Lecture on Cosmology

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Contents

Chapter 1

Introduction

1.1 Olber's Paradox

Olbers's paradox which is also called **the dark night sky paradox** and goes as follows: if the universe is static, infinitely large and old and with an infinite number of stars distributed uniformly, then the night sky should be bright.

Let us try to understand why. First of all, the stars are distributed uniformly, which means that their number density say n is a constant. Consider a spherical shell of thickness $\mathrm{d}R$ and radius R centred on the Earth. The number of stars inside this spherical shell is:

$$\mathrm{d}N = 4\pi n R^2 \,\mathrm{d}R \,. \tag{1.1}$$

The total luminosity of this spherical shell is $\mathrm{d}N$ multiplied by the luminosity say L of a single star, and we assume L to be the same for all the stars. By the inverse-square law, the total flux received on Earth is:

$$\mathrm{d}F_{\mathrm{tot}} = \frac{L\,\mathrm{d}N}{4\pi R^2} = nL\,\mathrm{d}R \,. \tag{1.2}$$

It does not depend on R and thus it diverges when integrated over R from zero to infinity. This means not only that the night sky should not be dark but also infinitely bright!

We can solve the problem of having an infinitely bright night sky by considering the fact that stars are not points and do eclipse each other, so that we do not really see all of them. Suppose that each star shows us a surface $\mathrm{d}A$. Therefore, if a star lies at a distance R, we receive from it the flux $\mathrm{d}F = \mathcal{L}/(4\pi R^2)\,\mathrm{d}A$, where $\mathcal{L}$ is the luminosity per unit area. But $\mathrm{d}A/R^2 = \mathrm{d}\Omega$ is the solid angle spanned by the star in the sky. Therefore:

$$\mathrm{d}F = \frac{\mathcal{L}}{4\pi}\,\mathrm{d}\Omega \,. \tag{1.3}$$

Once again, this does not depend on R! When we integrate it over the whole solid angle, we obtain that $F_{\mathrm{tot}} = \mathcal{L}$, i.e. the whole sky is as luminous as a star! In other words: it is true that

Figure 1.1: Pine forest. If the distribution of pine tress are uniform and extends indefinitely, any line-of-sight will hit a pine tree.

the farther a star is the fainter it appears, but we can pack more of them in the same patch of sky.

In order to solve Olbers's paradox, we can drop one or more of the initial assumptions. For example:

- The universe is not eternal so the light of some stars has not yet arrived to us. This is plausible, but even so we could expect a bright night sky and also to see some new star to pop out from time to time, without being a transient phenomenon such as a supernova explosion. There is no record of this.

- Maybe there is not an infinite number of stars. But we have showed that taking into account their dimension we do not need an infinite number and yet the paradox still exists.

- Are stars distributed not uniformly? Even so, we would expect still a bright night sky, even if not uniformly bright.

At the end, we must do something in order for the light of some stars not to reach us. A possibility is to drop the staticity assumption. The farther a spherical shell is, the faster it recedes from us (this is Hubble's law). In this way, beyond a certain distance (the Hubble radius) light from stars cannot reach us and the paradox is solved.

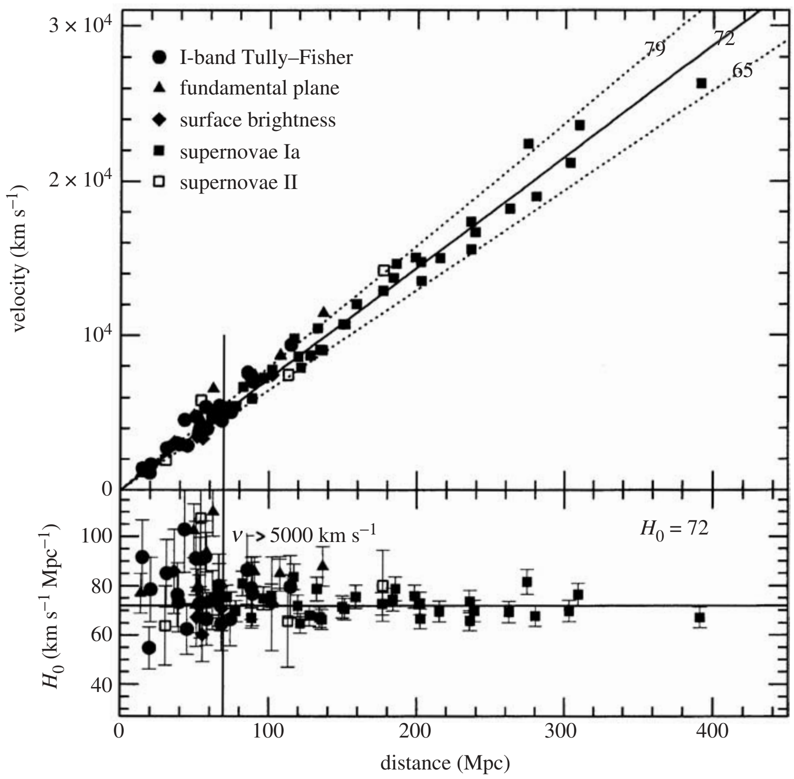

Figure 1.2: Hubble's relation showing a linear relationship between **v** and **r**

1.2 Hubble-Lemaître Law

The extraordinary evidence that we live in an expanding universe. This was a landmark discovery made in the 20th century, usually attributed to Edwin Hubble [1] and George Lemaître called **Hubble-Lemaître law**:

$$\mathbf{v} = H_0\,\mathbf{r}\ , \tag{1.4}$$

A recent measurement done by the Planck collaboration [2] gives

$$H_0 = 67.9^{+1.2}_{-1.3}\ \text{km s}^{-1}\ \text{Mpc}^{-1}\ . \tag{1.5}$$

This number means that for each Mpc away a source recedes 67.9 km/s faster. At a certain radius, the receding velocity attains the velocity of light and therefore we are unable to see further objects. This radius is called **Hubble radius**. The **Hubble time** is given by

$$t_{\text{H}} = \frac{1}{H_0}. \tag{1.6}$$

The Hubble time gives a good estimate of the age of the Universe.

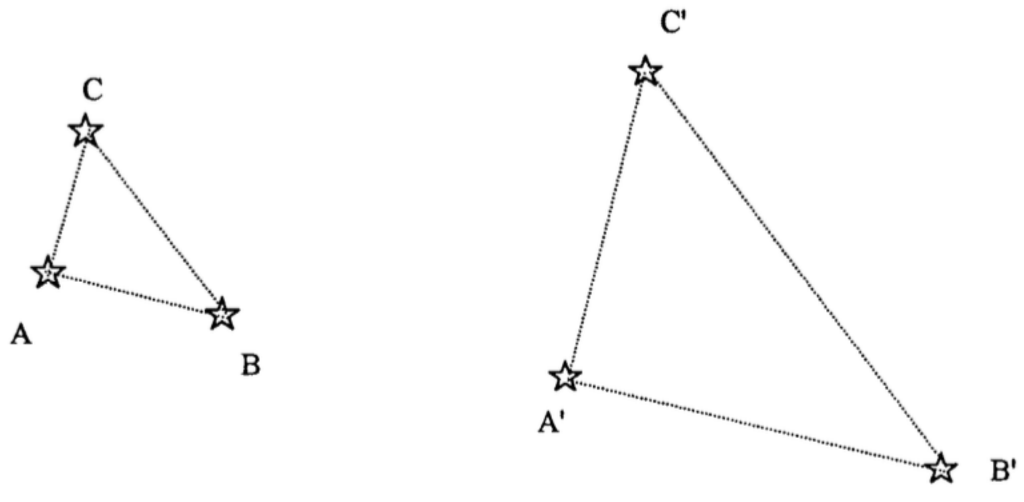

Figure 1.3: The relative position of galaxies during an expansion.

1.3 Homogeneity and Isotropy

The evidence that the Universe becomes smooth on large scales supports the use of the cosmological principle. It is therefore believed that our large-scale Universe possesses two important properties, **homogeneity** and **isotropy**. Homogeneity is the statement that the Universe looks the same at each point, while isotropy states that the Universe looks the same in all directions.

These do not automatically imply one another. For example, a universe with a uniform magnetic field is homogeneous, as all points are the same, but it fails to be isotropic because directions along the field lines can be distinguished from those perpendicular to them. Alternatively, a spherically-symmetric distribution, viewed from its central point, is isotropic but not necessarily homogeneous. However, if we require that a distribution is isotropic about *every point*, then that does enforce homogeneity as well.

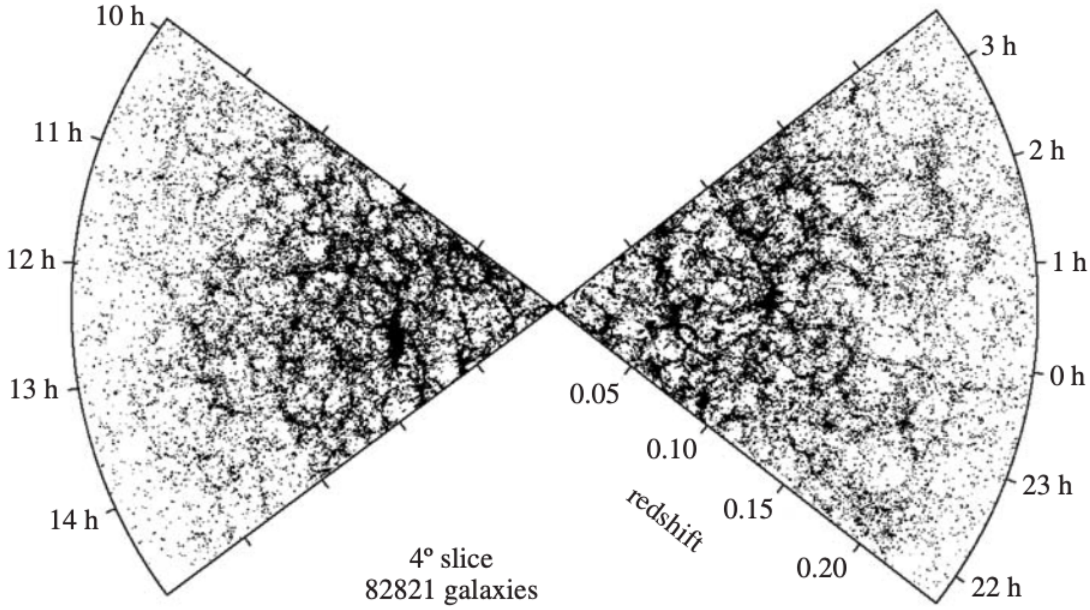

Figure 1.4: The distribution of galaxies in part of a redshift survey, drawn from a total of 213,703 galaxies.

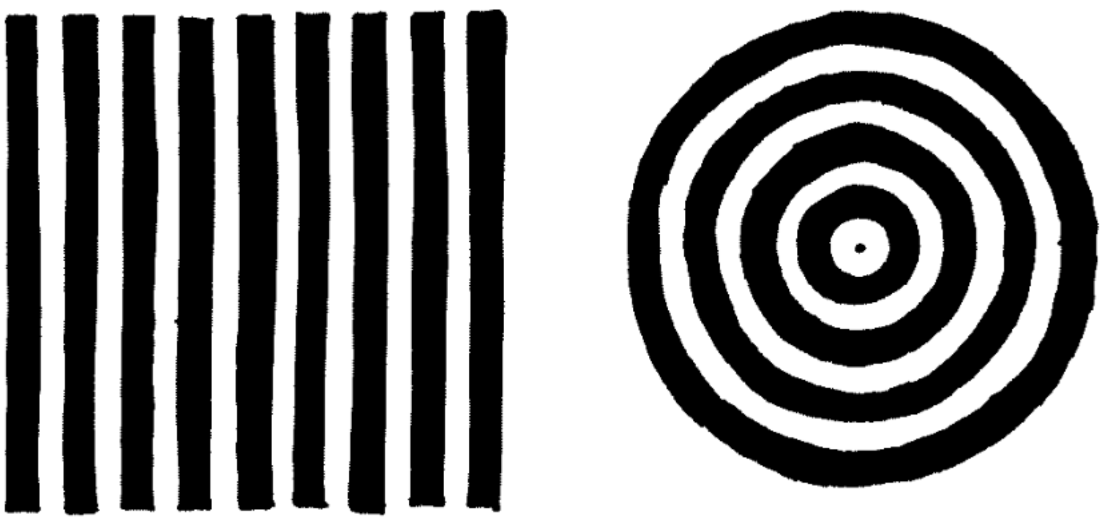

Figure 1.5: (a) A pattern which is anisotropic, but which is homogeneous on scales larger that the stripe width. (b) A pattern which is isotropic about the origin, but which is inhomogeneous.

1.4 Comoving Coordinates

We define a different coordinate system, known as **comoving coordinates**. These are coordinates which are carried along with the expansion. Because the expansion is uniform, the relationship between distance **r** and the comoving distance, which we can call **x** can be written as

$$\mathbf{r}(t) = a(t)\mathbf{x} \tag{1.7}$$

where the homogeneity property has been used to ensure that a is a function of time alone. Note that these distances have been written as vector distances. What you should think of when studying this equation is a coordinate grid which expands with time, as shown in Figure. 1.4. The galaxies remain at fixed locations in the **x** coordinates system. The original **r** coordinate system, which does not expand, is usually known as **physical coordinates**.

The quantity $a(t)$ is a crucial one, and is known as the **scale factor** of the Universe. It measures the universal expansion rate. It is a function of time alone, and it tells us how physical separations are growing with time, since the coordinate distance **x** are by definition fixed. For example, if, between times t_1 and t_2, the scale factor doubles in value, that tell us that the Universe has expanded in size by a factor of two, and it will take us twice as long to get from one galaxy to another.

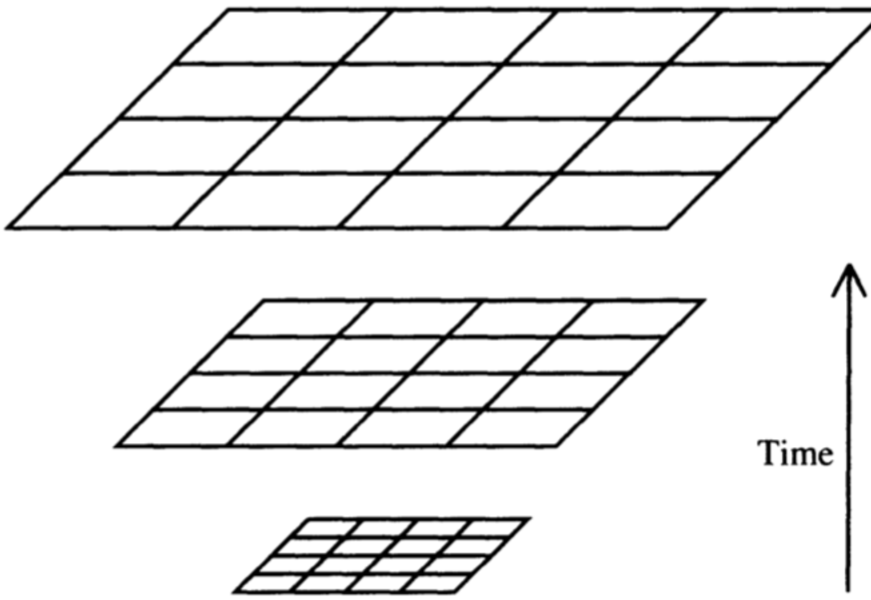

Figure 1.6: The comoving coordinate system is carried along with the expansion, so that any objects remain at fixed coordinates values.

1.5 Redshift

Redshift is a fundamental observable of cosmology. Its definition is the following:

$$z = \frac{\lambda_{\rm obs}}{\lambda_{\rm em}} - 1 = \frac{\Delta\lambda}{\lambda_{\rm em}} \tag{1.8}$$

It is always positive, i.e. observed radiation is redder than the emitted one, because the universe is in expansion. For the closest sources, such as Andromeda, it is negative, i.e. the observed radiation is bluer than the emitted one, because the Hubble flow is overcome by the peculiar motion due to local gravitational effects.

For the moment we can think of the redshift as a Doppler effect due to the relative motion of the sources; however, in reality it is call **cosmological redshift** which is due to the expansion of the universe. Redshift is measured in two ways: spectroscopically or photometrically. For the former one needs to do spectroscopy, i.e. detecting known emission or absorption lines from a source and comparing their wavelengths with the ones measured in a laboratory on Earth. Hence one uses Equation (1.8) and thus calculate z. Photometric redshifts are calculated by assuming certain spectral features for the sources and measuring their relative brightness in certain wavebands, using filters.

The relativistic formula of the redshift is given by

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}}. \tag{1.9}$$

For $v \ll c$, a good approximation is given by

$$z = \frac{v}{c}. \tag{1.10}$$

The energy of the photon is inversely proportional to the scale factor i.e. $E \propto 1/a$. Therefore, we can write:

$$\frac{E_{\rm obs}}{E_{\rm em}} = \frac{a_{\rm em}}{a_{\rm obs}} \ . \tag{1.11}$$

On the other hand the photon energy is $E = hf$, with f its frequency. Therefore:

$$\frac{a_{\rm em}}{a_{\rm obs}} = \frac{E_{\rm obs}}{E_{\rm em}} = \frac{f_{\rm obs}}{f_{\rm em}} = \frac{\lambda_{\rm em}}{\lambda_{\rm obs}} = \frac{1}{1+z} \ . \tag{1.12}$$

This is the relation between the redshift and the scale factor. We have connected observation with theory. Usually, $a_{\rm obs} = 1$ and the above relation is simply written as $1 + z = 1/a$ or $a = 1/(1 + z)$.

1.6 The Hubble Parameter

From the definition of the scale factor, one can show that

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{1.13}$$

It is clear from the equation that the Hubble parameter is not a constant. When the Hubble parameter H is evaluated at the present time t_0, it becomes a number: the Hubble constant H_0 which we already met in Section 1.2 in the Hubble's law Equation (1.4). Usually H_0 is conveniently written as

$$H_0 = 100\ h\ \mathrm{km\ s^{-1}\ Mpc^{-1}}\ . \tag{1.14}$$

The unit of measure of the Hubble constant is an inverse time:

$$H_0 = 3.24\ h \times 10^{-18}\ \mathrm{s^{-1}}\ , \tag{1.15}$$

whose inverse gives the order of magnitude of the age of the universe:

$$\frac{1}{H_0} = 3.09\ h^{-1} \times 10^{17}\ \mathrm{s} = 9.78\ h^{-1}\ \mathrm{Gyr}\ , \tag{1.16}$$

and multiplied by c gives the order of magnitude of the size of the visible universe, i.e. the Hubble radius.

$$\frac{c}{H_0} = 9.27\ h^{-1} \times 10^{25}\ \mathrm{m} = 3.00\ h^{-1}\ \mathrm{Gpc}\ . \tag{1.17}$$

But what does "present time" t_0 mean? Time flows, therefore t_0 cannot be a constant! That is true, but if we compare a time span of 100 years (the span of some human lives) to the age of the universe (about 14 billion years), we see that the ratio is about 10^{-8}. Since this is pretty small, we can consider t_0 to be a constant, also referred to as the age of the universe. We can calculate it as follows:

$$t_0 = c\int_0^{t_0} \mathrm{d}t = c\int_0^1 \frac{\mathrm{d}a}{\dot{a}} = c\int_0^1 \frac{\mathrm{d}a}{H(a)a} = c\int_0^\infty \frac{\mathrm{d}z}{H(z)(1+z)}\ . \tag{1.18}$$

1.7 Newtonian Cosmology

It is perfectly possible to discuss cosmology without having already learned general relativity. In fact, the most crucial equation, the Friedmann equation which describes the expansion of the Universe, turns out to be the same when derived from Newton's theory of gravity as it is when derived from the equations of general relativity. The Newtonian derivation is, however, some way from being completely rigorous, and general relativity is required to fully patch it up, a detail that need not concern us at this stage.

In Newtonian gravity all matter attracts, with the force exerted by an object of mass M on one of mass m given by the famous relationship

$$F = \frac{GMm}{r^2}, \tag{1.19}$$

where r is the distance between the objects and G is Newton's gravitational constant. That is, gravity obeys an inverse square law. Because a force on an object induces an acceleration which is also proportional to its mass, via $F = ma$, the acceleration an object feels under gravity is independent of its mass. The force exerted means there is a gravitational potential energy

$$V = -\frac{GMm}{r}, \tag{1.20}$$

with the force being in the direction which decreases the potential energy the fastest. Like the electric potential of two opposite charges, the gravitational potential is negative, favouring the two objects being close together. But with gravity there is no analogue of the repulsion of like charges. Gravity always attracts.

1.7.1 Friedmann Equation

Let's consider an observer in a uniform expanding medium, with mass density ρ, the mass density being the mass per unit volume. We begin by realizing that because the Universe looks the same from anywhere, we can consider any point to be its centre. Now consider a particle a distance r away with mass m. Due to Newton's theorem, this particle only feels a force from the material at smaller radii, shown as the shaded region. This material has total mass given by $M = 4\pi\rho r^3/3$, contributing force

$$F = \frac{GMm}{r^2} = \frac{4\pi G\rho r m}{3}, \tag{1.21}$$

and our particle has a gravitational potential energy

$$V = -\frac{GMm}{r} = -\frac{4\pi G\rho r^2 m}{3}. \tag{1.22}$$

The kinetic energy is

$$T = \frac{1}{2}m\dot{r}^2. \tag{1.23}$$

The equation describing how the separation r changes can now be derived from energy conservation for that particle, namely

$$E = T + V \tag{1.24}$$

where E is a a constant. Substituting gives

$$E = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m. \tag{1.25}$$

Converting from physical distance to comoving distance i.e. $r \to ax$ and $\dot{r} \to \dot{a}x$ gives

$$U = \frac{1}{2} m \dot{a}^2 x^2 - \frac{4\pi}{3} G \rho a^2 x^2 m. \tag{1.26}$$

Multiplying each side by $2/ma^2x^2$ and rearranging the terms then gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}, \tag{1.27}$$

where $Kc^2 = -2E/mx^2$ and k is called the curvature. This is the standard form of the **Friedmann equation**, and it will appear frequently.

1.7.2 The Fluid Equation

Fundamental though it is, the Friedmann equation is of no use without an equation to describe how the density ρ of material in the Universe is evolving with time. This involves the pressure P of the material, and is called the fluid equation. As we'll shortly see, the different types of material which might exist in our Universe have different pressures, and lead to different evolution of the density ρ.

We can derive the fluid equation by considering the first law of thermodynamics

$$\mathrm{d}E + P\,\mathrm{d}V = T\,\mathrm{d}S, \tag{1.28}$$

applied to an expanding volume V of unit comoving radius. This is exactly the same as applying thermodynamics to a gas in a cylinder. The volume has physical radius a, so the energy is given, using $E = mc^2$, by

$$E = \frac{4\pi}{3} a^3 \rho c^2. \tag{1.29}$$

The change of energy in a time $\mathrm{d}t$, using the product rule, is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 4\pi a^2 \rho c^2 \frac{\mathrm{d}a}{\mathrm{d}t} + \frac{4\pi}{3} a^3 \frac{\mathrm{d}\rho}{\mathrm{d}t} c^2, \tag{1.30}$$

while the rate of change in volume is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi a^2 \frac{\mathrm{d}a}{\mathrm{d}t}. \tag{1.31}$$

Assuming a reversible adiabatic expansion $\mathrm{d}S = 0$, putting these into Equation (1.28) and rearranging gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0, \tag{1.32}$$

where as always dots are shorthand for time derivatives. This is the **fluid equation** or **continuity equation**. As we see, there are two terms contributing to the change in the density. The first term in the brackets corresponds to the dilution in the density because the volume has increased, while the second corresponds to the loss of energy because the pressure of the material has done work as the Universe's volume increased. This energy has not disappeared entirely of course; energy is always conserved. The energy lost from the fluid via the work done has gone into gravitational potential energy.

1.7.3 The Acceleration Equation

The Friedmann and fluid equations can be used to derive a third equation, not independent of the first two of course, which describes the acceleration of the scale factor. By differentiating Equation (1.27) with respect to time we obtain

$$2\frac{\dot{a}}{a}\frac{a\ddot{a}-\dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{Kc^2\dot{a}}{a^3}. \tag{1.33}$$

Substituting in for $r\dot{h}o$ from Equation (1.32) and cancelling the factor $2\dot{a}/a$ in each term gives

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{P}{c^2}\right) + \frac{Kc^2}{a^2}. \tag{1.34}$$

and finally, using Equation (1.27) again, we arrive at an important equation known as the **acceleration equation** or **Raychaudhuri equation**

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right). \tag{1.35}$$

Notice that if the material has any pressure, this increases the gravitational force, and so further decelerates the expansion. The acceleration equation does not feature the constant K which appears in the Friedmann equation; it cancelled out in the derivation.

1.8 Friedmann-Lemaître-Robertson-Walker Metric

In cosmology, which is the metric which describes the universe and what is the matter content? It turns out that both questions are very difficult to answer and, indeed, there are no still clear answers.

The metric used to describe the universe on large scales is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric . This is based on the assumption of very high symmetry for the universe, called the **cosmological principle**, which is minimally stated as follows: the universe is isotropic and homogeneous, i.e. there is no preferred direction or preferred position.

The cosmological principle seems to be compatible with observations at very large scales. On a scale of about 100 h^{-1} Mpc the rms density fluctuations are at the level of $\sim$10% and on scales larger than 300 h^{-1} Mpc the distribution of both mass and luminous sources safely satisfies the cosmological principle of isotropy and homogeneity. Now, focusing on the 3-dimensional spatial case:

1. $\mathrm{d}s_3^2 = |\,\mathrm{d}\mathbf{x}|^2 \equiv \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$, i.e. the Euclidean space. The scalar curvature is zero, i.e. the space is flat. This metric is invariant under 3-translations and 3-rotations.

2. $\mathrm{d}s_3^2 = |\,\mathrm{d}\mathbf{x}|^2 + \mathrm{d}z^2$, with the constraint $z^2 + |\mathbf{x}|^2 = a^2$. This is a 3-sphere of radius a embedded in a 4-dimensional Euclidean space. It is invariant under the six 4-dimensional rotations.

3. $\mathrm{d}s_3^2 = |\,\mathrm{d}\mathbf{x}|^2 - \mathrm{d}z^2$, with the constraint $z^2 - |\mathbf{x}|^2 = a^2$. This is a 3-hypersphere, or a hyperboloid, in a 4-dimensional pseudo-Euclidean space. It is invariant under the six 4-dimensional pseudo-rotations (i.e. Lorentz transformations).

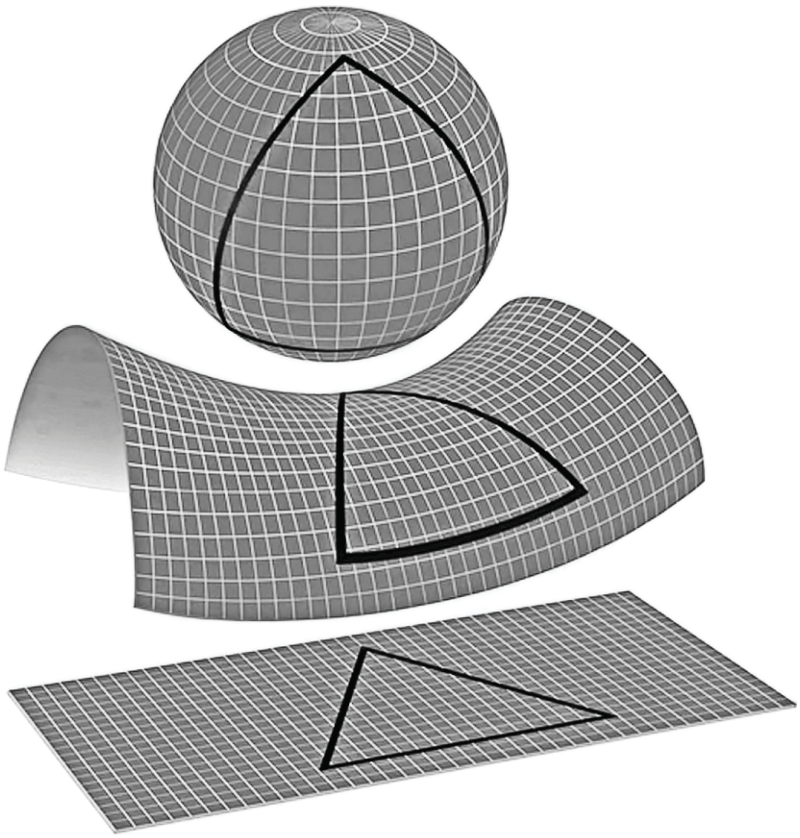

Figure 1.7: Universes with different geometry.

The general homogeneous and isotropic 3-dimensional space time line element is given by

$$\mathrm{d}s^2 = |\,\mathrm{d}\mathbf{x}|^2 + K\frac{(\mathbf{x}\cdot\,\mathrm{d}\mathbf{x})^2}{a^2 - K|\mathbf{x}|^2}\ , \tag{1.36}$$

with $K = 0$ for the Euclidean case, $K = 1$ for the spherical case and $K = -1$ for the hyperbolic case. In spherical polar coordinate,

$$\mathrm{d}s^2 = a^2\left[\frac{a^2\,\mathrm{d}r^2}{1 - Kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]. \tag{1.37}$$

With the time component and let a be a function of time we have the Friedmann-Lemaître-Robertson-Walker line element

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega^2\right), \tag{1.38}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2. \tag{1.39}$$

The time coordinate used here is called **cosmic time**, whereas the spatial coordinates are called **comoving coordinates**. For each t the spatial slices are maximally symmetric; $a(t)$ is called **scale factor**, since it tells us how the distance between two points scales with time.

1.9 Relativistic Cosmology

Given FLRW metric, Friedmann equations can be straightforwardly computed from the Einstein equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}\ , \tag{1.40}$$

where Λ is the cosmological constant. The the FLRW metric, the components of the Ricci tensor are

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}\ , \qquad R_{0i} = 0\ , \qquad R_{ij} = \frac{1}{c^2}g_{ij}\left(2H^2 + \frac{\ddot{a}}{a} + 2\frac{Kc^2}{a^2}\right)\ , \tag{1.41}$$

and the scalar curvature is:

$$R = \frac{6}{c^2}\left(\frac{\ddot{a}}{a} + H^2 + \frac{Kc^2}{a^2}\right)\ . \tag{1.42}$$

Finally, compute the Einstein equations:

$$H^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3c^2}T_{00} + \frac{\Lambda c^2}{3} \tag{1.43}$$

$$g_{ij}\left(H^2 + 2\frac{\ddot{a}}{a} + \frac{Kc^2}{a^2} - \Lambda c^2\right) = -\frac{8\pi G}{c^2}T_{ij} \tag{1.44}$$

Which stress-energy tensor $T_{\mu\nu}$ do we use in Equation (1.43) and Equation (1.44)? Having fixed the metric to be the FLRW one, we have some strong constraints:

- First of all: $G_{0i} = 0$ implies that $T_{0i} = 0$, i.e. there cannot be a flux of energy in any direction because it would violate isotropy;
- Second, since $G_{ij} \propto g_{ij}$, then $T_{ij} \propto g_{ij}$.
- Finally, since $G_{\mu\nu}$ depends only on t, then it must be so also for $T_{\mu\nu}$.

Therefore, let us stipulate that

$$T_{00} = \rho(t)c^2 = \varepsilon(t)\ , \qquad T_{0i} = 0\ , \qquad T_{ij} = g_{ij}P(t)\ , \tag{1.45}$$

where $\rho(t)$ is the rest mass density, $\varepsilon(t)$ is the energy density and $P(t)$ is the pressure. In tensorial notation we can write the following general form for the stress-energy tensor:

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu} \tag{1.46}$$

where u_μ is the four-velocity of the fluid element. In this form of Equation (1.46), the stress-energy tensor does not contain either viscosity or energy transport terms. Matter described by Equation (1.46) is known as **perfect fluid**.

Combine Equation (1.43), Equation (1.44) and Equation (1.45). The Friedmann equation becomes:

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{Kc^2}{a^2}. \tag{1.47}$$

while the acceleration equation is the following:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{1.48}$$

In the Friedmann and acceleration equations, ρ and P are the total density and pressure. Hence, they can be written as sums of the contributions of the individual components:

$$\rho \equiv \sum_x \rho_x\ , \qquad P \equiv \sum_x P_x\ . \tag{1.49}$$

The contribution from the cosmological constant can be considered either geometrically or as a matter component with the following density and pressure:

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G}\ , \qquad P_\Lambda \equiv -\rho_\Lambda c^2\ . \tag{1.50}$$

The scale factor a is, by definition, positive, but its derivative can be negative. This would represent a contracting universe. Note that the left hand side of the Friedmann equation Equation (1.47) is non-negative. Therefore, $\dot{a}$ can vanish only if $K > 0$, i.e. for a spatially closed universe. This implies that, if $K \leqslant 0$ and if there exists an instant for which $\dot{a} > 0$, then the universe will expand forever.

From, the energy conservation equation

$$\nabla_\nu T^{\mu\nu} = 0, \tag{1.51}$$

is encapsulated in GR through the Bianchi identities. Therefore, it is not independent from the Friedmann equations Equation (1.47) and Equation (1.48). For the FLRW metric and a perfect fluid, it has a particularly simple form:

$$\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0. \tag{1.52}$$

This is the $\mu = 0$ component of $\nabla_\nu T^{\mu\nu} = 0$ and it is also known from fluid dynamics as **continuity equation**. The continuity equation can be analytically solved if we assume an equation of state of the form $P = w\rho c^2$, with w constant. The general solution is:

$$\rho = \rho_0 a^{-3(1+w)} \qquad (w = \text{ constant}) \ , \tag{1.53}$$

where $\rho_0 \equiv \rho(a_0 = 1)$. There are three particular values of w which play a major role in cosmology:

Cold matter: $w = 0$, i.e. $P = 0$, for which $\rho = \rho_0 a^{-3}$. The adjective cold refers to the fact that particles making up this kind of matter have a kinetic energy much smaller than the mass energy, i.e. they are non-relativistic. If they are thermally produced, i.e. if they were in thermal equilibrium with the primordial plasma, they have a mass much larger than the temperature of the thermal bath.

Hot matter: $w = 1/3$, i.e. $P = \rho/3$, for which $\rho = \rho_0 a^{-4}$. The adjective hot refers to the fact that particles making up this kind of matter are relativistic.

Vacuum energy: $w = -1$, i.e. $P = -\rho$ and ρ is a constant. It behaves as the cosmological constant and provides the best (and the simplest) description that we have for dark energy.

1.10 The Conformal Time

A very useful form of rewriting FLRW metric Equation (1.38) is via the **conformal time** η:

$$a\mathrm{d}\eta = \mathrm{d}t \qquad \Rightarrow \qquad \eta - \eta_i = \int_{t_i}^{t} \frac{\mathrm{d}t'}{a(t')} \ . \tag{1.54}$$

As we shall see later, but as we already can guess from the above integration, $c(\eta - \eta_i)$ represents the comoving distance travelled by a photon between the times η_i and η, or t_i and t. The conformal time allows to rewrite FLRW metric Equation (1.38) as follows:

$$\mathrm{d}s^2 = a(\eta)^2 \left(-c^2 \mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \mathrm{d}\Omega^2 \right) . \tag{1.55}$$

i.e. the scale factor has become a conformal factor (hence the name for η). Recalling the earlier discussion about dimensionality, if a has dimensions then $c\eta$ is dimensionless. On the other hand, if a is dimensionless, then η is indeed a time. Note also that metric Equation (1.55) for $K = 0$ is Minkowski metric multiplied by a conformal factor.

1.11 Critical Density and Density Parameters

Let us now rewrite Equation (1.47) incorporating Λ in the total density ρ:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a^2} \, . \tag{1.56}$$

The value of the total ρ such that $K = 0$ is called **critical energy density** and has the following form:

$$\rho_{\rm cr} \equiv \frac{3H^2}{8\pi G}. \tag{1.57}$$

Its present value is:

$$\rho_{\rm cr,0} = 1.878 \; h^2 \times 10^{-29} \text{ g cm}^{-3} \tag{1.58}$$

It turns out that ρ_0 is very close to $\rho_{\rm cr,0}$, so that our universe is spatially flat. Such an extreme fine-tuning in K is a really surprising coincidence, known as the **flatness problem**.

Instead of densities, it is very common and useful to employ the density parameter Ω, which is defined as

$$\Omega \equiv \frac{\rho}{\rho_{\rm cr}} = \frac{8\pi G\rho}{3H^2}. \tag{1.59}$$

i.e. the energy density normalised to the critical one. We can then rewrite Friedmann equation Equation (1.47) as follows:

$$1 = \Omega - \frac{Kc^2}{H^2a^2} \, . \tag{1.60}$$

Defining

$$\Omega_K \equiv -\frac{Kc^2}{H^2a^2} \, , \tag{1.61}$$

i.e. associating the energy density

$$\rho_K \equiv -\frac{3Kc^2}{8\pi G a^2} \, , \tag{1.62}$$

to the spatial curvature, hence,

$$1 = \Omega + \Omega_K \, . \tag{1.63}$$

Therefore, the sum of all the density parameters, *the curvature one included*, is always equal to unity. In particular, if it turns out that $\Omega \simeq 1$, this implies that $\Omega_K \simeq 0$, i.e. the universe is spatially flat. From the latest Planck data, we know that:

$$\Omega_{K0} = 0.0008^{+0.0040}_{-0.0039}. \tag{1.64}$$

at the 95% confidence level. It is more widespread in the literature the normalisation of ρ to the **present-time** critical density, i.e.

$$\Omega \equiv \frac{\rho}{\rho_{\rm cr,0}} = \frac{8\pi G\rho}{3H_0^2}, \tag{1.65}$$

because it leaves more evident the dependence on a of each material component.

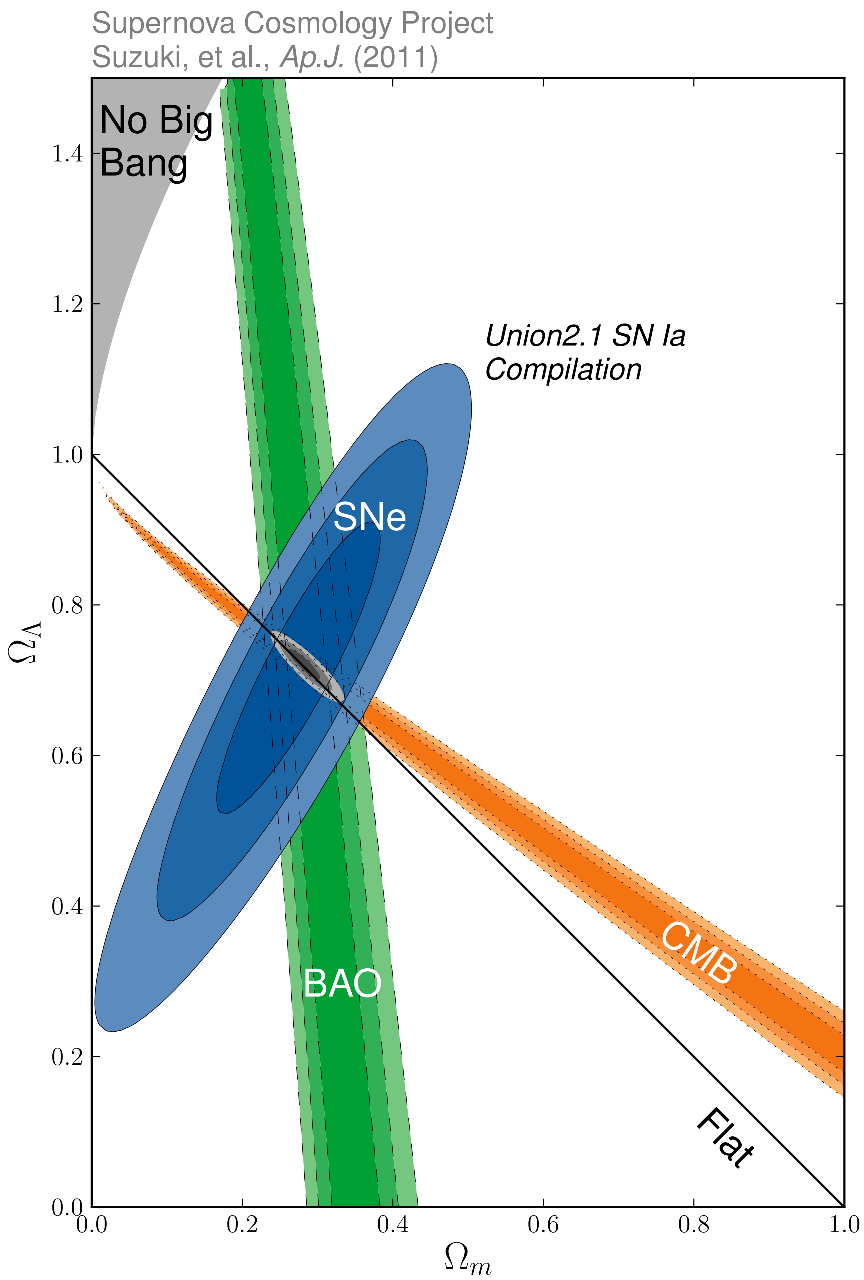

Figure 1.8: Constraint on Ω_Λ and $\Omega_{\rm M}$ from various observations.

1.12 The LCDM model

The most successful cosmological model is called ΛCDM and is made up of Λ, CDM, baryons and radiation (photons and massless neutrinos). The Friedmann equation for the ΛCDM model is the following:

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \frac{\Omega_{\rm c0}}{a^3} + \frac{\Omega_{\rm b0}}{a^3} + \frac{\Omega_{\rm r0}}{a^4} + \frac{\Omega_{K0}}{a^2}\ . \tag{1.66}$$

We already saw in Equation (1.64) the value of the spatial curvature contribution. From Planck data here are the other ones:

$$\Omega_\Lambda = 0.6911 \pm 0.0062 \, , \qquad \Omega_{\rm m0} = 0.3089 \pm 0.0062 \tag{1.67}$$

at 68% confidence level, where $\Omega_{\rm m0} = \Omega_{\rm c0} + \Omega_{\rm b0}$, i.e. it includes the contributions from both CDM and baryons, since they have the same dynamics (i.e. they are both cold). It is however possible to disentangle them and one observes:

$$\Omega_{\rm b0} h^2 = 0.02230 \pm 0.00014 \, , \qquad \Omega_{\rm c0} h^2 = 0.1188 \pm 0.0010 \tag{1.68}$$

also at 68% confidence level. The radiation content, i.e. photons plus neutrinos, can be easily calculated from the temperature of the CMB. It turns out that:

$$\Omega_{\gamma 0} h^2 \approx 2.47 \times 10^{-5} \, , \qquad \Omega_{\nu 0} h^2 \approx 1.68 \times 10^{-5} \tag{1.69}$$

Let us now calculate the age of the universe for the ΛCDM model. Using Equation (1.18), we get:

$$t_0 = \frac{c}{H_0} \int_0^1 {\rm d}a \frac{a}{\sqrt{\Omega_\Lambda a^4 + \Omega_{\rm m0} a + \Omega_{\rm r0} + \Omega_{K0} a^2}} \, . \tag{1.70}$$

Using the numbers shown insofar, we get upon numerical integration:

$$t_0 = \frac{0.95}{H_0} = 13.73 \text{ Gyr.} \tag{1.71}$$

Chapter 2

Cosmology

2.1 Distances in Cosmology

In this section, we will learn how to measure distances in the universe. We will use the unit where the speed of light c and reduced Planck's constant $\hbar$ are unity. According to homogeneity and isotropy of 3-dimensional space of the universe, the spacetime of the universe is describe by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right], \tag{2.1}$$

where $a(t)$ is a cosmic scale factor and r, θ and ϕ are comoving coordinates. The physical length l is related to the comoving length l_c via

$$l = al_c\,. \tag{2.2}$$

In GR, the speed of light in vacuum is constant so that light from a source at long distances was emitted long time ago before reaching us today. Since the universe is expanding, a scale factor at which the light was emitted from a long-distance source is smaller than the one at present. Hence, the physical wavelength of light traveling from a source to an observer is stretched due to the expansion of the universe as

$$\frac{\lambda_0}{\lambda_e} = \frac{a_0\lambda_c}{a_e\lambda_c}\frac{a_0}{a_e} \equiv 1 + z_e\,, \tag{2.3}$$

where subscripts $_e$ and $_0$ denote evaluation at emission time and at present. In the above expression λ_c is a comoving wavelength of light. Distance to a light source cannot be quantified using only redshift, because relation between redshift and time strongly depends on cosmological models.

2.1.1 Luminosity Distance

In addition to redshift, information about the distance of the light source is encoded in observed luminosity. We will learn that we can also use characteristic length scales in the universe to quantify distances in the universe. Distances can be implied from luminosity and observed flux of the light sources as follows: The flux measured by observer at distance r from the light

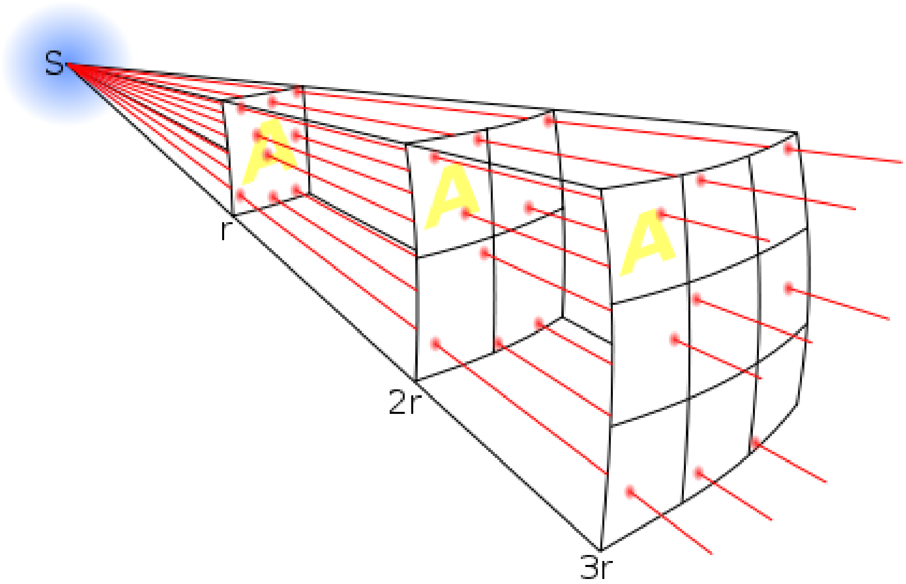

Figure 2.1: The flux is the energy per unit time interval per unit area.

source with luminosity L is

$$F = \frac{L}{4\pi r^2}\,, \tag{2.4}$$

In the expanding universe the energy of a single photon is redshifted by

$$\frac{E_{\gamma\, e}}{E_{\gamma\, 0}} = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e} = 1 + z_e\,, \tag{2.5}$$

where we have used $e_\gamma \propto 1/\lambda$ and subscript $_\gamma$ denotes photon. Physical distances between to adjacent photons at the emitting and detecting time are related by

$$\frac{\delta r_e}{\delta r_0} = \frac{a_e}{a_0} = \frac{1}{1+z_e}\,, \quad \Rightarrow \quad \frac{\delta t_0}{\delta t_e} = 1 + z_e\,, \tag{2.6}$$

where we have used $\delta r_e/\delta t_e = \delta r_0/\delta t_0 =$ speed of light. From the definition of luminosity:

$$L = \frac{E}{\delta t}\,, \quad \Rightarrow \quad L_0 = \frac{L_e}{(1+z_e)^2}\,. \tag{2.7}$$

Using the definition of flux:

$$F_0 = \frac{L_0}{4\pi r^2} = \frac{L_e}{4\pi r^2(1+z_e)^2}\,. \tag{2.8}$$

Luminosity distance is defined as

$$d_L = (1+z)r\,. \tag{2.9}$$

In the spatially flat universe, comoving distance propagated by photon is computed from

$$\mathrm{d}s^2 = 0 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}r^2, \tag{2.10}$$

where we have assumed that light emitted from a source has spherical symmetry. Hence, a comoving distance from a light source to observer today can be computed as

$$r = \int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_{t_e}^{t_0} \frac{\mathrm{d}a}{Ha^2} = \int_0^{z_e} \frac{\mathrm{d}z'}{H(z')}\,, \tag{2.11}$$

where the Hubble parameter is

$$H \equiv \frac{\dot{a}}{a}\,, \tag{2.12}$$

From Eq. (2.11), the luminosity distance is given by

$$d_L = (1+z)\int_0^z \frac{\mathrm{d}z'}{H(z')}\,, \quad \Rightarrow \quad d_L = (1+z)H_0^{-1}\int_0^z \frac{\mathrm{d}z'}{E(z')}\,,$$

where we have dropped subscript $_e$ and

$$E(z) \equiv \frac{H(z)}{H_0}\,. \tag{2.13}$$

The above equation gives a relation between luminosity and redshift.

We can use standard candle such as type I supernova to measure luminosity-redshift relation:

$$d_L^O = \sqrt{\frac{L}{4\pi F}}\,. \tag{2.14}$$

Comparing d_L from prediction of the model with d_L^O from SN data, we can constrain cosmological parameters. The data of observed d_L strongly indicate that the universe is expanding with acceleration.

2.1.2 Angular Diameter Distance

If we know the distance between two objects, we can determine a distance from us to those objects by measuring distance between the objects we seen in terms of as angular size as shown in the figure. The comoving angular diameter distance is defined as

$$d_A \equiv \frac{l}{\theta}\,, \tag{2.15}$$

where l is comoving length scales which can be compute model-independently, and θ can be precisely measured. The ratio l/θ can be computed as

$$\frac{l}{\theta} = d_A = r = \int_t^{t_0} \frac{\mathrm{d}t'}{a(t')} = \eta_0 - \eta\,, \tag{2.16}$$

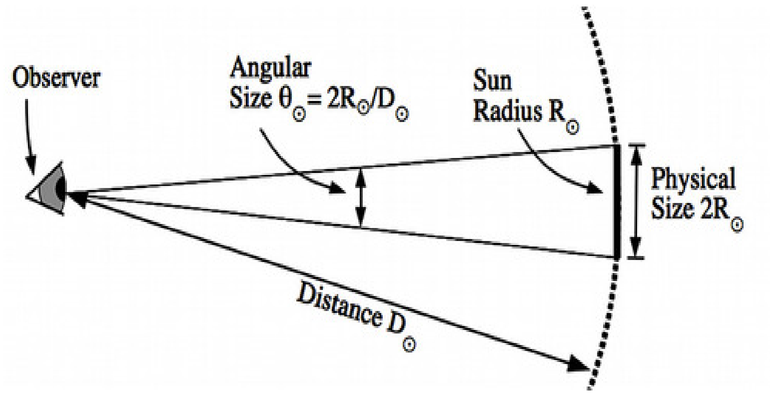

Figure 2.2: An angle θ is an angular size of a object.

where t is a time at which light is emitted from a source, and η is a conformal time. The above equation is related to the cosmological models through

$$\int_t^{t_0} \frac{\mathrm{d}t'}{a(t')} = \frac{1}{H_0} \int_0^z \frac{\mathrm{d}z'}{E(z')} \,. \tag{2.17}$$

The length scales associated with the pattern of the CMB and matter power spectrum are the important length scales that can be computed theoretically for given models of the universe and their angular size can be observed.

2.2 Conservation of Energy Density

2.2.1 Perfect Fluid

Matter and energy in the universe can be modeled by a fluid which a continuous substance. To satisfy homogeneity and isotropy of the background universe, the cosmic fluid cannot flow in the comoving frame of the universe. Since both energy and momentum do not flow, the properties of the fluid are characterized by energy density and pressure.

This type of fluid is a perfect fluid. From theory of relativity, the energy of a particle is given by

$$E = \sqrt{p^2 + m^2}\,, \tag{2.18}$$

where p^2 is a momentum square, m is a rest mass of a particle. For a non-relativistic particle, we have $m^2 \gg p^2$, and therefore

$$E \simeq m\,. \tag{2.19}$$

For a relativistic particle, we have $m^2 \ll p^2$, and therefore

$$E \simeq |p|\,. \quad \text{and for a massless particle} \quad E \propto \frac{1}{\lambda}\,, \tag{2.20}$$

where λ is a wavelength of a particle.

For a group of non-relativistic particles, the energy density can be computed as

$$\rho = \frac{NE}{V} \simeq \frac{Nm}{V} \propto \frac{1}{V} \propto \frac{1}{a^3 l_c^3} \propto a^{-3}\,, \tag{2.21}$$

where N is a total number of non-relativistic particle in a volume V, and l_c^3 corresponds to a comoving volume. For a group of relativistic particles, the energy density can be computed as

$$\rho = \frac{NE}{V} \propto \frac{N}{V\lambda} \propto \frac{1}{a^4 l_c^3 \lambda_c} \propto a^{-4}\,, \tag{2.22}$$

where N is total number of relativistic particle in a volume V. A group of non-relativistic particles in the universe is described by a perfect fluid known as matter or dust, while a group of relativistic particles is described by a perfect fluid known as radiation.

In the following consideration, quantities associated with matter are denoted by subscript $_m$ while those for radiation are denoted by subscript $_r$. We now have

$$\rho_m \propto a^{-3}\,, \quad \text{and} \quad \rho_r \propto a^{-3}\,, \tag{2.23}$$

2.2.2 Conservation of the Energy Density

In general, the relations in Eq. (2.23) can be derived from the conservation of the energy-momentum tensor:

$$\nabla_\mu T^\mu_\nu = 0\,, \tag{2.24}$$

where ∇_μ is a covariant derivative and T^μ_ν is the energy-momentum tensor. For the perfect fluid in the homogeneous and isotropic universe, we have

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 \\ 0 & \delta^j_i P \end{pmatrix}\,, \tag{2.25}$$

where ρ and P are the energy density and pressure of the fluid. Inserting Eq. (2.25) into Eq. (2.24), we get

$$\dot{\rho} = -3H(\rho + P)\,, \tag{2.26}$$

where a dot denotes derivative with respect to time, and $H \equiv \dot{a}/a$ is the Hubble parameter. One of the properties of a fluid is the relation between the energy density and pressure. This relation is the equation of state. For a simple perfect fluid, the equation of state can be written in the form

$$P = w\rho\,, \tag{2.27}$$

where w is an equation of state parameter.

From $\rho_m \propto a^{-3}$, we get $\dot{\rho}_m = -3H\rho_m$, so that

$$w_m = 0\,. \tag{2.28}$$

Inserting $P = w\rho$ into the Eq. (2.26), we get

$$\dot{\rho} = -3H(1+w)\rho\,. \tag{2.29}$$

For a constant w, Eq. (2.29) can be integrated as

$$\rho \propto a^{-3(1+w)}\,. \tag{2.30}$$

You can check that $w_m = 0$ yields $\rho_m \propto a^{-3}$ and $w_r = 1/3$ yields $\rho_r \propto a^{-4}$.

2.3 Cosmological Constant

Let us consider the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{2.31}$$

where $G_{\mu\nu}$ is the Einstein tensor and G is the Newton's gravitational constant. The above equation is the tensor equation. Hence, if we would like to add a constant term into this equation, this term has to be a tensor which the simplest form is

$$g_{\mu\nu}\Lambda\,, \tag{2.32}$$

where $g_{\mu\nu}$ is a metric tensor and Λ is a constant. Inserting the above term into Eq. (2.31), we get

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \underbrace{\frac{\Lambda g_{\mu\nu}}{8\pi G}}_{\equiv T_{\mu\nu\Lambda}}\right)\,, \tag{2.33}$$

where Λ in this equation is a cosmological constant, and $T_{\mu\nu\Lambda}$ is an energy momentum tensor of a cosmological constant. Comparing the energy-momentum tensor of Λ which is in the form

$$T_{\mu\nu\Lambda} = -\frac{\Lambda g_{\mu\nu}}{8\pi G}\,, \tag{2.34}$$

with that for a perfect fluid in the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & P g_{ij} \end{pmatrix}\,, \tag{2.35}$$

we get

$$P_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G}\,. \tag{2.36}$$

Inserting w_Λ into Eq. (2.29), we get

$$\dot{\rho}_\Lambda = 0\,, \quad \Rightarrow \quad \rho_\Lambda = \text{constant}\,. \tag{2.37}$$

This suggests that the energy associated with the cosmological constant is proportional to 3D spatial volume of spacetime, i.e.,

$$\rho_\Lambda = \text{constant}\,, \quad \Rightarrow \quad E_\Lambda \propto V\,. \tag{2.38}$$

2.4 Basic Cosmological Models

Using the FLRW metric for spatially flat spacetime and the perfect fluid model for the energy and matter in the universe, we obtain the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho_T\,, \tag{2.39}$$

where ρ_T is the total energy density in the universe.

2.4.1 De-Sitter Model

Let us first consider the universe containing only a cosmological constant. For this case, the Friedmann equation becomes

$$H^2 = \frac{8\pi G}{3}\rho_\Lambda = \text{constant}\,, \tag{2.40}$$

which suggests that the Hubble parameter is constant. If there is a cosmological constant only in the universe, one can show that

$$a(t) \propto \mathrm{e}^{Ht}\,. \tag{2.41}$$

This model of the universe is the de Sitter model.

2.4.2 Einstein-De Sitter Model

Let us next consider the universe containing only matter. If the universe contains only matter, the Friedmann equations becomes

$$H^2 = \frac{8\pi G}{3}\rho_m \propto a^{-3}\,. \tag{2.42}$$

The solution for the above equation is

$$a \propto t^{2/3}\,, \quad \Rightarrow \quad H = \frac{2}{3t}\,. \tag{2.43}$$

This model of the universe is the Einstein – de Sitter model.

2.5 Standard Model of the Universe

In the standard model of the universe, the energy and matter relevant to important phenomena in the universe are the follows:

× Photon ⇒ radiation

× Massless Neutrino ⇒ radiation

× Baryon ⇒ matter

× Cold Dark Matter (CDM) ⇒ matter

× Cosmological Constant ⇒ mysterious form of energy driving an accelerated expansion of the late-time universe

This standard model of the universe is ΛCDM model, where Λ represents that an accelerated expansion of the univer at late-time is driven by a cosmological constant, while dark matter in this model is a cold dark matter. The cosmological parameter which quantifies the energy fraction of each energy components in the universe is the density parameter defined as

$$\Omega_\alpha = \frac{8\pi G\rho_\alpha}{3H^2}\,, \quad \text{and} \quad \Omega_k = -\frac{k}{a^2H^2}\,, \tag{2.44}$$

where index α runs over γ, ν, b, c and Λ for photon, neutrino, baryon, CDM and cosmological constant, respectively. Here, Ω_k quantifies the contribution from the spatial curvature of the universe. Using the definition of the density parameter, we can write the Friedmann equation with spatial curvature,

$$H^2 = \frac{8\pi G}{3}\rho_T - \frac{k}{a^2}\,, \tag{2.45}$$

as

$$1 = \Omega_\gamma + \Omega_\nu + \Omega_b + \Omega_c + \Omega_\Lambda + \Omega_k\,. \tag{2.46}$$

According to observation, the present values of the density parameter of each species are

$$\begin{aligned} \Omega_\gamma^0 &\sim 10^{-4}\,, \quad \Omega_\nu^0 \sim 10^{-4}\,, \quad |\Omega_k^0| < 0.005\,, \\ \Omega_b^0 &\simeq 0.022\,, \quad \Omega_c^0 \simeq 0.3\,, \quad \Omega_\Lambda \simeq 0.68\,. \end{aligned} \tag{2.47}$$

Hence, usually we suppose that the universe is spatially flat.

2.5.1 Cosmic acceleration and dark Energy

From observation, supernova at particular redshift are dimmer than that predicted from the Einstein-de Sitter model in which the universe contains matter only. This implies that the observed distances to supernova at a given redshift are larger than those predicted from the Einstein-de Sitter model. Therefore, the expansion rate of the universe at late-time is larger than that in Einstein-de Sitter model. Fitting the cosmological models with the observed relation between luminosity and redshift show that the universe is expanding with acceleration at late-time.

We now discuss how an accelerated expansion of the universe at late-time can be achieved. Differentiating the Friedmann equation in Eq. (2.39) with respect to time, and using the conservation equation in Eq. (2.29), we get

$$\dot{H} = -\frac{3}{2}H^2\left(1 + w_T\right) \,, \tag{2.48}$$

where the equation of state parameter of the total fluid is $w_T = P_T/\rho_T$. Hence, we can compute the acceleration equation as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{2}H^2\left(1 + 3w_T\right) \,. \tag{2.49}$$

This suggests that the expansion of the universe will be accelerate if

$$w_T < -\frac{1}{3} \,. \tag{2.50}$$

From $w_T = P_T/\rho_T$ and the fact that the energy density of radiation can be ignored at late-time, we get

$$w_T = \frac{P_T}{\rho_T} \simeq \frac{w_m\rho_m + w_d\rho_d}{\rho_m + \rho_d} = w_d\Omega_d \,, \tag{2.51}$$

where w_d is the equation of state parameter and ρ_d is the energy density of unknown form of energy called dark energy. This energy component is introduced because matter and radiation have positive equation of state parameter and therefore Eq. (2.50) cannot be achieved. Inserting Eq. (2.51) into Eq. (2.50), we obtain

$$w_d < -\frac{1}{3\Omega_d} \,. \tag{2.52}$$

From observational constraint, we have

$$w_d \sim -1 \,. \tag{2.53}$$

The above constraint is satisfied well by a cosmological constant. Hence, a cosmological constant is the simplest model for dark energy. The model of the universe in which an accelerated expansion of the late-time universe is driven by a cosmological constant and dark matter is the cold dark matter is ΛCDM model. The ΛCDM is the standard model of the universe.

2.5.2 Galaxy rotation curve and dark matter

Galaxy rotation curve is a plot of orbital speeds of visible stars or gas those orbit around the galaxy as a function of radial distance from the center of galaxy. Let us compute the orbital speed of a point mass that rotates around a center of galaxy using Newtonian gravity. We suppose that a point mass has mass m while the mass M of galaxy is at its center. Matching the gravity force with the centrifugal force of a point mass, we get

$$\frac{GMm}{r^2} = \frac{mv^2}{r}\,, \quad v \propto \frac{1}{\sqrt{r}}\,, \tag{2.54}$$

where v is an orbital speed and r is a distance from the center of galaxy. However, the observed rotation curve is approximately "'flat', i.e., the orbital speed is approximately constant over a large range of distances. This seems to indicate that there is a non-visible matter distrivution in the galaxy. This non-visible matter is dark matter, which could be cold dark matter or hot dark matter, where cold dark matter is non-relativistic matter while hot dark matter is relativistic one. The word "non-visible" means that the dark matter does not emit electromagnetic waves.

2.6 Epochs of the Universe

In the standard model of the universe, the universe undergoes the following eras after the big bang.

Inflation	$t \sim 10^{-14} - 10^{-43}\,s;\ T \sim 10\,\mathrm{TeV} - 10^{19}$ Gev
Baryon & Lepton synthesis	$t < 10^{-14}\,s;\ T \gtrsim 10^{16}$ GeV
Neutrino decoupling & n_n/n_p freezes out	$t \sim 0.2\,s;\ T \sim 1 - 2$ MeV
electron-positron pairs annihilate	$t \sim 1\,s;\ T \sim 0.5$ MeV
Nucleosynthesis	$t \sim 200 - 300\,s;\ T \sim 0.05$ MeV
Recombination & Last scattering	$t \sim 10^{12} - 10^{13}\,s$
Structure formation	$t \sim 10^{16}\,s$
Acceleration	late time

The epochs of the universe can be classified into 3 types. From observations, the density parameter of dark energy is largest compared with others at present, so that the universe is in the acceleration epoch at a late-time. Since the energy density of matter decays faster than that of dark energy, and the energy density of radiation decays fastest, the energy density of the universe will be dominated by matter and radiation respectively when we look back in time.

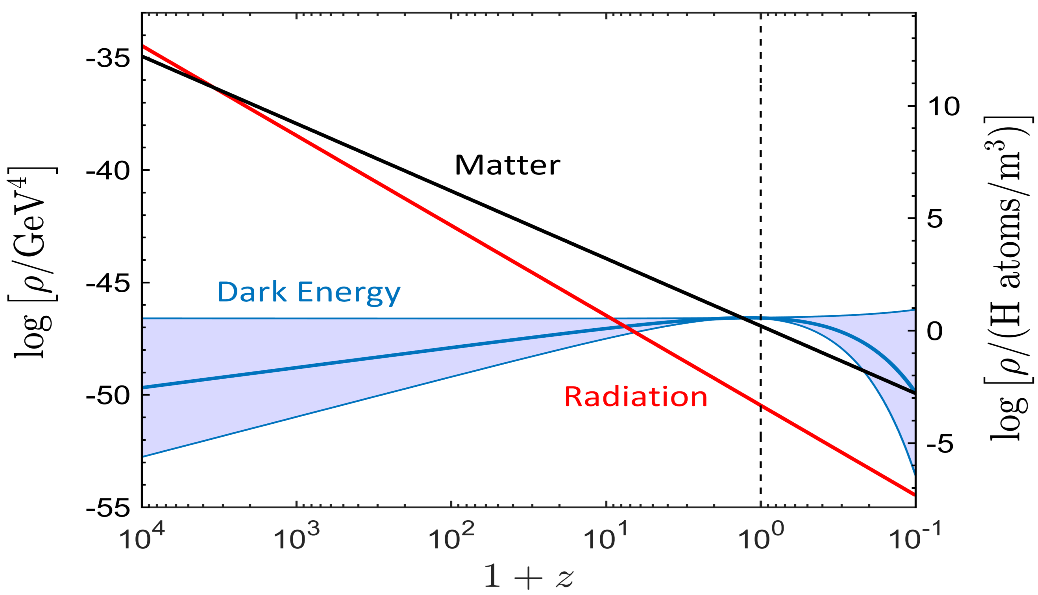

Figure 2.3: Plots of the energy density of radiation, matter and dark energy as function of redshift z.

1

2.7 Cosmic Microwave Background

Before recombination, photons are coupled to electrons through the Compton scattering and the effects of scattering are transferred to baryons through the Coulomb interaction between (charged) baryons and electrons. Since the number density of charged particles is high, photons and baryons are tightly coupled and can be treated as a single fluid known as a photon–baryon fluid.

Photons and baryons do not uniformly distribute over 3D space because there are small deviations from homogeneity and isotropy in the universe. Small inhomogeneities and anisotropies are created in the early universe during inflation. In the standard notion, such inhomogeneities and anisotropies are seeds of the large-scale structures in the present universe. Sizes of these inhomogeneities and anisotropies are small so that they can be treated as perturbations in the homogeneous and isotropic universe (Friedmann universe).

Due to these inhomogeneities and anisotropies, gradients of gravitational potential are developed. The evolution equation for the perturbations in energy density of the photon–baryon fluid takes a form of forced oscillation where a force relevant to the perturbations in photon–baryon fluid is the well of gravitational potential.

To get a basic picture of the oscillation of the photon-baryon fluid, we consider the evolution equation for the perturbations in photon temperature in the form

$$\Delta_T'' + \frac{r'}{1+r}\Delta_T' + k^2 c_s^2 \Delta_T = \mathcal{F}\,, \tag{2.55}$$

where a prime denotes derivative with respect to a conformal time $\tau = \int dt/a$, $\Delta_T \equiv \Delta t/\bar{T}$, $r \equiv 3\rho_b/(4\rho_\gamma)$ and $\mathcal{F}$ is the driving force produced by gravitational potential. Here, c_s^2 is the sound speed of the photon-baryon fluid defined as

$$c_s^2 \equiv \frac{1}{3}\frac{1}{1+r}\,. \tag{2.56}$$

The solution for this equation can be written in the form

$$\begin{aligned}[1+r]^{1/4}\Delta_T &= C_1 \cos kr_s + C_2 \sin kr_s \\ +\frac{\sqrt{3}}{k}&\int_{\tau_I}^{\tau} d\tau'[1+r(\tau')]^{3/4}\sin[kr_s(\tau) - kr_s(\tau')]F(\tau')\,,\end{aligned} \tag{2.57}$$

Here, r_s is the sound horizon defined as

$$r_s \equiv \int_0^{\tau} c_s d\tau'\,. \tag{2.58}$$

The quantity r_s is the one of important length scales in cosmology, e.g. the cosine term in the above equation has extrema at the wavelength $\lambda = 2nr_s$. The extrema of the cosine and sine term leave imprints on the angular power spectrum of the Cosmic Microwave Background (CMB).

After last scattering, photons freely propagate so that they cannot be described by a perfect fluid, e.g., shears are developed in the photon fluid.

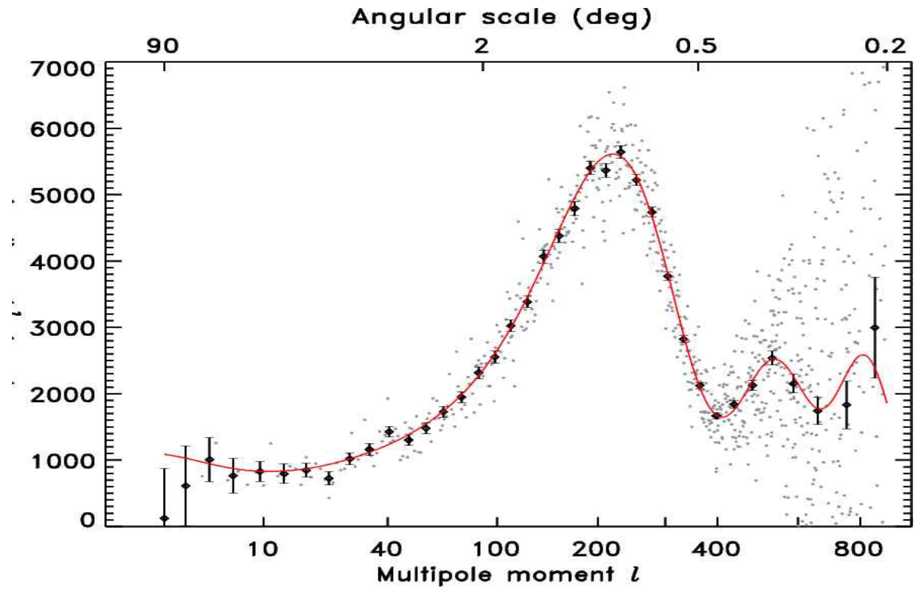

Figure 2.4: The angular power spectrum of the Cosmic Microwave Background .

Chapter 3

Inflation

3.1 Shortcomings of the Hot BB

According to observations, the universe is nearly homogeneous and isotropic on large scales, e.g., $\Delta_{T\text{CMB}} \sim 10^{-5}$. We will learn that the smooth universe cannot be realized by Big Bang (BB) scenario unless the initial conditions of the universe is fine tuned.

3.1.1 Horizon Problem

We first discuss the horizon problem. The horizon scales of the present universe can be represented by the side of the universe that we can observe today. can be computed from the particle horizon which The side of the observable universe is approximately given by ct_0 where t_0 is the age of the universe and c is a speed of light. At the initial time t_i, the scale factor is a_i and therefore the comoving size of this horizon can be written in terms of the quantities at the initial time as

$$\frac{l_i}{a_i} \sim \frac{ct_0}{a_0}\,, \quad \Rightarrow \quad l_i \sim ct_0 \frac{a_i}{a_0}\,. \tag{3.1}$$

Comparing this scale with the size of a causal region $l_c = ct_i$, we get

$$\frac{l_i}{l_c} = \frac{t_0 a_i}{t_i a_0}\,. \tag{3.2}$$

We suppose that the initial conditions for the universe are specified at the Planck time. Hence, $t_i = t_p \sim 10^{-43}$s, $T_i = T_p \sim 10^{32}$K, and then the ratio in Eq. (3.2) becomes

$$\frac{l_i}{l_c} = \frac{t_0 T_0}{t_p T_p} = \frac{10^{17} 2.7}{10^{-43} 10^{32}} \sim 10^{28}\,, \tag{3.3}$$

where we have used $a \propto T$, $T_0 \simeq T_{\text{CMB}} \simeq 2.7$K and $t_0 \sim 10^{17}$s. We see that at initial time the present horizon scale is 28 of magnitude larger than the size of causal region. Hence, the present horizon should be composed with 10^{84} causality disconnected regions. This leads to the puzzle why matter can smoothly distribute (up to $\Delta_{T\mathrm{CMB}} \sim 10^{-5}$) over a huge amount of causality disconnected regions. This is the horizon problem in the standard hot bigbang universe.

3.1.2 Flatness Problem

We now consider the flatness problem. Let us consider the Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho\,, \tag{3.4}$$

where ρ in the above equation is the total energy density. This equation can be written as

$$\Omega - 1 = \frac{k}{a^2H^2}\,, \tag{3.5}$$

and therefore

$$|\Omega_i - 1| = |\Omega_0 - 1|\frac{a_0^2H_0^2}{a_i^2H_i^2}\,. \tag{3.6}$$

If the dominant energy component of the universe has a constant equation of state parameter, one can show that $a \propto t^p$ where p is constant, so that $H \sim 1/t$. Hence, Eq. (3.6) can be approximate as

$$|\Omega_i - 1| \sim |\Omega_0 - 1|\frac{a_0^2t_i^2}{a_i^2t_0^2} \sim \frac{T_i^2t_i^2}{T_0^2t_0^2} \sim 10^{-56}\,, \tag{3.7}$$

where the ratio in Eq. (3.3) is used and we have supposed that $|\Omega_0 - 1| \sim \mathcal{O}(1)$ at present according to observations. This implies that in order to obtain $|\Omega - 1| \sim \mathcal{O}(1)$ at present, the initial amount of matter must be fine tuned such that the universe is extremely flat at the initial epoch. This is the flatness problem.

To understand how the mentioned problems can be solved, we write Eqs. (3.2) and (3.6) in the approximated forms as

$$\frac{l_i}{l_c} = \frac{t_0a_i}{t_ia_0} \sim \frac{H_ia_i}{H_0a_0} \sim \frac{\dot{a}_i}{\dot{a}_0}\,, \tag{3.8}$$

$$|\Omega_i - 1| = |\Omega_0 - 1|\frac{a_0^2H_0^2}{a_i^2H_i^2} = |\Omega_0 - 1|\frac{\dot{a}_0^2}{\dot{a}_i^2}\,, \tag{3.9}$$

where we have used $H \sim 1/t$ and $H = \dot{a}/a$. From the above approximation, and the numerical values given in Eq. (3.3), we have

$$\frac{l_i}{l_c} \sim \frac{\dot{a}_i}{\dot{a}_0} \gg 1\,, \tag{3.10}$$

$$|\Omega_i - 1| = |\Omega_0 - 1|\frac{\dot{a}_0^2}{\dot{a}_i^2} \ll 1\,. \tag{3.11}$$

The main problem is that the ratio$\dot{a}_0/\dot{a}_i$ is extremely small. This ratio is small because in the BB model the universe expands with deceleration during radiation and matter dominated epochs. This suggests that the problems can be solved if there is a mechanism that can enhance $\dot{a}$ before the radiation dominated epoch starts. This mechanism is the cosmic inflation in which an expansion of the universe is accelerated before radiation dominated epoch. The accelerated expansion of the early universe is known as inflationary epoch.

3.2 Graceful Exit

Let us consider the simplest model of inflation in which an accelerated expansion of the early universe is driven by a cosmological constant. We suppose that the universe contains only a cosmological constant, so that the Friedmann equation gives

$$H^2 = \frac{8\pi G}{3}\rho_\Lambda = \text{constant}\,, \quad \Rightarrow \quad a \propto \mathrm{e}^{Ht}\,. \tag{3.12}$$

From the above equation, we get

$$\ddot{a} = H^2 a\,, \tag{3.13}$$

which implies that an accelerated expansion of the universe never stop. A realistic inflationary model has to naturally stop after last sufficiently long otherwise the structures in the universe cannot be created. The natural end of inflation is known as graceful exit.

To find a necessary condition for the existence of the graceful exit, we consider a time derivative of the Hubble parameter as

$$\dot{H} = \frac{\ddot{a}}{a} - H^2\,. \quad \Rightarrow \quad \frac{\ddot{a}}{a} = \dot{H} + H^2\,. \tag{3.14}$$

A graceful exit can be realized if $\ddot{a}$ in the above equation can change from positive (acceleration) to negative (deceleration). Such change can be achieve when $\dot{H}$ is negative, i.e., H decreases with time. Moreover, $|\dot{H}|$ has to smaller than H^2 initially and then becomes larger than H^2 at the end of inflation. Hence, to get a graceful exit, H has to decrease with time during inflation.

3.3 Simple Model of Inflation

In the simplest model of inflation, the acceleration of the universe is driven by single scalar field called inflaton. The action of scalar field in gravitational field is

$$S = \int d^4x\sqrt{-g}\left\{\frac{M_p^2}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\}\,, \tag{3.15}$$

where $M_p \equiv 1/\sqrt{8\pi G}$ is the reduced Planck mass, g is the determinant of the metric tensor $g_{\mu\nu}$, R is the Ricci scalar and $V(\phi)$ is the potential of scalar field ϕ. Varying the above action with respect to the field ϕ, we get

$$\begin{aligned}
\delta S &= \int d^4x\delta\left(\sqrt{-g}\left\{\frac{M_p^2}{2}R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\right\}\right) \\
&= \int d^4x\sqrt{-g}\delta\left\{-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\right\} \\
&= \int d^4x\sqrt{-g}\left\{-\partial^\mu\phi\partial_\mu(\delta\phi)-\frac{dV}{d\phi}\delta\phi\right\} \\
&= \int d^4x\sqrt{-g}\left\{\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi)-\frac{dV}{d\phi}\right\}\delta\phi\,.
\end{aligned} \tag{3.16}$$

From $\delta S=0$, the above relation yields

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi)-\frac{dV}{d\phi}=0\,. \tag{3.17}$$

This is the equation of motion for scalar field in gravity. Now let us vary the action (3.15) with respect to the metric tensor $g^{\mu\nu}$:

$$\begin{aligned}
\delta S &= \int d^4x\delta\left(\sqrt{-g}\left\{\frac{M_p^2}{2}R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\right\}\right) \\
&= \int d^4x\left\{\underbrace{\delta\left(\sqrt{-g}\frac{M_p^2}{2}R\right)}_{=\sqrt{-g}M^2G_{\mu\nu}\delta g^{\mu\nu}/2}+\delta\left(\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V\right)\right)\right\} \\
&= \int d^4x\sqrt{-g}\delta g^{\mu\nu}\left\{\frac{M_p^2}{2}G_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+V\right)-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\right\}.
\end{aligned} \tag{3.18}$$

Here, $G_{\mu\nu}$ is the Einstein tensor defined by

$$G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\,, \tag{3.19}$$

where $R_{\mu\nu}$ is the Ricci tensor. To obtain the second term of the last line, we have used

$$\delta\sqrt{-g}=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{3.20}$$

From $\delta S=0$, Eq. (3.18) yields

$$G_{\mu\nu}=8\pi G\left(\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+V\right)\right)=8\pi GT_{\mu\nu}\,, \tag{3.21}$$

where $T_{\mu\nu}$ is the energy momentum tensor of the scalar field. We can write the energy-momentum tensor for scalar field as

$$T^\mu_\nu=\partial_\nu\phi\partial^\mu\phi+\delta^\mu_\nu\left(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi-V(\phi)\right)\,. \tag{3.22}$$

In the homogeneous and isotropic universe the energy momentum tensor of the scalar field satisfies the conditions for a perfect fluid, so that

$$-T^0_0 = \rho = \frac{1}{2}\dot{\phi}^2 + V\,, \quad \text{and} T^i_i = P = \frac{1}{2}\dot{\phi}^2 - V\,. \tag{3.23}$$

To drive an accelerated expansion of the universe, it follows from Eq. (2.50) that we demand $P < -\rho/3$ during inflation, so that Eq. (3.23) yields the condition $\dot{\phi}^2 \ll V$. The successful inflation requires a small kinetic energy $\dot{\phi}^2/2$ of scalar field compared to the potential V during inflation. For the spatially flat universe, the parameter K in the FLRW metric given by Eq. (2.1) vanishes, so that it can be written in cartesian coordinate as

$$\mathrm{d}s^2 = -dt^2 + a^2\delta_{ij}dx^i dx^j\,, \tag{3.24}$$

where δ_{ij} is the kronecker delta. Using the metric in Eq. (3.24), we get $\sqrt{-g} = a^3$ so that the field equation in Eq. (3.17) becomes

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0\,, \tag{3.25}$$

where $\partial_i\phi = 0$ in homogeneous space and $V_\phi \equiv dV/d\phi$. To solve this equation we need the Hubble parameter from the Friedmann equation

$$H^2 = \frac{8\pi}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right) \tag{3.26}$$

We will solve the above two equations for the case of slow-roll approximation and for specific models of inflation.

Slow-roll Inflation

Since $\dot{\phi}^2/2 \ll V$ during inflation, we can study the evolution of the universe during inflation by supposing that the inflaton field slowly rolls down its potential. In the slow-roll approximation, we suppose that

$$dot\phi^2 \ll V(\phi)\,, \qquad |\ddot{\phi}| \ll 3h|\dot{\phi}|\,. \tag{3.27}$$

The first relation in the above equations is required for $w \sim -1$ – accelerating universe, while the second relation implies the slow changing of $\dot{\phi}$. Based on these slow-roll approximation the evolution equation for the field and Friedmann equation become

$$\begin{aligned} 3H\dot{\phi} &\simeq -V_\phi\,, & (3.28)\\ H^2 &\simeq \frac{8\pi G}{3}V(\phi)\,. & (3.29) \end{aligned}$$

From these equations, we have

$$V_\phi^2 \simeq 9H^2\dot{\phi}^2 \ll 9H^2V(\phi) \simeq 24\pi G V^2(\phi)\,, \quad \Rightarrow \quad \frac{1}{24\pi G}\frac{V_\phi^2}{V^2} \ll 1\,. \tag{3.30}$$

This is the slow-roll condition in terms of the potential. This condition tells us that the field will slowly rolls down its potential if the potential is significantly flat. This condition can be rewritten in terms of Hubble parameter as

$$-\frac{\dot{H}}{H^2} \simeq -\frac{4\pi G}{3H^3} V_\phi \dot{\phi} \simeq \frac{4\pi G}{9H^4} V_\phi^2 \simeq \frac{1}{16\pi G} \frac{V_\phi^2}{V^2} \equiv \epsilon \,, \tag{3.31}$$

where ϵ is the slow-roll parameter. From this equation we see that during the slow-roll evolution $\epsilon \ll 1$, and hence we can suppose that inflation ends at $\epsilon \simeq 1$. The other slow-roll condition can be obtained by differentiating the slow-roll version of the evolution equation for the field eq. (**??**) as

$$-V_{\phi\phi}\dot{\phi} \simeq 3\dot{H}\dot{\phi} + 3H\ddot{\phi} \,, \qquad \Rightarrow \qquad \frac{V_{\phi\phi}}{3H^2} \simeq -\frac{\dot{H}}{H^2} - \frac{\ddot{\phi}}{H\dot{\phi}} \,, \tag{3.32}$$

where $V_{\phi\phi} = d^2V/d\phi^2$. Since $\epsilon \ll 1$ and $|\ddot{\phi}| \ll H|\dot{\phi}|$, we have

$$\eta \equiv \frac{1}{8\pi G} \left| \frac{V_{\phi\phi}}{V} \right| \ll 1 \,, \tag{3.33}$$

during inflation. Note that the slow-roll parameter η is related to ϵ by $\ddot{\phi}/H\dot{\phi}$ which is the other slow-roll parameter.

Using $H = \dfrac{\mathrm{d}\ln a}{\mathrm{d}t}$, we can write Eq. (3.28) as

$$\frac{\mathrm{d}\phi}{\mathrm{d}\ln a} = -\frac{V_\phi}{3H^2} \,. \tag{3.34}$$

Further using Eq. (3.29), we get

$$\frac{\mathrm{d}\ln a}{\mathrm{d}\phi} \simeq -\frac{8\pi G V}{V_\phi} \,. \tag{3.35}$$

This equation can be integrated as

$$a(t) \simeq a_i \exp\left(-8\pi G \int_{\phi_i}^{\phi} \frac{V}{V_\phi} \mathrm{d}\phi \right) \,. \tag{3.36}$$

Hence, during the slow-roll evolution, the number of e-foldings can be computed as

$$N = \ln\left(\frac{a}{a_i}\right) \simeq 8\pi G \int_{\phi}^{\phi_i} \frac{V}{V_{\phi'}} \mathrm{d}\phi' \,. \tag{3.37}$$

Thus, the total number of e-folding depends on the field value at the beginning ϕ_i and the end ϕ_f of inflation as

$$N \simeq 8\pi G \int_{\phi_f}^{\phi_i} \frac{V}{V_\phi} \mathrm{d}\phi \,. \tag{3.38}$$

According to observation, the horizon and flatness problems can be solved if the inflation has to last longer than $N > 60$.

3.4 The unsolved problem in cosmology

- × How can slow-roll inflation be realized in quantum theory?
- × The cosmolotical constant problem
- × The coincidence problem
- × The H_0 and σ_8 tensions

Chapter 4

Large-scale Structure of the Universe

4.1 Cosmological observations

4.1.1 The Cosmic Microwave Background

The cosmic microwave background (CMB) radiation provides a window onto the early universe, revealing its composition and structure. It is a relic, thermal radiation from a hot dense phase in the early evolution of our universe which has now been cooled by the cosmic expansion to just three degrees above absolute zero. Its existence had been predicted in the 1940s by Alpher and Gamow [3] and its discovery by Penzias and Wilson at Bell Labs in New Jersey, announced in 1965 [4] was convincing evidence for most astronomers that the cosmos we see today emerged from a **Hot Big Bang** more than 10 billion years ago.

Since its discovery, many experiments have been performed to observe the CMB radiation at different frequencies, directions and polarisations, mostly with ground- and balloon-based detectors. These have established the remarkable uniformity of the CMB radiation, at a temperature of 2.7 Kelvin in all directions, with a small ± 3.3 mK dipole due to the Doppler shift from our local motion (at 1 million kilometres per hour) with respect to this cosmic background.

However, the study of the CMB has been transformed over the last twenty years by three pivotal satellite experiments. The first of these was the *Cosmic Background Explorer* (CoBE, `https://lambda.gsfc.nasa.gov/product/cobe/`), launched by NASA in 1990. In 1992 CoBE reported the detection of statistically significant temperature anisotropies in the CMB, at the level of ± 30 μK on 10 degree scales [5] and it confirmed the black body spectrum with an astonishing precision, with deviations less than 50 parts per million [5]. CoBE was succeeded by the *Wilkinson Microwave Anisotropy Probe* (WMAP, `https://map.gsfc.nasa.gov/`) satellite, launched by NASA in 2001, which produced full sky maps in five frequencies (from 23 to 94 GHz) mapping the temperature anisotropies to sub-degree scales and determining the CMB polarisation on large angular scales for the first time.

The *Planck* satellite (http://sci.esa.int/planck/), launched by ESA in 2009, sets the current state of the art with nine separate frequency channels, measuring temperature fluctuations to a millionth of a degree at an angular resolution down to 5 arc-minutes.

Planck's mission ended in 2013 and the full-mission data were released in 2015 in [6] and in many companion papers. A fourth generation of full-sky, microwave-band satellite recently proposed to ESA within Cosmic Vision 2015-2025 is the **Cosmic Origins Explorer** (COrE, http://www.core-mission.org/).

4.1.2 Redshift Surveys

Redshift surveys are observations of certain patches of sky at certain wavelengths with the aim of determining mainly the angular positions (declination and right ascension) redshifts and spectra of galaxies.

The **Sloan Digital Sky Survey** (SDSS, http://www.sdss.org/) is a massive spectroscopic redshift survey which is ongoing since the year 2000 and it is now in its stage IV with 14 data releases available. It is ground-based and uses a telescope located in New Mexico (USA). The SDSS-IV is formed by three sub-experiment:

- **The Extended Baryon Oscillation Spectroscopic Survey** (eBOSS), focusing on redshifts $0.6 < z < 2.5$ and on the Baryon Acoustic Oscillations (BAO) phenomenon;
- The **Apache Point Observatory Galaxy Evolution Experiment** (APOGEE-2) is dedicated to the study of our Milky Way;
- The **Mapping Nearby Galaxies at Apache Point Observatory** (MaNGA) study instead nearby galaxies by measuring their spectrum along their extension and not only at the centre.

The **Dark Energy Survey** (DES, https://www.darkenergysurvey.org/) measures redshifts photometrically using a telescope situated in Chile and looking for Type Ia supernovae, BAO and weak lensing signals.

Planned surveys are the already mentioned satellite **Euclid** (http://sci.esa.int/euclid/), whose launch is due possibly in 2021 and the telescope LSST, which is being built in Chile and whose first light is due in 2019. We also cite the NASA satellite **Wide Field Infrared Survey Telescope** (WFIRST, https://www.nasa.gov/wfirst) and the **Javalambre Physics of the accelerating universe Astronomical Survey** (J-PAS, http://j-pas.org/). The main cosmological goals of these experiments relay on the detection of weak lensing, BAO and type Ia supernovae signals with high precision.

4.1.3 Gravitational Waves Observatories

The recent direct detection of gravitational waves (GW) by the **LIGO-Virgo collaboration** (https://www.ligo.org/) [7] has opened a new observational window on the universe. In particular, GW are relevant in cosmology because they could be a relic from inflation containing invaluable informations on the very early universe. As already mentioned, they are being searched via the detection of the B-mode polarisation of the CMB.

There are now three functioning ground-based GW observatories: **LIGO** (Hanford and Livingstone, USA) and **Virgo** (near Pisa, Italy). **KAGRA**, in Japan, is under construction and another one in India, **INDIGO**, is planned. The space-based **LISA** GW observer is still in a preliminary phase (**LISA** pathfinder).

In this chapter we recall current ideas about the physical origin of stochasticity in cosmic fields in different cosmological scenarios. We then present the statistical tools that are commonly used to describe random cosmic fields such as power spectra, probability distribution functions and give some mathematical properties of interest.

The current explanation of the large-scale structure of the universe is that the present distribution of matter on cosmological scales results from the growth of primordial, small, seed fluctuations on an otherwise homogeneous universe amplified by gravitational instability. Tests of cosmological theories which characterize these primordial seeds are not deterministic in nature but rather statistical, for the following reasons. First, we do not have direct observational access to primordial fluctuations (which would provide definite initial conditions for the deterministic evolution equations). In addition, the time-scale for cosmological evolution is so much longer than that over which we can make observations, that is not possible to follow the evolution of single systems. In other words, what we observe through our the past light cone is different objects at different times of their evolution, therefore testing the evolution of structure must be done statistically.

The observable universe is thus modeled as a stochastic realization of a statistical ensemble of possibilities. The goal is to make statistical predictions, which in turn depend on the statistical properties of the primordial perturbations leading to the formation of large-scale structures. Among the two classes of models that have emerged to explain the large-scale structure of the universe, the physical origin of stochasticity can be quite different and thus give rise to very different predictions.

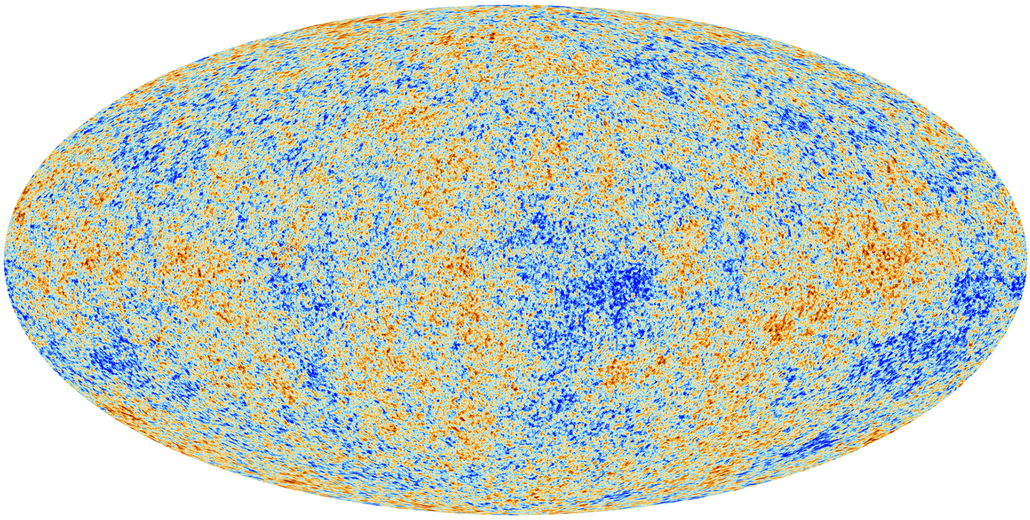

Figure 4.1: The cosmic microwave background radiation – the initial condition of the universe from quantum fluctuation left during the end of the inflation period.

4.2 Random Fields

A function $\delta(\mathbf{x})$ is a random field if its values are random variables for any $\mathbf{x}$ and distribution functions

$$\mathcal{F}_{1,2,\ldots,n}(\delta_1, \delta_2, \ldots, \delta_n) = \mathcal{F}[\delta(\mathbf{x}_1) < \delta_1, \delta(\mathbf{x}_2) < \delta_2, \ldots, \delta(\mathbf{x}_n) < \delta_n] \ , \tag{4.1}$$

exist for any n. We denote with $\delta(\mathbf{x})$ the random field, so it has $\mathbf{x}$ dependence, and with δ_n a certain value that it can assume at a given $\mathbf{x}_n$ among all the possible ones which form the **ensemble**. The subscripts in $\delta_1, \delta_2, \ldots, \delta_n$ refer to the different points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in which the random field is evaluated. In particular, at a given point $\mathbf{x}_1$ some probability density functional exists:

$$p_1(\delta_1)\mathrm{d}\delta_1 \ , \tag{4.2}$$

describing how probable is for $\delta(\mathbf{x})$ to assume a certain value δ_1 in $\mathbf{x}_1$. In terms of the distribution function, $p(\delta_1)$ is defined as:

$$p_1(\delta_1) = \frac{\mathrm{d}\mathcal{F}_1(\delta_1)}{\mathrm{d}\delta_1} \ . \tag{4.3}$$

Since $\mathcal{F}$ is a cumulative probability then $\mathcal{F}_1(-\infty) = 0$ and $\mathcal{F}_1(\infty) = 1$. In cosmology, $\delta(\mathbf{x})$ is a perturbative quantity, such as the density contrast. As usual, an expectation value of the random field is defined via an **ensemble average**:

$$\langle \delta(\mathbf{x}_1) \rangle \equiv \int_\Omega \delta_1 p_1(\delta_1)\mathrm{d}\delta_1 \tag{4.4}$$

where Ω denotes the ensemble. In general, one has:

$$p_1(\delta_1) \neq p_2(\delta_2) \ , \tag{4.5}$$

i.e. the probability distribution of the values which δ may assume in $\mathbf{x}_1$ is in general different from the one of the values which δ may assume in $\mathbf{x}_2$. When this does not happen, i.e. the probability of the realisation is translationally invariant, the random field is said to be **statistically homogeneous**. For a statistically homogeneous random field then Eq. (4.4) is independent of $\mathbf{x}$

$$\langle\delta\rangle \equiv \int_\Omega \delta p(\delta)\mathrm{d}\delta \tag{4.6}$$

Now we can think of more complicated and richer configurations of the random field δ asking for example what is the probability of $\delta(\mathbf{x}_1)$ and $\delta(\mathbf{x}_2)$ being δ_1 and δ_2, respectively. This is given by

$$p_{12}(\delta_1, \delta_2)\mathrm{d}\delta_1\mathrm{d}\delta_2 \ , \tag{4.7}$$

which can be written again as a derivative of the distribution function $\mathcal{F}_{12}$. In general:

$$p_{12}(\delta_1, \delta_2) \neq p_1(\delta_1)p_2(\delta_2) \ , \tag{4.8}$$

unless the realisations are independent, in which case the random process is said to be Poissonian. The 2-dimensional probability density allows us to define the **2-point correlation function** as follows:

$$\xi(\mathbf{x}_1, \mathbf{x}_2) \equiv \langle\delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\rangle \equiv \int_\Omega \delta_1\delta_2 p_{12}(\delta_1, \delta_2)\mathrm{d}\delta_1\mathrm{d}\delta_2 \tag{4.9}$$

In a similar fashion, one can define N-point correlation functions as

$$\begin{aligned} \xi^{(N)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) &\equiv \langle\delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\ldots\delta(\mathbf{x}_N)\rangle \\ \equiv \int_\Omega \delta_1\delta_2\ldots\delta_N p_{12\ldots N}(\delta_1, \delta_2, \ldots, \delta_N)\mathrm{d}\delta_1\mathrm{d}\delta_2\ldots\delta_N \ . \end{aligned} \tag{4.10}$$

The order of the points matters, i.e. in general $p_{12\ldots N} \neq p_{21\ldots N}$ for example, and we are using the same ensemble Ω for each point.

Following the standard definition that we learn in the first course of statistics, the **ensemble variance** of the random field is defined as:

$$\sigma^2(\mathbf{x}_1, \mathbf{x}_2) \equiv \langle\delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\rangle - \langle\delta(\mathbf{x}_1)\rangle\langle\delta(\mathbf{x}_2)\rangle \tag{4.11}$$

Therefore, if the random field is statistically homogeneous and isotropic, we get:

$$\sigma^2(r_{12}) = \xi(r_{12}) - \langle\delta\rangle^2 \ , \tag{4.12}$$

where recall that $\langle\delta\rangle^2$ does not depend on the position. Note that if random process is Poissonian, i.e. $p_{12}(\delta_1, \delta_2) = p_1(\delta_1)p_2(\delta_2)$, then the variance is zero, since:

$$\xi(r_{12}) = \langle\delta\rangle^2 \ . \tag{4.13}$$

If δ is a variable representing the distribution of galaxies, we do not expect it to be a Poissonian random variable because gravity turns the odds for the galaxies to be closer to each other rather than farther. Hence, we attribute deviations from the Poissonian behaviour to gravity and for this reason it is very important to study the 2-point correlation function.

Note that in the above description of a random field δ we have used the configuration space whereas we have mostly used Fourier-transformed quantities representing our cosmological perturbations. We assume that the Fourier transform of a random field is also a random field, and all the above described properties apply.

Observationally, we are able to probe a realisation in a certain finite volume, say a box of volume L^3. Hence, the Fourier transform is defined here as a Fourier series:

$$\delta(\mathbf{x}) = \frac{1}{L^3} \sum_n \delta_n e^{\mathrm{i}\mathbf{k}_n \cdot \mathbf{x}} \tag{4.14}$$

where the coefficients are given by:

$$\delta_n = \int \mathrm{d}^3\mathbf{x}\ \delta(\mathbf{x}) e^{-\mathrm{i}\mathbf{k}_n \cdot \mathbf{x}} \ , \tag{4.15}$$

and the wavenumbers are quantised, because of the periodic boundary conditions that we must impose, which are equivalent to ask that G vanishes on the sides of the box:

$$\mathbf{k}_n = \frac{2\pi}{L}\mathbf{n} \ , \tag{4.16}$$

where $\mathbf{n}$ is a generic vector whose components are integers. If the side of the box goes to infinity, we recover the usual FT, i.e.

$$\delta(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ \tilde{\delta}(\mathbf{k}) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \ , \qquad \tilde{\delta}(\mathbf{k}) = \int \mathrm{d}^3\mathbf{x}\ \delta(\mathbf{x}) e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \ . \tag{4.17}$$

Note that if $\delta(\mathbf{x})$ is a real field, then $\tilde{\delta}(-\mathbf{k}) = \tilde{\delta}^*(\mathbf{k})$, because of the following: The relation $\tilde{\delta}(-\mathbf{k}) = \tilde{\delta}^*(\mathbf{k})$, is called **reality condition**.

4.3 Correlation Functions and Power Spectra

From now on, we consider a cosmic scalar field whose statistical properties we want to describe. This field can either be the cosmic density field, $\delta(\mathbf{x})$, the cosmic gravitational potential, the velocity divergence field, or any other field of interest.

A random field is called **statistically homogeneous** if all the joint multipoint **probability distribution functions** $p(\delta_1, \delta_2, \dots)$ or its **moments**, ensemble averages of local density products, remain the same under translation of the coordinates $\mathbf{x}_1, \mathbf{x}_2, \dots$ in space(here $\delta_i \equiv \delta(\mathbf{x}_i)$).

Thus the probabilities depend only on the relative positions. A stochastic field is called **statistically isotropic** if $p(\delta_1, \delta_2, \dots)$ is invariant under spatial rotations. We will assume that cosmic fields are statistically homogeneous and isotropic, as predicted by most cosmological theories. The validity of this assumption can and should be tested against the observational data.

The two-point correlation function is defined as the joint ensemble average of the density at two different locations,

$$\xi(r) = \big\langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\big\rangle, \tag{4.18}$$

which depends only on the norm of $\mathbf{r}$ due to statistical homogeneity and isotropy. The density contrast $\delta(\mathbf{x})$ is usually written in terms of its Fourier components,

$$\delta(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, \delta(\mathbf{k})\, \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}). \tag{4.19}$$

The quantities $\delta(\mathbf{k})$ are then complex random variables. As $\delta(\mathbf{x})$ is real, it follows that

$$\delta(\mathbf{k}) = \delta^*(-\mathbf{k}). \tag{4.20}$$

The density field is therefore determined entirely by the statistical properties of the random variable $\delta(\mathbf{k})$. We can compute the correlators in Fourier space,

$$\big\langle \delta(\mathbf{k})\delta(\mathbf{k}')\big\rangle = \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{r}\, \big\langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\big\rangle \, \exp[-\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} - \mathrm{i}\mathbf{k}'\cdot\mathbf{r}] \tag{4.21}$$

which gives,

$$\begin{aligned} \big\langle \delta(\mathbf{k})\delta(\mathbf{k}')\big\rangle &= \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{r}\, \xi(r)\, \exp[-\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} - \mathrm{i}\mathbf{k}'\cdot\mathbf{r}] \\ &= (2\pi)^3 \delta_D(\mathbf{k}+\mathbf{k}') \int \mathrm{d}^3\mathbf{r}\, \xi(r)\, \exp(-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}) \\ &\equiv (2\pi)^3 \delta_D(\mathbf{k}+\mathbf{k}')\, P(k), \end{aligned} \tag{4.22}$$

where $P(k)$ is by definition the density **power spectrum**. The inverse relation between two-point correlation function and power spectrum thus reads

$$\xi(r) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, P(k) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}). \tag{4.23}$$

Similarly,

$$P(k) \equiv \int d^3\mathbf{r}\; \xi(r) e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{4.24}$$

Performing the integral gives

$$P(k) = 4\pi \int_0^\infty dr\; r^2\; \xi(r) \frac{\sin(kr)}{kr}\;. \tag{4.25}$$

Similarly,

$$\xi(r) = \int_0^\infty dk\, \frac{k^2 P(k)}{2\pi^2} \frac{\sin(kr)}{kr}. \tag{4.26}$$

It is customary then to define a dimensionless power spectrum as

$$\Delta_G^2(k) \equiv \frac{k^3 P_G(k)}{2\pi^2}. \tag{4.27}$$

so that Eq. (4.26) becomes:

$$\xi_G(r) = \int_0^\infty \frac{dk}{k}\, \Delta_G^2(k) \frac{\sin(kr)}{kr}. \tag{4.28}$$

There are basically two conventions in the literature regarding the definition of the power spectrum, which differ by a factor of $(2\pi)^3$. In this lecture we use the convention

$$f(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, \tilde{f}(\mathbf{k}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}). \tag{4.29}$$

and

$$\tilde{f}(\mathbf{k}) = \int \mathrm{d}^3\mathbf{x}\, f(\mathbf{x}) \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}). \tag{4.30}$$

Another popular choice is to reverse the role of $(2\pi)^3$ factors in the Fourier transforms, i.e. $\delta(\mathbf{k}) \equiv \int \mathrm{d}^3\mathbf{r}/(2\pi)^3 \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r})\delta(\mathbf{r})$, and then modify Eq. (4.22) to read $\left\langle \delta(\mathbf{k})\delta(\mathbf{k}')\right\rangle \equiv \delta_D(\mathbf{k}+\mathbf{k}')\, P(k)$, which leads to $4\pi k^3 P(k)$ being the contribution per logarithmic wavenumber to the variance, rather than $k^3P(k)/(2\pi^2)$ as in our case.

4.4 Non-Gaussian perturbations

For non-Gaussian perturbations the odd-order correlators are non-vanishing. For example:

$$\langle \tilde{\delta}(\mathbf{k}_1)\tilde{\delta}(\mathbf{k}_2)\tilde{\delta}(\mathbf{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) B_\delta(k_1,k_2,k_3)\ , \tag{4.31}$$

where $B_\delta(k_1,k_2,k_3)$ is called **bispectrum** and can be rewritten as:

$$B_\delta(k_1,k_2,k_3) = \mathcal{B}_\delta(k_1,k_2,k_3)\left[P_\delta(k_1,k_2)+P_\delta(k_2,k_3)+P_\delta(k_1,k_3)\right]\ , \tag{4.32}$$

i.e. in terms of a purely FT of a 3-point correlation function, called **reduced bispectrum**, multiplied by all the possible combinations of the FT transforms of the 2-point correlation functions (i.e. the PS). This formula can be obtained using the following expansion:

$$\delta(\mathbf{x}) = \delta_{\mathrm{G}}(\mathbf{x}) + f_{\mathrm{NL}}\left(\delta_{\mathrm{G}}^2(\mathbf{x}) - \langle\delta_{\mathrm{G}}^2(\mathbf{x})\rangle\right) + \dots\ , \tag{4.33}$$

where $\langle\delta_{\mathrm{G}}^2(\mathbf{x})\rangle \equiv \sigma_{\mathrm{G}}^2$. This expansion is called **local type non-Gaussianity** and is based on the fact that the square of a Gaussian random field $\delta_G(\mathbf{x})$ is not Gaussian. The amount of non-Gaussianity is indicated by the free parameter f_{NL}, which Planck has constrained to be:

$$\boxed{f_{NL} = 2.5 \pm 5.7} \qquad \text{(68\% CL, statistical)} \tag{4.34}$$

The huge relative error shows how difficult is to extract this kind of information and at the same time how Gaussianity ($f_{NL} = 0$) is fully consistent with data. Note that we are talking here of primordial non-Gaussianity. In the structure formation process, non-Gaussianity naturally arises in the non-linear regime of evolution.

4.5 The Wick Theorem for Gaussian Fields

The power spectrum is a well defined quantity for almost all homogeneous random fields. This concept becomes however extremely fruitful when one considers a **Gaussian** field. It means that any joint distribution of local densities is Gaussian distributed. Any ensemble average of product of variables can then be obtained by product of ensemble averages of pairs. We write explicitly this property for the Fourier modes as it will be used extensively in this work,

$$\left\langle \delta(\mathbf{k}_1)\dots\delta(\mathbf{k}_{2p+1})\right\rangle = 0 \tag{4.35}$$

$$\left\langle \delta(\mathbf{k}_1)\dots\delta(\mathbf{k}_{2p})\right\rangle = \sum_{\text{all pair associations}} \prod_{p \text{ pairs (i,j)}} \left\langle \delta(\mathbf{k}_i)\delta(\mathbf{k}_j)\right\rangle \tag{4.36}$$

This is the **Wick theorem**, a fundamental theorem for classic and quantum field theories.

The statistical properties of the random variables $\delta(\mathbf{k})$ are then entirely determined by the shape and normalization of $P(k)$. A specific cosmological model will eventually be determined e.g. by the power spectrum in the linear regime, by Ω_m and Ω_Λ only as long as one is only interested in the dark matter behavior.

4.5.1 Higher-Order Correlators: Diagrammatics

In general it is possible to define higher-order correlation functions. They are defined as the **connected** part (denoted with subscript c) of the joint ensemble average of the density in an arbitrarily number of locations. They can be formally written,

$$\begin{aligned}\xi_N(\mathbf{x}_1,\dots,\mathbf{x}_N) &= \langle \delta(\mathbf{x}_1),\dots,\delta(\mathbf{x}_N)\rangle_c \\ &\equiv \langle \delta(\mathbf{x}_1),\dots,\delta(\mathbf{x}_N)\rangle - \sum_{\mathcal{S}\in\mathcal{P}(\{\mathbf{x}_1,\dots,\mathbf{x}_n\})} \prod_{s_i\in\mathcal{S}} \xi_{\#s_i}(\mathbf{x}_{s_i(1)},\dots,\mathbf{x}_{s_i(\#s_i)})\end{aligned} \tag{4.37}$$

where the sum is made over the proper partitions (any partition except the set itself) of $\{\mathbf{x}_1,\dots,\mathbf{x}_N\}$ and s_i is thus a subset of $\{\mathbf{x}_1,\dots,\mathbf{x}_N\}$ contained in partition $\mathcal{S}$. When the average of $\delta(\mathbf{x})$ is defined as zero, only partitions that contain no singlets contribute.

The decomposition in connected and non-connected parts can be easily visualized. It means that any ensemble average can be decomposed in a product of connected parts. They are defined for instance in Figure. 4.2. The tree-point moment is "written" in Figure. 4.3 and the four-point moment.

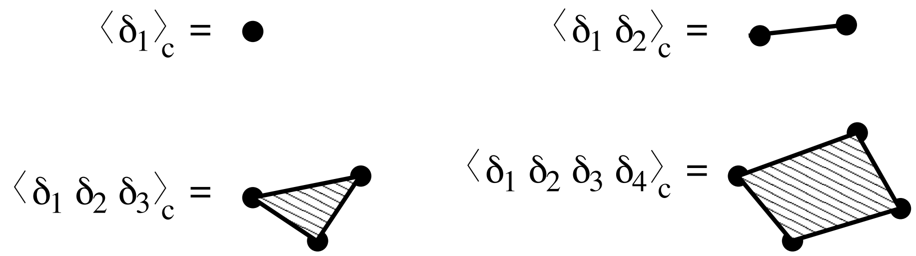

Figure 4.2: Representation of the connected part of the moments.

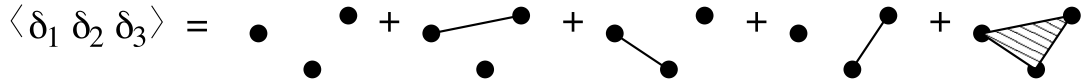

Figure 4.3: Writing of the three-point moment in terms of connected parts.

In case of a Gaussian field all connected correlation functions are zero except ξ_2. This is a consequence of Wick's theorem. As a result the only non-zero connected part is the two-point correlation function. An important consequence is that the statistical properties of any field, not necessarily linear, built from a Gaussian field δ can be written in terms of combinations of two-point functions of δ. Note that in a diagrammatic representation the connected moments of any of such field is represented by a **connected** graph. This is illustrated in Figure. 4.4 for the field $\delta = \phi^2$: the connected part of the 2-point function of this field is obtained by all the diagrams that explicitly join the two points. The other ones contribute to the moments, but not to its connected part.

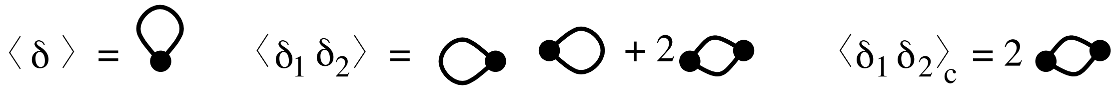

Figure 4.4: Disconnected and connected part of the two-point function of the field δ assuming it is given by $\delta = \phi^2$ with ϕ Gaussian.

The connected part has the important property that it vanishes when one or more points are separated by infinite separation. In addition, it provides a useful way of characterizing the statistical properties, since unlike unconnected correlation functions, each connected correlation provides independent information.

These definitions can be extended to Fourier space. Because of homogeneity of space $\langle \delta(\mathbf{k}_1) \ldots \delta(\mathbf{k}_N) \rangle_c$ is always proportional to $\delta_D(\mathbf{k}_1 + \cdots + \mathbf{k}_N)$. Then we can define $P_N(\mathbf{k}_1, \ldots, \mathbf{k}_N)$ with

$$\langle \delta(\mathbf{k}_1) \ldots \delta(\mathbf{k}_N) \rangle_c = \delta_D(\mathbf{k}_1 + \cdots + \mathbf{k}_N)\, P_N(\mathbf{k}_1, \ldots, \mathbf{k}_N). \tag{4.38}$$

One particular case that will be discussed in the following is for $n = 3$, the bispectrum, which is usually denoted by $B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$.

4.5.2 Probabilities and Correlation Functions

Correlation functions are directly related to the multi-point probability function, in fact they can be defined from them. Here we illustrate this for the case of the density field, as these results are frequently used in the literature. The physical interpretation of the two-point correlation function is that it measures the excess over random probability that two particles at volume elements $\mathrm{d}V_1$ and $\mathrm{d}V_2$ are separated by distance $x_{12} \equiv |\mathbf{x}_1 - \mathbf{x}_2|$,

$$\mathrm{d}P_{12} = n^2[1 + \xi(x_{12})]\mathrm{d}V_1\mathrm{d}V_2, \tag{4.39}$$

where n is the mean density. If there is no clustering (random distribution), $\xi = 0$ and the probability of having a pair of particles is just given by the mean density squared, independently of distance. Since the probability of having a particle in $\mathrm{d}V_1$ is $n\mathrm{d}V_1$, the conditional probability that there is a particle at $\mathrm{d}V_2$ given that there is one at $\mathrm{d}V_1$ is

$$\mathrm{d}P(2|1) = n[1 + \xi(x_{12})]\mathrm{d}V_2. \tag{4.40}$$

The nature of clustering is clear from this expression; if objects are clustered ($\xi(x_{12}) > 0$), then the conditional probability is enhanced, whereas if objects are anti-correlated ($\xi(x_{12}) < 0$) the conditional probability is suppressed over the random distribution case, as expected. Similarly to Eq. (4.39), for the three-point case the probability of having three objects is given by

$$\mathrm{d}P_{123} = n^3[1 + \xi(x_{12}) + \xi(x_{23}) + \xi(x_{31}) + \xi_3(x_{12}, x_{23}, x_{31})]\mathrm{d}V_1\mathrm{d}V_2\mathrm{d}V_3, \tag{4.41}$$

where ξ_3 denotes the three-point (connected) correlation function. If the density field were Gaussian, $\xi_3 = 0$, and all probabilities are determined by $\xi(r)$ alone. Analogous results hold for higher-order correlations.

4.5.3 Moments, Cumulants and their Generating Functions

One particular case for Eq. (4.37) is when all points are at the same location. Because of statistical homogeneity $\xi_p(\mathbf{x}, \ldots, \mathbf{x})$ is independent on the position $\mathbf{x}$ and it reduces to the cumulants of the one-point density probability distribution functions, $\langle \delta^p \rangle_c$. The relation (4.37) tells us also how the cumulants are related to the moments $\langle \delta^p \rangle$. For convenience we write here the first few terms,

$$
\begin{aligned}
\langle\delta\rangle_c &= \langle\delta\rangle \\
\langle\delta^2\rangle_c &= \sigma^2 = \langle\delta^2\rangle - \langle\delta\rangle_c^2 \\
\langle\delta^3\rangle_c &= \langle\delta^3\rangle - 3\langle\delta^2\rangle_c\langle\delta\rangle_c - \langle\delta\rangle_c^3 \\
\langle\delta^4\rangle_c &= \langle\delta^4\rangle - 4\langle\delta^3\rangle_c\langle\delta\rangle_c - 3\langle\delta^2\rangle_c^2 - 6\langle\delta^2\rangle_c\langle\delta\rangle_c^2 - \langle\delta\rangle_c^4 \\
\langle\delta^5\rangle_c &= \langle\delta^5\rangle - 5\langle\delta^4\rangle_c\langle\delta\rangle_c - 10\langle\delta^3\rangle_c\langle\delta^2\rangle_c - 10\langle\delta^3\rangle_c\langle\delta\rangle_c^2 - 15\langle\delta^2\rangle_c^2\langle\delta\rangle_c \\
&\quad -10\langle\delta^2\rangle_c\langle\delta\rangle_c^3 - \langle\delta\rangle_c^5
\end{aligned}
\tag{4.42}
$$

In most cases $\langle\delta\rangle = 0$ and the above equations simplify considerably. In the following we usually denote σ^2 the local second order cumulant. The Wick theorem then implies that in case of a Gaussian field σ^2 is the only non-vanishing cumulant.

It is important to note that the local PDF is essentially characterized by its *cumulants* which constitute a set of independent quantities. This is important since in most of applications that follow the higher-order cumulants are small compared to their associated moments. Finally, let's note that a useful mathematical property of cumulants is that $\langle(b\delta)^n\rangle_c = b^n\langle\delta^n\rangle_c$, and $\langle(b+\delta)^n\rangle_c = \langle\delta^n\rangle_c$ where b is an ordinary number.

The density distribution is usually smoothed with a filter W_R of a given size, R, commonly a top-hat or a Gaussian window. Indeed, this is required by the discrete nature of galaxy catalogs and N-body experiments used to simulate them. Moreover, we shall see later that the scale-free nature of gravitational clustering implies some remarkable properties about the scaling behavior of the smoothed density distribution. The quantities of interest are then the moments $\langle\delta_R^p\rangle$ and the cumulants $\langle\delta_R^p\rangle_c$ of the smoothed density field

$$\delta_R(\mathbf{x}) = \int W_R(\mathbf{x}' - \mathbf{x})\delta(\mathbf{x}')\mathrm{d}^3\mathbf{x}'. \tag{4.43}$$

Note that for the top hat window,

$$\langle\delta_R^p\rangle_c = \int_{v_R} \xi_p(\mathbf{x}_1, \ldots, \mathbf{x}_p)\frac{\mathrm{d}^{\mathcal{D}}\mathbf{x}_1 \ldots \mathrm{d}^{\mathcal{D}}\mathbf{x}_p}{v_R^p} \tag{4.44}$$

(where $\mathcal{D} = 2$ or 3 is the dimension of the field) is nothing but the average of the N-point correlation function over the corresponding cell of volume v_R.

4.6 Power Spectrum Evolution in Linear Perturbation Theory

The simplest (trivial) application of the gravitational perturbation theory is the leading order contribution to the evolution of the power spectrum. Since we are dealing with the two-point

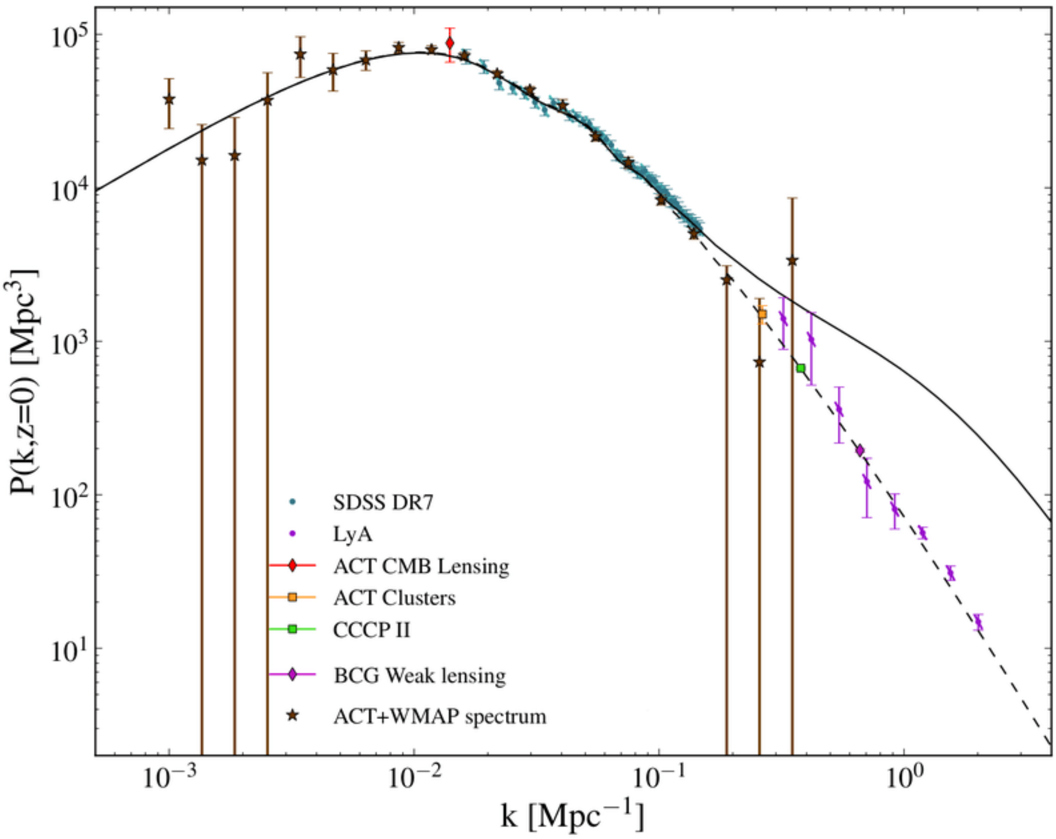

Figure 4.5: The matter power spectrum.

function in Fourier space ($N = 2$), only linear theory is required, that is, the connected part is just given by a single line joining the two points.

In this lecture we are concerned about time evolution of the cosmic fields during the matter domination epoch. In this case, as we discussed previously, diffusion effects are negligible and the evolution can be cast in terms of perfect fluid equations that describe conservation of mass and momentum. In this case, the evolution of the density field is given by a simple time-dependent scaling of the "linear" power spectrum

$$P(k, \tau) = [D_1^{(+)}(\tau)]^2 \; P_L(k) \tag{4.45}$$

where $D_1^{(+)}(\tau)$ is the growing part of the linear growth factor. One must note, however, that the "linear" power spectrum specified by $P_L(k)$ derives from the linear evolution of density fluctuations through the radiation domination era and the resulting decoupling of matter from radiation. This evolution must be followed by using general relativistic Boltzmann numerical codes, although analytic techniques can be used to understand quantitatively the results. The end result is that

$$P_L(k) \propto k^{n_s}\, T^2(k) \tag{4.46}$$

where n_s is the primordial spectral index ($n_s = 1$ denotes the canonical scale-invariant spectrum), $T(k)$ is the transfer function that describes the evolution of the density field perturbations through decoupling ($T(0) \equiv 1$). It depends on cosmological parameters in a complicated way, although in simple cases (where the baryonic content is negligible) it can be approximated by a fitting function that depends on the shape parameter $\Gamma \equiv \Omega_m h$. For the adiabatic cold dark matter (CDM) scenario, $T^2(k) \to \ln^2(k)/k^4$ as $k \to \infty$, due to the suppression of fluctuations growth during the radiation dominated era.

4.7 CMB power spectra

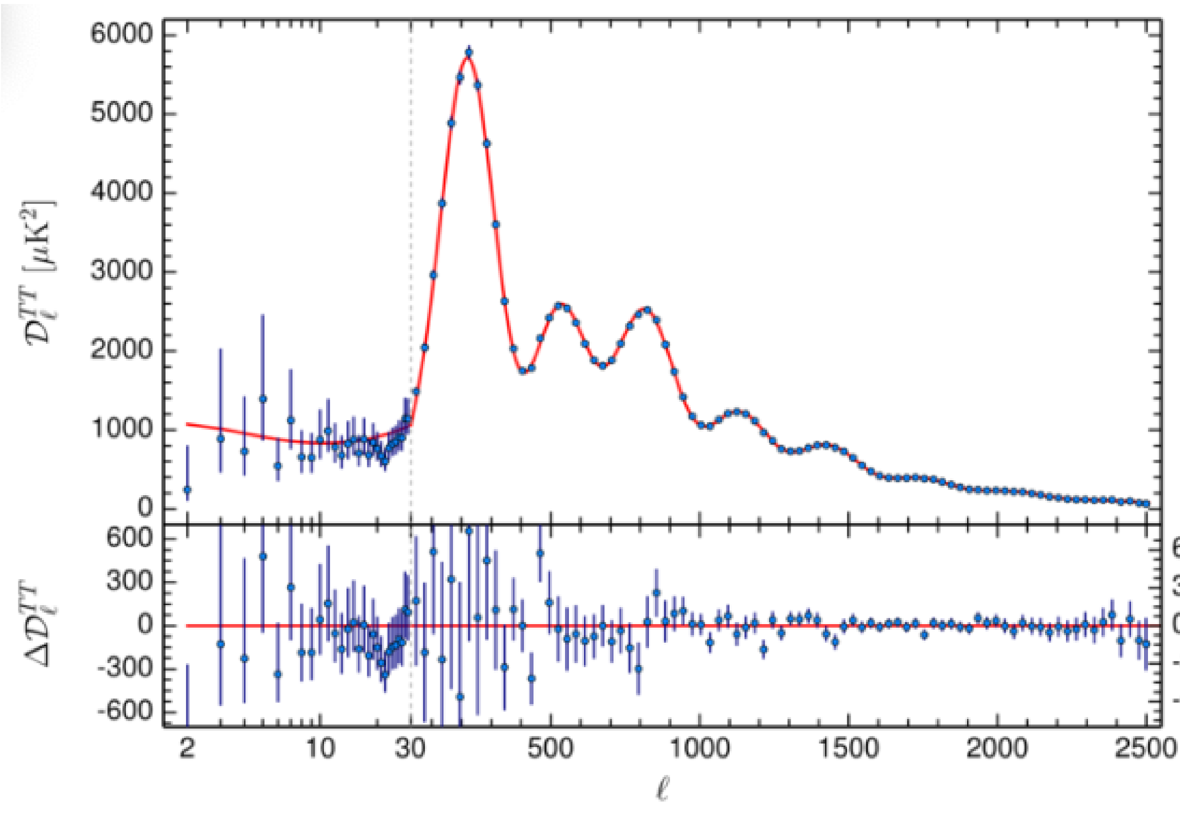

Figure 4.6: CMB TT spectrum. The redline is the best fit ΛCDM model.

CMB photons do not come from a localised source but from the whole celestial sphere and from a finite distance, the last scattering surface, which corresponds to a redshift of $z \sim 1100$. Hence, one usually approaches the CMB power spectrum differently from the matter one (this coming from the observation of galaxies).

We shall define the relative fluctuation in the temperature as $\Theta(\hat{n}) = \delta T(\hat{n})/T$. Define the **two-point angular correlation function** as

$$C(\theta) \equiv \langle \Theta(\hat{n})\Theta(\hat{n}') \rangle = \sum_{\ell} C_{\ell}^{TT} \frac{2\ell+1}{4\pi} \mathcal{P}_{\ell}(\hat{n}\cdot\hat{n}') = \sum_{\ell} C_{\ell}^{TT} \frac{2\ell+1}{4\pi} \mathcal{P}_{\ell}(\cos\theta), \tag{4.47}$$

where $\cos\theta = \hat{n}\cdot\hat{n}'$ and the expansion is in terms of Legendre polynomial $P_{\ell}(\theta)$. The factor $(2\ell+1)/4\pi$ is chosen for later convenient.

It is customary to expand a function on the sphere in spherical harmonics $Y_{\ell m}(\theta,\phi)$. Hence, for our temperature fluctuation we have:

$$\Theta(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}^{T} Y_{\ell m}(\theta,\phi) \tag{4.48}$$

where we have written $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ (i.e. employed spherical coordinates) and $Y_{\ell m}(\theta,\phi)$ are the spherical harmonics.

The spherical harmonics is

$$\begin{aligned} Y_{\ell m}(\hat{n}) &= Y_{\ell m}(\theta,\phi), \\ &= \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_{\ell m}(\cos\theta)\exp(im\phi), \end{aligned} \tag{4.49}$$

where $P_{\ell m}(\cos\theta)$ is the associated Legendre function. The normalization convention for $Y_{\ell m}(\hat{n})$ guarantees orthonormality, i.e.

$$\int d^2\hat{n}\; Y_{\ell m}(\hat{n}) Y^{*}_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\delta_{mm'}\ . \tag{4.50}$$

A useful relation is the spherical harmonic addition theorem:

$$P_{\ell}(\cos\theta) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n}) Y^{*}_{\ell m}(\hat{n}'). \tag{4.51}$$

Note that the spherical harmonics $Y_{\ell m}$ are in general complex and so the $a_{\ell m}$ must also be such, in order for $\Theta(\theta,\phi)$ to be real. Note that the expansion (4.48) can be done at each point of spacetime, but then the angular dependence changes because the (θ,ϕ) measured from one point in space correspond to different celestial coordinates as seen from another spot in space.

The expansion of Eq. (4.48) can be inverted as follows:

$$a_{\ell m}^{T} = \int d^2\hat{n}\; Y^{*}_{\ell m}(\hat{n})\Theta(\hat{n})\ , \tag{4.52}$$

and the $a_{\ell m}$'s are promoted to stochastic variable, just as Θ is. Again, it is the initial conditions for cosmological perturbations which are actual stochastic variables for which inflation predicts a power spectrum.

For Gaussian perturbations, the expectation value and variance of the $a_{\ell m}^{T}$'s are:

$$\langle a_{\ell m}^{T} \rangle = 0, \qquad \langle a_{\ell m}^{T} a_{\ell' m'}^{T*} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_{\ell}^{TT} \tag{4.53}$$

and the $C_{\ell}^{TT} = \langle |a_{\ell m}^{T}|^2 \rangle$ form the CMB power spectrum.

4.8 Cosmic Variance of angular power spectrum

In this section we compute the cosmic variance of the angular power spectra. The result can be applied to any field defined on the sphere and for Gaussian perturbations, which are characterised by a correlation function which depends only on the angular separation and thus the power spectrum is C_ℓ, depending only on ℓ.

In order to compute the cosmic variance, let us do first an estimate. Our objective is to determine C_ℓ observationally, in order to compare it with our theoretical prediction. In order to do that, we probe $\langle a_{\ell m} a^*_{\ell m}\rangle$ for different values of m, which are $2\ell+1$. Hence, we have $2\ell+1$ possible sampling of C_ℓ for any given ℓ and a sampling error

$$\Delta C_\ell \propto \sqrt{2\ell+1}\,. \tag{4.54}$$

The relative error associated to the sampling, i.e. the cosmic variance, is thus:

$$\sigma_{C_\ell} = \frac{\Delta C_\ell}{2\ell+1} \propto \frac{1}{\sqrt{2\ell+1}}\,. \tag{4.55}$$

Now, let us make a more precise calculation. Consider the temperature-temperature correlation function, as an example:

$$\langle \Theta(\hat{n})\Theta(\hat{n}')\rangle = \sum_{\ell m \ell' m'} \langle a_{\ell m} a_{\ell' m'}\rangle Y_{\ell m}(\hat{n}) Y_{\ell' m'}(\hat{n}') = \sum_{\ell m} C_\ell Y_{\ell m}(\hat{n}) Y_{\ell,-m}(\hat{n}')\,, \tag{4.56}$$

where $\cos\theta \equiv \hat{n}\cdot\hat{n}'$ and we have performed the ensemble average assuming Gaussian perturbations. The $-m$ of the second spherical harmonics comes from the reality condition by which $a^*_{\ell m} = a_{\ell,-m}$.

Using the addition theorem of the spherical harmonics, we can do the sum over m in the above formula and obtain:

$$C(\theta) \equiv \langle \Theta(\hat{n})\Theta(\hat{n}')\rangle = \sum_\ell C_\ell \frac{2\ell+1}{4\pi}\mathcal{P}_\ell(\hat{n}\cdot\hat{n}') = \sum_\ell C_\ell \frac{2\ell+1}{4\pi}\mathcal{P}_\ell(\cos\theta)\,. \tag{4.57}$$

So we have explicitly proved that for Gaussian perturbations the correlation function depends only on the angle between the two directions. Inverting this relation using the orthonormality of the Legendre polynomials, we get:

$$C_\ell^{\rm th} = \frac{1}{4\pi}\int d^2\hat{n}\, d^2\hat{n}'\, \mathcal{P}_\ell(\hat{n}\cdot\hat{n}')\langle \Theta(\hat{n})\Theta(\hat{n}')\rangle\,. \tag{4.58}$$

The integral is of course on the whole sky. These are the theoretical C_ℓ^{th}'s, and the average is the ensemble one. Observationally, the only average that we can do is the angular one, i.e.

$$C_\ell^{\rm obs} = \frac{1}{4\pi}\int d^2\hat{n}\, d^2\hat{n}'\, \mathcal{P}_\ell(\hat{n}\cdot\hat{n}')\Theta(\hat{n})\Theta(\hat{n}')\,. \tag{4.59}$$

One can show that

$$C_\ell^{\rm th} = \langle |a_{\ell m}|^2 \rangle, \tag{4.60}$$

and

$$C_\ell^{\rm obs} = \frac{1}{2\ell+1} \sum_m |a_{\ell m}|^2. \tag{4.61}$$

Here it appears more clearly that for each value of ℓ we have $2\ell+1$ possible realisations and thus we expect that the counting error is $\sqrt{2\ell+1}$. We can calculate this exactly, and the cosmic variance is the following ensemble average:

$$\sigma^2_{C_\ell} = \left\langle \left(\frac{C_\ell - C_\ell^{\rm obs}}{C_\ell} \right)^2 \right\rangle = 1 - 2\frac{\langle C_\ell^{\rm obs} \rangle}{C_\ell} + \frac{1}{C_\ell^2} \langle C_\ell^{\rm obs^2} \rangle \ . \tag{4.62}$$

Of course, the ensemble average of $\langle C_\ell^{\rm obs} \rangle$ is C_ℓ, just look at the integral which defines it, and the ensemble average of C_ℓ is C_ℓ, since it is already averaged. Therefore, let us focus on

$$\langle C_\ell^{\rm obs^2} \rangle = \frac{1}{(2\ell+1)^2} \sum_{mm'} \langle a_{\ell m} a_{\ell,-m} a_{\ell m'} a_{\ell,-m'} \rangle \ . \tag{4.63}$$

Since the perturbations are Gaussian, this 4-point correlator can be split into the following sum:

$$\begin{aligned} \langle a_{\ell m} a_{\ell,-m} a_{\ell m'} a_{\ell,-m'} \rangle = {} & \langle a_{\ell m} a_{\ell,-m} \rangle \langle a_{\ell m'} a_{\ell,-m'} \rangle + \langle a_{\ell m} a_{\ell m'} \rangle \langle a_{\ell,-m} a_{\ell,-m'} \rangle \\ & + \langle a_{\ell m} a_{\ell,-m'} \rangle \langle a_{\ell m'} a_{\ell,-m} \rangle \ . \end{aligned} \tag{4.64}$$

Using Eq. (4.53), we finally get:

$$\boxed{\sigma^2_{C_\ell} = \frac{2}{2\ell+1}} \tag{4.65}$$

as expected.

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Index