

Feynman integrals, black holes, resurgent structure and the cosmohedron

David Broadhurst, orcid.org/0009-0009-4632-1997

at *UK-Japan Workshop on Quantum Gravity*, Durham, 27 August 2026

I discuss a family of massive **Feynman integrals** whose algebraic geometry involves an **elliptic curve** at 2 loops, a **K3 surface** at 3 loops and a **Calabi-Yau manifold** at 4 loops, where they contribute to the **magnetic moment** of the **electron** and also yield the areas of **black holes** in compactifications of 10-dimensional **supergravity**. At 2 and 3 loops, quasi-modularity leads to **resurgence** of structure at large and small momenta, which **Daniele Dorigoni** and I extended to strong and weak couplings in **topological string** theory. I conclude with a resurgent study from the **cosmohedron**.

1. **Ramanujan's** transformation of a 3-loop **Feynman** integral
2. **Black-hole** areas from compactifying **supergravity**
3. **Resurgent** Lambert series and **topological strings**
4. A pair of **Stokes constants** from combinatorics of the **cosmohedron**

1. Ramanujan's transformation of a 3-loop Feynman integral.

Ramanujan's notebooks record this convenient evaluation

$$\zeta(3) = \frac{7\pi^3}{180} - 2 \sum_{n>0} \frac{1}{n^3} \frac{q^n}{1 - q^n}, \quad q = \exp(-2\pi) < \frac{1}{\mathbf{353}}$$

which I have recently improved [2401.08997] to obtain $\zeta(3)$ in terms of $\sqrt{15}\pi^3$ and **Lambert series** with positive $q \leq \exp(-12\pi/\sqrt{15}) < 1/\mathbf{16879}$.

His value of $\zeta(3)$ comes from a family of **quasi-modular** transformations of

$$T_s(\tau) = \frac{1}{2}\zeta(s) + \sum_{n>0} \frac{1}{n^s} \frac{q^n}{1 - q^n}, \quad q = \exp(2\pi i\tau) = \exp(-2\pi y)$$

with $\Im\tau = \Re y > 0$. This converges well at large y . To access $q \rightarrow 1^-$, as $y \rightarrow 0^+$, we use a **Fricke involution** $\tau \rightarrow -1/\tau$, when s is **odd**, for which he obtained

$$T_{2k+1}\left(\frac{-1}{\tau}\right) = \frac{i}{\pi} \sum_{j=0}^{k+1} \frac{\zeta(2j)\zeta(2k+2-2j)}{\tau^{2j-1}} + \frac{T_{2k+1}(\tau)}{\tau^{2k}}$$

which evaluates $\zeta(3)$ when we set $k = 1$ and $\tau = i$.

The **Schwinger** parametrization of the equal-mass **3-loop banana** integral is

$$J(t) = \int_0^\infty \int_0^\infty \int_0^\infty \left(\frac{1}{(1+a+b+c)(1+1/a+1/b+1/c)-t} \right) \frac{da}{a} \frac{db}{b} \frac{dc}{c}$$

whose denominator defines a **K3 surface**. It is given by the **Bessel moment**

$$J(t) = 8 \int_0^\infty I_0(\sqrt{t}x) K_0^4(x) x \, dx$$

for $t < 16$. **Stefano Laporta** and I discovered the **on-shell** evaluation

$$J(1) = \frac{\pi^3}{2} \left(1 - \frac{1}{\sqrt{5}} \right) \left(1 + 2 \sum_{n>0} \exp(-\sqrt{15}\pi n^2) \right)^4 = \frac{\Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right)}{30\sqrt{5}}.$$

Jon Borwein and I developed the **Taylor series** $J(t) = \sum_{n \geq 0} C_n t^n$, with coefficients $C_0 = 7\zeta(3)$, $C_1 = C_0 - 6$ and for $n > 1$ the **recursion**

$$C_n = \left(1 - \frac{1}{2n} \right) \left(5 - \frac{5}{n} + \frac{2}{n^2} \right) C_{n-1} - 4 \left(1 - \frac{1}{n} \right)^3 C_{n-2}.$$

This recurrence relation determines the inhomogeneous **Picard-Fuchs** equation

$$t^2(t-4)(t-16)J''' + 6t(t^2-15t+32)J'' + (7t^2-68t+64)J' + (t-4)J + \mathbf{24} = 0$$

with **singularities** for $t \in \{0, 4, 16, \infty\}$. The **Feynman** integral is **regular** at $t = 4$. A **modular** parametrization of this equation was given by **Helena Verrill** in 1996 using the **Dedekind eta** function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24} = \frac{\eta(-1/\tau)}{\sqrt{-i\tau}}, \text{ with } q = \exp(2\pi i\tau).$$

This was exploited by Spencer **Bloch**, Matt **Kerr** and Pierre **Vanhove**. Let

$$t = -64 \left(\frac{\eta_2 \eta_6}{\eta_1 \eta_3} \right)^6, \quad \psi = \frac{\eta_1^2 \eta_3^2}{\eta_2 \eta_6}, \quad h = \frac{\eta_2^{16}}{\eta_1^8} - 9 \frac{\eta_6^{16}}{\eta_3^8}$$

where $\eta_n = \eta(n\tau)$, ψ^2 is the **square** of a elliptic integral, obtained by **Geoff Joyce** in 1973 as a solution to the **homogeneous** equation, and h is a weight-4 **cusppform**. Then the **inhomogeneous** differential equation for the Feynman integral becomes

$$\left(q \frac{d}{dq} \right)^3 \frac{J(t)}{\psi^2} = 24h = \mathbf{24} \sum_{n=1}^{\infty} \frac{n^3(q^n - 8q^{3n} + q^{5n})}{1 - q^{6n}}.$$

This is easily integrated to give **Lambert series** in

$$\frac{J(t)}{\psi^2} = H(\tau) = 24T_3(\tau) - 3T_3(2\tau) - 8T_3(3\tau) + T_3(6\tau)$$

which converges well for $|t| \leq 8$. Note that at $t = 0$ we obtain the correct value of

$$J(0) = \frac{1}{2}(24 - 3 - 8 + 1)\zeta(3) = \mathbf{7}\zeta(3).$$

The **Fricke** involution $\tau \rightarrow -1/(6\tau)$ gives $t \rightarrow 64/t$ and **Ramanujan** delivers

$$48\tau^2 H\left(\frac{-1}{6\tau}\right) = 16(3T_3(\tau) - 6T_3(2\tau) - T_3(3\tau) + 2T_3(6\tau)) + 4\log^3(q)$$

for fast evaluation of $J(t)$ with $|t| \geq 8$. Then, at large $|t|$, we obtain

$$-tJ(t) = \mathbf{16}\zeta(3) + 4\log^3(-t)(1 + \mathcal{O}(1/t))$$

whose **asymptotic** constant $\mathbf{16}\zeta(3)$ is determined by $J(0) = \mathbf{7}\zeta(3)$.

Thus, Ramanujan's notebooks, from more than 100 years ago, provide us with a **connection** between expansions at **large and small momenta**. In Part 3, I shall describe a method to relate expansions at **strong and weak couplings**.

Cheshire cat resurgence.

*I've often seen a cat without a grin, thought Alice, but a grin without a cat!
It's the most curious thing I ever saw in my life!* (Charles Lutwidge Dodgson, 1865)

Ramanujan's very useful formula

$$T_{2k+1}\left(\frac{-1}{\tau}\right) = \frac{i}{\pi} \sum_{j=0}^{k+1} \frac{\zeta(2j)\zeta(2k+2-2j)}{\tau^{2j-1}} + \frac{T_{2k+1}(\tau)}{\tau^{2k}}$$

has a **terminating** Laurent series and an **exponentially suppressed** tail, as $\tau \rightarrow i\infty$. By contrast, Lambert series $\sum_{n>0} n^{-s} q^n / (1 - q^n)$ for **even** s , give **divergent** series, requiring directional **Borel resummation** whose ambiguities are eventually resolved by careful treatment of the exponentially suppressed tail.

Daniele Dorigoni and **Axel Kleinschmidt** described the **quasi-modularity** of Ramanujan's formula for **odd integer** s , as an example of what they called “**Cheshire cat** resurgence” [2001.11035].

Amusingly, we know of a Cheshire cat with **even** s , at **two** loops.

A characterful Cheshire cat, at sunrise.

The 2-loop **sunrise diagram** produces Lambert series with $s = 2$. So how can it **avoid** a divergent asymptotic series at large momenta?

For the sunrise integral $I(w^2) = 4 \int_0^\infty I_0(wx) K_0^3(x) x dx$, the **modular** parameterization

$$w = 3 \frac{\eta_2^2 \eta_3^4}{\eta_1^4 \eta_6^2}, \quad f = \frac{\eta_1^6 \eta_6}{\eta_2^3 \eta_3^2}, \quad g = \frac{\eta_3^9}{\eta_1^3} + \frac{\eta_6^9}{\eta_2^3}$$

neatly reduces the problem to solving

$$-\left(q \frac{d}{dq}\right)^2 \frac{I(w^2)}{f} = 6g = 6 \sum_{n=1}^{\infty} \frac{n^2(q^n - q^{5n})}{1 - q^{6n}}$$

and the integration produces, inter alia, a **Lambert series with a character**

$$\mathcal{L}_2(\chi_{6,5}; q) = \sum_{n=1}^{\infty} \frac{\chi_{6,5}(n)}{n^2} \left(\frac{q^n}{1 - q^n} \right)$$

with $\chi_{6,5}(n) = \pm 1$ for $n = \pm 1 \bmod 6$ and $\chi_{6,5}(n) = 0$ otherwise. Then quasi-modularity allows one to expand about any of the singular points $w \in \{0, 1, 3, \infty\}$ corresponding to **cusps** of modular forms on the congruence subgroup $\Gamma_0(6)$, with **level 6**.

Two types of divisor sums.

$$\begin{aligned}
\mathcal{L}_s(\chi_r; \tau) &= \sum_{n>0} \frac{\chi_r(n)}{n^s} \frac{q^n}{1-q^n} \\
&= \sum_{n>0} \frac{\sigma'_{s,\chi_r}(n) q^n}{n^s}, \quad \sigma'_{s,\chi_r}(n) = \sum_{d|n} \chi_r(n/d) d^s, \\
\tilde{\mathcal{L}}_s(\chi_r; \tau) &= \sum_{n>0} \frac{\sigma_{s,\chi_r}(n) q^n}{n^s}, \quad \sigma_{s,\chi_r}(n) = \sum_{d|n} \chi_r(d) d^s.
\end{aligned}$$

If χ_r is a primitive **quadratic** character with fundamental discriminant D , then we have quasi-modularity for **odd** s when $D > 1$, and for **even** s , when $D < 0$. With conductor $N = |D|$, the modularity gap is given by **Bernoulli** polynomials in

$$\begin{aligned}
&\sqrt{D} \left(2\mathcal{L}_s(\chi_D; \tau) + \sum_{n>0} \frac{\chi_D(n)}{n^s} \right) + 2(-\tau)^{s-1} \tilde{\mathcal{L}}_s \left(\chi_D; \frac{-1}{N\tau} \right) = \\
&\frac{(2\pi i)^s}{(s+1)!} \sum_{j=0}^{\lfloor s/2 \rfloor} \tau^{2j-1} \binom{s+1}{2j} B_{2j}(0) \sum_{m=1}^{N-1} \chi_D(m) B_{s+1-2j}(m/N).
\end{aligned}$$

2. Black-hole areas from compactifying supergravity.

The **Bessel moments**

$$M_{m,n}(z) = \int_0^\infty I_0(xz)[I_0(x)]^m[K_0(x)]^{5-m}x^{2n+1}dx$$

were intensively studied because of their relevance to **four-loop** corrections to the **magnetic moment** of the **electron**, where **Stefano Laporta** found contributions from $M_{0,0}(\mathbf{1})$ and $M_{0,1}(\mathbf{1})$. **Jon Borwein** and I derived the **Calabi-Yau** equation

$$\sum_{k=0}^3 (-z^2)^{1-k} P_k(\theta + 2 - k) M_{m,0}(z) = 5! \delta_{m,0}, \quad \theta = z \frac{d}{dz},$$

$$P_0(t) = t^4, \quad P_1(t) = 35t^4 + 42t^2 + 3, \quad P_2(t) = 259t^4 + 104t^2, \quad P_3(t) = 225(t^2 - 1)^2$$

for $m = 0, 1, 2$, with an **inhomogeneous** constant in the **Feynman** case $m = 0$.

Philip **Candelas**, Xenia **de la Ossa**, Mohamed **Elmi** and Duco **van Straten** encountered this Calabi-Yau equation in their construction of **black holes** in 4 spacetime dimensions by compactifying a 10-dimensional **supergravity** theory on a Calabi-Yau threefold with complex structure. Remarkably, their constructions involve the **special values** $z = \sqrt{-7}$ and $z = \sqrt{17} - 4$, where Bessel moments determine **areas** of black holes.

Modular forms of level 6 at $z = 1$: I discovered and **Yajun Zhou** proved that

$$\mathcal{M} = \frac{2}{\pi^2} \begin{bmatrix} 4M_{0,0}(1) & \frac{36}{5}(M_{0,0}(1) + M_{0,1}(1)) \\ \frac{5}{3}M_{1,0}(1) & 3(M_{1,0}(1) + M_{1,1}(1)) \end{bmatrix} = \begin{bmatrix} -\Omega_2 & \widehat{\Omega}_2 \\ -\Omega_1 & \widehat{\Omega}_1 \end{bmatrix}$$

is a matrix with elements determined by **Eichler integrals**

$$\frac{\Omega_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} f_{4,6} \left(\frac{1+iy}{2} \right) y^{s-1} dy, \quad \frac{\widehat{\Omega}_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} \widehat{f}_{4,6} \left(\frac{1+iy}{2} \right) y^{s-1} dy,$$

of **modular forms** of weight 4 and **level 6**, given by the **eta quotients**

$$f_{4,6}(\tau) = (\eta_1\eta_2\eta_3\eta_6)^2, \quad \frac{\widehat{f}_{4,6}}{f_{4,6}} = \frac{1}{32} \left(w + \frac{3}{w} \right)^4 - \frac{9}{16} \left(w + \frac{3}{w} \right)^2, \quad w = 3 \frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2}.$$

The 4-loop **periods** $\Omega_{1,2}$, from $f_{4,6}$, and the **quasi-periods** $\widehat{\Omega}_{1,2}$, from the **weakly** holomorphic modular form $\widehat{f}_{4,6}$, have a rational **intersection number**, namely $\Omega_1\widehat{\Omega}_2 - \Omega_2\widehat{\Omega}_1 = \det \mathcal{M} = \frac{1}{12}$. David **Roberts** and I conjectured explicit forms of such **quadratic relations** for **all loops**. Javier **Fresán**, Claude **Sabbah** and Jeng-Daw **Yu** then proved our formulas up to **20 loops**, as did Yajun **Zhou**.

Modular forms of level 14 at $z = \sqrt{-7}$: Here the weight-4 **cusppform**

$$f_{4,14}(\tau) = \frac{(\eta_2\eta_7)^6}{(\eta_1\eta_{14})^2} - 4(\eta_1\eta_2\eta_7\eta_{14})^2 + \frac{(\eta_1\eta_{14})^6}{(\eta_2\eta_7)^2}$$

has **periods** given by **critical values** of the L -function

$$L(f_{4,14}, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{4,14}(iy) y^{s-1} dy$$

which are easily evaluated by the **Bessel moments**

$$M_{2,0}(\sqrt{-7}) = \frac{1}{2}L(f_{4,14}, 2), \quad M_{1,0}(\sqrt{-7}) = L(f_{4,14}, 3) = \frac{\pi^2}{7}L(f_{4,14}, 1),$$

with the **black-hole area** determined by their **ratio** $M_{2,0}(\sqrt{-7})/M_{1,0}(\sqrt{-7})$.

I determined the corresponding **quasi-periods**, using an Eichler integral of

$$\widehat{f}_{4,14} = (253 + 645h + 1446h^2 + 2064h^3 + 1024h^4)(\eta_1\eta_2\eta_7\eta_{14})^2, \quad h = \left(\frac{\eta_2\eta_{14}}{\eta_1\eta_7} \right)^3.$$

Modular forms of level 34 at $z = \sqrt{17} - 4$: Bessel moments determine the **area** of the event horizon of a **black hole**, with **charges** specified by (k, ℓ) , constructed by **Candelas** et al. I found that the area is given by

$$A = 34\pi \left(\frac{k^2}{v} + \ell^2 v \right), \quad v = 4\pi \frac{M_{2,0}(\sqrt{17} - 4)}{M_{1,0}(\sqrt{17} - 4)}.$$

The **Bessel moments** also determine the **periods** of a **cusppform** $f_{4,34}$ whose **complex** Fourier coefficients are **Gaussian** integers. The **critical values** of its L -function are

$$\begin{aligned} L(f_{4,34}, 1) &= \left(\frac{7 - 11z + (11 + 7z)i}{\pi^2} \right) M_{1,0}(z) \\ L(f_{4,34}, 2) &= 4 \left(\frac{5 - 3z + (3 + 5z)i}{17} \right) M_{2,0}(z) \\ L(f_{4,34}, 3) &= \left(\frac{13 - z + (1 + 13z)i}{17} \right) M_{1,0}(z) \end{aligned}$$

with $z = \sqrt{17} - 4$. I identified the corresponding **quasi-periods** as Bessel moments. **Kilian Bönisch**, with assistance from **Don Zagier**, identified a **weakly** holomorphic modular form $\widehat{f}_{4,34}$ whose Eichler integrals also give those quasi-periods.

Arithmetical observations

Francesco **Aprile** showed that **6**-dimensional supergravity with **21** tensor multiplets is consistent with unitarity, thanks to the arithmetical identity

$$\mathbf{21} = \left(\frac{\mathbf{21}}{3} - 2 \right)^2 - 2^2.$$

The result of **Candelas** et al. compactifying **10**-dimensional supergravity on a Calabi-Yau threefold at $z = \sqrt{\mathbf{17}} - \mathbf{4}$ relies on the **powers of 2** in the following **7 identities**

$$\begin{aligned} \mathbf{17} &= (\mathbf{4} + \mathbf{5})^2 - 2^6 \\ &= (\mathbf{4} + \mathbf{3})^2 - 2^5 \\ &= (\mathbf{4} + \mathbf{1})^2 - 2^3 \\ &= (\mathbf{4} + \mathbf{0})^2 + 2^0 \\ &= (\mathbf{4} - \mathbf{1})^2 + 2^3 \\ &= (\mathbf{4} - \mathbf{3})^2 + 2^4 \\ &= (\mathbf{4} - \mathbf{5})^2 + 2^4. \end{aligned}$$

At $z = \sqrt{-\mathbf{7}}$, we have **powers of 2** in $-\mathbf{7} = 5^2 - 2^5 = 3^2 - 2^4 = 1^2 - 2^3$.

3. Resurgent Lambert series and topological strings.

Daniel Dorigoni and I generalized the work of Part 1 to include **pairs** of characters and indices in

$$\Xi_{s_1, s_2}(\chi_{r_1}, \chi_{r_2}; \tau) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\chi_{r_1}(n_1)}{n_1^{s_1}} \frac{\chi_{r_2}(n_2)}{n_2^{s_2}} q^{n_1 n_2}.$$

With $\tau = iy$ and $y \rightarrow 0^+$ we found that resummed perturbative terms require, in general, an **exponentially suppressed** tail that is a **sum** of transformed terms:

$$\sum_{n=0}^{\infty} \frac{(s_1)_n (s_2)_n}{n!} \frac{(r_1 r_2 \tau)^{s_1 + s_2 + n - 1}}{(-2\pi i)^n} \Xi_{s_1 + n, s_2 + n} \left(\bar{\chi}_{r_2}, \bar{\chi}_{r_1}; \frac{-1}{r_1 r_2 \tau} \right)$$

with only one term surviving in the single character cases

$$\begin{aligned} \mathcal{L}_s(\chi_r; \tau) &= \Xi_{s,0}(\chi_r, \chi_{1,1}; \tau) = \Xi_{0,s}(\chi_{1,1}, \chi_r; \tau), \\ \tilde{\mathcal{L}}_s(\chi_r; \tau) &= \Xi_{0,s}(\chi_r, \chi_{1,1}; \tau) = \Xi_{s,0}(\chi_{1,1}, \chi_r; \tau), \end{aligned}$$

relevant to work by **Veronica Fantini** and **Claudia Rella**, on quantum modularity of resurgent **topological strings** [2212.10606; 2404.10695; 2404.11550; 2506.08265].

They studied a fermionic **spectral trace** in topological string theory compactified on a **Calabi-Yau** manifold with $\mathbb{P}^2$ geometry, obtaining q -**Pochhammer** symbols in

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{3\sqrt{\tau}} \exp\left(-\frac{\pi i}{12}\tau + \frac{\pi i}{36}\tau^{-1} + \frac{\pi i}{4}\right) \frac{(q^2; q^3)_\infty^2}{(q; q^3)_\infty} \frac{(\omega_3; \tilde{q})_\infty}{(\omega_3^{-1}; \tilde{q})_\infty^2},$$

$$(xq^\alpha; q)_\infty := \prod_{n=0}^{\infty} (1 - xq^{\alpha+n}), \quad q = \exp(2\pi i\tau), \quad \tilde{q} = \exp(-2\pi i/(3\tau)), \quad \omega_3 = \exp(2\pi i/3)$$

which involve the **odd** character $\chi_{3,2}(n) = \pm 1$ for $n = \pm 1 \bmod 3$. The behaviour of

$$\log \left[\mathrm{Tr}(\rho_{\mathbb{P}^2}) \right] = -\frac{1}{2} \log(3^{\frac{5}{2}}\tau) - \frac{\pi i}{4} + \frac{3}{2} \left(\tilde{\mathcal{L}}_1(\chi_{3,2}; \tau) - \sqrt{-3} \mathcal{L}_1(\chi_{3,2}; -\frac{1}{3\tau}) \right)$$

at strong or weak **coupling** $y = -i\tau$ requires **Borel resummation** of an infinite number of perturbative terms, with exponentially suppressed **tails** in

$$\begin{pmatrix} \mathcal{L}_1(\chi_{3,2}; \tau) \\ \tilde{\mathcal{L}}_1(\chi_{3,2}; \tau) \end{pmatrix} = \begin{pmatrix} \mathcal{S}_\mp [\mathcal{L}_1^{\mathrm{Pert}}] (\chi_{3,2}; \tau) \\ \mathcal{S}_\mp [\tilde{\mathcal{L}}_1^{\mathrm{Pert}}] (\chi_{3,2}; \tau) \end{pmatrix} \pm i \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_1(\chi_{3,2}; -\frac{1}{3\tau}) \\ \tilde{\mathcal{L}}_1(\chi_{3,2}; -\frac{1}{3\tau}) \end{pmatrix}$$

that **resolve** the ambiguity of **directional** Borel resummation, indicated by $S_\mp$.

Spectral trace of local $\mathbb{P}^{m,n}$. Here we need to consider characters with **conductor** $N = m + n + 1$. With $(m, n) = (2, 1)$, we encounter the **odd** character $\chi_{4,3}(n) = \pm 1$ for $n = \pm 1 \bmod 4$ and $\chi_{4,3}(n) = 0$ otherwise. The logarithm of the spectral trace involves the combination

$$\tilde{\mathcal{L}}_1(\chi_{4,3}; \tau) - \sqrt{-4} \mathcal{L}_1(\chi_{4,3}; -\frac{1}{4\tau})$$

and hence directional Borel resummation is required at strong and weak coupling. With $(m, n) = (3, 1)$, or $(m, n) = (2, 2)$, there are **three** non-principal characters:

$n \bmod 5$	1	2	3	4
$\chi_{5,2}(n)$	1	i	-i	-1
$\chi_{5,3}(n)$	1	-i	i	-1
$\chi_{5,4}(n)$	1	-1	-1	1

The **odd** complex characters give contributions that **require** Borel resummation. The **even** real character gives a contribution

$$\tilde{\mathcal{L}}_1(\chi_{5,4}; \tau) - \sqrt{5} \mathcal{L}_1(\chi_{5,4}, -\frac{1}{5\tau})$$

with perturbative expansions that **terminate**, at both large and small coupling. Yet a non-perturbative tail **persists**, like the grin of Lewis Carroll's **Cheshire cat**.

4. A pair of Stokes constants from combinatorics of the cosmohedron.

In a recent article *Combinatorics of the cosmohedron* [2603.03425], Federico **Ardila**, Nima **Arkani-Hamed**, Carolina **Figueiredo** and Francisco **Vařao** encountered the integer sequence

1, 2, 10, 72, 644, 6704, 78408, 1008480, 14065744, 210682080...

enumerating **nestings of polygons** within polygons and satisfying a **recursion**

$$A_n = \sum_{k=1}^{n-1} (k+1)A_k A_{n-k} \text{ for } n > 1, \text{ with } A_1 = 1,$$

resulting from patterns of energies in denominators of **old-fashioned, time-ordered** perturbation theory, applied to **scalar field theory** in the background of a

Friedmann-Lemaître-Robertson-Walker metric evolving in **cosmological** time.

They conjecture that there is a **finite constant** S_1 such that

$$\lim_{n \rightarrow \infty} \frac{A_n}{n^4 n!} = S_1 \approx 0.005428 \dots$$

and suggest that this lies on “the **border** between decidability and **undecidability**”.

In fact, this integer sequence yields **readily** to the methods of **Jean Écal**e.

I now show how to develop the **asymptotic expansion**

$$A_n \sim S_1 \sum_{k>0} \Gamma(n + \mathbf{6} - k) B_k = S_1 (n + 4)! \left(\mathbf{1} - \frac{\mathbf{8}}{n + 4} + \frac{\mathbf{4}}{(n + 4)(n + 3)} + O\left(\frac{1}{n^3}\right) \right)$$

with **decreasing factorials** and **rational coefficients** B_k forming the sequence

$$1, -8, 4, -36, -388, -\frac{22992}{5}, -\frac{894152}{15}, -\frac{87775088}{105}, -\frac{1318809728}{105} - \frac{7025275808}{35}, -\frac{1781593980128}{525} \dots$$

This is the first of an infinite series of **hyperasymptotic** expansions. The **next** is

$$B_n \sim \mathbf{2}S_1 \sum_{k>0} \Gamma(n + \mathbf{6} - k) C_k = 2S_1 (n + 4)! \left(-\mathbf{1} + \frac{\mathbf{16}}{n + 4} - \frac{\mathbf{74}}{(n + 4)(n + 3)} + O\left(\frac{1}{n^3}\right) \right)$$

whose coefficients C_k form the sequence

$$-1, 16, -74, 160, 168, \frac{17584}{5}, \frac{762688}{15}, \frac{1921536}{105}, \frac{264968176}{21}, \frac{22528464128}{105}, \frac{669680351104}{175}, \frac{58928184776704}{825} \dots$$

with asymptotic behaviour $C_n = \mathbf{3}S_1(n + 4)!(\mathbf{3}/\mathbf{2} - \mathbf{37}/(n + 4) + \mathcal{O}(1/n^2))$.

*Large A's need smaller B's, especially to guide them,
and larger B's need smaller C's, and so ad infinitum.*

Step 1: a non-linear differential equation. The **formal** power series $G_0(x) = \sum_{n>0} A_n x^{n-1} = 1 + 2x + 10x^2 + 72x^3 + O(x^4)$ solves

$$G = 1 + G \frac{d}{dx}(x^2 G).$$

Step 2: an instanton correction. We posit a solution of the form

$$G(x) = G_0(x) + \sigma x^{-\beta} e^{-\mu/x} G_1(x) + O(\sigma^2)$$

with an **arbitrary** constant σ and a power series $G_1(x) = \sum_{n>0} B_n x^{n-1}$. Then

$$x^2 G_1' + \left(\mu + (2 - \beta)x - \frac{1}{G_0^2} \right) G_1 = 0$$

is soluble by a formal power series $G_1 = 1 + O(x)$ if and only if $\mu = 1$ and $\beta = 6$.

Step 3: the complete trans-series. This takes the form

$$G(x) = G_0(x) + \sum_{m>0} \left(\sigma x^{-6} e^{-1/x} \right)^m G_m(x).$$

Step 4: numerical evaluation of the first Stokes constant.

With $\beta = 6$, it takes **one second** to obtain **500 decimal digits** of

$S_1 = 0.005428317993266202636748034138132075286101589263688307574601631285 \dots$

by truncating the series $A_n \sim S_1 \sum_{k>0} \Gamma(n + 6 - k) B_k$ at $k \approx n/2 \approx 850$, using

$$B_1 = 1, \quad (k-1)B_k = \sum_{j=1}^{k-1} t_{j+2} B_{k-j} \text{ for } k > 1, \quad \sum_{n>0} t_n x^{n-1} = \frac{1}{G_0^2}.$$

I determined **5000 digits** of S_1 in **40 minutes**, finding no fit to products of powers of rationals, π , e , $\Gamma(1/3)$, $\Gamma(1/4)$, etc. **Stokes constants** are **rarely** determined by simpler transcendentals.

Step 5: continuation of the trans-series. For $m > 1$, $G_m(x)$ satisfies a linear **inhomogeneous** differential equation

$$x^2 G'_m + \left(m + (2 - 6m)x - \frac{1}{G_0^2} \right) G_m = \frac{T_m}{G_0^{m+1}}$$

with $G_2 = \sum_{n>0} C_n x^{n-1}$, $G_3 = \sum_{n>0} D_n x^{n-1}$, etc. At $m = 1$ we have $T_1 = 0$.

The **inhomogeneous** term T_m is **polynomial** in $\{G_k | k < m\}$. For example, we have

$$\begin{aligned} T_2 &= -G_1^2, & T_3 &= -2G_2G_1G_0 + G_1^3, & T_4 &= -2(G_3G_1 + G_2^2)G_0^2 + 3G_2G_1^2G_0 - G_1^4, \\ T_5 &= -2(G_4G_1 + G_3G_2)G_0^3 + 3(G_3G_1^2 + G_2^2G_1)G_0^2 - 4G_2G_1^3G_0 + G_1^5 \end{aligned}$$

whose monomials correspond to **partitions** of m .

Step 6: the second Stokes constant. As in previous work with **Michael Borinsky** a **backwards** Stokes constant, S_2 , is encountered in the asymptotic behaviour of the coefficients C_n of $G_2(x) = \sum_{n>0} C_n x^{n-1}$ at action $m = 2$, for which I found that

$$C_n \sim 3S_1 \sum_{k>0} \Gamma(n + \mathbf{6} - k) D_k - S_2 \sum_{k>0} \Gamma(n - \mathbf{6} - k) (-1)^{n-k} B_k$$

with $S_2 = 5.1147474676274685743670822179733767878555798612703925509331 \dots$

With $G_m(x) = \sum_{n>0} a_m(n) x^{n-1}$, I found the hyperasymptotic behaviour

$$\begin{aligned} a_m(n) &\sim (m+1)S_1 \sum_{k>0} \Gamma(n + \mathbf{6} - k) a_{m+1}(k) \\ &\quad - (m-1)S_2 \sum_{k>0} \Gamma(n - \mathbf{6} - k) (-1)^{n-k} a_{m-1}(k) \end{aligned}$$

for $m > 0$. This has been checked up to $m = 12$ at more than 300-digit precision.

A family of trans-series: Following **Paul Balduf**, I consider the recursion

$$A_n(\lambda) = \sum_{k=1}^{n-1} (k + \lambda) A_k(\lambda) A_{n-k}(\lambda) \text{ for } n > 1, \text{ with } A_1(\lambda) = 1.$$

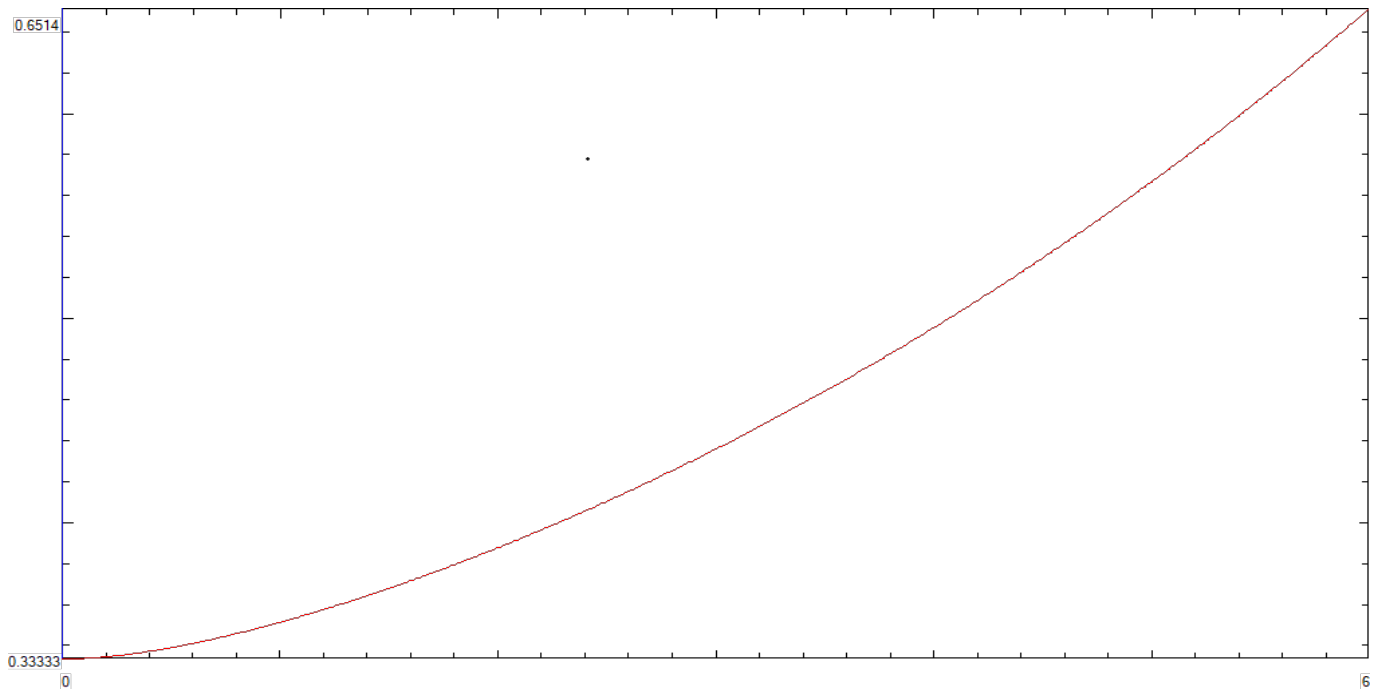
For rational $\lambda = c/d$, this generates an integer sequence $d^{n-1} A_n(c/d)$ with notable cases in the Online Encylopdia (OEIS): **A177384**, for the **cosmohedron** with $\lambda = 1$; **A088716**, for **tree-tubings** with $\lambda = 0$; **A000699**, for **chord diagrams** with $\lambda = -\frac{1}{2}$; **A397588**, with $\lambda = \frac{1}{2}$; **A376086**, with $\lambda = -\frac{1}{3}$; **A215648**, with $\lambda = -\frac{2}{3}$.

The corresponding differential equation leads to a **trans-series** of the form

$$G(x) = G_0(x) + \sum_{m>0} \left(\sigma x^{-\beta} e^{-1/x} \right)^m G_m(x), \quad \beta = 3(1 + \lambda)$$

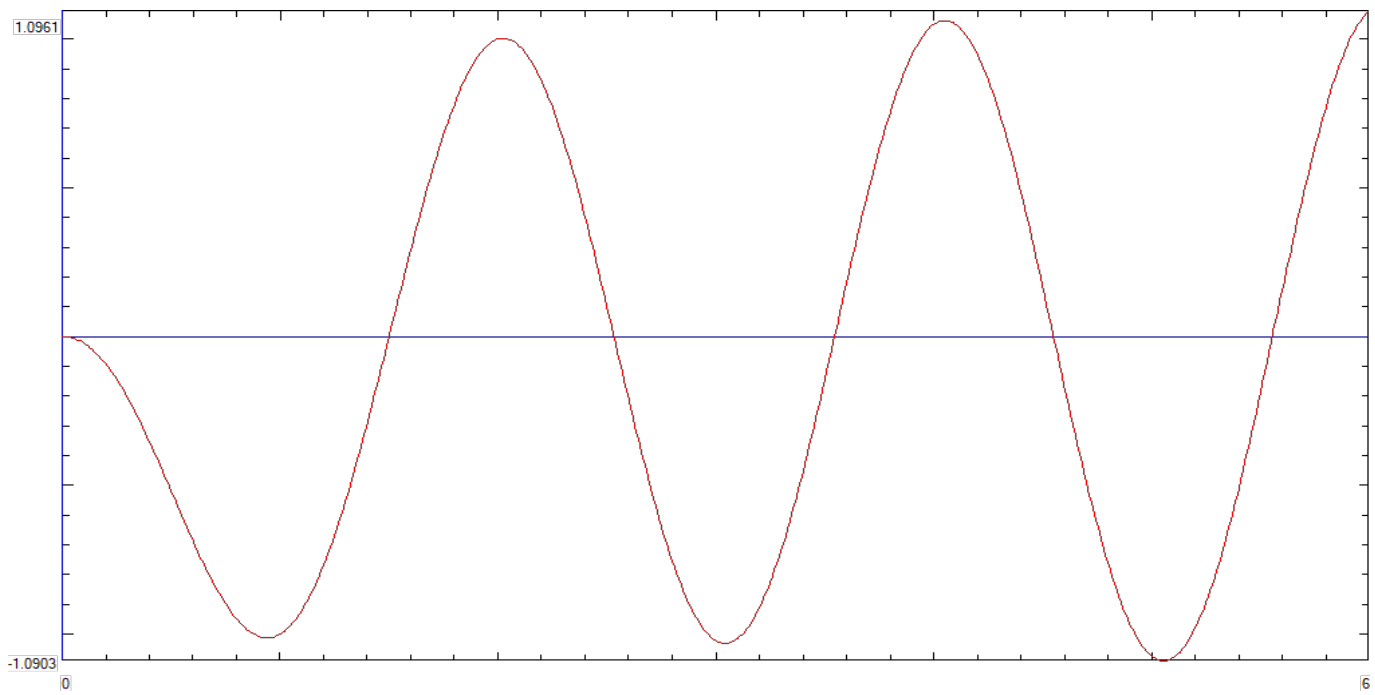
with the hyperasymptotics of $G_m(x) = \sum_{n>0} a_m(n) x^{n-1}$ given by

$$\begin{aligned} a_m(n) \sim (m+1) S_1(\lambda) \sum_{k>0} \Gamma(n + \beta - k) a_{m+1}(k) \\ - (m-1) S_2(\lambda) \sum_{k>0} \Gamma(n - \beta - k) (-1)^{n-k} a_{m-1}(k). \end{aligned}$$



Plot of $P(\beta) = \Gamma(\beta)S_1(\lambda) \in [\frac{1}{3}, 0.7)$, for $\beta = 3(1 + \lambda) \in [0, 6]$, with values

$$P(0) = \frac{1}{3}, \quad P(1) = \frac{3}{\pi e}, \quad P\left(\frac{3}{2}\right) = \frac{1}{e}, \quad P(2) = \frac{9}{\pi e^2}.$$



Plot of $Q(\beta) = (2\pi)^2 S_1(\lambda) S_2(\lambda) \in (-1.1, 1.1)$, for $\beta = 3(1 + \lambda) \in [0, 6]$, with values

$$Q(0) = 0, \quad Q(1) = -1, \quad Q\left(\frac{3}{2}\right) = 0, \quad Q(2) = 1.$$

The case $\lambda = -\frac{1}{2}$ comes from a **Dyson-Schwinger** equation that **Dirk Kreimer** and I encountered in quantum field theory. It is notable that $S_2(-\frac{1}{2})$ **vanishes**. Here, only **one** Stokes constant is needed, namely $S_1(-\frac{1}{2}) = 2/(\sqrt{\pi}e)$, as was shown by **Michael Borinsky** and **Gerald Dunne**.

From numerical work, **Paul Balduf** conjectured that $S_1(-\frac{1}{3}) = 9/(\pi e^2)$ and **Vaclav Kotesovec** conjectured that $S_1(-\frac{2}{3}) = 3/(\pi e)$. To my knowledge, neither of these evaluations has been proved. It would be interesting to know of combinatoric problems that generate the OEIS sequences oeis.org/A376086 and oeis.org/A215648.

Recently I made the surprising **numerical** discoveries that the relations

$$\begin{aligned} S_1(-\tfrac{1}{3})S_2(-\tfrac{1}{3}) &= \frac{+1}{(2\pi)^2} \\ S_1(-\tfrac{2}{3})S_2(-\tfrac{2}{3}) &= \frac{-1}{(2\pi)^2} \end{aligned}$$

hold at **100-digit** precision, yet lack proofs. For $\lambda \notin \{-1, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}\}$, I found no relation between Stokes constants and transcendentals such as π , e and Γ values.

Summary

1. **Ramanujan's** transformation of the **3-loop banana** is an example of **Cheshire cat** resurgence. The exponentially suppressed tail comes from **quasi-modularity**.
2. 4-loop integrals give **black-hole areas** from compactified **supergravity**.
3. **Topological string** observables have exponentially suppressed tails that **resolve** ambiguities of **Borel resummation** of Lambert series with **characters**.
4. The **cosmodedron** yields an example of resurgence with **two Stokes constants**.

Acknowledgements. My interest in resurgence came from a programme on *Applicable resurgent asymptotics* at the **Isaac Newton Institute** in 2022. I am grateful for discussions with **Ovidiu Costin**, **Daniele Dorigoni** and **Gerald Dunne** at a workshop on *Resurgence and modularity in QFT and string theory* at the **Galileo Galilei Institute** in 2024. I also benefited from programmes on *Amplitudes and algebraic geometry*, at the **Erwin Schrödinger Institute** in 2026, and on *New connections between physics and number theory*, at the Isaac Newton Institute in 2026.