Lagrangian and Hamiltonian Mechanics

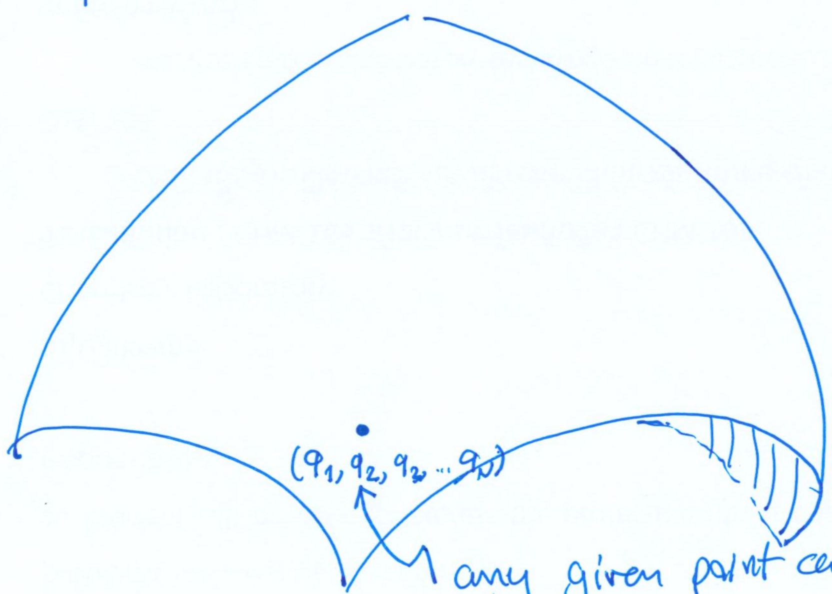
Sitarin Yoo-kong

2026 IF Summer school

May 26 – June 5, 2026

1. Lagrangian Mechanics.

Given a set of generalised coordinates (or good coordinates) $\{q_1, q_2, q_3, \dots, q_N\}$, where N is a number of degree of freedom, We define a space called a configuration space with N-dimensions : M



any given point can be identified by N variables.

Given an initial point $(q_1(t'), q_2(t'), \dots, q_N(t'))$ and a final point $(q_1(t''), q_2(t''), \dots, q_N(t''))$, the system will take a path, connected between two points, in which the action is "critical".

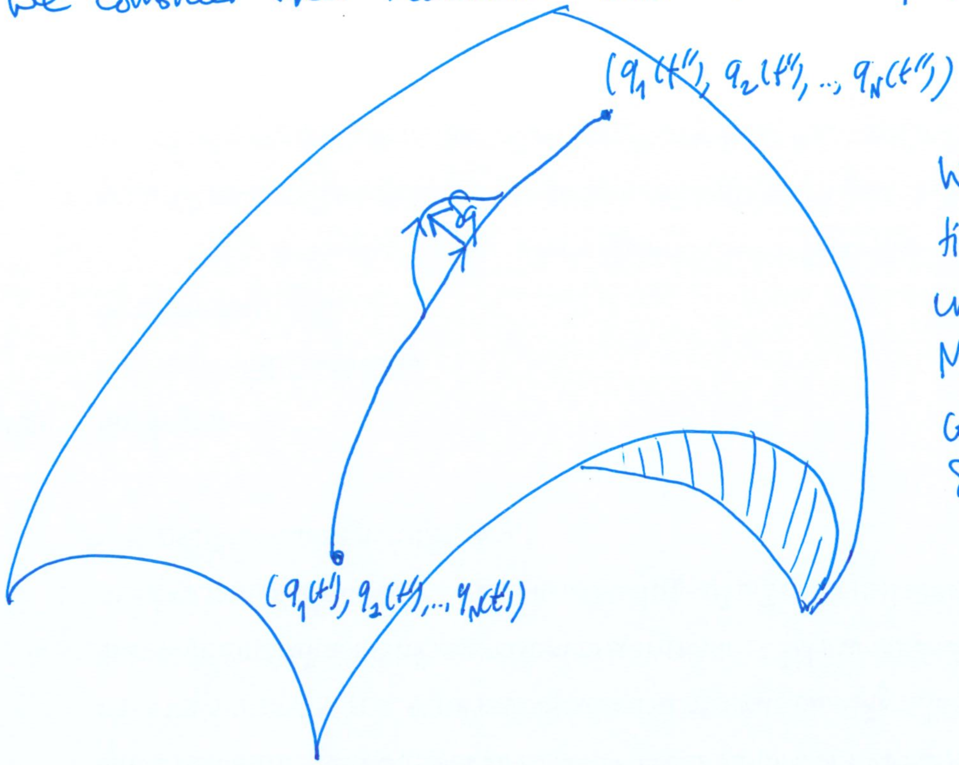
$$S[q(t)] \equiv \int_{t'}^{t''} dt\, L(q, \dot{q}; t), \qquad (1)$$

where L is a Lagrangian given by

$$L(q, \dot{q}; t) = T(\dot{q}) - V(q). \qquad (2)$$

Next, we shall consider the critical condition of the action. ②

First, we consider the variation such that $q(t) \to \underbrace{q(t) + \delta q(t)}$.



We note that the time is being fixed under this deformation. Moreover, we have conditions

$\delta q(t') = \delta q(t'') = 0$. —(☆)

A new action is

$$S[q+\delta q] = \int_{t'}^{t''} dt\, L(q+\delta q, \dot{q}+\delta\dot{q}; t)$$

Taylor series

$$= \int_{t'}^{t''} dt \left\{ L(q,\dot{q};t) + \frac{\partial L}{\partial q}\delta q + \underbrace{\frac{\partial L}{\partial \dot{q}}\delta\dot{q}} + \underbrace{\cdots}_{\text{higher orders}} \right\}$$

integrating by parts, we obtain

$$\cancel{\frac{\partial L}{\partial \dot{q}}\delta q \Big|_{t'}^{t''}} - \int_{t'}^{t''} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$$

since applied ☆ 0

Then, what we habe now is

$$S[q+\delta q] \approx \int_{t'}^{t''} dt \left[L(q,\dot{q};t) + \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)\delta q \right]$$

we ignore all higher orders, since δq is extreamly small.

The first term is nothing but the action for the path before derformachun.

$$\delta S \equiv S[q+\delta q] - S[q] = \int_{t'}^{t''} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right] \delta q .$$

↳ the variation of the action (up to the first order ~~of~~ expansion)

Taking the critical condition $\delta S = 0$, we have

$$0 = \int_{t'}^{t''} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right] \delta q .$$

since $\delta q \neq 0$, we have

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 ,$$

which is known as the Euler–Lagrange equation.

$\boxed{\text{The path, that satisfies the critical condition } \delta S = 0 \text{, called a "classical path".}}$

gives.

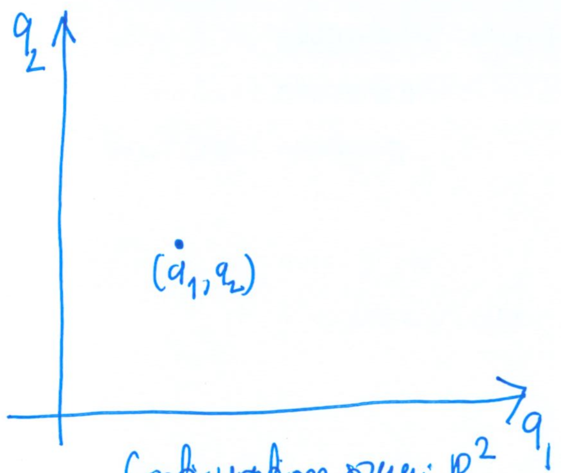
$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

would describe the dynamics of the system subjects to the initial conditions.

Example 1: Let's consider two particles with mass m, interacting with each other through the Hooke's law. The Lagrangian is given by

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2; t) = \frac{m\dot{q}_1^2}{2} + \frac{m\dot{q}_2^2}{2} - \frac{k}{2}(q_1 - q_2)^2 ,$$

where k is a coupling constant.



Configuration space: $\mathbb{R}^2$

We introduce a new set of coordinates

$$Q \equiv q_1 + q_2 \Rightarrow \dot{Q} = \dot{q}_1 + \dot{q}_2$$

$$q = q_1 - q_2 \Rightarrow \dot{q} = \dot{q}_1 - \dot{q}_2 .$$

Then Lagrangian becomes

$$L(Q, \dot{Q}, q, \dot{q}; t) = \underbrace{\frac{m}{4}\dot{Q}^2}_{\text{free particle part}} + \underbrace{\frac{m}{4}\dot{q}^2 - \frac{k}{2}q^2}_{\text{harmonic oscillator part}}$$

Lagrangian is not <u>unique</u>! ⑤

We can modify the Lagrangian in some certain ways and the Euler-Lagrange equation is still intact.

1) $L \rightarrow \alpha L + \beta$, when α and β are constant.

2) $L \rightarrow L + \underbrace{\frac{d}{dt} f(q;t)}_{\text{total derivative term.}}$

$$\int_{t'}^{t''} dt \left(L + \frac{d}{dt} f(q;t)\right) = \int_{t'}^{t''} dt\, L + \int_{t'}^{t''} dt \left(\frac{d}{dt} f(q;t)\right)$$

$$\Rightarrow \delta\left[\int_{t'}^{t''} dt\, L(q;\dot{q};t) + \underbrace{f(q'';t'') - f(q';t')}_{\substack{\text{will not contribute}\\ \text{or add up anything to}\\ \text{the variational process.}}}\right] \qquad \underbrace{f(q'';t'') - f(q';t)}_{\substack{\text{defined at}\\ \text{end-points.}}}$$

<u>EX2</u>: Consider $L(q,\dot{q};t) = \dot{q}^2 - \dot{q}$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \Rightarrow 0 - \frac{d}{dt}\left(2\dot{q} - 1\right)$$

$$\ddot{q} = 0 \quad (\text{free particle})$$

Therefore, $L(q,\dot{q};t) = \dot{q}^2$ and $L(q,\dot{q};t) = \dot{q}^2 - \underbrace{\dot{q}}_{\frac{d}{dt}q}$ give us exactly EOM.

<u>EX3</u>: Consider $L = \frac{1}{12}\dot{q}^4 + \dot{q}^2 V - V^2$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \dot{q}^2 V' - 2VV' - \left[\dot{q}^2\ddot{q} + 2\ddot{q}V + 2\dot{q}^2 V'\right] = 0$$

$$\dot{q}^2\ddot{q} + 2\ddot{q}V + \dot{q}^2 V' + 2VV' = 0$$

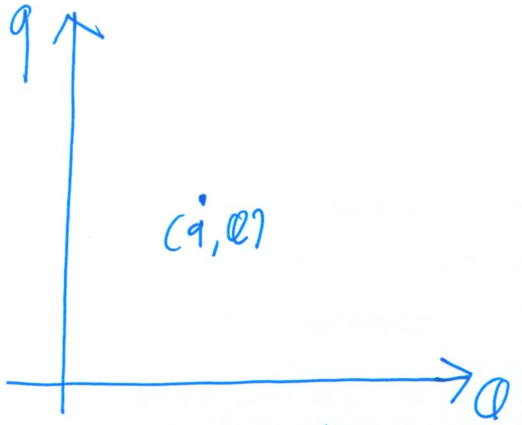
$$\ddot{q}(\dot{q}^2 + 2V) + V'(\dot{q}^2 + 2V) = 0 \rightarrow (\ddot{q} + V')(\dot{q}^2 + 2V) = 0$$

$$\hookrightarrow \ddot{q} = -V' \quad \#$$

<u>EX4</u>: Consider

$$L_\lambda(q,\dot{q};t) = m\lambda^2\left[e^{-\frac{\dot{q}^2}{2\lambda^2}} + \frac{\dot{q}}{\lambda^2}\int_0^{\dot{q}} d\xi\, e^{-\frac{\xi^2}{2\lambda^2}}\right] e^{-\frac{V(q)}{m\lambda^2}}$$

$$\hookrightarrow e^{-\frac{\dot{q}^2}{2\lambda^2}}(+V' + m\ddot{q}) = 0$$



New configuration space : $\mathbb{R}^2$

$$\cancel{\frac{\partial L}{\partial Q}}^{\,0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right) = 0$$

$$-\frac{d}{dt}\left(\frac{m}{2}\dot{Q}\right) = 0 \Rightarrow \frac{m}{2}\ddot{Q} = 0$$

$$Q(t) = \underbrace{A\,t + B}_{\text{must be determined.}}$$

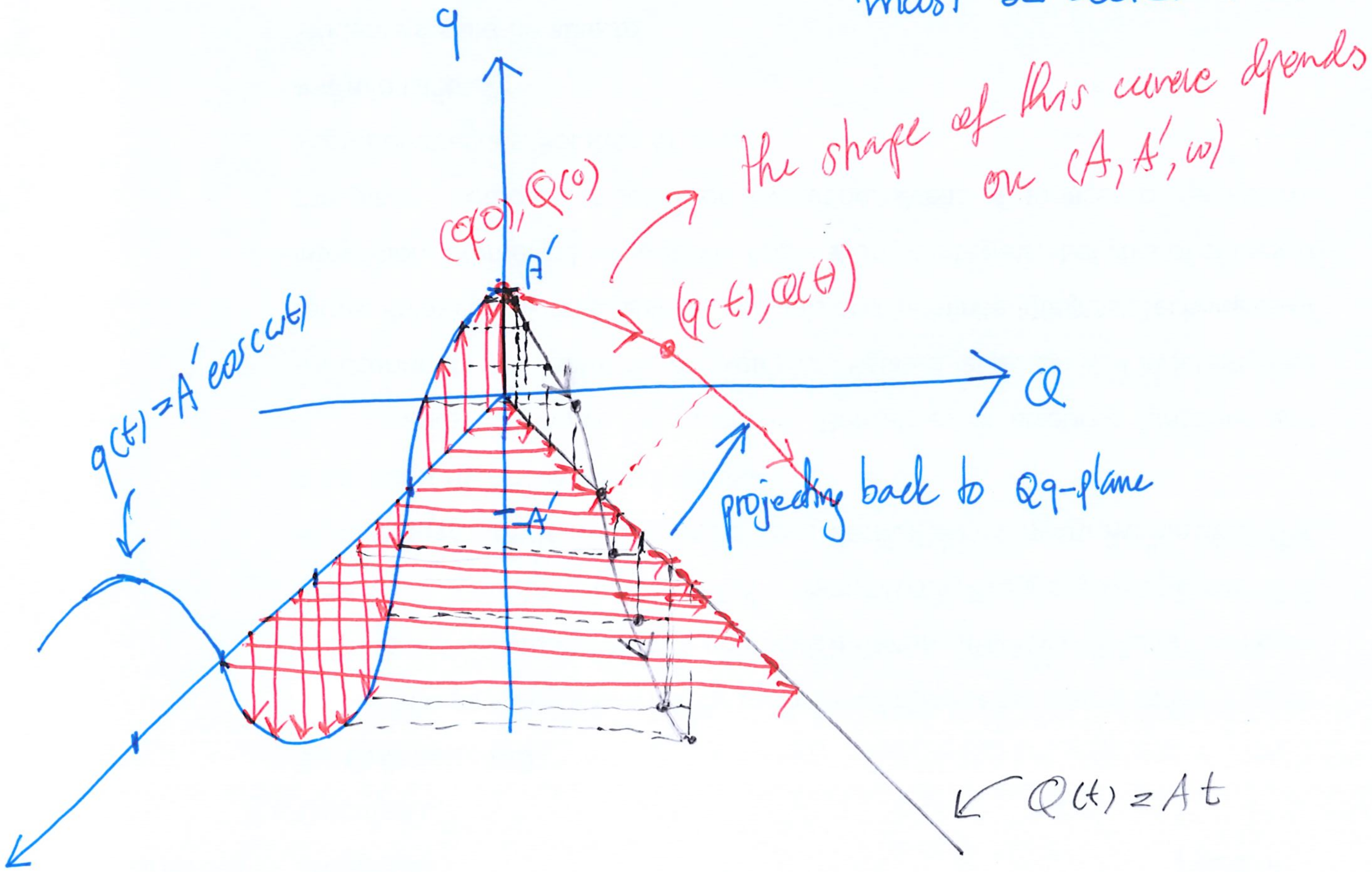
$\dot{Q}(t) = A \equiv$ center of mass velocity

Next, we consider

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \Rightarrow -\frac{k}{2}(2)q - \frac{d}{dt}\left(\frac{m}{2}\dot{q}\right) = 0$$

$$\ddot{q} + \omega^2 q = 0 \ , \quad \omega^2 = \frac{2k}{m}$$

$$q(t) = \underbrace{A'\cos(\omega t + B')}_{\text{must be determined.}}$$



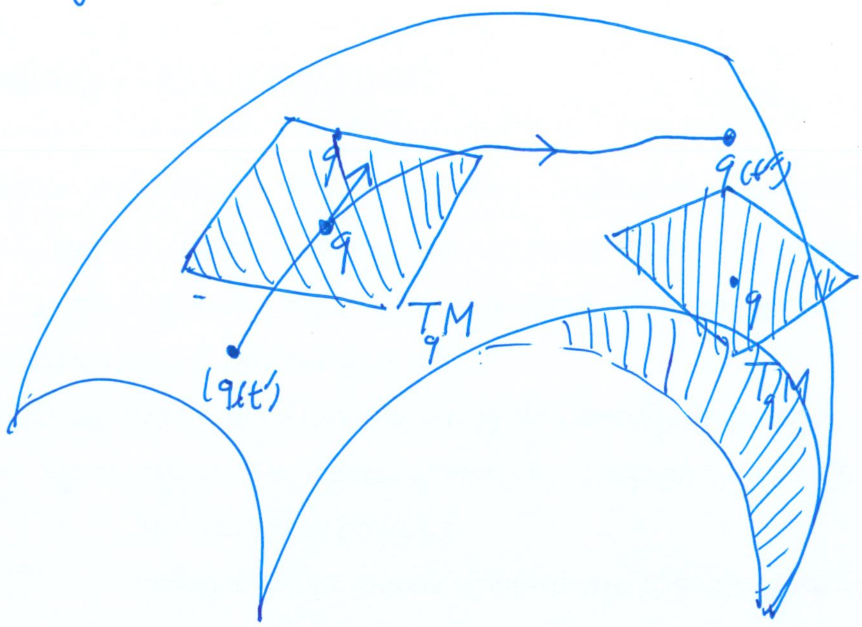
$\Rightarrow$ Transforming back to original variables

$$\left.\begin{aligned} q_1 + q_2 &= At \\ q_1 - q_2 &= A'\cos\omega t \end{aligned}\right\} \quad \begin{aligned} q_1 &= \tfrac{1}{2}\left(At + A'\cos(\omega t)\right) \\ q_2 &= \tfrac{1}{2}\left(At - A'\cos(\omega t)\right) \end{aligned}$$

#

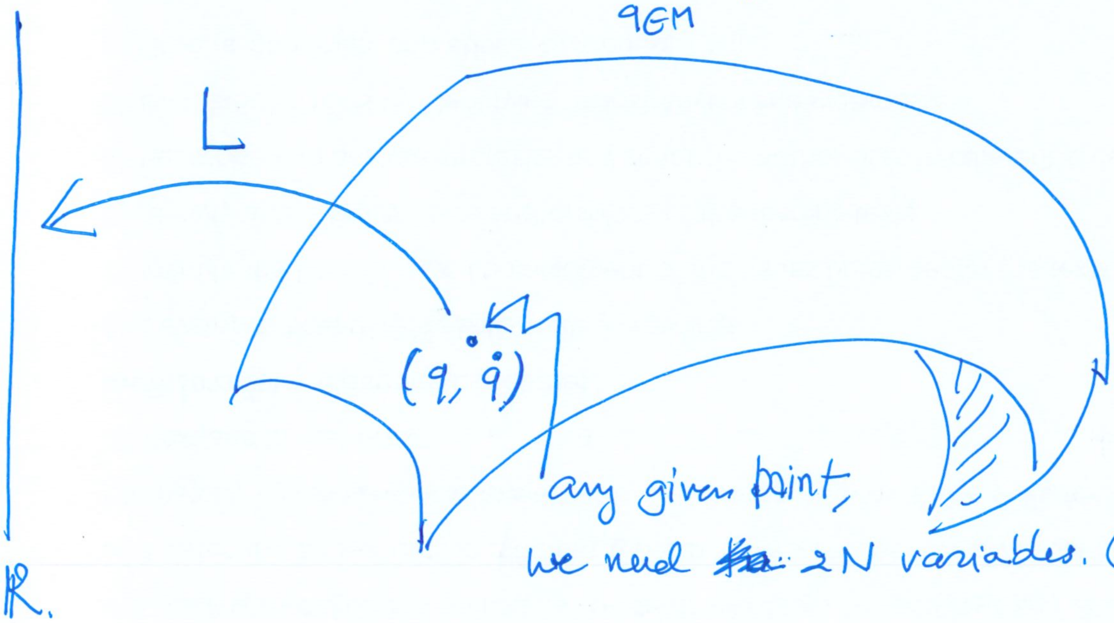
Tangent bundle

At any given point on the configuration space, there is a tangent space with dimension N.



Now, we introduce a new space called "tangent bundle"

$$TM = \bigsqcup_{q \in M} T_q M.$$



any given point,
we need ~~...~~ $2N$ variables. (N from q, N from $\dot{q}$)

Now, we can think that the Lagrangian is nothing but a map (from any point) from tangent bundle to the real line $\mathbb{R}$.

$$(q, \dot{q}) \longrightarrow \boxed{L} \longrightarrow \text{numerical value.}$$

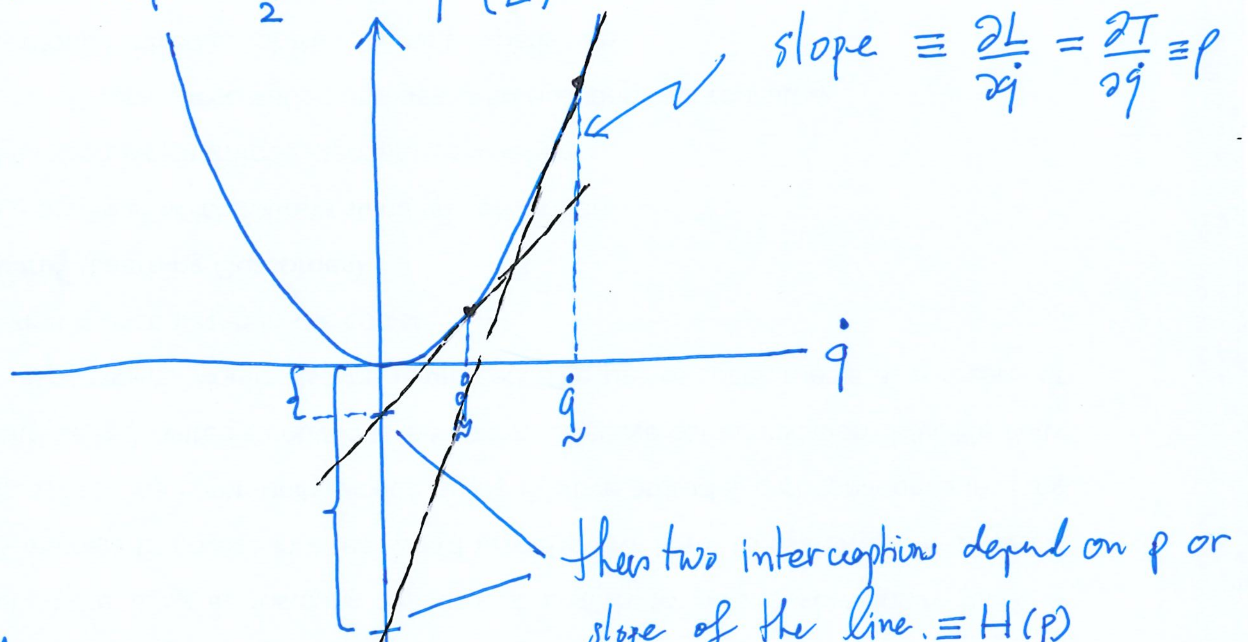
Generalised momentum.

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

If $\frac{\partial L}{\partial q} = 0$, it means that the associated momentum is a constant, and "q" ~~con~~ can be called a cyclic variable. (see Ex.1)

2. Hamiltonian mechanics

We shall first consider the shape of kinetic term of the Lagrangian. $T(\dot{q}) = \frac{m\dot{q}^2}{2}$



Line equation reads $y = mx + C$
↑ slope ↑ interception.

Therefore, we have $L(\dot{q}, q; t) = \frac{\partial L}{\partial \dot{q}}\dot{q} - H(p, q; t)$.

↑ slope ↑ interception. ← Hamiltonian; a set of generalised momentum $(p_1, p_2, \ldots, p_N)$

This relation is known as the Legendre transformation.

Ex 5: Given $L(\dot{q}, q; t) = \frac{\dot{q}^2}{2} - \frac{q^2}{2}$, find the Hamiltonian.

We first compute $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$. Then, applying the Legendre transformation, we obtain

$$H(p, q; t) = \cancel{\dot{q}\dot{q}}\, p\,p - \left(\frac{p^2}{2} - \frac{q^2}{2}\right) = \frac{p^2}{2} + \frac{q^2}{2}$$

#

Next we consider

$$dH = d(p\dot q) - dL \Rightarrow \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial t} = \dot q\,dp + p\,d\dot q - \cancelto{p}{\frac{\partial L}{\partial \ddot q}}\,d\dot q - \cancelto{-\frac{d}{dt}(p)=\dot p}{\frac{\partial L}{\partial q}}\,dq - \frac{\partial L}{\partial t}$$

Grouping terms, we have

$$\left(\frac{\partial H}{\partial p} - \dot q\right)dp + \left(\frac{\partial H}{\partial q} + \dot p\right)dp + \frac{\partial H}{\partial t} - \frac{\partial L}{\partial t} = 0$$

Since p and q are independent, we have

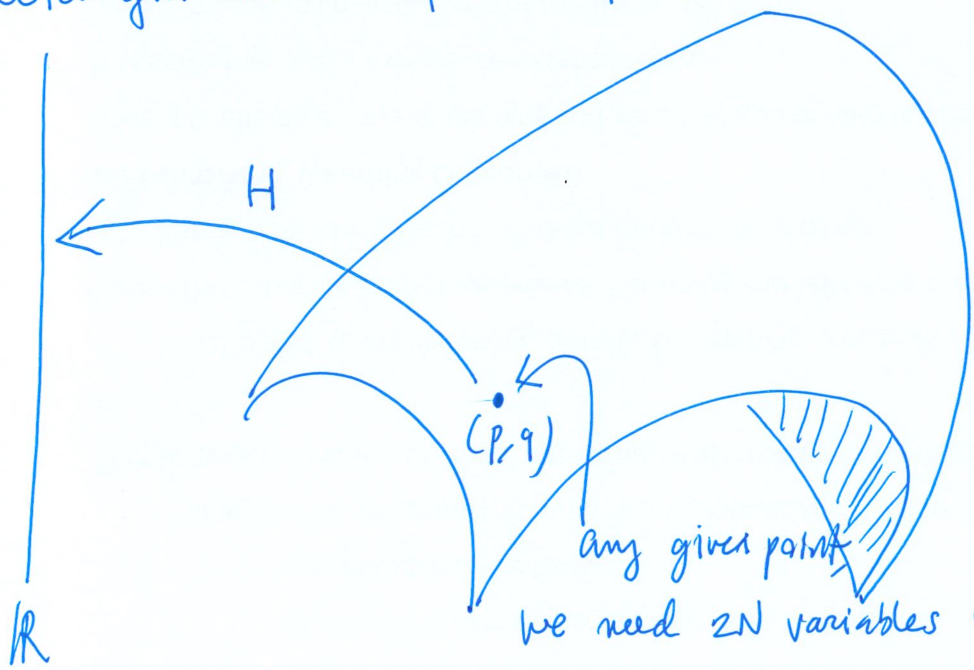
$$\dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q} \qquad \left[\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}\right]$$

which are known as the Hamilton's equations.

EX6: Given $H(p,q) = \frac{p^2}{2} + \frac{q^2}{2}$, find the equation of motion

$$\Rightarrow \dot q = \frac{\partial H}{\partial p} = p, \quad \dot p = -\frac{\partial H}{\partial q} = -q \Rightarrow \ddot q = \dot p = -q \Rightarrow \ddot q + q = 0$$

Cotangent bundle (phase-space)



we need 2N variables (N from p, N from q)

Let F be a function defined on the cotangent bundle. We find that

$$\frac{dF}{dt} = \frac{\partial F}{\partial p}\cancelto{-\frac{\partial H}{\partial q}}{\dot p} + \frac{\partial F}{\partial q}\cancelto{\frac{\partial H}{\partial p}}{\dot q} \quad \text{(F does not depend on time explicitly)}$$

$$= \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q}.$$

Now, we introduce a new mathematical object $\{F, H\} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q}$, known as the Poisson bracket.

Then, the change of F with respect to time is

$$\frac{dF}{dt} = \{F, H\}.$$

Hamiltonian can be treated as a time generator.

$$\{*, H\} = \frac{d}{dt} *$$

any function defined on Cotangent bundle.

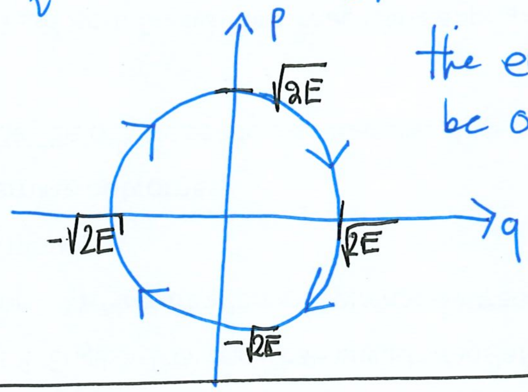
The Hamilton's equations can be reexpressed as

$$\dot{q} = \{q, H\} \quad , \quad \dot{p} = \{p, H\}$$

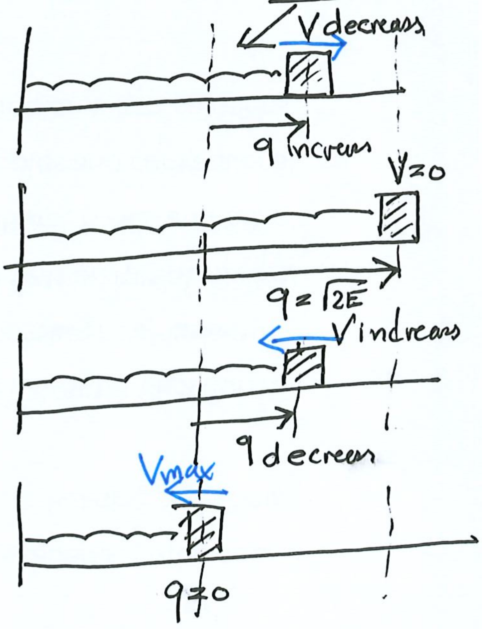
$$\Rightarrow \{p, q\} = 1 \quad , \quad \{p, p\} = \{q, q\} = 0$$

__Ex 7:__ Given $H(p,q) = \frac{p^2}{2} + \frac{q^2}{2}$, find the trajectory of system on the phase-space.

Let $E \equiv \frac{p^2}{2} + \frac{q^2}{2}$ be an energy function.

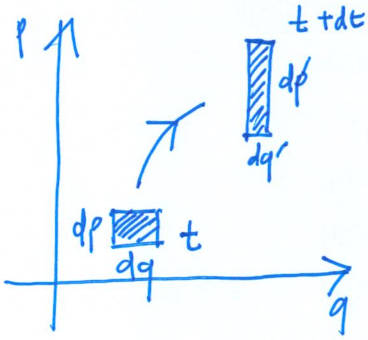


the evolution of the system will be on the circle with radius $\sqrt{2E}$ and rotate clockwise



Constant hypervolume

We first consider a simple case, 2D space.



$$q' \equiv q(t+dt) = q + \dot{q}\,dt$$

$$dq' = dq + d\dot{q}\,dt = dq + \frac{d\dot{q}}{dq} dq\,dt$$

$$p' = p(t+dt) = p + \dot{p}\,dt$$

$$dp' = dp + d\dot{p}\,dt = dp + \frac{d\dot{p}}{dp} dp\,dt$$

$$\Rightarrow dp'dq' = \left(dp + \frac{d\dot{p}}{dp} dp\,dt\right)\left(dq + \frac{d\dot{q}}{dq} dq\,dt\right) \cong dpdq + \frac{d\dot{p}}{dp} dpdq\,dt + \frac{d\dot{q}}{dq} dqdp\,dt$$

$$= dpdq\left[1 + \left(\frac{d\dot{p}}{dp} + \frac{d\dot{q}}{dq}\right)dt\right] = dpdq\left[1 + \left(\cancel{\frac{d}{dp}\left(-\frac{\partial H}{\partial q}\right)} + \cancel{\frac{d}{dq}\left(\frac{\partial H}{\partial p}\right)}\right)dt\right]$$

Therefore

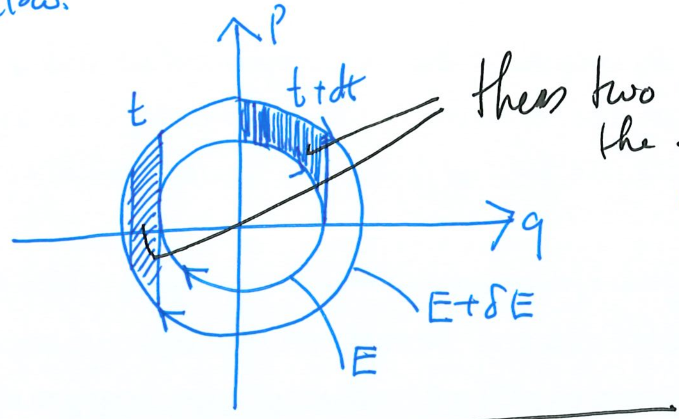
$$dp'dq' = dpdq.$$

the area ~~is~~ preserves under the time evolution.

Now we introduce $\omega \equiv dp \wedge dq$ which a symplectic 2-form.

wedge product.

Ex8: Consider a ring shell bounded by E and $E+dE$ given in figure below.



these two bounded areas are the same.

$$E = \frac{p^2}{2} + \frac{q^2}{2}$$

In general, the symplectic 2-form reads

$$\omega = \sum_{i=1}^{N} dp_i \wedge dq_i$$

Canonical transformation

In Hamiltonian mechanics, a set of conjugate variables (p, q) is not unique!!. This means that we can work with a new set (P, Q) of the Hamilton's equations are invariant.

$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ -\dot{p} &= \frac{\partial H}{\partial q} \end{aligned}}\ \boxed{1} \longrightarrow \boxed{\begin{aligned} \dot{Q} &= \frac{\partial K}{\partial P} \\ -\dot{P} &= \frac{\partial K}{\partial Q} \end{aligned}}\ \boxed{2} \quad , K \equiv K(P, Q)$$

$$P(q,p), \ Q(q,p)$$

With this condition, $(p, q) \mapsto (P, Q)$ is called "canonical transformation".

We knew that $\boxed{1}$ could be derive from

$$\delta \int_{t'}^{t''} dt\, L(\dot{q}, q; t) = \delta \int_{t'}^{t''} dt \left[p\dot{q} - H(p, q; t) \right].$$

Similarly, $\boxed{2}$

$$\delta \int_{t'}^{t''} dt\, \mathcal{L}(\dot{Q}, Q, t) = \delta \int_{t'}^{t''} dt \left[P\dot{Q} - K(P, Q, t) \right]$$

(Note: $Q = Q(q, t)$ is point transformation)

Recalling the property

$$L(\dot q, q; t) = \mathcal{L}(\dot Q, Q; t) + \frac{d}{dt} F(Q, q; t),$$

therefore, we have

$$p\dot q - H(p,q) = P\dot Q - K(P,Q) + \underbrace{\frac{d}{dt} F_1(Q,q;t)}_{\frac{\partial F_1}{\partial q}\dot q + \frac{\partial F_1}{\partial Q}\dot Q + \frac{\partial F_1}{\partial t}}$$

$$\Rightarrow \left[p - \frac{\partial F_1}{\partial q}\right]\dot q - H = \left[P + \frac{\partial F_1}{\partial Q}\right]\dot Q - K + \frac{\partial F_1}{\partial t}$$

Assume that F_1 determines p and P such that

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}, \qquad ※$$

then we have

$$K(P, Q; t) = H(p, q; t) + \frac{\partial}{\partial t} F_1(Q, q; t) \qquad ※$$

Here is a thing. There are two slots in an argument of F.
There are four possible combination

$$\begin{tikzcd}
F_1(Q,q;t) \arrow[rr, "{F_1(Q,q;t) = F_2(P,q;t) - QP \;\; (\text{left} \leftarrow \text{right})}"] \arrow[d, "{F_1 = F_3(Q,p,t) + qp}"'] & & F_2(P,q;t) \arrow[d, "{F_1 = F_4 + qp - QP}"] \\
F_3(Q,p;t) \arrow[rr, "{F_1(Q,q;t) - qp = F_3(Q,p;t) \;\; (\text{left} \rightarrow \text{right})}", "{F_1 = F_4 + qp - QP}"'] & & F_4(P,p;t)
\end{tikzcd}$$

Generating function	function derivatives.
$F_1(Q,q;t)$	$p = \frac{\partial F_1}{\partial q}$, $P = -\frac{\partial F_1}{\partial Q}$
$F_2(q,P;t) - Q\cdot P$	$p = \frac{\partial F_2}{\partial q}$, $Q = \frac{\partial F}{\partial P}$
$F_3(p,Q;t) + qp$	$q = \frac{\partial F_3}{\partial p}$, $P = -\frac{\partial F}{\partial Q}$
$F_4(p,P;t) + qp - QP$	$q = -\frac{\partial F_4}{\partial p}$, $Q = \frac{\partial F}{\partial P}$

The symplectic structure presurves under canonical transformation

$$\omega = dp\wedge dq = dP\wedge dQ = dP(q,p)\wedge dQ(q,p)$$

$$= \left(\frac{dP}{dq}dq+\frac{dP}{dp}dp\right)\wedge\left(\frac{dQ}{dq}dq+\frac{dQ}{dp}dp\right)$$

$$= \frac{dP}{dq}\frac{dQ}{dq}\cancelto{0}{dq\wedge dq} + \frac{dP}{dp}\frac{dQ}{dq}dp\wedge dq + \frac{dP}{dq}\frac{dQ}{dp}dq\wedge dp - \frac{dP}{dp}\frac{dQ}{dp}\cancelto{0}{dp\wedge dp}$$

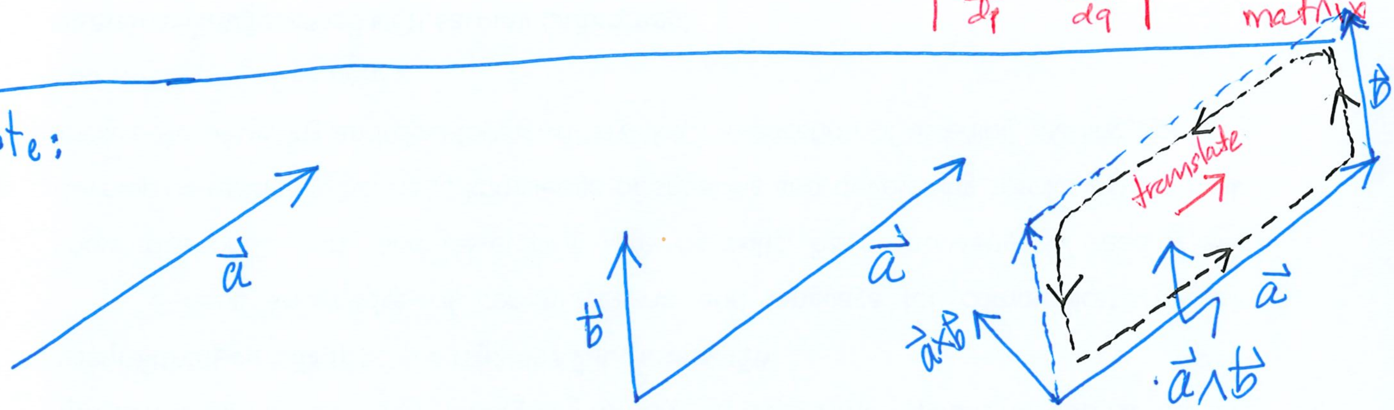
$$= \left[\frac{dP}{dp}\frac{dQ}{dq} - \frac{dP}{dq}\frac{dQ}{dp}\right]dp\wedge dq = \cancelto{1}{\{P,Q\}}\,dp\wedge dq$$

$$= dp\wedge dq \quad ※$$

alternative view

$$\begin{vmatrix} \frac{dP}{dp} & \frac{dP}{dq} \\ \frac{dQ}{dp} & \frac{dQ}{dq} \end{vmatrix} \equiv \text{Jacobian matrix}$$

Note:



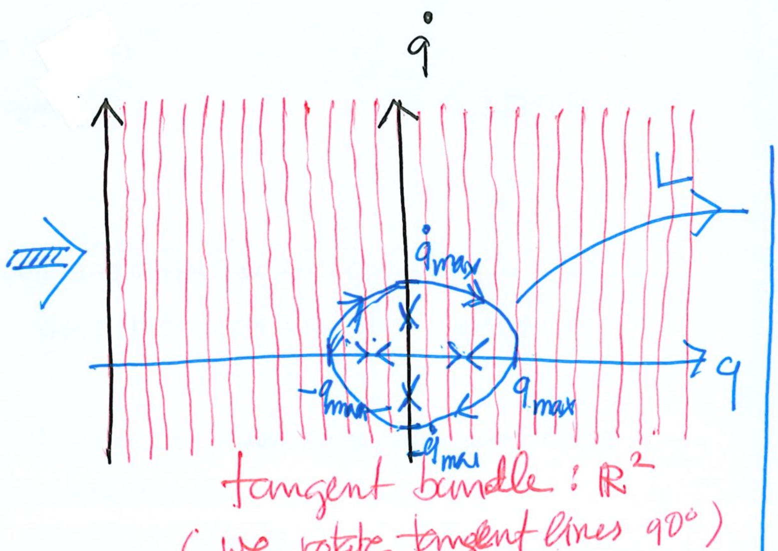
Ex 7: Consider the harmonic oscillator again

$$L(\dot q, q; t) = \frac{\dot q^2}{2} - \frac{q^2}{2}$$

every single point, there is a tangent line: $\mathbb{R}$, where $\dot q$ lives on.

$\rightarrow q$

configuration space: $\mathbb{R}$



$$H(p,q;t) = p\dot q - L(\dot q, q; t) = \frac{p^2}{2} + \frac{q^2}{2}$$

Using $Q = p$, $P = -q$, we find that

$$\frac{p^2}{2} + \frac{q^2}{2} \Rightarrow \frac{Q^2}{2} + \frac{P^2}{2} \equiv K(P,Q)$$

$(F_2 = -qP)$