Quantum Information IF Summer School 2026

Problem Set 2

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1. Bell states.

(a) Prove that the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is entangled.

(b) Show that the singlet state $(|01\rangle - |10\rangle)/\sqrt{2}$ can be written as $(|\hat{\mathbf{n}}\rangle|-\hat{\mathbf{n}}\rangle - |-\hat{\mathbf{n}}\rangle|\hat{\mathbf{n}}\rangle)/\sqrt{2}$ for any spin direction $\hat{\mathbf{n}}$. This means that the state is invariant under any local unitaries that act the same on the two qubits i.e. unitary transformations of the form $\widehat{U} \otimes \widehat{U}$. (**Hint:** start from $(|\hat{\mathbf{n}}\rangle|-\hat{\mathbf{n}}\rangle - |-\hat{\mathbf{n}}\rangle|\hat{\mathbf{n}}\rangle)/\sqrt{2}$ and work backward.)

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2. Schmidt decomposition.

If Bell states are maximally entangled, there must also exist states that are "less entangled". In this exercise, we develop a systematic way to quantify entanglement for bipartite pure states.

Consider a bipartite state written in product bases:

$$|\psi\rangle = \sum_{j,k} A_{jk} \left|e_j\right\rangle \otimes |f_k\rangle ,$$

where $\{|e_j\rangle\}$ and $\{|f_k\rangle\}$ are ONBs for the two subsystems. Since local unitary transformations do not change entanglement, we are free to change these bases independently. One might try to simplify the expression by defining

$$\left|\overline{\phi_j}\right\rangle := \sum_k A_{jk} |f_k\rangle ,$$

so that

$$|\psi\rangle = \sum_j \left|e_j\right\rangle \otimes \left|\overline{\phi_j}\right\rangle .$$

However, the vectors $\left|\overline{\phi_j}\right\rangle$ are generally neither normalized nor orthogonal. A better approach leads to the *Schmidt decomposition*, which reveals the full structure of bipartite entanglement.

(a) Assume the two subsystems have the same dimension and that

$$A_{jk} = A^*_{kj},$$

so that A is a Hermitian matrix. Show that $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_\nu \lambda_\nu |\nu\rangle \otimes \left|\nu'\right\rangle ,$$

where λ_ν are the eigenvalues of A, $\{|\nu\rangle\}$ is an orthonormal eigenbasis of A, and $|\nu'\rangle$ is related to $|\nu\rangle$ (make this relation explicit!).

(b) To treat the general case (possibly having different subsystem dimensions), we develop the **singular value decomposition (SVD)**. Let A be an arbitrary $n \times m$ matrix. Show that $A^\dagger A$ and $AA^\dagger$ are Hermitian, and that both are positive semidefinite, i.e.

$$\langle x|A^\dagger A|x\rangle \geq 0, \quad \langle y|AA^\dagger|y\rangle \geq 0$$

for all vectors $|x\rangle, |y\rangle$.

(c) Show that if $|v\rangle$ is an eigenvector of $A^\dagger A$ with eigenvalue δ^2, then $A|v\rangle$ is either zero or an eigenvector of $AA^\dagger$ with the same eigenvalue. Similarly, $A^\dagger$ maps eigenvectors of $AA^\dagger$ to those of $A^\dagger A$.

Conclude that the nonzero eigenvalues of $A^\dagger A$ and $AA^\dagger$ coincide. If A is not square, the larger space has additional eigenvalues equal to 0.

(d) Use the previous parts to show that

$$A = U\Sigma V^\dagger,$$

where U is an $n \times n$ unitary matrix, V is an $m \times m$ unitary matrix, and Σ is an $n \times m$ diagonal matrix whose diagonal entries $\delta_\nu \geq 0$ are the **singular values** of A, i.e. the square roots of the nonzero eigenvalues of $A^\dagger A$.

(e) Using the SVD, show that

$$|\psi\rangle = \sum_\nu \delta_\nu |\nu\rangle \otimes |\nu'\rangle,$$

where $\{|\nu\rangle\}$ and $\{|\nu'\rangle\}$ are orthonormal bases for the two subsystems, and the number of nonzero terms is at most $\min(n, m)$. This is called the **Schmidt decomposition**.

Applying the Schmidt decomposition to a two-qubit system, any pure state can be written as

$$|\psi\rangle = \cos\theta\,|0\rangle|0'\rangle + \sin\theta\,|1\rangle|1'\rangle$$

where $|0\rangle, |1\rangle$ and $|0'\rangle, |1'\rangle$ are suitably chosen ONBs, and $\theta \in [0, \pi/4]$ (up to relabeling and phases). Thus, $\theta = 0$ corresponds to a product state, and entanglement increases monotonically with θ and is maximized at the Bell state ($\theta = \pi/4$).

(f) Schmidt basis and RDMs.

Let

$$|\psi\rangle = \sum_\nu \delta_\nu |\nu\rangle \otimes |\nu'\rangle$$

be the Schmidt decomposition of a bipartite pure state, and let $\rho = |\psi\rangle\langle\psi|$ be the corresponding density matrix. Show that the reduced density matrices

$$\rho_A = \mathrm{Tr}_B(\rho), \qquad \rho_B = \mathrm{Tr}_A(\rho)$$

take the form

$$\rho_A = \sum_\nu \delta_\nu^2 |\nu\rangle\langle\nu|, \qquad \rho_B = \sum_\nu \delta_\nu^2 |\nu'\rangle\langle\nu'|.$$

Thus, performing the Schmidt decomposition automatically diagonalizes the RDM of both subsystems.

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3. Correlations for arbitrary measurement directions.

For any unit vectors $\hat{\mathbf{n}}, \hat{\mathbf{m}}$, define the two-point correlation function to be the expectation value

$$E(\hat{\mathbf{n}}, \hat{\mathbf{m}}) = \langle\Phi^+|(\hat{\mathbf{n}}\cdot\sigma)\otimes(\hat{\mathbf{m}}\cdot\sigma)|\Phi^+\rangle$$

in the state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

(a) Show that $E(\hat{\mathbf{n}}, \hat{\mathbf{m}}) = \hat{\mathbf{n}} \cdot \hat{\mathbf{m}}$

(b) Now, restrict the measurements to be in the X-Z plane, so that $\hat{\mathbf{n}} = \begin{bmatrix} \sin\alpha & 0 & \cos\alpha \end{bmatrix}^T$ and $\hat{\mathbf{m}} = \begin{bmatrix} \sin\beta & 0 & \cos\beta \end{bmatrix}^T$, and

$$E(\hat{\mathbf{n}}, \hat{\mathbf{m}}) \equiv E(\alpha, \beta) = \cos(\alpha - \beta).$$

Find the four unit vectors that gives the *Tsirelson quantity*

$$S = E(\alpha_0, \beta_0) + E(\alpha_0, \beta_1) + E(\alpha_1, \beta_0) - E(\alpha_1, \beta_1)$$

the value of $2\sqrt{2}$, violating the CHSH inequality $S \leq 2$.