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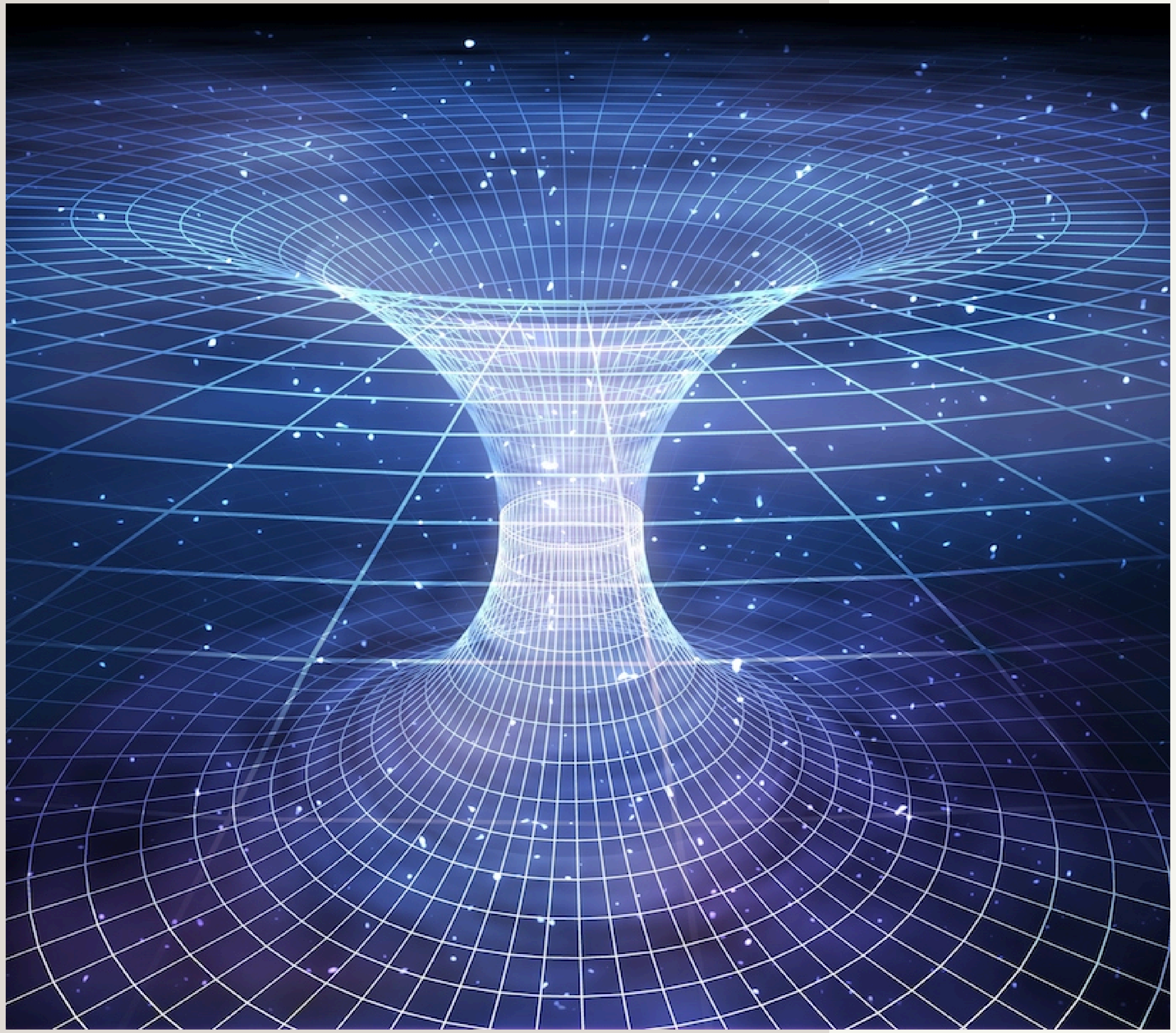

# Jacobson's thermodynamic approach to classical gravity applied to non-Riemmanian geometries: remarks on the simplicity of Nature


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Based on: J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.

# 1.RG:Review

The Metric

$$g_{\mu\nu},$$

tensor comparison at different points,

$$\Gamma^{\rho}_{\mu\nu}(g_{\mu\nu}), \quad \Rightarrow R^{\lambda}{}_{\rho\mu\nu},$$

and field equations are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu},$$

A. Einstein, A. der Wissenschaften, 1915.

# 1.1.Laws of black hole dynamics

Zeroth law:

$$\kappa^2 \equiv -\frac{1}{2}\xi^{\alpha;\beta}\xi_{\alpha;\beta}, \qquad \kappa_{;\alpha} = 0,$$

First law:

$$(M, J, A) \to (M + \delta M, J + \delta J, A + \delta A),$$

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J,$$

J. M. Bardeen, B. Carter, S. W. Hawking, Commun. Math. Phys., 1973.

# 1.1.Laws of black hole dynamics

second law (generalized):

$$T = \frac{\hbar}{2\pi}\kappa, \qquad S = \frac{k_{\rm B} A}{4\ell_{\rm P}^2},$$

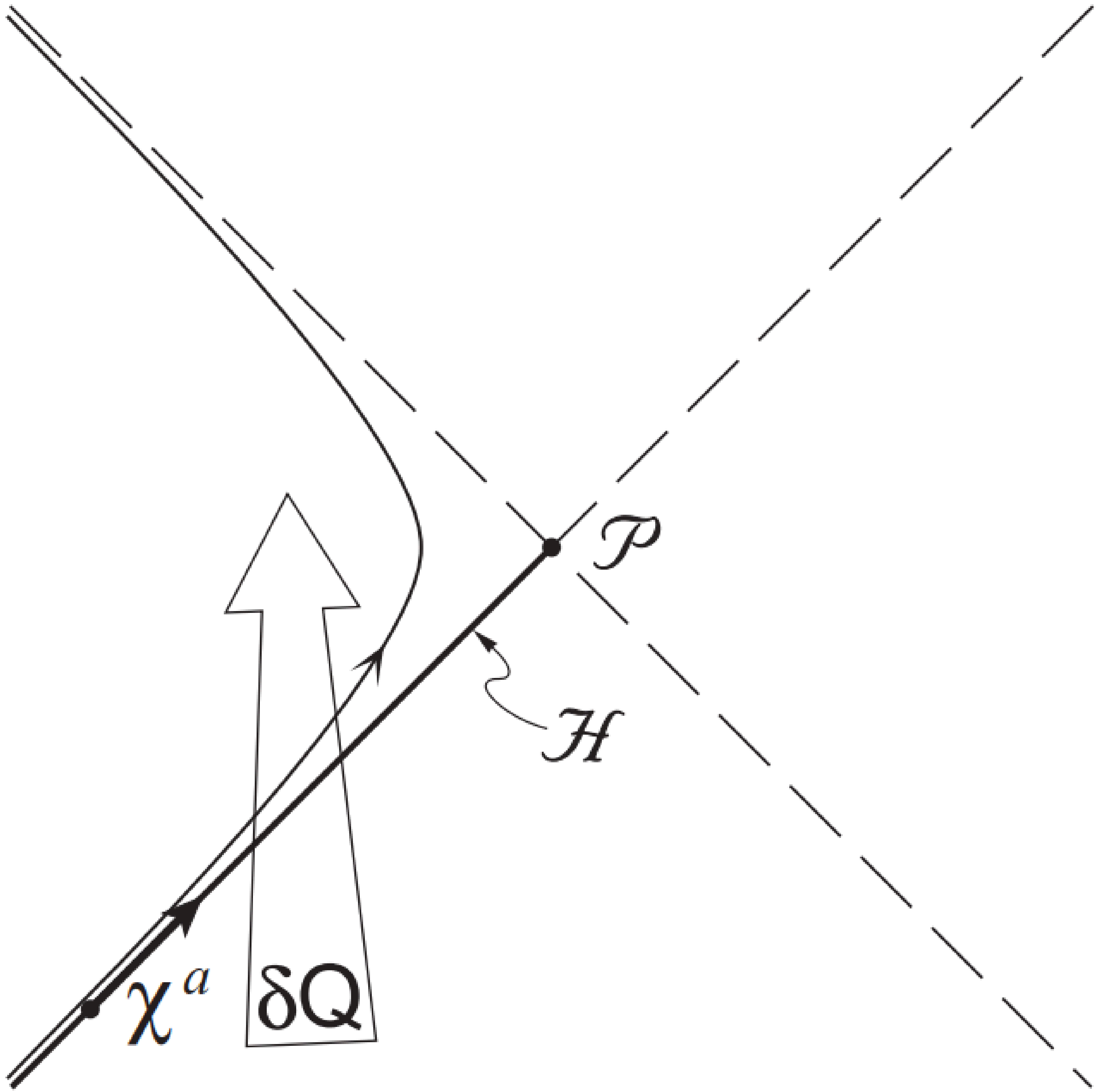
therefore,

$$\Delta S_{Tot} = \Delta S + \Delta S_M \geq 0,$$

S. W. Hawking, Nature, 1974.

J. D. Bekenstein, Phys. Rev. D, 1974.

# 1.2.RG:Thermodynamics




T. Jacobson, Phys. Rev. Lett., 1995.

# 1.2.1.Unruh effect

In Minkowski spacetime, accelerated observers measure a thermal bath of temperature:

$$T = \frac{\hbar \kappa}{2\pi},$$

where $\kappa$ is the fourth-acceleration norm.

T. Jacobson, Phys. Rev. Lett., 1995.

# 1.2.2. Energy flux

$$\delta Q = \int_{\mathcal{H}} T_{\mu\nu}\chi^{\mu}\mathrm{d}\Sigma^{\nu},$$

$$\chi^{\mu} = -\kappa\lambda k^{\mu}, \qquad d\Sigma^{\mu} = k^{\mu} d\lambda d\mathcal{A},$$

therefore,

$$\delta Q = -\kappa \int_{\mathcal{H}} \lambda T_{\mu\nu} k^{\mu} k^{\nu} d\lambda d\mathcal{A},$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 1.2.3.Horizon entropy

$$dS = \eta \delta \mathcal{A},$$

$$\delta \mathcal{A} = \int_{\mathcal{H}} \theta \mathrm{d}\lambda \mathrm{d}\mathcal{A},$$

using the Raychaudhuri equation,

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{\mu\nu}k^\mu k^\nu,$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 1.2.4. Area variation

$$\theta = -\lambda R_{\mu\nu} k^{\mu} k^{\nu},$$

then,

$$\delta \mathcal{A} = -\int_{\mathcal{H}} \lambda R_{\mu\nu} k^{\mu} k^{\nu} d\lambda d\mathcal{A},$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 1.2.5.Clausius relation

$$\delta Q = TdS = \frac{\hbar\kappa}{2\pi}\eta\delta\mathcal{A},$$

$$\kappa\int_{\mathcal{H}}\lambda T_{\mu\nu}k^{\mu}k^{\nu}d\lambda d\mathcal{A} = \frac{\hbar\kappa}{2\pi}\eta\int_{\mathcal{H}}\lambda R_{\mu\nu}k^{\mu}k^{\nu}d\lambda d\mathcal{A},$$

but, for an arbitrary horizon:

$$T_{\mu\nu}k^{\mu}k^{\nu} = \frac{\hbar\eta}{2\pi}R_{\mu\nu}k^{\mu}k^{\nu}, \qquad \Rightarrow \qquad \frac{2\pi}{\hbar\eta}T_{\mu\nu} = R_{\mu\nu} + fg_{\mu\nu},$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 1.2.6.Field equation

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \Rightarrow \qquad f = -R/2 + \Lambda,$$

D. Lovelock, J. Math. Phys., 1971.

D. Iosifidis and F. W. Hehl, Phys. Lett. B, 2024.

finally,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{2\pi}{\hbar\eta} T_{\mu\nu},$$

where,

$$\eta = \frac{1}{4\hbar G}, \qquad \Rightarrow \qquad \eta^{-1/2} = 2 l_p,$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 1.2.7.¿Field equation or state equation?

- Viewed in this way, RG is an equation of state.
- It may be no more appropriate to canonically quantize the Einstein equation than it would be to quantize the wave equation for sound in air.
- The cosmological constant remains as enigmatic as ever.
- High amplitude and frequency perturbations of the gravitational field could no longer be described by GR due to the lack of local equilibrium conditions.
- Entropy production in the free expansion of a gas is not described by Clausius' Relation.
- f(R) and Einstein-Cartan has been found by non-equilibrium thermodynamics.

T. Jacobson, Phys. Rev. D, 1995.
F. Falk, J. Elast., 1981.
R. Kupferman et. al., J. Geom. Mech., 2015.
R. Kupferman et. al., Isr. J. Math., 2017.
T. De Lorenzo et. al., Phys. Rev. D., 2018.
R. Dey, S. Liberati, D. Pranzetti, Phys. Rev. D, 2017.

# 2.RG: reasons to modify it

- Dark matter: $25.89\% \pm 0.57\%.$

N. Aghanim et. al., Astron. Astrophys.,2020.

- Dark energy: $69.11\% \pm 0.62\%.$

N. Aghanim et. al., Astron. Astrophys.,2020.

- Hubble tension: $H_0 = (67.4 \pm 0.5)\mathrm{km/s/Mpc},$

$$H_0 = (73.52 \pm 1.62)\mathrm{km/s/Mpc}.$$

A. G. Riess et al., Astrophys. J., 2018.

# 2.RG: reasons to modify it

- Singularities.

R. Penrose, Phys. Rev. Lett., 1965.

- Renormalizability and unitarity.

R. M. Wald, University of Chicago Press, 2010.
K. S. Stelle, Phys. Rev. D, 1977.

# 2.1.General Manifold

Beyond the metric, we also have

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\nu\beta} (\partial_{\alpha} g_{\beta\mu} + \partial_{\mu} g_{\alpha\beta} - \partial_{\beta} g_{\mu\alpha})$$
$$+ \frac{1}{2} (T^{\nu}{}_{\alpha\mu} - T_{\alpha\mu}{}^{\nu} - T_{\mu\alpha}{}^{\nu}) + \frac{1}{2} (Q^{\nu}{}_{\alpha\mu} - Q_{\mu\alpha}{}^{\nu} - Q_{\alpha\mu}{}^{\nu})$$

where

$$Q_{\mu\alpha\beta} \equiv \nabla_{\mu} g_{\alpha\beta}, \qquad T^{\alpha}{}_{\mu\nu} \equiv \Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu}.$$

# 3.MAG

$$\mathbf{R}^{1,3} \rtimes \mathrm{O}(1,3)\,, \quad \Rightarrow \quad \mathbf{R}^{1,3} \rtimes \mathrm{GL}(1,3)\,,$$

the non metricity tensor is defined as

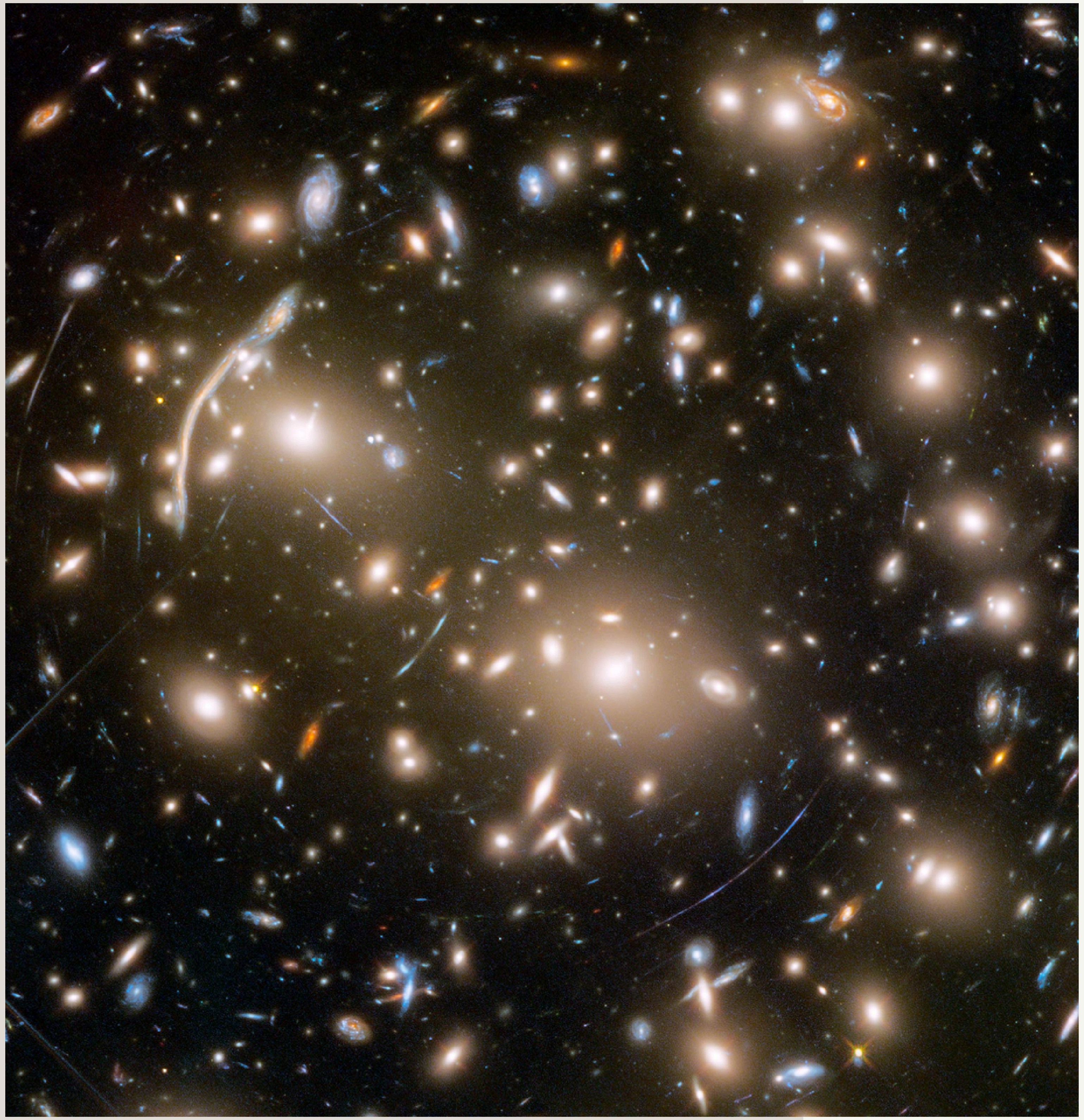
$$Q_{\mu\alpha\beta} \equiv \nabla_\mu g_{\alpha\beta}\,,$$

and modifies the inner product under parallel transport as

$$\partial_\alpha(g_{\mu\nu}V^\mu W^\nu) = \nabla_\alpha(g_{\mu\nu}V^\mu W^\nu) = Q_{\alpha\mu\nu}V^\mu W^\nu\,,$$

deformations of ideal crystal can be described by a metric like in GR, but local deffects, require structures like torsion and non metricity to be described.

F. W. Hehl et. al., Phys. Rept., 1995.
F. Falk, Journal of Elasticity, 1981.



# 4.Gravitational equation of state (vanishing Q)

# 4.1.Unruh effect

In Minkowski spacetime, accelerated observers measure a thermal bath of temperature:

$$T = \frac{\hbar\kappa}{2\pi},$$

where $\kappa$ is the fourth-acceleration norm.

T. Jacobson, Phys. Rev. Lett., 1995.

# 4.2.Energy flux

$$\delta Q = \int_{\mathcal{H}} T_{\mu\nu} \chi^{\mu} \mathrm{d}\Sigma^{\nu},$$

$$\chi^{\mu} = -\kappa \lambda k^{\mu}, \qquad d\Sigma^{\mu} = k^{\mu} d\lambda d\mathcal{A},$$

therefore,

$$\delta Q = -\kappa \int_{\mathcal{H}} \lambda T_{\mu\nu} k^{\mu} k^{\nu} d\lambda d\mathcal{A},$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 4.3.Horizon entropy (vanishing Q)

$$dS = \eta \delta \mathcal{A},$$

where

$$\delta \mathcal{A} = \int_{\mathcal{H}} \theta \mathrm{d}\lambda \mathrm{d}\mathcal{A},$$

and by definition

$$\boldsymbol{\theta} = \frac{1}{2} \boldsymbol{q}^{\mu\nu} \boldsymbol{\mathcal{L}}_u \boldsymbol{q}_{\mu\nu},$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 4.4. Area variation (vanishing Q)

we find that

$$\left.\frac{d\Theta}{d\lambda}\right|_{\mathcal{P}} = - R_{\alpha\beta}l^{\alpha}l^{\beta} - l^{\alpha}l^{\beta}\nabla_{\alpha}T_{\beta} + l^{\alpha}l^{\beta}T_{\alpha\beta\nu}T^{\nu}$$
$$- l^{\alpha}l^{\beta}\nabla_{\nu}T_{\beta\alpha}{}^{\nu} - T_{\mu\rho\nu}\nabla^{\nu}(l^{\mu}l^{\rho})$$
$$- (\nabla_{\mu}l^{\alpha})(\nabla_{\alpha}l^{\mu}) + T_{\beta\mu\alpha}l^{\alpha}\nabla^{\beta}l^{\mu}\,.$$

where

$$T^{\alpha}{}_{\mu\nu} \equiv \Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu}.$$

J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.

# 4.5.Clausius relation (vanishing Q)

$$\delta Q = TdS = \frac{\hbar\kappa}{2\pi}\eta\delta\mathcal{A},$$

from which we have ***the fundamental relation***

$$\frac{2\pi}{\hbar\eta}\mathcal{T}_{(\alpha\beta)} = R_{(\alpha\beta)} + fg_{\alpha\beta} + (\nabla_\nu - T_\nu)T_{(\alpha\beta)}{}^{\nu} + \nabla_{(\alpha}T_{\beta)}\,.$$

Tidal heating term:

$$\delta S_i = \eta\int_{\mathcal{H}}\Big(-T_{\mu\rho\nu}\nabla^{\nu}(l^{\mu}l^{\rho}) - (\nabla_\mu l^{\alpha})(\nabla_\alpha l^{\mu}) + T_{\beta\mu\alpha}l^{\alpha}\nabla^{\beta}l^{\mu}\Big)\lambda d\lambda d\mathcal{A}.$$

J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.

# 4.6.What if...? (vanishing Q)

Choosing the metric energy-momentum tensor we have:

$$\mathcal{T}_{\mu\nu} \equiv \frac{-2}{\sqrt{-\det g}} \frac{\delta(\sqrt{-\det g}\mathcal{L}_M)}{\delta g^{\mu\nu}},$$

and, therefore,

$$\begin{aligned} S &= S_{\text{grav}} + S_M \\ &= \frac{1}{16\pi G} \int d^4x \sqrt{-\det g}\,[R - 2\Lambda + T_\alpha T^\alpha] \quad + \int d^4x \sqrt{-\det g}\mathcal{L}_M \,, \end{aligned}$$

J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.

D. Lovelock, J. Math. Phys., 1971.

D. Iosifidis and F. W. Hehl, Phys. Lett. B, 2024.

# 4.7. What if...? (vanishing Q)

Choosing the metric energy-momentum tensor:

$$\mathcal{T}_{\mu\nu} \equiv \Sigma_{\mu\nu},$$

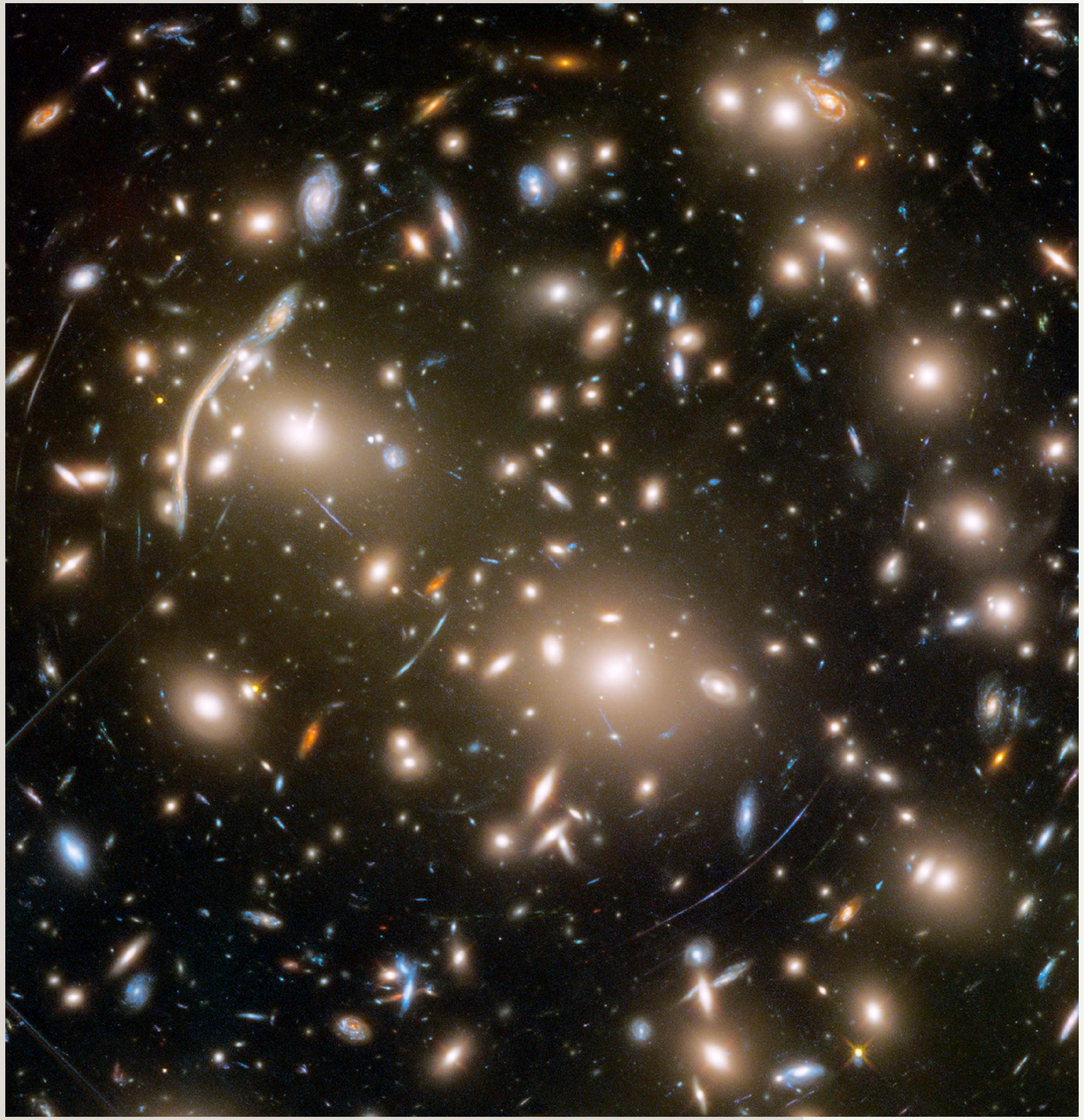
the action would look like

$$S_{\text{grav}} \supset S_{EH} + \frac{1}{16\pi G}\int d^4x\sqrt{-\det g}\;(2\nabla_\sigma - T_\sigma)T^\sigma\,,$$

which sadly do not generate a term like

$$(\nabla_\sigma - T_\sigma)T_{(\mu\nu)}{}^{\sigma}.$$

J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.
D. Lovelock, J. Math. Phys., 1971.
D. Iosifidis and F. W. Hehl, Phys. Lett. B, 2024.



# 5.On the general gravitational equation of state

# 5.1.Unruh effect

In Minkowski spacetime, accelerated observers measure a thermal bath of temperature:

$$T = \frac{\hbar\kappa}{2\pi},$$

where $\kappa$ is the fourth-acceleration norm.

T. Jacobson, Phys. Rev. Lett., 1995.

# 5.2.Energy flux

$$\delta Q = \int_{\mathcal{H}} T_{\mu\nu}\chi^{\mu}\mathrm{d}\Sigma^{\nu},$$

$$\chi^{\mu} = -\kappa\lambda k^{\mu}, \qquad d\Sigma^{\mu} = k^{\mu}d\lambda d\mathcal{A},$$

therefore,

$$\delta Q = -\kappa\int_{\mathcal{H}} \lambda T_{\mu\nu}k^{\mu}k^{\nu}d\lambda d\mathcal{A},$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 5.3.Horizon entropy (full)

$$dS = \eta \delta \mathcal{A},$$

where

$$\delta \mathcal{A} = \int_{\mathcal{H}} \theta \mathrm{d}\lambda \mathrm{d}\mathcal{A},$$

and by definition

$$\boldsymbol{\theta} = \frac{1}{2} q^{\mu\nu} \mathcal{L}_u q_{\mu\nu},$$

T. Jacobson, Phys. Rev. Lett., 1995.

# 5.4.Area variation (full)

we find that

$$\frac{d\Theta}{d\lambda}\bigg|_{\mathcal{P}} = - R_{\alpha\beta}l^{\alpha}l^{\beta} - l^{\alpha}l^{\beta}\nabla_{\alpha}T_{\beta} + l^{\alpha}l^{\beta}T_{\alpha\beta\nu}T^{\nu} - l^{\alpha}l^{\beta}\nabla_{\nu}T_{\beta\alpha}{}^{\nu} + \frac{1}{2}l^{\alpha}l^{\beta}\nabla_{\alpha}\tilde{Q}_{\beta} - \frac{1}{2}l^{\alpha}l^{\beta}Q_{\alpha\beta\nu}\tilde{Q}^{\nu}$$
$$- l^{\alpha}l^{\beta}\nabla_{\nu}Q_{\beta\alpha}{}^{\nu} + l^{\alpha}l^{\beta}T^{\nu}Q_{\alpha\beta\nu} - \frac{1}{2}l^{\alpha}l^{\beta}T_{\alpha\beta\nu}\tilde{Q}^{\nu} - (\nabla_{\beta}l^{\alpha})(\nabla_{\alpha}l^{\beta}) - T_{\mu\rho\nu}\nabla^{\nu}(l^{\mu}l^{\rho})$$
$$+ T_{\beta\mu\alpha}l^{\alpha}\nabla^{\beta}l^{\mu} + T^{\nu}l_{\mu}\nabla_{\nu}l^{\mu} - \frac{1}{2}\tilde{Q}_{\nu}l_{\mu}\nabla^{\nu}l^{\mu} - Q_{\rho\mu\nu}\nabla^{\nu}(l^{\mu}l^{\rho}) - \nabla_{\nu}(l_{\mu}\nabla^{\nu}l^{\mu}) \,.$$

where

$$Q_{\mu} \equiv Q^{\nu}{}_{\nu\mu}, \qquad \tilde{Q}_{\mu} \equiv Q_{\mu\nu}{}^{\nu}.$$

J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.

# 5.5. Clausius relation (full)

Therefore we have ***the fundamental relation***

$$\frac{2\pi}{\hbar\eta}\mathcal{T}_{(\alpha\beta)} = R_{(\alpha\beta)} + f g_{\alpha\beta} + \nabla_{(\alpha}\left(T_{\beta)} - \frac{1}{2}\tilde{Q}_{\beta)}\right) + \left[\nabla_{\nu} - T_{\nu} + \frac{1}{2}\tilde{Q}_{\nu}\right]\left({T_{(\beta\alpha)}}^{\nu} + {Q_{(\beta\alpha)}}^{\nu}\right),$$

and the tidal heating term

$$\delta S_i = \eta \int_{\mathcal{H}} \Big( - T_{\mu\rho\nu}\nabla^{\nu}(l^{\mu}l^{\rho}) - (\nabla_{\mu}l^{\alpha})(\nabla_{\alpha}l^{\mu}) + T_{\beta\mu\alpha}l^{\alpha}\nabla^{\beta}l^{\mu} + T^{\nu}l_{\mu}\nabla_{\nu}l^{\mu} - \frac{1}{2}\tilde{Q}_{\nu}l_{\mu}\nabla^{\nu}l^{\mu} - Q_{\rho\mu\nu}\nabla^{\nu}(l^{\mu}l^{\rho}) - \nabla_{\nu}(l_{\mu}\nabla^{\nu}l^{\mu})\Big)\lambda d\lambda d\mathcal{A}\,.$$

J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.

# 5.6. What if...? (full)

Choosing the metric energy-momentum tensor we have:

$$\mathcal{T}_{\mu\nu} \equiv \frac{-2}{\sqrt{-\det g}} \frac{\delta(\sqrt{-\det g}\mathcal{L}_M)}{\delta g^{\mu\nu}},$$

and, therefore,

$$S_{\text{grav}} \supset S_{EH} + \frac{1}{16\pi G}\int \Big[ -\left(T^\alpha - \frac{1}{2}\tilde{Q}^\alpha\right)\left(T_\alpha - \frac{1}{2}\tilde{Q}_\alpha\right) + \left(\nabla_\alpha - T_\alpha + \frac{1}{2}\tilde{Q}_\alpha\right)\left(-\frac{1}{2}\tilde{Q}^\alpha\right) - \frac{1}{2}\tilde{Q}_\alpha(2T^\alpha - \tilde{Q}^\alpha + Q^\alpha)\Big]\sqrt{-\det g}\, d^4x.$$

J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.
D. Lovelock, J. Math. Phys., 1971.
D. Iosifidis and F. W. Hehl, Phys. Lett. B, 2024.

# 6.Conclusions, prospects...

- Gravitational phenomena is emergent, many gravitational theories can be found by this procedure.
- Even here, the cosmological constant remains as enigmatic as ever: vacuum energy.
- In the studied case, non metricity is not favoured by nature.
- Dilation density modifies clausius relation!

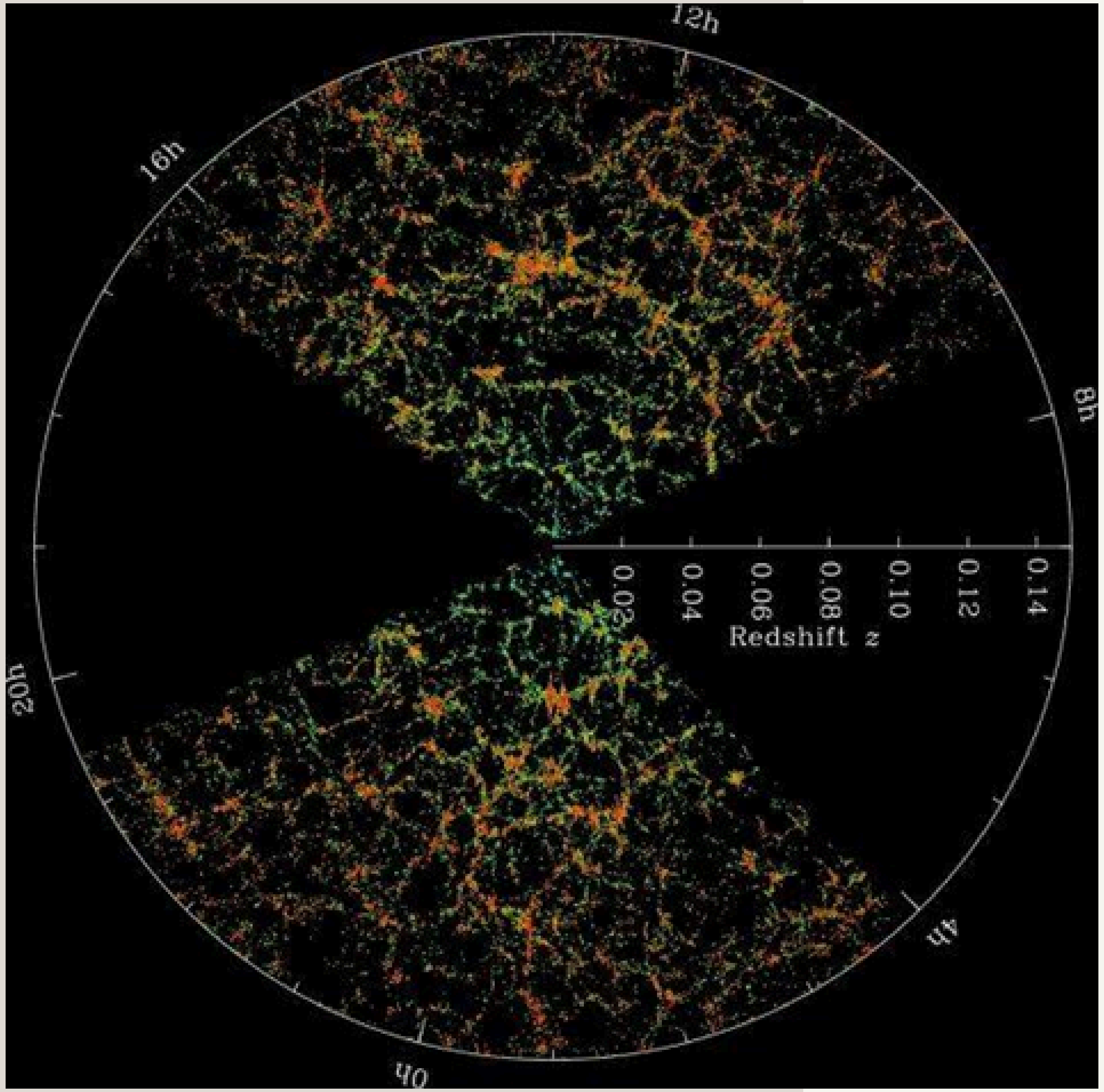
D. Iosifidis and T. S. Koivisto, JCAP, 2024.

- What about the canonical EM tensor when modifyng clausius relation?
- This “state” equations hints to be the statistical limit of a more fundamental description of the spacetime quanta (e.g. Loop quantum gravity).

T. Jacobson, Phys. Rev. Lett., 1995.

- It is surprising that we have been able to find out actions that give way to the fundamental relation for some of the cases considered. What this is telling us about the nature of the gravitational interaction, we have not been able to interpret it yet.

J. Martínez, J. Rodríguez, Y. Rodríguez, Ann.Phys.(Berlin), 2026.




# Thanks a lot!

# State of the art

| | Lovelock | Gauge | Thermodynamics |
|---|---|---|---|
| $g_{\mu\nu}$ | $R$ <br> D. Lovelock, J. Math. Phys., 1971. | $R + \mathcal{O}\left(R^2_{\mu\nu\alpha\beta}\right)$ <br> F. W. Hehl, Nato science series B, 1979. | $R$ <br> T. Jacobson, Phys. Rev. Lett., 1995. |
| $g_{\mu\nu}, T^{\mu}{}_{\alpha\beta}$ | $R$ <br> A. Mardones et. al., Class Quant. Grav., 1991. | $R + \mathcal{O}\left(R^2_{\mu\nu\alpha\beta}, T^2_{\alpha\mu\nu}\right)$ <br> F. W. Hehl, Nato science series B, 1979. | $R$ <br> T. De Lorenzo et. al., Phys. Rev. D., 2018. |

$$\bar{G}_{\mu\nu} = 8\pi G\left(T^M_{\mu\nu} + \left(\bar{\nabla}_\sigma + T_\sigma\right)\left(\tau^{\sigma}{}_{\mu\nu} - \tau_{\mu\nu}{}^{\sigma} - \tau_{\nu\mu}{}^{\sigma}\right)\right),$$

$$S^{\sigma}{}_{\mu\nu} = 16\pi G\tau^{\sigma}{}_{\mu\nu},$$

N. J. Poplawski, Astron. Rev., 2013.
lM. Gasperini, Phys. Rev. Lett., 1986.
A. Kasem et. al., Int. J. Mod. Phys. A, 2021.
Y. Bonder, Int. J. Mod. Phys. D, 2016.
B. A. Costa et. al., Phys. Lett. B, 2024.
F. Cabral et. al., Class Quant. Grav., 2021.

# MAG

| | Lovelock | Gauge | Thermodynamics |
|---|---|---|---|
| $g_{\mu\nu}$ | $R$<br>D. Lovelock, J. Math. Phys., 1971. | $R + \mathcal{O}\left(R^2_{\mu\nu\alpha\beta}\right)$<br>F. W. Hehl, Nato science series B, 1979. | $R$<br>T. Jacobson, Phys. Rev. Lett., 1995. |
| $g_{\mu\nu}, T^{\mu}{}_{\alpha\beta}$ | $R$<br>A. Mardones et. al., Class Quant. Grav., 1991. | $R + \mathcal{O}\left(R^2_{\mu\nu\alpha\beta}, T^2_{\alpha\mu\nu}\right)$<br>F. W. Hehl, Nato science series B, 1979. | $R$<br>T. De Lorenzo et. al., Phys. Rev. D., 2018. |
| $g_{\mu\nu}, T^{\mu}{}_{\alpha\beta}, Q^{\mu}{}_{\alpha\beta}$ | $R + \mathcal{O}\left(Q^2_{\alpha\mu\nu}\right)$<br>A. Jiménez-Cano, Theor. Phys. A., 2022. | $R + \mathcal{O}\left(R^2_{\mu\nu\alpha\beta}, T^2_{\alpha\mu\nu}, Q^2_{\alpha\mu\nu}\right)$<br>F. W. Hehl et. al., Phys. Rept., 1995. | ¿? |

# A.Field equations

$$S = \int d^4x \sqrt{-g}(R - 2\Lambda),$$
$$= \int d^4x \sqrt{-g}(g^{\nu\beta} R_{\beta\nu} - 2\Lambda),$$

varying the action,

$$\delta S = \int d^4x \sqrt{-g} \Big[ R_{\beta\nu} \delta g^{\nu\beta} - \frac{1}{2} g_{\beta\nu} (R - 2\Lambda) \delta g^{\nu\beta} + g^{\nu\beta} \delta R_{\beta\nu} \Big],$$
$$= \int d^4x \sqrt{-g} \Big[ \Big( R_{\beta\nu} - \frac{1}{2} g_{\beta\nu} (R - 2\Lambda) \Big) \delta g^{\nu\beta} + g^{\nu\beta} \delta R_{\beta\nu} \Big].$$

# A.Field equations

For the Ricci tensor

$$\delta R_{\sigma\nu} = \nabla_\mu\left(\delta\Gamma^\mu{}_{\nu\sigma}\right) - \nabla_\nu\left(\delta\Gamma^\mu{}_{\mu\sigma}\right) + T^\lambda{}_{\mu\nu}\delta\Gamma^\mu{}_{\lambda\sigma}$$
$$= \mathring{\nabla}_\mu(\delta\Gamma^\mu{}_{\nu\sigma}) - \mathring{\nabla}_\nu(\delta\Gamma^\mu{}_{\mu\sigma}) + \left(\delta^\alpha_\lambda\tilde{\Gamma}^\mu{}_{\nu\sigma} - \delta^\alpha_\nu\tilde{\Gamma}^\mu{}_{\lambda\sigma} - \delta^\mu_\sigma\tilde{\Gamma}^\alpha{}_{\nu\lambda} + \delta^\alpha_\nu\delta^\mu_\sigma\tilde{\Gamma}^\beta{}_{\beta\lambda}\right)\delta\Gamma^\lambda{}_{\alpha\mu},$$

and from,

$$\delta\mathring{\Gamma}^\lambda{}_{\alpha\mu} = \frac{1}{2}g^{\lambda\gamma}\left(\mathring{\nabla}_\alpha\delta g_{\mu\gamma} + \mathring{\nabla}_\mu\delta g_{\gamma\alpha} - \mathring{\nabla}_\gamma\delta g_{\alpha\mu}\right)$$
$$= \frac{1}{2}g^{\lambda\gamma}\left(g_{\alpha\beta}g_{\mu\nu}\mathring{\nabla}_\gamma\delta g^{\nu\beta} - g_{\gamma\nu}g_{\alpha\beta}\mathring{\nabla}_\mu\delta g^{\nu\beta} - g_{\mu\nu}g_{\gamma\beta}\mathring{\nabla}_\alpha\delta g^{\nu\beta}\right),$$

# A.Field equations

is found that

$$\delta\Gamma^{\lambda}{}_{\alpha\mu} = \frac{1}{2}g^{\lambda\gamma}\Big(g_{\alpha\beta}g_{\mu\nu}\nabla_{\gamma}\delta g^{\nu\beta} - g_{\gamma\nu}g_{\alpha\beta}\nabla_{\mu}\delta g^{\nu\beta} - g_{\mu\nu}g_{\gamma\beta}\nabla_{\alpha}\delta g^{\nu\beta}\Big)$$
$$+ \frac{1}{2}\Big(\delta T^{\lambda}{}_{\alpha\mu} - g^{\lambda\gamma}g_{\mu\beta}\delta T^{\beta}{}_{\alpha\gamma} - g^{\lambda\gamma}g_{\alpha\beta}\delta T^{\beta}{}_{\mu\gamma}\Big) + \frac{1}{2}\Big(g^{\lambda\gamma}g_{\alpha\beta}g_{\mu\nu}\delta Q_{\gamma}{}^{\nu\beta} - g_{\alpha\beta}\delta Q_{\mu}{}^{\lambda\beta} - g_{\mu\nu}\delta Q_{\alpha}{}^{\nu\lambda}\Big).$$

and therefore from

$$\delta S = \int d^4x\sqrt{-g}\Big[\Big(R_{\beta\nu} - \frac{1}{2}g_{\beta\nu}(R-2\Lambda)\Big)\delta g^{\nu\beta} + g^{\nu\sigma}\Big(\delta^{\alpha}_{\lambda}\tilde{\Gamma}^{\mu}{}_{\nu\sigma} - \delta^{\alpha}_{\nu}\tilde{\Gamma}^{\mu}{}_{\lambda\sigma} - \delta^{\mu}_{\sigma}\tilde{\Gamma}^{\alpha}{}_{\nu\lambda} + \delta^{\alpha}_{\nu}\delta^{\mu}_{\sigma}\tilde{\Gamma}^{\beta}{}_{\beta\lambda}\Big)\delta\Gamma^{\lambda}{}_{\alpha\mu}\Big]$$
$$+ \int d^4x\sqrt{-g}\Big[\mathring{\nabla}_{\mu}(g^{\nu\sigma}\delta\Gamma^{\mu}{}_{\nu\sigma}) - \mathring{\nabla}_{\nu}(g^{\nu\sigma}\delta\Gamma^{\mu}{}_{\mu\sigma})\Big],$$

# A.Field equations

is found for the variation with respect to the torsion

$$\frac{1}{2}\Big(\delta^{\alpha}_{\lambda}\tilde{\Gamma}^{\mu\theta}{}_{\theta}-\tilde{\Gamma}^{\mu}{}_{\lambda}{}^{\alpha}-\tilde{\Gamma}^{\alpha\mu}{}_{\lambda}+g^{\alpha\mu}\tilde{\Gamma}^{\theta}{}_{\theta\lambda}\Big)\Big(\delta^{\lambda}_{\beta}\delta^{\nu}_{\alpha}\delta^{\gamma}_{\mu}-g^{\lambda\gamma}g_{\mu\beta}\delta^{\nu}_{\alpha}-g^{\lambda\gamma}g_{\alpha\beta}\delta^{\nu}_{\mu}\Big)=-16\pi G\frac{\delta\mathcal{L}_M}{\delta T^{\beta}{}_{\nu\gamma}},$$

for the variation with respect to the non metricity

$$\frac{1}{2}\Big(\delta^{\alpha}_{\lambda}\tilde{\Gamma}^{\mu\theta}{}_{\theta}-\tilde{\Gamma}^{\mu}{}_{\lambda}{}^{\alpha}-\tilde{\Gamma}^{\alpha\mu}{}_{\lambda}+g^{\alpha\mu}\tilde{\Gamma}^{\theta}{}_{\theta\lambda}\Big)\Big(g^{\lambda\gamma}g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\beta}\delta^{\lambda}_{\nu}\delta^{\gamma}_{\mu}-g_{\mu\nu}\delta^{\gamma}_{\alpha}\delta^{\lambda}_{\beta}\Big)=-16\pi G\frac{\delta\mathcal{L}_M}{\delta Q_{\gamma}{}^{\nu\beta}},$$

and

$$\begin{aligned}\delta S=\int d^4x\sqrt{-g}\Big[&\Big(R_{\beta\nu}-\frac{1}{2}g_{\beta\nu}R\Big)\delta g^{\nu\beta}+\mathring{\nabla}_{\mu}(g^{\nu\sigma}\delta\Gamma^{\mu}{}_{\nu\sigma})-\mathring{\nabla}_{\nu}(g^{\nu\sigma}\delta\Gamma^{\mu}{}_{\mu\sigma})\\&+\frac{1}{2}\Big(T^{\gamma}{}_{\nu\beta}-T_{\nu\beta}{}^{\gamma}-T_{\beta\nu}{}^{\gamma}+Q^{\gamma}{}_{\nu\beta}-2Q_{\beta\nu}{}^{\gamma}-2\delta^{\gamma}_{\nu}\Big(T_{\beta}-\frac{1}{2}\tilde{Q}_{\beta}\Big)+g_{\beta\nu}\Big(2T^{\gamma}-\tilde{Q}^{\gamma}+Q^{\gamma}\Big)\Big)\nabla_{\gamma}\delta g^{\nu\beta}\Big],\end{aligned}$$

# A.Field equations

and finally for the metric variation

$$8\pi G\tau_{\beta\nu} = R_{(\beta\nu)} - \frac{1}{2}g_{\beta\nu}(R-2\Lambda)$$
$$+\left(\nabla_\gamma - T_\gamma + \frac{1}{2}\tilde{Q}_\gamma\right)\left[T_{(\nu\beta)}{}^{\gamma} + Q_{(\nu\beta)}{}^{\gamma} + \delta^{\gamma}_{(\nu}\left(T_{\beta)} - \frac{1}{2}\tilde{Q}_{\beta)}\right) - \frac{1}{2}Q^{\gamma}{}_{\beta\nu} - \frac{1}{2}g_{\beta\nu}\left(2T^{\gamma} - \tilde{Q}^{\gamma} + Q^{\gamma}\right)\right],$$

and a remaining total divergence

$$\delta S = \int d^4x\sqrt{-g}\Big[\mathring{\nabla}_\mu(g^{\nu\sigma}\delta\Gamma^{\mu}{}_{\nu\sigma}) - \mathring{\nabla}_\nu(g^{\nu\sigma}\delta\Gamma^{\mu}{}_{\mu\sigma})$$
$$+\frac{1}{2}\mathring{\nabla}_\gamma\left(\left[T^{\gamma}{}_{\nu\beta} - T_{\nu\beta}{}^{\gamma} - T_{\beta\nu}{}^{\gamma} + Q^{\gamma}{}_{\nu\beta} - 2Q_{\beta\nu}{}^{\gamma} - 2\delta^{\gamma}_{\nu}\left(T_\beta - \frac{1}{2}\tilde{Q}_\beta\right) + g_{\beta\nu}\left(2T^{\gamma} - \tilde{Q}^{\gamma} + Q^{\gamma}\right)\right]\delta g^{\nu\beta}\right)\Big].$$