the diagram in which the lightcone angle is fixed to 45°, one can choose the coordinate transformation such that

$$v \;=\; \tilde{\tilde{p}} + \tilde{\tilde{q}} = \tan^{-1} \tilde{p} \pm \tan^{-1} \tilde{q} = \tan^{-1}\left(e^{\frac{ct+\tilde{r}}{4\mu}}\right) \mp \tan^{-1}\left(e^{-\frac{ct-\tilde{r}}{4\mu}}\right), \quad (190)$$
$$u \;=\; (\tilde{\tilde{p}} - \tilde{\tilde{q}}) = \pm(\tan^{-1} \tilde{p} \mp \tan^{-1} \tilde{q}) = \left(\tan^{-1}\left(e^{\frac{ct+\tilde{r}}{4\mu}}\right) \pm \tan^{-1}\left(e^{-\frac{ct-\tilde{r}}{4\mu}}\right)\right). \quad (191)$$

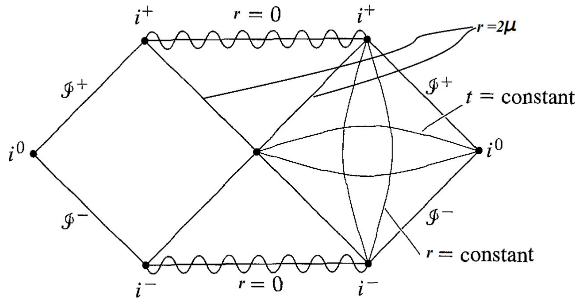

Figure 14: Conformal diagram for Schwarzschild spacetime [2].

Note that the range of the coordinates can be written as

$$-\frac{\pi}{2} < v < \frac{\pi}{2}, \ -\pi < u < \pi, \ \text{and} \ -\pi < u + v < \pi. \tag{192}$$

This leads to the line element as

$$ds^2 = 8\mu^3 \frac{e^{-\frac{r}{2\mu}}}{r(\cos v + \cos u)^2} \left(-dv^2 + du^2\right) + r^2 d\Omega^2, \tag{193}$$

In order to sketch the diagram, the important relation that does not depend on ct can be written in terms of new coordinates as

$$\tilde{p}\tilde{q} = -\left(\frac{r}{2\mu} - 1\right) e^{\frac{r}{2\mu}} = \tan\tilde{\tilde{p}}\ \tan\tilde{\tilde{q}} = \tan\left(\frac{v \mp u}{2}\right)\ \tan\left(\frac{v \pm u}{2}\right) = \frac{\cos u - \cos v}{\cos u + \cos v}, \tag{194}$$

For $r = 2\mu$, equation (194) is reduced to

$$\frac{\cos u - \cos v}{\cos u + \cos v} = 0 \ \Rightarrow \ \cos u - \cos v = 0 \ \Rightarrow \ v = \pm u. \tag{195}$$

These lines represent the crossing line at the origin of the diagram in figure 14.

For $r = 0$, equation (194) becomes

$$\frac{\cos u - \cos v}{\cos u + \cos v} = 1 \ \Rightarrow \ \cos v = 0 \ \Rightarrow \ v = \pm\frac{\pi}{2}. \tag{196}$$

These represent the line of singularity shown in figure 14. For $r \to \infty$, one obtains $\cos u + \cos v$ where $\cos u - \cos v \neq 0$. Since $-\pi/2 < v < \pi/2$, one found that these points correspond to $(u, v) = (\pm\pi, 0)$ represented by i^0 as found in figure 14.

In order to find the locus by fixing t, one has to find the relation that does not depend on r. As a result, we have

$$e^{\frac{ct}{2\mu}} = \pm \frac{\tan\left(\frac{u+v}{2}\right)}{\tan\left(\frac{u-v}{2}\right)}. \tag{197}$$

For i^+, this corresponds to the point $t \to \infty$. From equation (197), these points can be obtained by

$$\tan\left(\frac{u+v}{2}\right) \to \infty \;\; \Rightarrow \;\; \frac{u+v}{2} = \pm\frac{\pi}{2} \;\; \Rightarrow \;\; (u,v) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) \tag{198}$$

Note that the condition $u - v \neq \pm\pi$ is used for the points i^+. By using the same strategy, the points $i^-(r \to -\infty)$ can be obtained as

$$\tan\left(\frac{u-v}{2}\right) \to \infty \;\; \Rightarrow \;\; \frac{u-v}{2} = \pm\frac{\pi}{2} \;\; \Rightarrow \;\; (u,v) = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{199}$$

It is important to note that it is convenient to switch $i+$ and i^- in the left part of the diagram. For this convenience, one does not need to switch the future lightcone from up to down when the trajectory crosses from the left to the right of the diagram.

5.6 Exercise

1. Considering the Einstein equation with cosmological constant

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{200}$$

   Find the solution of this equation and the behavior of particle motion in a plane for both massive and massless cases by using the effective potential analysis.

2. Using the coordinate transformation $p = ct + r + 2\mu \ln |\frac{r}{2\mu} - 1|$
   (a) Find line element in these coordinates
   (b) Find a solution for the radial motion of a photon in both incoming and outgoing directions
   (c) Sketch the spacetime diagram and lightcone structure

3. Using the coordinate transformation $q = ct - r - 2\mu \ln |\frac{r}{2\mu} - 1|$
   (a) Find line element in these coordinates
   (b) Find a solution for the radial motion of a photon in both incoming and outgoing directions
   (c) Sketch the spacetime diagram and lightcone structure

4. Considering line element $ds^2 = c^2 dt^2 - t^{2\alpha}(dr^2 + r^2 d\Omega)$, find conformal diagram for this spacetime.

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[3] http://ckw.phys.ncku.edu.tw/public/pub/Notes/TheoreticalPhysics/Lawrie_2/Chap04/4.5.4._EddingtonFinkelsteinCoordinates.htm