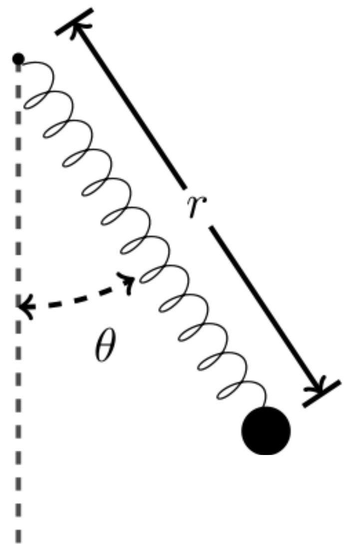
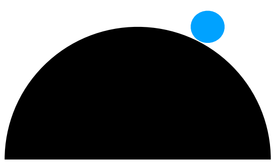
Problem set 1: Lagrangian and Hamiltonian mechanics

Question 1: Consider a string pendulum as shown in the figure. The object mass m is suspended from a spring with constant k and an unextended length l_0. Note that motion is constrained on the xy plane.



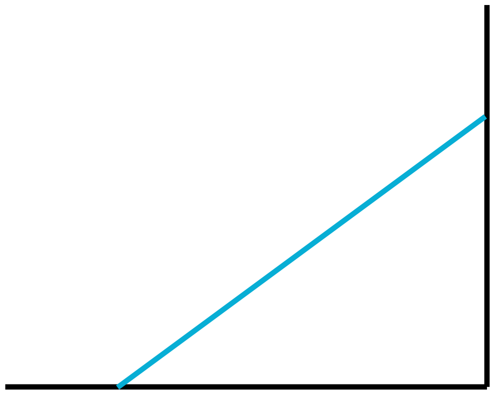
1.1) What is a set of generalised coordinates?
1.2) How many degrees of freedom do we have?
1.3) Use Newtonian mechanics to determine the equation of motion of the mass m.
1.4) Show that the Euler-Lagrange equation produces the same equation with the Newton method.

Question 2: A particle mass m starts at rest on top of smooth fixed hemisphere of radius R.



2.1) Use Newtonian mechanics to find the equation of motion of mass m and also find the angle at which the particle leaves the hemisphere.
2.2) What is the set of generalised coordinates? and what is the constraint equation?
2.3) Use Lagrangian multiplier method to determine the force of constraint and the angle at which the particle leaves the hemisphere.

Question 3: A uniform plank initially leans against a smooth wall and rests on the frictionless floor.



3.1) Use the Newtonian mechanics to find the equation of motion together with all forces acting on the plank.
3.2) Use Lagrangian mechanics and constraints to obtain the result in 3.1)

Question 4: Consider a uniformly charged ring with a radius R and total charge Q with the line charge density λ.
4.1) The charge -q is placed at the centre of the ring. What will happen to the charge -q if it is pulled out from the centre of the ring with an infinitesimal displacement x $(x << R)$ along the x-axis. Use the Hamilton's equations to determine the motion of charge -q.
4.2) Draw and explain the trajectory of the system on the phase space.

Question 5: Let p and q be canonical variables for a particle with Hamiltonian $H = p^2/2m$. Under the canonical transformation

$$G(p, P) = (g - at)(P + b),$$

where a and b are constants, find P and Q and a new Hamiltonian.

Question 6: Consider the transformation

$$Q = \ln\left(\frac{\sin p}{q}\right), \qquad P = q \cot p\,.$$

Compute the four major types of generating functions associated with this transformation.

Question 7: Given the action functional

$$S[q, p] = \int_{t'}^{t''} dt \left[p \cdot \dot{q} - H(p, q)\right],$$

where $p = (p_1, p_2, \ldots, p_N)$ and $q = (q_1, q_2, \ldots, q_N)$ are generalised momenta and coordinates. Show that, under the critical condition $\delta S = 0$, we obtain

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \text{ and } -p_i = \frac{\partial H}{\partial q_i}.$$

Question 8: Consider the double pendulum given in figure 8.1

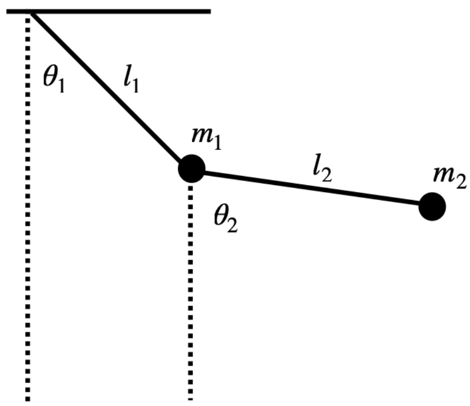


Figure 8.1 Double pendulum

8.1 What is a set of good coordinates ?
8.2 Using Newtonian approach, Lagrangian approach and Hamiltonian approach to show that the equations of motion are
$l_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + l_2\ddot{\theta}_2 - l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin\theta_2 = 0$
$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \theta_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1 = 0\,.$

Question 9: Given Hamiltonian $H(p_1, p_2, q_1, q_2) = p_1^2/2 + p_2^2/2 + \cos(2q_1 + q_2)$ with a set of conical transformations

$Q_1 = (2q_1 + q_2),\ Q_2 = q_2,\ P_1 = p_1/2,\ P_2 = p_2 - p_1/2,$

show that the symplectic 2-form is preserved

$$\omega = \sum_{i=1}^{2} dp_i \wedge dq_i = \sum_{i=1}^{2} dP_i \wedge dQ_i\,.$$