Quantum Information	IF Summer School 2026

Problem Set 1

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1. Pauli operator basis.

In quantum information, we will very often work with so-called *Pauli operators*. You should memorize their matrix forms, as they show up everywhere:

$$\widehat{X} \simeq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \widehat{Y} \simeq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \widehat{Z} \simeq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(a) Show that each of the Pauli operators $\widehat{X}, \widehat{Y}, \widehat{Z}$ is both Hermitian and unitary.

(b) Show that $\{\hat{\mathbb{1}}, \widehat{X}, \widehat{Y}, \widehat{Z}\}$ forms a basis the real vector space of 2×2 Hermitian matrices. Moreover, show that $\{\hat{I}, \widehat{X}, \widehat{Y}, \widehat{Z}\}$ forms a basis for the complex vector space of all 2×2 matrices.

(c) The Hilbert–Schmidt inner product between matrices A, B is defined as

$$\langle A, B \rangle = \mathrm{Tr}(A^\dagger B).$$

Show that this satisfies the axioms of an inner product.

(d) Show that the operators $\hat{\mathbb{1}}, \widehat{X}, \widehat{Y}, \widehat{Z}$ are orthogonal under the Hilbert–Schmidt inner product. Are they orthonormal? If not, can you make them orthonormal?

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2. Operator exponentials.

(a) Suppose $\hat{A}$ is invertible and $\hat{A}\widehat{B}\hat{A}^{-1} = \widehat{C}$, show that $\hat{A}e^{\widehat{B}}\hat{A}^{-1} = e^{\widehat{C}}$.

(b) Show that if $\hat{A}$ is a linear operator that squares to the identity: $\hat{A}^2 = \hat{\mathbb{1}}$, then

$$e^{i\theta\hat{A}} = \cos(\theta)\hat{\mathbb{1}} + i\sin(\theta)\hat{A}.$$

From this, compute the matrix form of

$$\exp\left(i\frac{\pi}{2}\frac{\widehat{X} + \widehat{Z}}{\sqrt{2}}\right).$$

(c) By applying the result of **(b)** to Pauli operators, show that

$$e^{i\theta(\widehat{X}+\widehat{Z})} \neq e^{i\theta\widehat{X}}e^{i\theta\widehat{Z}}$$

for a general value of θ.

(d) Local unitary operators are defined to be of the form $\widehat{U} \otimes \widehat{V}$. In contrast, when it comes to observables and Hamiltonians, the local ones are of the form $\hat{A} \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \widehat{B}$. Show that

$$e^{i(\hat{A}\otimes\hat{\mathbb{1}}+\hat{\mathbb{1}}\otimes\widehat{B})} = e^{i\hat{A}} \otimes e^{i\widehat{B}}.$$

That is, the exponential of local observables factorizes into local unitary operators.

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3. Projection operators (If time permits).

A standard characterization of a projection operator $\widehat{P}$ is that it satisfies

$$\widehat{P}^2 = \widehat{P} \quad \text{and} \quad \widehat{P}^\dagger = \widehat{P}.$$

(a) One might try to define a projection operator using only the condition $\widehat{P}^2 = \widehat{P}$. What goes wrong if we omit the condition $\widehat{P}^\dagger = \widehat{P}$? Give a counterexample and explain what fails in terms of eigenvectors.

(b) Show that the eigenvalues of a projection operator must be 0 or 1.

(c) Let $\widehat{P}$ be a projector onto a subspace S, and $\widehat{P}'$ a projector onto a subspace S'. What is the subspace fixed by the product $\widehat{P}\widehat{P}'$?

(d) Let $\hat{A}$ be a Hermitian operator on a finite-dimensional Hilbert space with distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Show that the projector $\widehat{P}_k$ onto the eigenspace associated with λ_k can be expressed as a polynomial in $\hat{A}$, namely

$$\widehat{P}_k = \prod_{j \neq k} \frac{\hat{A} - \lambda_j \hat{\mathbb{1}}}{\lambda_k - \lambda_j}.$$