NARESUAN UNIVERSITY
The Institute for Fundamental Study (IF)

LINEAR SPACES, EXAMPLES, THE DIFFERENT HILBERT SPACES IN QUANTUM MECHANICS

FOURIER SERIES, FOURIER TRANSFORM, DIRAC'S DELTA

SUMMER SCHOOL
Einstein's Term 2026

Homework assignment

(Final update: May 25, 2026)

1st) Consider the following discrete orthonormal basis in $\mathcal{L}^2(\Omega)$, where Ω is the interval $[0, L]$:

$$\psi_j(x) = \sqrt{\frac{1}{L}} \exp\left(\mathrm{i}\frac{2\pi j}{L} x\right), \quad j = \cdots, -2, -1, 0, +1, +2, \cdots . \tag{1.1}$$

Every function $\psi(x) \in \mathcal{L}^2(\Omega)$ can be expanded in one and only one way in terms of the $\{\psi_j(x)\}$:

$$\psi(x) = \sum_{j=-\infty}^{\infty} C_j \, \psi_j(x). \tag{1.2}$$

As you know, the coefficients of the latter series are given by the formula $C_j = \langle \psi_j, \psi \rangle$. Also note that $\psi(x)$ satisfies the periodic boundary condition, i.e., $\psi(0) = \psi(L)$. **(a)** Try to obtain the typical Fourier series from the series given above, namely,

$$\psi(x) = A_0 + \sum_{j=1}^{\infty} A_j \, \cos\left(\frac{2\pi j}{L} x\right) + \sum_{j=1}^{\infty} B_j \, \sin\left(\frac{2\pi j}{L} x\right), \tag{1.3}$$

with the coefficients A_0, A_j and B_j written in terms of the coefficients C_j, namely,

$$A_0 = \sqrt{\frac{1}{L}} \, C_0 = \frac{1}{L} \int_0^L \mathrm{d}y \, \psi(y) \, , \tag{1.4}$$

$$A_j = \sqrt{\frac{1}{L}} \, (C_j + C_{-j}) = \frac{2}{L} \int_0^L \mathrm{d}y \, \cos\left(\frac{2\pi j}{L} y\right) \psi(y) \, , \tag{1.5}$$

$$B_j = \sqrt{\frac{1}{L}} \, \mathrm{i}(C_j - C_{-j}) = \frac{2}{L} \int_0^L \mathrm{d}y \, \sin\left(\frac{2\pi j}{L} y\right) \psi(y) \, . \tag{1.6}$$

(b) Demonstrate that $\psi(x)$ is real if and only if $C_{-j} = C_j^*$ (the asterisk * denotes the complex conjugate, as usual), and therefore, the coefficients A_0, A_j and B_j are real.

2nd) Try to demonstrate the following integral representation for the Dirac delta:

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} = \lim_{a\to 0} \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} \mathrm{e}^{-a|x|} = 2\pi\, \delta(k). \tag{2.1}$$

Note that, I am proposing you to add a regularizing factor to derive the expression. You will also need to use the following representation of the Dirac delta:

$$\lim_{a\to 0} \frac{1}{\pi} \frac{a}{a^2 + k^2} = \delta(k). \tag{2.2}$$

Note: In both limits, the parameter a approaches zero from the positive side.

3rd) As presented in class, if $\psi(x)$ is a (real or complex) function of the variable x, its Fourier transform $\phi(k)$, if it exists, is defined by

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x)\, \mathrm{e}^{-\mathrm{i}kx} \equiv \mathrm{FT}[\psi(x)], \tag{3.1}$$

and the inverse formula is:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k\, \phi(k)\, \mathrm{e}^{+\mathrm{i}kx} \equiv (\mathrm{FT})^{-1}[\phi(k)]. \tag{3.2}$$

(a) Demonstrate the following property:

$$\mathrm{FT}[\psi(cx)] = \frac{1}{|\,c\,|}\, \phi\left(\frac{k}{c}\right), \tag{3.3}$$

where c is a (real) nonzero constant. **(b)** In particular, $\mathrm{FT}[\psi(-x)] = \phi(-k)$, and together with $\mathrm{FT}[\psi(x)] = \phi(k)$ (see Eq. (3.1)), it follows that if the function $\psi(x)$ has a definite parity, its Fourier transform $\phi(k)$ has the same parity. Prove it!

4th) (a) Let us consider the following function:

$$\psi(x) = \frac{1}{\sqrt{2a}} \left[\, \Theta(x+a) - \Theta(x-a) \,\right], \tag{4.1}$$

where $a > 0$ and $\Theta(x)$ is the Heaviside step function, namely, $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$. Find the Fourier transform of $\psi(x)$, namely,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \psi(x)\, \mathrm{e}^{-\mathrm{i}kx}. \tag{4.2}$$

(b) Is the function $\psi(x)$ normalized to unity? In other words, is $\psi(x)$ a square-integrable function with norm equal to one? (c) But furthermore, is also $\phi(k)$ a square-integrable function with norm equal to one? Hint: You may find the following integral useful:

$$\int_0^{+\infty} \mathrm{d}u \, \frac{\sin^2(u)}{u^2} = \frac{\pi}{2}. \tag{4.3}$$

5th) Prove the following result (in fact, it is a theorem): If $\phi(k)$ and $\varphi(k)$ are the respective Fourier transforms of the square-integrable functions $\psi(x)$ and $\chi(x)$, one has that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \chi^*(x) \, \psi(x) = \int_{-\infty}^{+\infty} \mathrm{d}k \, \varphi^*(k) \, \phi(k). \tag{5.1}$$

A particular case of this result is the conservation of the norm

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, |\psi(x)|^2 = \int_{-\infty}^{+\infty} \mathrm{d}k \, |\phi(k)|^2 \,, \tag{5.2}$$

that is, a function and its Fourier transform have the same norm. This result is called the Parseval-Plancherel formula.

6th) Using the so-called sifting property, that is,

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \delta(x - x_0) f(x) = \int_{-\infty}^{+\infty} \mathrm{d}x \, \delta(x - x_0) f(x_0) = f(x_0) \int_{-\infty}^{+\infty} \mathrm{d}x \, \delta(x - x_0) = f(x_0), \tag{6.1}$$

prove the following properties of the Dirac delta: **(a)** $\delta(cx) = \delta(x)/\,|c|$, where $c \neq 0$ is a scalar (this is a scaling property). Note that this result shows that $\delta(-x) = \delta(x)$, i.e., the Dirac delta is an even distribution, or generalized function. **(b)** $x\delta(x) = 0$ (in fact, it can be demonstrated that the equation $xg(x) = 0$ has the general solution $g(x) = \text{const} \times \delta(x)$). **(c)** $\delta'(x) = -\delta(x)/x$, i.e., $x\delta'(x) = -\delta(x)$ (Roughly speaking, what would the graph of $\delta'(x)$ look like?). **(d)** $x^2\delta'(x) = 0$. **(e)** $\delta(x) = \mathrm{d}\Theta(x)/\mathrm{d}x$; and therefore, **(f)** $\Theta(x) = \int_{-\infty}^{x} \mathrm{d}y \, \delta(y)$.

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