2026 IF Summer School
Field theory 1-2

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Recommended exercises for tutorial: 1.1, 1.2, 1.3, 1.4, 2.1, 2.2, 2.3, 2.4

1 Path Integral in Quantum Mechanics

Later in this school, we will be learning path integral quantization in field theory. The purpose of this lecture note is to prepare students for two main topics: path integral quantisation and field theory. In particular, this section will first discuss path integral quantization as applied to quantum mechanics, then the next section will be on classical field theory. After studying these students should then be ready to learn path integral quantisation on field theory which is a topic to be given later on in this school.

In order to describe quantum mechanics and quantum field theory, there are two important alternative methods which are canonical quantisation and path integral quantisation.

The key idea of canonical quantisation is that observables for example x and p are promoted to operators. As for the path integral quantisation, the key idea is that a particle follows every possible paths. Important physical quantities can be obtained by summing "phase", described by classical action, of all possible paths.

Path integral quantization has some advantages over canonical quantisation. One important advantage is that path integral quantization only involves complex-valued functions but not operators as encountered in canonical quantization. So in path integral quantisation, we normally do not need to worry about commutators. Another advantage is seen especially in quantum field theory. Path integral quantization is more beautiful than canonical quantisation. For example, Lorentz invariance can easily be kept while working with the path integral quantisation

1.1 Propagator and its phase-space path integral expression

Consider an isolated quantum system, in which the Hamiltonian is time-independent (in the Schrödinger picture). This system describes a particle moving in one dimension.

Suppose that at time t_a, the system is described by the state ket in the Schrödinger picture as

$$|\psi, t_a\rangle. \tag{1.1}$$

The state is evolved in time such that at time t_b it is given by

$$|\psi, t_b\rangle = U(t_b, t_a)|\psi, t_a\rangle, \tag{1.2}$$

where

$$U(t_b, t_a) = e^{\frac{i}{\hbar}(t_b - t_a)H}. \tag{1.3}$$

It can be shown that $U(t_b, t_a)$ is unitary, i.e. $U^\dagger(t_b, t_a)U(t_b, t_a) = \mathbb{1}$.

Exercise 1.1. Show that $U(t_b - t_a)$ as defined in the above equation is unitary. $\diamond$

Inserting the completeness relation

$$\int dx_a |x_a\rangle\langle x_a| = \mathbb{1} \tag{1.4}$$

into the equation (1.2) gives

$$|\psi, t_b\rangle = \int dx_a U(t_b, t_a)|x_a\rangle\langle x_a|\psi, t_a\rangle. \tag{1.5}$$

Applying $\langle x_b|$ gives

$$\langle x_b|\psi, t_b\rangle = \int dx_a \langle x_b|U(t_b, t_a)|x_a\rangle\langle x_a|\psi, t_a\rangle. \tag{1.6}$$

Since $\psi(x, t) = \langle x|\psi, t\rangle$, we obtain

$$\psi(x_b, t_b) = \int dx_a K(x_b, t_b; x_a, t_a)\psi(x_a, t_a), \tag{1.7}$$

where the quantity

$$K(x_b, t_b; x_a, t_a) = \langle x_b|U(t_b, t_a)|x_a\rangle \tag{1.8}$$

is called the propagator. We may also use the shorthand

$$(x_b t_b|x_a t_a) \equiv K(x_b, t_b; x_a, t_a). \tag{1.9}$$

In this section, physics that we are interested in is that we prepare the state of the system at time t_a, we want to learn how the state evolves as time passes. The state of the system at time t_b is given by

$$|\psi, t_b\rangle = U(t_b, t_a)|\psi, t_a\rangle, \tag{1.10}$$

or

$$\psi(x_b, t_b) = \int dx_a K(x_b, t_b; x_a, t_a)\psi(x_a, t_a). \tag{1.11}$$

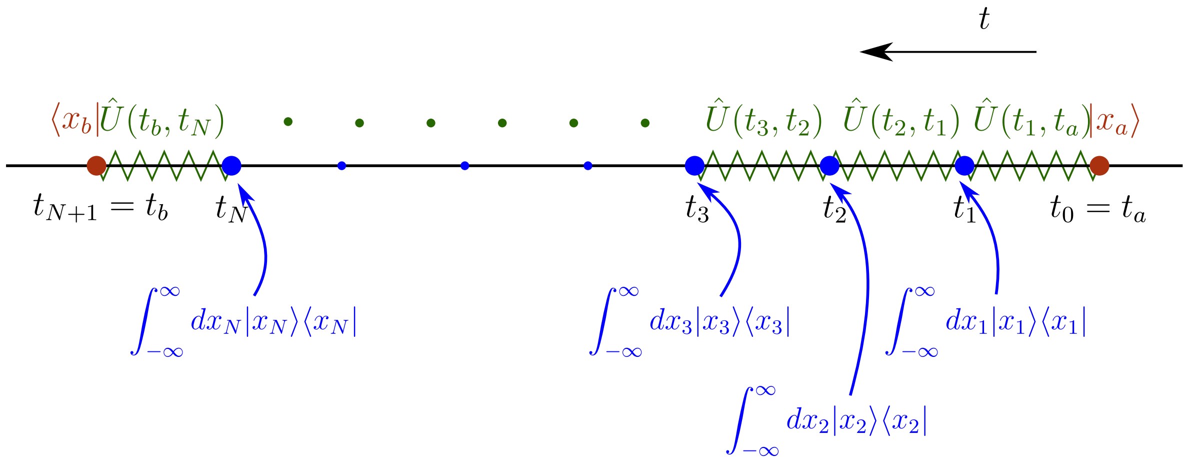

Figure 1: The illustration of dividing the time interval. The operator U is inserted on each link. A Completeness relation is inserted at each point.

The operator $U(t_b,t_a)$ or the propagator $K(x_b,t_b;x_a,t_a)$ encodes the information on how states evolves in time. One way to compute the propagator is by using path integral.

We may compute the propagator by using path integral as follows.

In figure 1, we divide the time interval $t_a \leq t \leq t_b$ into $N+1$ equal pieces. This allows us to rewrite the transition amplitude as

$$(x_b t_b|x_a t_a) = \langle x_b|U(t_b,t_N)\cdots U(t_3,t_2)U(t_2,t_1)U(t_1,t_a)|x_a\rangle \tag{1.12}$$

Pictorially, $U(t_n,t_{n-1})$ is on the link joining the points t_n, t_{n-1}. Next let us insert on each point the expression

$$\int_{-\infty}^{\infty} dx_n|x_n\rangle\langle x_n| = 1. \tag{1.13}$$

From the figure, we note that there are N internal sites and $N+1$ links. So it is easily seen that the transition amplitude now becomes

$$\begin{aligned}(x_b t_b|x_a t_a) &= \left(\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right)\prod_{n=1}^{N+1}\langle x_n|U(t_n,t_{n-1})|x_{n-1}\rangle \\ &= \left(\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right)\prod_{n=1}^{N+1}(x_n t_n|x_{n-1}t_{n-1}).\end{aligned} \tag{1.14}$$

So in order to obtain the amplitude for the whole interval, we need to compute the amplitude on each small intervals. Consider

$$\begin{aligned}U(t_n,t_{n-1}) &= \exp\left(-\frac{i}{\hbar}(t_n - t_{n-1})H\right) \\ &= \exp\left(-\frac{i}{\hbar}\epsilon H\right),\end{aligned} \tag{1.15}$$

where $\epsilon = t_n - t_{n-1}$. Suppose that the Hamiltonian is of the form

$$H = T(p) + V(x). \tag{1.16}$$

Then

$$\begin{aligned} \exp\left(-\frac{i}{\hbar}\epsilon H\right) &= \exp\left(-\frac{i}{\hbar}\epsilon(T+V)\right) \\ &= \exp\left(-\frac{i}{\hbar}\epsilon V\right)\exp\left(-\frac{i}{\hbar}\epsilon T\right)\exp\left(-\frac{i}{\hbar^2}\epsilon^2 X\right). \end{aligned} \tag{1.17}$$

Then since $t_b - t_a = \epsilon(N+1)$, we consider the limit where $N \to \infty, \epsilon \to 0$, while keeping $\epsilon(N+1) = t_b - t_a$. So

$$\begin{aligned} \exp\left(-\frac{i}{\hbar}(t_b - t_a)H\right) &= \lim_{N\to\infty} \exp\left(-\frac{i}{\hbar}\epsilon(N+1)H\right) \\ &= \lim_{N\to\infty}\left(\exp\left(-\frac{i}{\hbar}\epsilon V\right)\exp\left(-\frac{i}{\hbar}\epsilon T\right)\exp\left(-\frac{i}{\hbar^2}\epsilon^2 X\right)\right)^{N+1} \\ &= \lim_{N\to\infty}\left(\exp\left(-\frac{i}{\hbar}\epsilon V\right)\exp\left(-\frac{i}{\hbar}\epsilon T\right)\right)^{N+1}, \end{aligned} \tag{1.18}$$

where the last equality can be rigorously shown, but we omit the proof. The above equation is known as the Trotter product formula

$$\exp\left(-\frac{i}{\hbar}(t_b - t_a)H\right) = \lim_{N\to\infty}\left(\exp\left(-\frac{i}{\hbar}\epsilon V\right)\exp\left(-\frac{i}{\hbar}\epsilon T\right)\right)^{N+1}. \tag{1.19}$$

So essentially, by comparing eq.(1.19) with

$$U(t_b, t_a) = U(t_b, t_N)\cdots U(t_3, t_2)U(t_2, t_1)U(t_1, t_a), \tag{1.20}$$

and using eq.(1.15), we have that in the limit $N \to \infty$ and for $1 \leq n \leq N+1$,[1]

$$U(t_n, t_{n-1}) \approx \exp\left(-\frac{i}{\hbar}\epsilon V\right)\exp\left(-\frac{i}{\hbar}\epsilon T\right). \tag{1.21}$$

[1]The approximation symbol we use does not mean that we are not being accurate. It simply means that after substituting each side into the full expression and taking the limit $N \to \infty$, they will completely agree with each other.

Then,

$$\begin{aligned}
(x_n t_n|x_{n-1}t_{n-1}) &\approx \langle x_n|\exp\left(-\frac{i}{\hbar}\epsilon V\right)\exp\left(-\frac{i}{\hbar}\epsilon T\right)|x_{n-1}\rangle \\
&= \int dx \langle x_n|\exp\left(-\frac{i}{\hbar}\epsilon V\right)|x\rangle\langle x|\exp\left(-\frac{i}{\hbar}\epsilon T\right)|x_{n-1}\rangle \\
&= \int dx \langle x_n|x\rangle \exp\left(-\frac{i}{\hbar}\epsilon V(x)\right)\langle x|\exp\left(-\frac{i}{\hbar}\epsilon T\right)|x_{n-1}\rangle \\
&= \int dx \delta(x_n - x)\exp\left(-\frac{i}{\hbar}\epsilon V(x)\right)\langle x|\exp\left(-\frac{i}{\hbar}\epsilon T\right)|x_{n-1}\rangle \\
&= \exp\left(-\frac{i}{\hbar}\epsilon V(x_n)\right)\langle x_n|\exp\left(-\frac{i}{\hbar}\epsilon T\right)|x_{n-1}\rangle \\
&= \int \frac{dp_n}{2\pi\hbar}\exp\left(-\frac{i}{\hbar}\epsilon V(x_n)\right)\langle x_n|p_n\rangle\langle p_n|\exp\left(-\frac{i}{\hbar}\epsilon T\right)|x_{n-1}\rangle \\
&= \int \frac{dp_n}{2\pi\hbar}\exp\left(-\frac{i}{\hbar}\epsilon V(x_n)\right)\langle x_n|p_n\rangle\langle p_n|x_{n-1}\rangle\exp\left(-\frac{i}{\hbar}\epsilon T(p_n)\right) \\
&= \int \frac{dp_n}{2\pi\hbar}\exp\left(\frac{i}{\hbar}p_n(x_n - x_{n-1})\right)\exp\left(-\frac{i}{\hbar}\epsilon V(x_n)\right)\exp\left(-\frac{i}{\hbar}\epsilon T(p_n)\right) \\
&= \int \frac{dp_n}{2\pi\hbar}\exp\left(\frac{i}{\hbar}\left(p_n(x_n - x_{n-1}) - \epsilon(T(p_n) + V(x_n))\right)\right).
\end{aligned} \tag{1.22}$$

Therefore,

$$\begin{aligned}
(x_b t_b|x_a t_a) &\approx \left(\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right)\prod_{n=1}^{N+1}\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\exp\left(\frac{i}{\hbar}\left(p_n(x_n - x_{n-1}) - \epsilon(T(p_n) + V(x_n))\right)\right) \\
&= \left(\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right)\left(\prod_{n=1}^{N+1}\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right)\prod_{n=1}^{N+1}\exp\left(\frac{i}{\hbar}\left(p_n(x_n - x_{n-1}) - \epsilon(T(p_n) + V(x_n))\right)\right) \\
&= \left(\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right)\left(\prod_{n=1}^{N+1}\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right)\exp\left(\frac{i}{\hbar}\sum_{n=1}^{N+1}\left(p_n(x_n - x_{n-1}) - \epsilon(T(p_n) + V(x_n))\right)\right) \\
&\equiv \left(\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right)\left(\prod_{n=1}^{N+1}\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right)\exp\left(\frac{i}{\hbar}A^N\right)
\end{aligned} \tag{1.23}$$

where

$$A^N \equiv \sum_{n=1}^{N+1}\left(p_n(x_n - x_{n-1}) - \epsilon H(x_n, p_n)\right). \tag{1.24}$$

In the limit $N \to \infty$, we obtain Feynman's path integral formula

$$\boxed{(x_b t_b|x_a t_a) = \lim_{N\to\infty}\left(\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right)\left(\prod_{n=1}^{N+1}\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right)\exp\left(\frac{i}{\hbar}A^N\right),} \tag{1.25}$$

or in practice we write

$$(x_bt_b|x_at_a) = \int \mathcal{D}x(t) \int \mathcal{D}p(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt(p(t)\dot{x}(t) - H(x(t),p(t)))\right). \tag{1.26}$$

1.2 Path integral expression for propagator

Consider the Hamiltonian of the form

$$H(\hat{x},\hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.27}$$

The propagator in eq.(1.26) is then

$$(x_bt_b|x_at_a) = \int \mathcal{D}x(t) \int \mathcal{D}p(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt\left(p(t)\dot{x}(t) - \frac{p^2(t)}{2m} - V(x(t))\right)\right). \tag{1.28}$$

We may integrate over $p(t)$ by using Gaussian integral. For this, let us first consider

$$\int_{-\infty}^{\infty} dx\ e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}, \tag{1.29}$$

where $a > 0$, $b \in \mathbb{R}$. The integral can actually be extended to the case where $\Re(a) \geq 0$ but $a \neq 0$. For this, we write $a = |a|e^{i\theta}$, where $\theta \in [-\pi/2, \pi/2]$.

An important example is

$$\int_{-\infty}^{\infty} dx\ e^{-i\alpha x^2+i\beta x} = e^{-i\pi/4}\sqrt{\frac{\pi}{\alpha}} e^{\frac{i\beta^2}{4\alpha}}, \tag{1.30}$$

where $\alpha > 0$.

Exercise 1.2. For, $a > 0$, Show that

$$\int_{-\infty}^{\infty} dx\ e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{1.31}$$

then show that for $a > 0$, $b \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} dx\ e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}. \tag{1.32}$$

◇

Exercise 1.3. Suppose we want to evaluate

$$\int_{-\infty}^{\infty} dx\ e^{-x^2} \tag{1.33}$$

using eq.(1.29) with $a = e^{2\pi i}, b = 0$. Investigate what happens and explain why our approach is incorrect. ◇

The integral eq.(1.29) can also be extended to higher dimensions. This is

$$\int d^n x \; e^{-\vec{x}^T A\vec{x}+\vec{b}\cdot\vec{x}} = \frac{\pi^{n/2}}{\sqrt{\det A}} \exp\left(\frac{1}{4}\vec{b}^T A^{-1}\vec{b}\right). \qquad (1.34)$$

We may extend to the case we are interested in, where the number of variables is continuous. We have

$$\int \mathcal{D}p(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left(p(t)\dot{x}(t) - \frac{p^2(t)}{2m}\right)\right) \propto \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt \frac{1}{2}m\dot{x}^2(t)\right). \qquad (1.35)$$

So eq.(1.28) becomes

$$(x_b t_b|x_a t_a) \propto \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left(\frac{1}{2}m\dot{x}^2(t) - V(x(t))\right)\right), \qquad (1.36)$$

or

$$(x_b t_b|x_a t_a) \propto \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt \; \mathcal{L}(x(t),\dot{x}(t))\right). \qquad (1.37)$$

1.3 Fluctuation around classical solution

One way to evaluate eq.(1.37) is to write

$$x(t) = x_{cl}(t) + h(t), \qquad (1.38)$$

where $x_{cl}(t)$ is the classical solution such that $x_a = x(t_a) = x_{cl}(t_a), x_b = x(t_b) = x_{cl}(t_b)$. Therefore, $h(t_b) = h(t_a) = 0$. Since $x_{cl}(t)$ is the classical solution, it satisfies

$$\frac{d}{dt}\frac{\partial \mathcal{L}(x_{cl}(t),\dot{x}_{cl}(t))}{\partial \dot{x}_{cl}(t)} - \frac{\partial \mathcal{L}(x_{cl}(t),\dot{x}_{cl}(t))}{\partial x_{cl}(t)} = 0. \qquad (1.39)$$

Next, let us expand $\mathcal{L}(x(t),\dot{x}(t))$. For this, let us denote

$$\mathcal{L}_{cl} \equiv \mathcal{L}(x_{cl}(t),\dot{x}_{cl}(t)), \qquad (1.40)$$

$$\mathcal{L}_x^{cl} \equiv \frac{\partial \mathcal{L}_{cl}}{\partial x_{cl}(t)}, \qquad \mathcal{L}_{\dot{x}}^{cl} \equiv \frac{\partial \mathcal{L}_{cl}}{\partial \dot{x}_{cl}(t)}, \qquad (1.41)$$

$$\mathcal{L}_{xx}^{cl} \equiv \frac{\partial^2 \mathcal{L}_{cl}}{\partial x_{cl}^2(t)}, \qquad \mathcal{L}_{x\dot{x}}^{cl} \equiv \frac{\partial^2 \mathcal{L}_{cl}}{\partial x_{cl}(t)\partial \dot{x}_{cl}(t)}, \qquad \mathcal{L}_{\dot{x}\dot{x}}^{cl} \equiv \frac{\partial^2 \mathcal{L}_{cl}}{\partial \dot{x}_{cl}^2(t)}. \qquad (1.42)$$

Now, the expansion gives

$$
\begin{aligned}
\mathcal{L}(x(t),\dot{x}(t)) &= \mathcal{L}(x_{cl}(t)+h(t),\dot{x}_{cl}(t)+\dot{h}(t)) \\
&= \mathcal{L}_{cl} + \mathcal{L}^{cl}_{x}h(t) + \mathcal{L}^{cl}_{\dot{x}}\dot{h}(t) + \frac{1}{2}\mathcal{L}^{cl}_{xx}h^2(t) + \mathcal{L}^{cl}_{x\dot{x}}h(t)\dot{h}(t) + \frac{1}{2}\mathcal{L}^{cl}_{\dot{x}\dot{x}}\dot{h}^2(t) + \mathcal{O}(h^3) \\
&= \mathcal{L}_{cl} + \left(\mathcal{L}^{cl}_{x} - \frac{d}{dt}\mathcal{L}^{cl}_{\dot{x}}\right)h(t) + h(t)\left(\frac{1}{2}\mathcal{L}^{cl}_{xx} + \mathcal{L}^{cl}_{x\dot{x}}\frac{d}{dt} - \frac{1}{2}\mathcal{L}^{cl}_{\dot{x}\dot{x}}\frac{d^2}{dt^2} + \frac{1}{4}\ddot{\mathcal{L}}^{cl}_{\dot{x}\dot{x}}\right)h(t) \\
&\quad + \frac{d}{dt}\left(\mathcal{L}^{cl}_{\dot{x}}h(t) + \frac{1}{2}\mathcal{L}^{cl}_{\dot{x}\dot{x}}h(t)\dot{h}(t) - \frac{1}{4}\dot{\mathcal{L}}^{cl}_{\dot{x}\dot{x}}h^2(t)\right) + \mathcal{O}(h^3) \\
&= \mathcal{L}_{cl} + h(t)\left(\frac{1}{2}\mathcal{L}^{cl}_{xx} + \mathcal{L}^{cl}_{x\dot{x}}\frac{d}{dt} - \frac{1}{2}\mathcal{L}^{cl}_{\dot{x}\dot{x}}\frac{d^2}{dt^2} + \frac{1}{4}\ddot{\mathcal{L}}^{cl}_{\dot{x}\dot{x}}\right)h(t) \\
&\quad + \frac{d}{dt}\left(\mathcal{L}^{cl}_{\dot{x}}h(t) + \frac{1}{2}\mathcal{L}^{cl}_{\dot{x}\dot{x}}h(t)\dot{h}(t) - \frac{1}{4}\dot{\mathcal{L}}^{cl}_{\dot{x}\dot{x}}h^2(t)\right) + \mathcal{O}(h^3),
\end{aligned}
\tag{1.43}
$$

where in the last step, we use eq.(1.39). Next, integrating the above equation gives

$$
\int_{t_a}^{t_b} dt\ \mathcal{L}(x(t),\dot{x}(t)) = \int_{t_a}^{t_b} dt\ \mathcal{L}_{cl} + \int_{t_a}^{t_b} dt\ h(t)\left(\frac{1}{2}\mathcal{L}^{cl}_{xx} + \mathcal{L}^{cl}_{x\dot{x}}\frac{d}{dt} - \frac{1}{2}\mathcal{L}^{cl}_{\dot{x}\dot{x}}\frac{d^2}{dt^2} + \frac{1}{4}\ddot{\mathcal{L}}^{cl}_{\dot{x}\dot{x}}\right)h(t) + \mathcal{O}(h^3),
\tag{1.44}
$$

where we used $h(t_a) = h(t_b) = 0$. Then eq.(1.37) becomes

$$
\begin{aligned}
(x_b t_b|x_a t_a) \propto\ & e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\ \mathcal{L}_{cl}} \\
& \times \int \mathcal{D}h(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt\ h(t)\left(\frac{1}{2}\mathcal{L}^{cl}_{xx} + \mathcal{L}^{cl}_{x\dot{x}}\frac{d}{dt} - \frac{1}{2}\mathcal{L}^{cl}_{\dot{x}\dot{x}}\frac{d^2}{dt^2} + \frac{1}{4}\ddot{\mathcal{L}}^{cl}_{\dot{x}\dot{x}}\right)h(t) + \mathcal{O}(h^3)\right).
\end{aligned}
\tag{1.45}
$$

If the Lagrangian is quadratic in $x, \dot{x}$, we do not have the terms $\mathcal{O}(h^3)$. On the contrary, if the Lagrangian contains higher order terms in $x, \dot{x}$, the terms $\mathcal{O}(h^3)$ would also appear. The presence of the terms $\mathcal{O}(h^3)$ make it difficult to directly do the path integration.

In fact, we may further simplify eq.(1.45). Since

$$
\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x),
\tag{1.46}
$$

the equation eq.(1.45) becomes

$$
\begin{aligned}
(x_b t_b|x_a t_a) \propto\ & \exp\left(\frac{i}{\hbar}S_{cl}\right) \\
& \times \int \mathcal{D}h(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt\ h(t)\left(-\frac{1}{2}V''(x_{cl}(t)) - \frac{1}{2}m\frac{d^2}{dt^2}\right)h(t)\right),
\end{aligned}
\tag{1.47}
$$

where now

$$
S_{cl} = \left(\frac{1}{2}m(x_b\dot{x}_{cl}(t_b) - x_a\dot{x}_{cl}(t_a)) + \int_{t_a}^{t_b} dt\left(\frac{1}{2}x_{cl}(t)V'(x_{cl}(t)) - V(x_{cl}(t))\right)\right)
\tag{1.48}
$$

1.4 Propagator for free particle

In the case of free particle, i.e. $V(x) = 0$, we have

$$(x_b t_b | x_a t_a) \propto \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt\ \frac{1}{2}m\dot{x}^2\right). \tag{1.49}$$

In order to evaluate this integral, we may consider the discretised form

$$(x_b t_b | x_a t_a) \propto \lim_{N\to\infty}\left(\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right)\exp\left(\sum_{n=1}^{N+1}\frac{i}{\hbar}\frac{1}{2}\frac{m}{\epsilon}(x_n - x_{n-1})^2\right). \tag{1.50}$$

From Gaussian integral, we see the pattern

$$\begin{aligned}(x_b t_b | x_a t_a) &\propto \lim_{N\to\infty}\left(\prod_{n=k}^{N}\int_{-\infty}^{\infty} dx_n\right)\\ &\qquad\times\exp\left(\sum_{n=k+2}^{N+1}\frac{i}{\hbar}\frac{1}{2}\frac{m}{\epsilon}(x_n - x_{n-1})^2 + \frac{i}{\hbar}\frac{1}{2}\frac{m}{\epsilon}\left(\frac{1}{k}x_a^2 - \frac{2x_k(x_a + kx_{k+1})}{k} + \frac{k+1}{k}x_k^2 + x_{k+1}^2\right)\right)\\ &\propto \lim_{N\to\infty}\exp\left(\frac{i}{\hbar}\frac{1}{2}\frac{m}{\epsilon}\frac{x_b^2 + x_a^2 - 2x_a x_b}{N+1}\right)\\ &= \exp\left(\frac{im}{\hbar}\frac{1}{2}\frac{(x_b - x_a)^2}{t_b - t_a}\right).\end{aligned} \tag{1.51}$$

Therefore

$$(x_b t_b | x_a t_a) \propto \exp\left(\frac{im}{\hbar}\frac{1}{2}\frac{(x_b - x_a)^2}{t_b - t_a}\right). \tag{1.52}$$

We expect that

$$\lim_{t_b - t_a \to 0}(x_b t_b | x_a t_a) = \langle x_b | x_a\rangle = \delta(x_b - x_a). \tag{1.53}$$

Dirac delta function can be expressed as a limit of Gaussian distribution

$$\delta(x_b - x_a) = \lim_{\eta\to 0}\frac{1}{\eta\sqrt{2\pi}}\exp\left(-\frac{(x_b - x_a)^2}{2\eta^2}\right). \tag{1.54}$$

By setting

$$\eta = \sqrt{\frac{i\hbar(t_b - t_a)}{m}}, \tag{1.55}$$

we obtain

$$\lim_{t_b - t_a\to 0}(x_b t_b | x_a t_a) = \lim_{t_b - t_a \to 0}\sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}}\exp\left(\frac{im}{\hbar}\frac{1}{2}\frac{(x_b - x_a)^2}{t_b - t_a}\right). \tag{1.56}$$

Comparing with eq.(1.52), we obtain

$$(x_b t_b | x_a t_a) = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}}\exp\left(\frac{im}{\hbar}\frac{1}{2}\frac{(x_b - x_a)^2}{t_b - t_a}\right). \tag{1.57}$$

Exercise 1.4. Let $x_b = x, t_b = t, t_a = 0$.

a) Show that wavefunction for free particle is

$$\psi(x,t) = \int_{-\infty}^{\infty} dx_a \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{im}{2\hbar t}(x - x_a)^2\right) \psi(x_a, 0) \tag{1.58}$$

b) Suppose that initially the wavefunction is a Gaussian wave packet

$$\psi(x_a, 0) = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma}} \exp\left(-\frac{x_a^2}{4\sigma^2} + ikx_a\right), \qquad \sigma > 0. \tag{1.59}$$

Show that

$$\int_{-\infty}^{\infty} dx_a |\psi(x_a, 0)|^2 = 1. \tag{1.60}$$

c) Then show that

$$\psi(x,t) = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma}} \frac{\sigma}{\sqrt{\sigma^2 + \frac{i\hbar t}{2m}}} \exp\left(-\frac{\left(x - \frac{\hbar k t}{m}\right)^2}{4\left(\sigma^2 + \frac{i\hbar t}{2m}\right)} + ikx - i\frac{\hbar k^2}{2m}t\right). \tag{1.61}$$

d) Show that

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2 + \frac{\hbar^2 t^2}{4m^2\sigma^2}}} \exp\left(-\frac{\left(x - \frac{\hbar k t}{m}\right)^2}{2\left(\sigma^2 + \frac{\hbar^2 t^2}{4m^2\sigma^2}\right)}\right). \tag{1.62}$$

e) Explain the qualitative behaviour of $|\psi(x,t)|^2$.

◇

Let us consider another approach, which is the fluctuation around classical solution. Since classical free particle has constant velocity, we have

$$\frac{x_{cl}(t) - x_a}{t - t_a} = \frac{x_b - x_a}{t_b - t_a}. \tag{1.63}$$

So

$$x_{cl}(t) = x_a + \frac{x_b - x_a}{t_b - t_a}(t - t_a). \tag{1.64}$$

So

$$\mathcal{L}_{cl} = \frac{1}{2}m\left(\frac{x_b - x_a}{t_b - t_a}\right)^2, \tag{1.65}$$

$$\mathcal{L}^{cl}_{\dot{x}\dot{x}} = m. \tag{1.66}$$

So eq.(1.45) becomes

$$(x_b t_b | x_a t_a) \propto \exp\left(\frac{i}{\hbar}\frac{1}{2}m\frac{(x_b - x_a)^2}{t_b - t_a}\right) \int \mathcal{D}h(t) \exp\left(-\frac{i}{\hbar}\frac{1}{2}m \int_{t_a}^{t_b} dt\ h(t)\frac{d^2}{dt^2}h(t)\right). \quad (1.67)$$

The integral over $h(t)$ would be proportional to $1/\sqrt{\det(-d^2/dt^2)}$. To evaluate $\det(-d^2/dt^2)$, we first note that the determinant is the product of eigenvalues. So we consider the eigenvalue equation

$$-\frac{d^2}{dt^2}\phi(t) = \mu\phi(t), \quad (1.68)$$

subject to $\phi(t_a) = \phi(t_b) = 0$. This is analogous to the problem of infinite square well in quantum mechanics. Let us denote $T \equiv t_b - t_a$. We know that the eigenvalues are

$$\mu_n = \frac{(n\pi)^2}{T^2}, \qquad n = 1, 2, \cdots \quad (1.69)$$

with the corresponding eigenfunctions

$$\phi_n(t) = \sin\left(\frac{n\pi(t - t_a)}{T}\right). \quad (1.70)$$

Then the determinant is

$$\begin{aligned} D &\equiv \det(-d^2/dt^2) \\ &= \prod_{n=1}^{\infty} \mu_n \\ &= \prod_{n=1}^{\infty} \frac{n^2\pi^2}{T^2}, \end{aligned} \quad (1.71)$$

which is actually divergent. Strict mathematicians would say that this results indicate that the way we take continuum limit is problematic. However, quantum field theorists would keep proceeding and say "when in(finity leads to) doubt, regularise". Consider

$$\xi(s) = \sum_{n=1}^{\infty} \mu_n^{-s} = \left(\frac{T}{\pi}\right)^{2s} \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \left(\frac{T}{\pi}\right)^{2s} \zeta(2s), \quad (1.72)$$

where $\zeta(2s)$ is the Riemann zeta function. Consider

$$\left.\frac{d}{ds}\xi(s)\right|_{s=0} = -\sum_{n=1}^{\infty} \log(\mu_n) = -\log(D). \quad (1.73)$$

On the other hand,

$$\left.\frac{d}{ds}\xi(s)\right|_{s=0} = 2\log(T/\pi)\zeta(0) + 2\zeta'(0). \quad (1.74)$$

The values of $\zeta(0)$ and $\zeta'(0)$ are

$$\zeta(0) = -\frac{1}{2}, \qquad \zeta'(0) = -\frac{1}{2}\log(2\pi). \tag{1.75}$$

Therefore, by comparing eq.(1.73) with eq.(1.74), we obtain

$$D = 2T. \tag{1.76}$$

Therefore,

$$\int \mathcal{D}h(t) \exp\left(-\frac{i}{\hbar}\frac{1}{2}m\int_{t_a}^{t_b} dt\ h(t)\frac{d^2}{dt^2}h(t)\right) \propto \frac{1}{\sqrt{t_b - t_a}}, \tag{1.77}$$

and hence eq.(1.67) becomes

$$(x_b t_b | x_a t_a) \propto \frac{1}{\sqrt{t_b - t_a}} \exp\left(\frac{i}{\hbar}\frac{1}{2}m\frac{(x_b - x_a)^2}{t_b - t_a}\right). \tag{1.78}$$

Then after fixing the proportionality constant, we see that eq.(1.78) eventually leads to eq.(1.57).

1.5 Propagator for simple harmonic oscillator

In the case of simple harmonic oscillator (SHO), $V(x) = (1/2)m\omega^2 x^2$. So

$$(x_b t_b | x_a t_a) \propto \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} dt\ \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right)\right). \tag{1.79}$$

The classical SHO satisfies

$$\frac{d^2}{dt^2}x_{cl}(t) = -\omega^2 x_{cl}(t) \tag{1.80}$$

subject to $x_{cl}(t_a) = x_a, x_{cl}(t_b) = x_b$. The solution is

$$x_{cl}(t) = \frac{x_a \sin(\omega(t_b - t)) + x_b \sin(\omega(t - t_a))}{\sin(\omega T)}. \tag{1.81}$$

Substituting this into eq.(1.48), we obtain

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}\left((x_a^2 + x_b^2)\cos(\omega T) - 2x_a x_b\right). \tag{1.82}$$

Therefore, eq.(1.47) becomes

$$\begin{aligned}(x_b t_b | x_a t_a) \propto \exp&\left(\frac{mi\omega}{2\hbar\sin(\omega T)}\left((x_a^2 + x_b^2)\cos(\omega T) - 2x_a x_b\right)\right) \\ &\times \int \mathcal{D}h(t) \exp\left(-\frac{mi}{2\hbar}\int_{t_a}^{t_b} dt\ h(t)\left(\frac{d^2}{dt^2} + \omega^2\right)h(t)\right).\end{aligned} \tag{1.83}$$

The integral is proportional to $(\det(-d^2/dt^2 - \omega^2))^{-1/2}$. Since we already know the eigenvalues μ_n of $-d^2/dt^2$, the eigenvalues of $-d^2/dt^2 - \omega^2$ are just

$$\lambda_n = \mu_n - \omega^2 = \frac{(n\pi)^2}{T^2} - \omega^2. \tag{1.84}$$

So

$$\det(-d^2/dt^2 - \omega^2) = \prod_{n=1}^{\infty} \left(\frac{(n\pi)^2}{T^2} - \omega^2 \right). \tag{1.85}$$

Consider

$$\frac{\det(-d^2/dt^2 - \omega^2)}{\det(-d^2/dt^2)} = \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2 T^2}{n^2\pi^2} \right). \tag{1.86}$$

The product is recognised as

$$\prod_{n=1}^{\infty} \left(1 - \frac{\omega^2 T^2}{n^2\pi^2} \right) = \frac{\sin(\omega T)}{\omega T}. \tag{1.87}$$

Therefore

$$\begin{aligned} \det(-d^2/dt^2 - \omega^2) &= \det(-d^2/dt^2)\frac{\sin(\omega T)}{\omega T} \\ &= \frac{2\sin(\omega T)}{\omega}. \end{aligned} \tag{1.88}$$

Then eq.(1.83) becomes

$$(x_b t_b | x_a t_a) \propto \sqrt{\frac{\omega}{2\sin(\omega T)}} \exp\left(\frac{mi\omega}{2\hbar\sin(\omega T)} \left((x_a^2 + x_b^2)\cos(\omega T) - 2x_a x_b \right) \right). \tag{1.89}$$

In the limit $t_b - t_a \to 0$, we denote

$$\eta \equiv \sqrt{\frac{i\hbar\sin(\omega T)}{m\omega}}. \tag{1.90}$$

So

$$\begin{aligned} (x_b t_b | x_a t_a) &\propto \lim_{\eta\to 0} \sqrt{\frac{i\hbar}{2m}} \frac{1}{\eta} \exp\left(-\frac{(x_b - x_a)^2}{2\eta^2} \right) \\ &= \sqrt{\frac{\pi i\hbar}{m}} \delta(x_b - x_a). \end{aligned} \tag{1.91}$$

Therefore, the proportionality constant of eq.(1.89) is

$$\sqrt{\frac{m}{\pi i\hbar}}. \tag{1.92}$$

So

$$(x_b t_b | x_a t_a) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega T)}} \exp\left(\frac{mi\omega}{2\hbar\sin(\omega T)} \left((x_a^2 + x_b^2)\cos(\omega T) - 2x_a x_b \right) \right). \tag{1.93}$$

2 Free scalar field

2.1 Interlude: Relativistic wavefunction

Recall that the Schrödinger equation for a free point particle of mass m is given by

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(t,\vec{x}), \tag{2.1}$$

where $\nabla^2 \equiv \partial/\partial_x^2 + \partial/\partial_y^2 + \partial/\partial_z^2$. In order to write down this equation, one may consider the equation describing energy of a free point particle:

$$E = \frac{\vec{p}^2}{2m}. \tag{2.2}$$

Then one makes a replacement

$$E \mapsto i\hbar\frac{\partial}{\partial t}, \qquad p_j \mapsto -i\hbar\partial_j, \tag{2.3}$$

and apply everything to the wavefunction.

In order to write down the wave equation for a free relativistic point particle of mass m, one considers the energy-momentum-mass relation

$$E^2 = \vec{p}^2c^2 + m^2c^4. \tag{2.4}$$

By making use of eq.(2.3), we obtain the Klein-Gordon equation

$$-\hbar^2\frac{\partial^2}{\partial t^2}\phi(t,\vec{x}) = -\hbar^2c^2\vec{\nabla}^2\phi(t,\vec{x}) + m^2c^4\phi(t,\vec{x}), \tag{2.5}$$

or

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right)\phi(t,\vec{x}) = \frac{m^2c^2}{\hbar^2}\phi(t,\vec{x}). \tag{2.6}$$

In quantum field theory, it is often convenient to set $\hbar = c = 1$. Well, in fact, we have not yet discussed quantum field theory (this subsection is on relativistic quantum mechanics). But we will use this convention anyway.

With $\hbar = c = 1$, the Klein-Gordon equation reads

$$\left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right)\phi(t,\vec{x}) = m^2\phi(t,\vec{x}). \tag{2.7}$$

It is convenient to make use of index notation where we label $t = x^0, x = x^1, y = x^2, z = x^3$. We also express $\phi(x) \equiv \phi(t,\vec{x})$. Then, using mostly plus signature $\eta_{\mu\nu} = (-1,+1,+1,+1)$, we obtain

$$\partial_\mu\partial^\mu\phi(x) = m^2\phi(x). \tag{2.8}$$

A solution to eq.(2.8) is in the form of plane wave $e^{-i\omega t+i\vec{k}\cdot\vec{x}}$. After substituting the plane wave into eq.(2.8), we obtain the condition

$$\omega^2 = \vec{k}^2 + m^2. \tag{2.9}$$

Denote

$$E_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}. \tag{2.10}$$

Therefore, the plane wave solutions are $e^{-iE_{\vec{k}}t\pm i\vec{k}\cdot\vec{x}}$, $e^{iE_{\vec{k}}t\pm i\vec{k}\cdot\vec{x}}$.

In order to understand these solutions geometrically, let us first consider an equation

$$\vec{k}\cdot\vec{x} = a, \tag{2.11}$$

where $\vec{k}$ is a given non-zero vector and a is a given real number. Positions $\vec{x}$ which satisfy the above equation form a plane perpendicular to $\vec{k}$. On this plane, the point closest to the origin is

$$\vec{x}_0 = a\frac{\vec{k}}{|\vec{k}|^2}. \tag{2.12}$$

If we fix $\vec{k}$, different planes (perpendicular to $\vec{k}$) are characterised by the value a. If $a_2 > a_1$, the vector pointing from the plane a_1 to the plane a_2 is in the same direction as $\vec{k}$. This means that $\vec{k}$ can be used to determine the direction that plane increases the value of a. Consider a plane wave $e^{-iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}$. The phase 0 are at the positions $\vec{x}$ which satisfy

$$E_{\vec{k}}t - \vec{k}\cdot\vec{x} = 0, \tag{2.13}$$

or

$$\vec{k}\cdot\vec{x} = E_{\vec{k}}t. \tag{2.14}$$

At each fixed t, the solution $\vec{x}$ forms a plane perpendicular to $\vec{k}$ such that the point closest to the origin is

$$\vec{x}_0 = E_{\vec{k}}t\frac{\vec{k}}{|\vec{k}|^2}. \tag{2.15}$$

As t increases, the plane is shifted in the same direction as $\vec{k}$. The same argument also applies for other values of phase. So the planes of constant phase move in the same direction as $\vec{k}$. The motion of the plane wave $e^{iE_{\vec{k}}t-i\vec{k}\cdot\vec{x}}$ is also the same as described above. However, for the plane waves $e^{-iE_{\vec{k}}t-i\vec{k}\cdot\vec{x}}$, and $e^{iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}$, the planes of constant phase move in the opposite direction as $\vec{k}$.

Exercise 2.1. Consider sine waves $e^{iEt\pm ikx}$ where $E > 0$ and $k > 0$. Explain why the wave $e^{iEt-ikx}$ is moving to the right, whereas the wave $e^{iEt+ikx}$ is moving to the left. ◇

The energy of the plane waves can be obtained by applying $i\partial/\partial t$. This gives

$$i\frac{\partial}{\partial t}e^{-iE_{\vec{k}}t\pm i\vec{k}\cdot\vec{x}} = E_{\vec{k}}e^{-iE_{\vec{k}}t\pm i\vec{k}\cdot\vec{x}}, \tag{2.16}$$

$$i\frac{\partial}{\partial t}e^{iE_{\vec{k}}t\pm i\vec{k}\cdot\vec{x}} = -E_{\vec{k}}e^{-iE_{\vec{k}}t\pm i\vec{k}\cdot\vec{x}}. \tag{2.17}$$

Therefore, the plane waves $e^{-iE_{\vec{k}}t\pm i\vec{k}\cdot\vec{x}}$ has energy $E_{\vec{k}}$, whereas the plane waves $e^{iE_{\vec{k}}t\pm i\vec{k}\cdot\vec{x}}$ has energy $-E_{\vec{k}}$. We now encounter a problem. The spectrum (allowed values of energy) of Klein-Gordon equation is $(-\infty, -m] \cup [m, \infty)$. This means that there is no ground state. As the quantum particle moves to a lower energy state, the system releases the energy. Since there is no ground state, the particle can keep moving down and generating energy indefinitely. This indicates that the system is not stable.

There is also another problem which is that the probability density can be negative. We will not talk about this in this school.

The problems of infinitely negative energy state and of negative probablilty can be solved by changing the framework. Instead of viewing Klein-Gordon equation as an equation describing a quantum relativistic point particle, we will now think about it as being an equation describing a classical scalar field, which is to be quantised, giving a quantum scalar field.

2.2 Why study (quantum) field theory

Fields are functions of space and time. For example, a scalar field can be described as follows. At each specific time, there is one number corresponding to each point on the space. As time changes, these numbers generically change. As another example, consider a vector field. At each specific time, there is one vector (not necessarily three-dimensional (!)) corresponding to each point on the space. As time changes, these vectors generically change.

It turns out that in order to describe fundamental phenomena related to elementary particles, one needs to use field theory. This is because fields are more fundamental than particles. Particles can be created out of “nothing”, destroyed into “nothing”, or change numbers and types. From the field theory perspective, particles are only result from the quantisation of fields.

2.3 Scalar field action

A scalar field is a function of space and time. It can be written as $\phi(x) \equiv \phi(t, \vec{x})$, where $(x) \equiv (x^\mu) \equiv (t, \vec{x})$. The dependence of the field on the space can be thought of as the promotion of the discrete label i to continuous label $\vec{x}$. The Lagrangian usually takes the form

$$L = \int d^3\vec{x}\ \mathcal{L}[\phi(x), \partial_\mu\phi(x)], \tag{2.18}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$, and $\mathcal{L}[\phi(x), \partial_\mu\phi(x)]$ is called the **Lagrangian density**. We may abbreviate $\mathcal{L}[\phi(x), \partial_\mu\phi(x)]$ as $\mathcal{L}(x)$ or even $\mathcal{L}$.

With the above set up, an action for a scalar field theory takes the form

$$S = \int d^4x\ \mathcal{L}[\phi(x), \partial_\mu\phi(x)]. \tag{2.19}$$

Under a variation $\phi \mapsto \phi + \delta\phi$, we obtain

$$\begin{aligned} \delta S &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu \delta\phi \right) \\ &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta\phi + \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right). \end{aligned} \tag{2.20}$$

After the divergence theorem is applied to the last term, it becomes an expression which depends on the value of field and its variation at infinity. If the field decays sufficiently quickly at infinity, then this term vanishes. So we are left with

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta\phi. \tag{2.21}$$

Requiring that $\delta S = 0$ for arbitrary field variation $\delta\phi$, we obtain the Euler-Lagrange equation for scalar field:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0. \tag{2.22}$$

Exercise 2.2. Derive equations of motion for the action of the form

$$S = \int d^4x \; \mathcal{L}[\phi(x), \partial_\mu \phi(x), \partial_\mu \partial_\nu \phi(x)]. \tag{2.23}$$

◇

2.4 Solution for free scalar field

The action for a free scalar field theory is given by

$$S = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right). \tag{2.24}$$

So the Lagrangian density for the free scalar field theory is given by

$$\mathcal{L}(x) = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \tag{2.25}$$

In order to obtain equation of motion, let us vary the action. Under the field variation $\phi \mapsto \phi + \delta\phi$, the action transforms as $S \mapsto S + \delta S$, where

$$\begin{aligned} \delta S &= \int d^4x \left(-\partial_\mu \phi \partial^\mu \delta\phi - m^2 \phi \delta\phi \right) \\ &= \int d^4x \left(\partial^\mu \partial_\mu \phi - m^2 \phi \right) \delta\phi - \int d^4x \partial_\mu (\partial^\mu \phi \delta\phi). \end{aligned} \tag{2.26}$$

The second term can be made vanished by requiring that the scalar field dies off quick enough at infinity. We are left with

$$\delta S = \int d^4x \left(\partial^\mu \partial_\mu \phi - m^2 \phi \right) \delta\phi. \tag{2.27}$$

Demanding $\delta S = 0$ gives scalar field equation of motion

$$\partial^\mu \partial_\mu \phi - m^2 \phi = 0. \tag{2.28}$$

We have already seen that there are plane wave solutions to the above equation. Since this equation is also linear, superpositions of solutions are also solutions. So we may consider a superposition of $e^{-iE_{\vec{k}}t \pm i\vec{k}\cdot\vec{x}}$, $e^{iE_{\vec{k}}t \pm i\vec{k}\cdot\vec{x}}$. However, we are making over counting. Selecting $e^{\pm iE_{\vec{k}}t + i\vec{k}\cdot\vec{x}}$ for all $\vec{k}$ is the same as selecting $e^{\pm iE_{\vec{k}}t - i\vec{k}\cdot\vec{x}}$ for all $\vec{k}$. So positive energy solutions can be expressed either as

$$\phi^{(+)}(t, \vec{x}) = \int d^3\vec{k} a^{(+)}_{\vec{k}} e^{-iE_{\vec{k}}t + i\vec{k}\cdot\vec{x}}, \tag{2.29}$$

or

$$\phi^{(+)}(t, \vec{x}) = \int d^3\vec{k} b^{(+)}_{\vec{k}} e^{-iE_{\vec{k}}t - i\vec{k}\cdot\vec{x}}. \tag{2.30}$$

Similarly, negative energy solutions can be expressed either as

$$\phi^{(-)}(t, \vec{x}) = \int d^3\vec{k} a^{(-)}_{\vec{k}} e^{-iE_{\vec{k}}t + i\vec{k}\cdot\vec{x}}, \tag{2.31}$$

or

$$\phi^{(-)}(t, \vec{x}) = \int d^3\vec{k} b^{(-)}_{\vec{k}} e^{-iE_{\vec{k}}t - i\vec{k}\cdot\vec{x}}. \tag{2.32}$$

To understand this better, attempt the exercise.

Exercise 2.3. Consider a wave in a one dimensional space which is confined between $x = 0, x = L$. The wave satisfies the Klein-Gordon equation

$$-\partial_t^2 \phi + \partial_x^2 \phi - m^2 \phi = 0. \tag{2.33}$$

We have known that there are "plane wave" solutions. For definiteness, consider

$$u_k(t, x) = e^{-iE_k t + ikx}, \tag{2.34}$$

where $E_k > 0$. A standard practice is to impose periodic boundary condition

$$\phi(t, 0) = \phi(t, L). \tag{2.35}$$

a) Show that $E_k = \sqrt{k^2 + m^2}$.

b) Show that $k = 2\pi n/L$.

c) Explain that a general positive energy solution is

$$\psi(t, x) = \sum_{n=-\infty}^{\infty} a_n u_{2\pi n/L}(t, x). \tag{2.36}$$

d) Repeat $(a)-(c)$ but this time, start with

$$v_k(t,x)=e^{-iE_kt-ikx}. \tag{2.37}$$

Show that a general positive energy solution is

$$\chi(t,x)=\sum_{n=-\infty}^{\infty} b_n v_{2\pi n/L}(t,x). \tag{2.38}$$

e) Since $\psi(t,x)$ and $\chi(t,x)$ are both general positive energy solutions, their coefficients should be related. Find how $\{a_n\}$ is related to $\{b_n\}$.

◇

There are therefore four ways to write a general solution of the Klein-Gordon equation. That is, two ways to write positive energy solutions and two ways to write negative energy solutions. However, we will often use only the following two ways:

$$\phi(x)=\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}\left(e^{-iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}a_{\vec{k}}+e^{iE_{\vec{k}}t-i\vec{k}\cdot\vec{x}}a^*_{\vec{k}}\right). \tag{2.39}$$

$$\phi(x)=\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}\left(e^{-iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}a_{\vec{k}}+e^{iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}a^*_{-\vec{k}}\right). \tag{2.40}$$

Exercise 2.4. Show that by imposing $\phi(x)=\phi^*(x)$ on

$$\phi(x)=\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}\left(e^{-iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}\tilde{\phi}(E_{\vec{k}},\vec{k})+e^{iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}\tilde{\phi}(-E_{\vec{k}},\vec{k})\right), \tag{2.41}$$

we obtain

$$\tilde{\phi}^*(E_{\vec{k}},-\vec{k})=\tilde{\phi}(-E_{\vec{k}},\vec{k}). \tag{2.42}$$

◇

For integrals in free field theories, we often encounter the integration measure

$$\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}. \tag{2.43}$$

The significance is that it is Lorentz invariant. The verification is left as exercises for readers.

Exercise 2.5.

(a) Show that for time-like vector v^μ, (orthochronous) Lorentz transformation will not change the sign of v^0. Hint: use $\Lambda^0_0\geq 0$, $\Lambda\eta^{-1}\Lambda^T=\eta^{-1}$, and inequality $-|\vec{a}||\vec{b}|\leq \vec{a}\cdot\vec{b}\leq|\vec{a}||\vec{b}|$.

(b) Analyse how

$$\int \frac{d^4k}{(2\pi)^4}\Theta(k^0)2\pi\delta(k^2+m^2) \tag{2.44}$$

transforms under (orthochronous) Lorentz transformation.

(c) Show that

$$\int \frac{d^4k}{(2\pi)^4}\Theta(k^0)2\pi\delta(k^2+m^2) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}. \tag{2.45}$$

◇

2.5 Hamiltonian formulation

Let us turn to Hamiltonian formulation. Let us start by computing conjugate momenta

$$\Pi(x) = \frac{\partial\mathcal{L}(x)}{\partial\dot{\phi}(x)}, \tag{2.46}$$

where $\dot{}$ stands for derivative with respect to t. Direct computation gives

$$\Pi(x) = \dot{\phi}(x). \tag{2.47}$$

Hamiltonian density is given by

$$\begin{aligned}\mathcal{H}(x) &= \Pi(x)\dot{\phi}(x) - \mathcal{L}(x) \\ &= \frac{1}{2}\Pi^2(x) + \frac{1}{2}\vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(x) + \frac{1}{2}m^2\phi^2(x).\end{aligned} \tag{2.48}$$

Poisson's bracket is defined such that

$$\{\phi(t,\vec{x}),\Pi(t,\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y}), \tag{2.49}$$

$$\{\phi(t,\vec{x}),\phi(t,\vec{y})\} = \{\Pi(t,\vec{x}),\Pi(t,\vec{y})\} = 0, \tag{2.50}$$

where we note that the two fields have to be evaluated at the same time.

Let us now consider the solution (2.39) to the equation of motion (2.28). For this, the conjugate momentum is given by

$$\begin{aligned}\Pi(x) &= \dot{\phi}(x) \\ &= -\frac{i}{2}\int\frac{d^3\vec{k}}{(2\pi)^3}\left(a_{\vec{k}}e^{-iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}} - a^*_{\vec{k}}e^{iE_{\vec{k}}t-i\vec{k}\cdot\vec{x}}\right),\end{aligned} \tag{2.51}$$

or alternatively,

$$\Pi(x) = -\frac{i}{2}\int\frac{d^3\vec{k}}{(2\pi)^3}\left(a_{\vec{k}}e^{-iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}} - a^*_{-\vec{k}}e^{iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}\right). \tag{2.52}$$

Let us invert the equations (2.40) and (2.52) to write $a_{\vec{k}}$ and $a^*_{\vec{k}}$ in terms of $\phi(x)$ and $\Pi(x)$. This gives

$$a_{\vec{k}} = \int d^3\vec{x}\, \big(E_{\vec{k}}\phi(x) + i\Pi(x)\big)\, e^{iE_{\vec{k}}t - i\vec{k}\cdot\vec{x}}, \tag{2.53}$$

$$a^*_{\vec{k}} = \int d^3\vec{x}\, \big(E_{\vec{k}}\phi(x) - i\Pi(x)\big)\, e^{-iE_{\vec{k}}t + i\vec{k}\cdot\vec{x}}. \tag{2.54}$$

This allows us to obtain Poisson's brackets for $a_{\vec{k}}$ and $a^*_{\vec{k}}$. After calculations, we obtain

$$\{a_{\vec{k}}, a_{\vec{k}'}\} = 0, \qquad \{a^*_{\vec{k}}, a^*_{\vec{k}'}\} = 0, \qquad \{a_{\vec{k}}, a^*_{\vec{k}'}\} = -2iE_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'). \tag{2.55}$$

Exercise 2.6. Consider eq.(2.53). By using equation of motion for free scalar field, show that $\dot{a}_{\vec{k}} = 0$. ◇

Exercise 2.7. Prove eq.(2.55). ◇

2.6 Noether's theorem

Noether's theorem states that for each continuous symmetry, there is a corresponding conservation law. A symmetry is a transformation which leaves the action invariant, whereas a conservation law usually corresponds to the presence of a conserved charge, a quantity which does not change in time.

Consider a theory of a scalar field $\phi(x)$. Suppose that the theory has a symmetry whose infinitesimal form is given by

$$\phi(x) \mapsto \phi(x) + \epsilon\Delta\phi(x), \tag{2.56}$$

where ϵ is a constant parameter. In the calculations, we will only keep to order ϵ. The Lagrangian should be invariant up to a total divergence. That is,

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon\partial_\mu K^\mu. \tag{2.57}$$

On the other hand, let us consider how an action is transformed under a generic transformation. For this, let us suppose that the action is of the form

$$S[\phi] = \int d^4x\; \mathcal{L}(\phi, \partial_\mu\phi). \tag{2.58}$$

So

$$S[\phi + \epsilon\Delta\phi] = \int d^4x\; \mathcal{L}(\phi + \epsilon\Delta\phi, \partial_\mu\phi + \epsilon\partial_\mu\Delta\phi). \tag{2.59}$$

Expanding this in ϵ gives

$$S[\phi + \epsilon\Delta\phi] = S[\phi] + \left.\frac{\partial S[\phi + \epsilon\Delta\phi]}{\partial\epsilon}\right|_{\epsilon=0} \epsilon. \tag{2.60}$$

To compute the second term, we can use chain rule:

$$\left.\frac{\partial S[\phi+\epsilon\Delta\phi]}{\partial\epsilon}\right|_{\epsilon=0} = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\Delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\mu\Delta\phi\right). \tag{2.61}$$

Let us now impose, on the first term, the equation of motion

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}. \tag{2.62}$$

This gives

$$\left.\frac{\partial S[\phi+\epsilon\Delta\phi]}{\partial\epsilon}\right|_{\epsilon=0} = \int d^4x\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Delta\phi\right). \tag{2.63}$$

So

$$S[\phi+\epsilon\Delta\phi] = S[\phi] + \epsilon\int d^4x\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Delta\phi\right), \tag{2.64}$$

which is equivalent to

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Delta\phi\right). \tag{2.65}$$

Equating eq.(2.57) with eq.(2.65) gives

$$\partial_\mu K^\mu = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Delta\phi\right). \tag{2.66}$$

This implies that the quantity

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Delta\phi - K^\mu \tag{2.67}$$

satisfies

$$\partial_\mu j^\mu = 0. \tag{2.68}$$

The quantity j^μ is called the conserved current. By using the conserved current, it is possible to define a quantity which does not change over time. Let us define a quantity called the conserved charge:

$$q = \int d^3x j^0. \tag{2.69}$$

This quantity satisfies

$$\frac{dq}{dt} = 0. \tag{2.70}$$

Exercise 2.8. Show that eq.(2.69) satisfies eq.(2.70). ◇

2.7 Energy momentum tensor

Let us turn to the scalar field theory and consider a transformation on spacetime

$$x^\mu \mapsto x'^\mu. \tag{2.71}$$

This induces the transformation on the scalar field ϕ so that it changes to ϕ' such that

$$\phi'(x') = \phi(x). \tag{2.72}$$

Then,

$$\begin{aligned} \partial'_\mu \phi'(x') &= \partial'_\mu \phi(x) \\ &= \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi(x). \end{aligned} \tag{2.73}$$

In particular, let us consider spacetime translation

$$x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu, \tag{2.74}$$

where ϵ^μ is a constant 4−vector. This induces the transformation on the fields as

$$\begin{aligned} \phi(x) \mapsto \phi'(x) &= \phi(x - \epsilon) \\ &= \phi(x) - \epsilon^\mu \partial_\mu \phi(x). \end{aligned} \tag{2.75}$$

Next, in order to work out the induced transformation on $\partial_\mu \phi(x)$, let us consider

$$\begin{aligned} \partial'_\mu \phi'(x') &= \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi(x) \\ &= \delta^\nu_\mu \partial_\nu \phi(x) \\ &= \partial_\mu \phi(x). \end{aligned} \tag{2.76}$$

On the other hand,

$$\begin{aligned} \partial'_\mu \phi'(x') &= \frac{\partial}{\partial x'^\mu} \phi'(x + \epsilon) \\ &= \frac{\partial}{\partial x'^\mu} (\phi'(x) + \epsilon^\nu \partial_\nu \phi'(x)) \\ &= \frac{\partial}{\partial x'^\mu} (\phi'(x) + \epsilon^\nu \partial_\nu \phi(x)) \\ &= \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu (\phi'(x) + \epsilon^\nu \partial_\nu \phi(x)) \\ &= \partial_\mu \phi'(x) + \epsilon^\nu \partial_\mu \partial_\nu \phi(x). \end{aligned} \tag{2.77}$$

So

$$\partial_\mu \phi'(x) = \partial_\mu \phi(x) - \epsilon^\nu \partial_\mu \partial_\nu \phi(x). \tag{2.78}$$

In all of the above calculations, we have kept only up to the first order in ϵ^μ; this will also be implemented in subsequent calculations. Then, under spacetime translation transformation, the Lagrangian density of free scalar field transforms as

$$\begin{aligned}\mathcal{L} \mapsto & -\frac{1}{2}(\partial_\mu\phi - \epsilon^\nu\partial_\mu\partial_\nu\phi)(\partial^\mu\phi - \epsilon^\rho\partial^\mu\partial_\rho\phi) - \frac{1}{2}m^2(\phi - \epsilon^\nu\partial_\nu\phi)(\phi - \epsilon^\rho\partial_\rho\phi) \\ &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \partial_\mu\phi\epsilon^\nu\partial^\mu\partial_\nu\phi + m^2\phi\epsilon^\nu\partial_\nu\phi \\ &= \mathcal{L} - \epsilon^\nu\partial_\nu\mathcal{L}.\end{aligned} \tag{2.79}$$

The transformation of the Lagrangian density gives a total derivative at the first order in ϵ^μ. So indeed the spacetime translation is a symmetry of the theory.

Note that there are four parameters in the spacetime translation (2.74). Each of the parameter corresponds to the translation along each spacetime direction. So their label is the spacetime index. So the symmetry transformation on the Lagrangian takes the form

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon^\nu\partial_\mu K^\mu{}_{(\nu)} \tag{2.80}$$

and the conserved current is given by

$$\begin{aligned} T^\mu{}_\nu &\equiv j^\mu{}_{(\nu)} \\ &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}(\Delta\phi)_{(\nu)} - K^\mu{}_{(\nu)}.\end{aligned} \tag{2.81}$$

The conserved currents for spacetime translation are combined to form a tensor, which is called an energy-momentum tensor. This tensor is important in the studies of physics of the system.

For our case, we have

$$(\Delta\phi)_{(\nu)} = -\partial_\nu\phi, \tag{2.82}$$

$$K^\mu{}_{(\nu)} = -\delta^\mu_\nu\mathcal{L}. \tag{2.83}$$

and

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = -\partial^\mu\phi. \tag{2.84}$$

So

$$T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi + \delta^\mu_\nu\mathcal{L}. \tag{2.85}$$

Raising the index ν gives

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}\eta^{\mu\nu}m^2\phi^2. \tag{2.86}$$

The energy-momentum tensor encodes Hamiltonian density, momentum density, and stress density.

Having obtained conserved currents, the conserved charges are given by

$$\int d^3\vec{x}\; T^{0\nu}. \tag{2.87}$$

They are identified with Hamiltonian and momentum as

$$H = \int d^3\vec{x}\; T^{00}, \qquad \vec{P} = \int d^3\vec{x}\; T^{0i}. \tag{2.88}$$

So T^{00} is Hamiltonian density, whereas T^{0i} is momentum density.

To see that T^{00} is Hamiltonian density, consider

$$\begin{aligned} T^{00} &= \partial^0\phi\partial^0\phi - \frac{1}{2}\eta^{00}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}\eta^{00}m^2\phi^2 \\ &= \frac{1}{2}\Pi^2 + \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{1}{2}m^2\phi^2, \end{aligned} \tag{2.89}$$

which indeed agrees with the Hamiltonian density $\mathcal{H}$ defined from the Legendre transformation of Lagrangian density. Let us express the Hamiltonian in terms of oscillation amplitudes. For this, let us first compute several terms. Let us recall the expression of $\phi(x)$ and $\Pi(x)$ from eq.(2.40) and eq.(2.52). By denoting

$$a_{\vec{k}}(t) \equiv a_{\vec{k}} e^{-iE_{\vec{k}}t}, \tag{2.90}$$

we have

$$\phi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}} e^{i\vec{k}\cdot\vec{x}} \left(a_{\vec{k}}(t) + a^*_{-\vec{k}}(t)\right). \tag{2.91}$$

$$\Pi(x) = -\frac{i}{2}\int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(a_{\vec{k}}(t) - a^*_{-\vec{k}}(t)\right). \tag{2.92}$$

It is also convenient to consider the identity

$$\int d^3\vec{x} \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} f(\vec{k})g(\vec{k}') = \int \frac{d^3\vec{k}}{(2\pi)^3} f(\vec{k})g(-\vec{k}). \tag{2.93}$$

Therefore,

$$\begin{aligned} \int d^3\vec{x}\frac{1}{2}\Pi^2 = -\frac{1}{8}\int d^3\vec{x} \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} \\ \times \left(a_{\vec{k}}(t) - a^*_{-\vec{k}}(t)\right)\left(a_{\vec{k}'}(t) - a^*_{-\vec{k}'}(t)\right) \\ = -\frac{1}{8}\int \frac{d^3\vec{k}}{(2\pi)^3} \left(a_{\vec{k}}(t) - a^*_{-\vec{k}}(t)\right)\left(a_{-\vec{k}}(t) - a^*_{\vec{k}}(t)\right). \end{aligned} \tag{2.94}$$

Next,

$$\begin{aligned} \int d^3\vec{x}\frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi = \frac{1}{2}\int d^3\vec{x} \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} \\ \times \frac{-\vec{k}\cdot\vec{k}'}{2E_{\vec{k}}2E_{\vec{k}'}} \left(a_{\vec{k}}(t) + a^*_{-\vec{k}}(t)\right)\left(a_{\vec{k}'}(t) + a^*_{-\vec{k}'}(t)\right) \\ = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\vec{k}\cdot\vec{k}}{(2E_{\vec{k}})^2} \left(a_{\vec{k}}(t) + a^*_{-\vec{k}}(t)\right)\left(a_{-\vec{k}}(t) + a^*_{\vec{k}}(t)\right). \end{aligned} \tag{2.95}$$

Next,

$$\begin{aligned}\int d^3\vec{x}\frac{1}{2}m^2\phi^2 &= \frac{1}{2}m^2\int d^3\vec{x}\int\frac{d^3\vec{k}}{(2\pi)^3}\int\frac{d^3\vec{k}'}{(2\pi)^3}e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} \\ &\quad\times\frac{1}{2E_{\vec{k}}2E_{\vec{k}'}}\left(a_{\vec{k}}(t)+a^*_{-\vec{k}}(t)\right)\left(a_{\vec{k}'}(t)+a^*_{-\vec{k}'}(t)\right) \\ &= \frac{1}{2}m^2\int\frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{1}{2E_{\vec{k}}}\right)^2\left(a_{\vec{k}}(t)+a^*_{-\vec{k}}(t)\right)\left(a_{-\vec{k}}(t)+a^*_{\vec{k}}(t)\right).\end{aligned} \tag{2.96}$$

We may combine eq.(2.95) and eq.(2.96) to get

$$\begin{aligned}&\int d^3\vec{x}\left(\frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi+\frac{1}{2}m^2\phi^2\right) \\ &\quad=\frac{1}{8}\int\frac{d^3\vec{k}}{(2\pi)^3}\left(a_{\vec{k}}(t)+a^*_{-\vec{k}}(t)\right)\left(a_{-\vec{k}}(t)+a^*_{\vec{k}}(t)\right).\end{aligned} \tag{2.97}$$

Then after combining with eq.(2.94), we obtain

$$H=\frac{1}{4}\int\frac{d^3\vec{k}}{(2\pi)^3}\left(a_{\vec{k}}a^*_{\vec{k}}+a^*_{-\vec{k}}a_{-\vec{k}}\right). \tag{2.98}$$

For the second term on RHS, let us transform $\vec{k}\to-\vec{k}$. Then

$$H=\frac{1}{4}\int\frac{d^3\vec{k}}{(2\pi)^3}\left(a_{\vec{k}}a^*_{\vec{k}}+a^*_{\vec{k}}a_{\vec{k}}\right). \tag{2.99}$$

It can be seen that in fact the second term equals to the first term. So

$$\begin{aligned}H &= \frac{1}{2}\int\frac{d^3\vec{k}}{(2\pi)^3}a_{\vec{k}}a^*_{\vec{k}} \\ &= \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}E_{\vec{k}}a_{\vec{k}}a^*_{\vec{k}}.\end{aligned} \tag{2.100}$$

Let us turn to momentum density. It is given by

$$\begin{aligned}\mathcal{P}^i &= T^{0i} \\ &= \partial^0\phi\partial^i\phi \\ &= -\dot{\phi}\partial^i\phi \\ &= -\Pi\partial^i\phi.\end{aligned} \tag{2.101}$$

The integral of momentum density over space gives a quantity called a momentum. Let us express it in terms of oscillation amplitudes. So consider

$$\begin{aligned}
P_j &= \int d^3\vec{x}\mathcal{P}_j \\
&= \frac{i}{2}\int d^3\vec{x}\int \frac{d^3\vec{k}}{(2\pi)^3}\int \frac{d^3\vec{k}'}{(2\pi)^3}e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} \\
&\qquad \times \left(a_{\vec{k}}(t) - a^*_{-\vec{k}}(t)\right)\frac{1}{2E_{\vec{k}'}}\left(a_{\vec{k}'}(t) + a^*_{-\vec{k}'}(t)\right)ik'_j \\
&= \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{k_j}{2E_{\vec{k}}}\left(a_{\vec{k}}(t)a_{-\vec{k}}(t) - a^*_{-\vec{k}}(t)a^*_{\vec{k}}(t) + a_{\vec{k}}a^*_{\vec{k}} - a^*_{-\vec{k}}a_{-\vec{k}}\right).
\end{aligned} \tag{2.102}$$

Note that the first two terms on RHS vanish because after changing the variable $\vec{k} \to -\vec{k}$, these two terms become minus of themselves. So we are left with

$$P_j = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{k_j}{2E_{\vec{k}}}\left(a_{\vec{k}}a^*_{\vec{k}} - a^*_{-\vec{k}}a_{-\vec{k}}\right). \tag{2.103}$$

Let us make a change of variable $\vec{k} \to -\vec{k}$ on the second term. We see that the second term become equals to the first term. So

$$P_j = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}k_j a_{\vec{k}}a^*_{\vec{k}}. \tag{2.104}$$

Exercise 2.9. Verify by explicit calculations that eq.(2.103) indeed implies eq.(2.104). ⋄

For completeness, we could also consider the component T^{ij} which gives stress densities. However, this is out of the scope of our course. So we will not discuss these quantities.