

Quantum Modularity of False Theta Functions & Asymptotics of Witten-Reshetikhin-Turaev invariants



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Studies of Quantum Modularity



Abstract

Main objects: • **WRT invariants**: partition functions of $SU(2)$ Chern-Simons theory on M_3
• **False θ -functions**: coming from reduction on M_3 of 6d $(2,0)$ theory

Problems: ① Topology: derive **asymptotic expansions** of **WRT invariants** for 3-manifolds.
② Number theory: establish **quantum modularity** for **false θ -functions**.



Main results: ① for **negative definite plumbed manifolds**.
② for **general false theta functions**.

New technique: **Poisson summation formula with sign**.

Contents

§1 Asymptotics of Witten-Reshetikhin-Turaev invariants

§2 Proof strategy

§3 Gukov-Pei-Putrov-Vafa ($\hat{\mathbb{Z}}$ -) invariants

§4 Quantum modularity & asymptotics for false theta functions

§5 Poisson summation formula with sign

§1 Asymptotics of Witten-Reshetikhin-Turaev invariants

- Fix
- M : a closed oriented 3-manifold (e.g., S^3 , $S^1 \times S^2$, ...)
 - $k \in \mathbb{Z}_{>0}$

Def The Witten-Reshetikhin-Turaev invariant is defined as

$$Z_k(M) := \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i(k-2)CS(A)} \mathcal{D}A \quad (\text{path integral})$$

where

- $\mathcal{A} := \Omega^1(M, \mathfrak{su}(2)) := \{A: M \rightarrow T^*M \otimes \mathfrak{su}(2) : C^\infty\}$
"su(2)-connection"

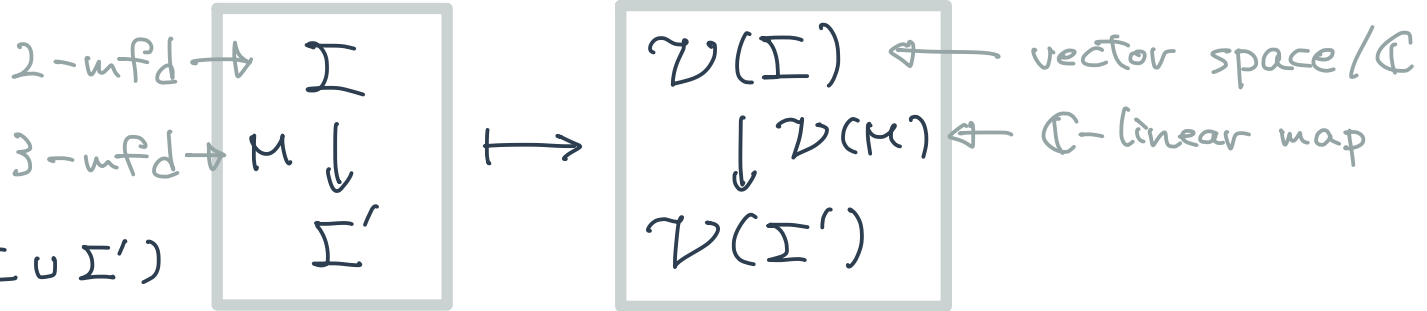
- $\mathcal{G} := C^\infty(M, SU(2)) := \{g: M \rightarrow SU(2) : C^\infty\}$ "gauge transformation"

- $CS(A) := \frac{1}{8\pi^2} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \in \mathbb{Q}/\mathbb{Z}$
"Chern-Simons action"

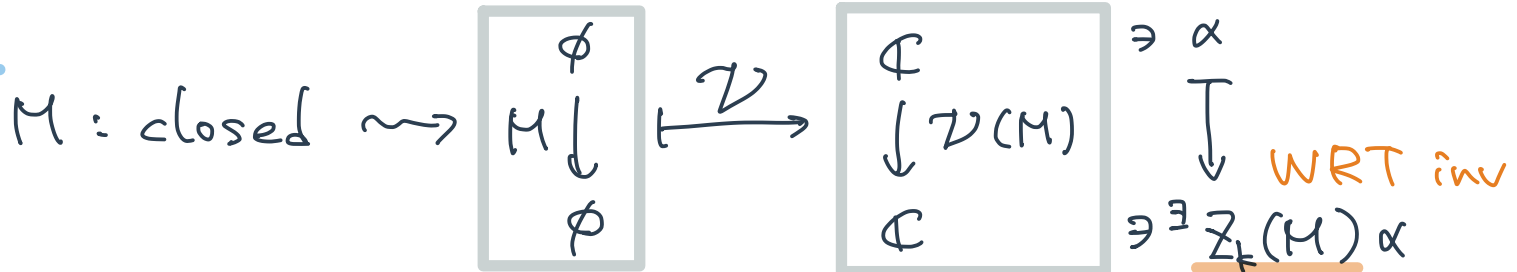
● A viewpoint from TQFT

Def A TQFT (Topological Quantum Field Theory) is

a functor $\mathcal{V} : \underline{\text{Cob}}_3 \rightarrow \underline{\text{Vect}} / \mathbb{C}$



Rem



Asymptotics of quantum invariants

Formally applying the stationary phase approximation to

$$Z_k(M) := \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i (k-2) CS(A)} \mathcal{D}A \quad \text{yields:}$$

Asymptotic expansion conjecture

$$Z_k(M) \underset{k \rightarrow \infty}{\sim} \int_{\theta \in \mathbb{C}/\mathbb{Z}} e^{2\pi i k \theta} \quad \exists Z_\theta(k) \\ \text{Chern-Simons invariant of } M \quad \uparrow \\ \bigcup_{p \in \mathbb{Z}} \mathbb{C}(k^{-\frac{1}{p}})$$

Proved for Seifert homology spheres:

Lawrence-Rozansky (1999) / Hikami (2006) /

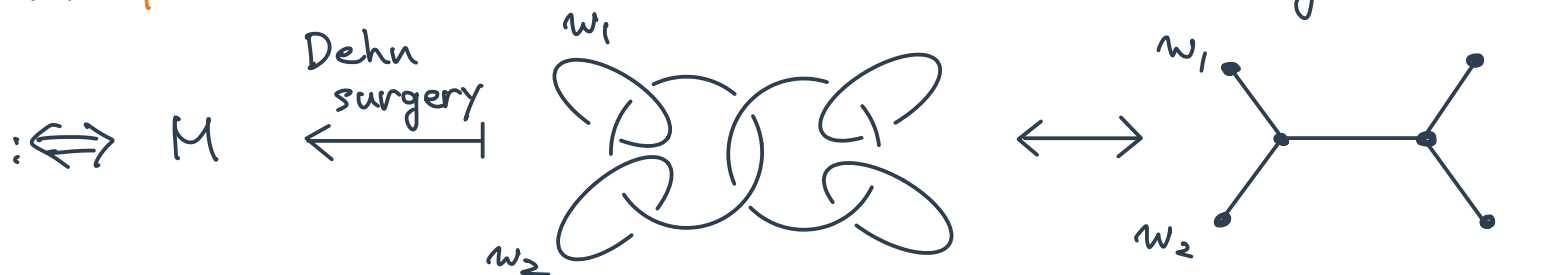
Andersen-Mistegård (2022) / Matsusaka-Terashima (2024)

Main thm 1 M : negative definite plumbed manifold

$$\Rightarrow Z_k(M) \underset{k \rightarrow \infty}{\sim} \sum_{\theta \in \mathcal{J} \subset \mathbb{Q}/\mathbb{Z}} e^{2\pi i k \theta} \exists Z_\theta(k) \underset{\cap}{\sim} \mathbb{C}((k^{-\frac{1}{2}}))$$

fin, explicit

• M : plumbed



• M : negative definite $\Leftrightarrow$ the linking matrix is neg. def.

non-hyperbolic,
 $H_i(M, \mathbb{Q}) = 0$

∪

Seifert homology spheres, lens spaces

● Comparison with asymptotic expansion conjecture

Q (joint work with Terashima)

$$\mathcal{S} = \{0\} \cup \{ \text{Chern-Simons invariants of } M \}$$

Rem True for Seifert homology spheres.

Explicit form of $\mathcal{I}$

$$\mathcal{I} = \left\{ \begin{array}{l} lk(\alpha, \alpha) - \frac{1}{4} \sum_{1 \leq s' < s} t_{n_{s'}} W_{s'}^* n_{s'} \pmod{\mathbb{Z}} ; \\ \alpha \in H^1(M, \mathbb{Z}), \quad V_1 \cup \dots \cup V_s = V^\vee, \\ n_{s'} \in \mathbb{Z}^{V_{s'}} \text{ satisfying some congruences} \end{array} \right\}$$

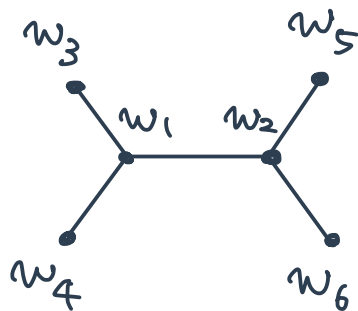
where $\cdot V^\vee := \{ \text{vertices with } \deg \geq 3 \}$.

- W : the linking matrix.

- $(W^\vee)^{-1} \in \text{Sym}(\mathbb{Q}^{V^\vee})$: $V^\vee \times V^\vee$ -submatrix of W^{-1} .

- $W_{s'}^* \in \text{Sym}(\mathbb{Q}^{V_{s'}})$: defined by using $W^\vee$.

Ex H-graph & $H_1(M, \mathbb{Z}) = 0$ case.



$$\rightsquigarrow \mathcal{L} = \left\{ 0, \frac{1}{4} {}^t n W^{\vee} n, \frac{n_2^2}{4 w_1^{\vee} P_1 P_2}, \frac{n_1^2}{4 w_2^{\vee} P_1 P_2} \mid n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathcal{P}_1 \times \mathcal{P}_2 \right\}$$

where $w_1^{\vee} := w_1 - \frac{1}{w_3} - \frac{1}{w_4}$, $w_2^{\vee} := w_2 - \frac{1}{w_5} - \frac{1}{w_6}$,

$$W^{\vee} = \begin{pmatrix} w_1^{\vee} & 1 \\ 1 & w_2^{\vee} \end{pmatrix}, \quad P_1 := w_3 w_4, \quad P_2 := w_5 w_6,$$

$$\mathcal{P}_1 := \mathbb{Z} \setminus (w_3 \mathbb{Z} \cup w_4 \mathbb{Z}), \quad \mathcal{P}_2 := \mathbb{Z} \setminus (w_5 \mathbb{Z} \cup w_6 \mathbb{Z})$$

§2 Proof strategy

● Why difficult?

WRT inv

$$Z_k(M) \doteq \sum_{\mu \in (\mathbb{Z} - k\mathbb{Z})^r / 2k\mathbb{Z}^r} \zeta_{4k}^{Q(\mu)} F(\zeta_{2k}^{\mu_1}, \dots, \zeta_{2k}^{\mu_r}) \quad (\text{in our case})$$



Many k 's appear



where $\cdot \zeta_k := e^{2\pi i / k}$,

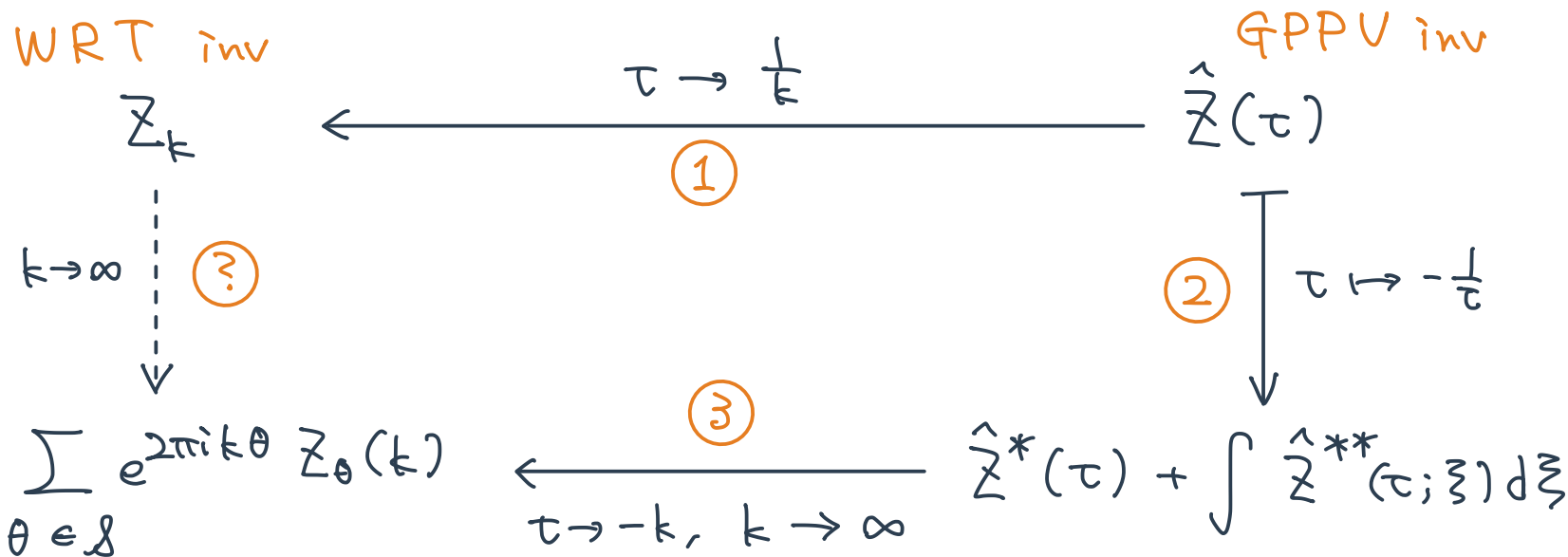
$\cdot Q : \mathbb{Z}^r \rightarrow \mathbb{Z} : \text{a quadratic form,}$

$\cdot F(z_1, \dots, z_r) \in \mathbb{Q}(z_1, \dots, z_r)$

$\leadsto$ This expression does NOT yield the asymptotic expansion.

$\leadsto$ Need: Good expression of $Z_k(M)$.

● Proof strategy



① : Radial limit conjecture, conj'd by Gukov-Peri-Putrov-Vafa (2020)
proved by YM (2024)

② : Quantum modularity

③ : Asymptotic expansions

This talk !

§3 Gukov-Peri-Putrov-Vafa ($\hat{\mathbb{Z}}$ -) invariants

WRT inv $Z_k(M)$

The partition function of
SU(2) Chern-Simons
theory on M

①

$$q \rightarrow e^{2\pi i/k}$$



GPPV inv

$$\hat{Z}(q; M)$$

② ||

$$\sum_{i,j} q^i (-1)^j \dim \mathcal{H}^{i,j}(M)$$

Categorification

③? An expected

homological invariant

$$\mathcal{H}_{\text{TEM}}^{\text{BPS}}(D^2) = \bigoplus_{i,j} \mathcal{H}^{i,j}(M)$$

Euler

characteristic

● Physical meaning

$$\mathcal{H}_{\text{[TM]}}^{\text{BPS}}(D^2) = \bigoplus_{i,j} \mathcal{H}^{i,j}(M)$$



Hilbert space of BPS states of [TM] on D^2 , where:

- [TM] : 3d $\mathcal{N}=2$ SCFT $\xleftarrow{\text{reduction on } M}$ 6d $(2,0)$ theory
- i : the charge under the $U(1)_q$ rotation of D^2
- j : R-charge of the $U(1)_R$ R-symmetry

● A construction of $\hat{Z}(q; M)$ by GPPV (2020)

Def M : a negative definite plumbed manifold

$\leadsto$ Gukov-Pei-Putrov-Vafa invariant

$$\hat{Z}_b(q; M) := q^\Delta \sum_{\ell \in b + 2W(\mathbb{Z}^V)} \boxed{F_\ell} q^{-\ell W^{-1} \ell / 4} \in q^\Delta \mathbb{Z}[[q]]$$

where

- $V := \{\text{vertices}\}$
- W : the adjacency matrix
- $b \in \text{Spin}^c(M) \cong \frac{(\deg v)_v + 2\mathbb{Z}^V}{2W(\mathbb{Z}^V)}$

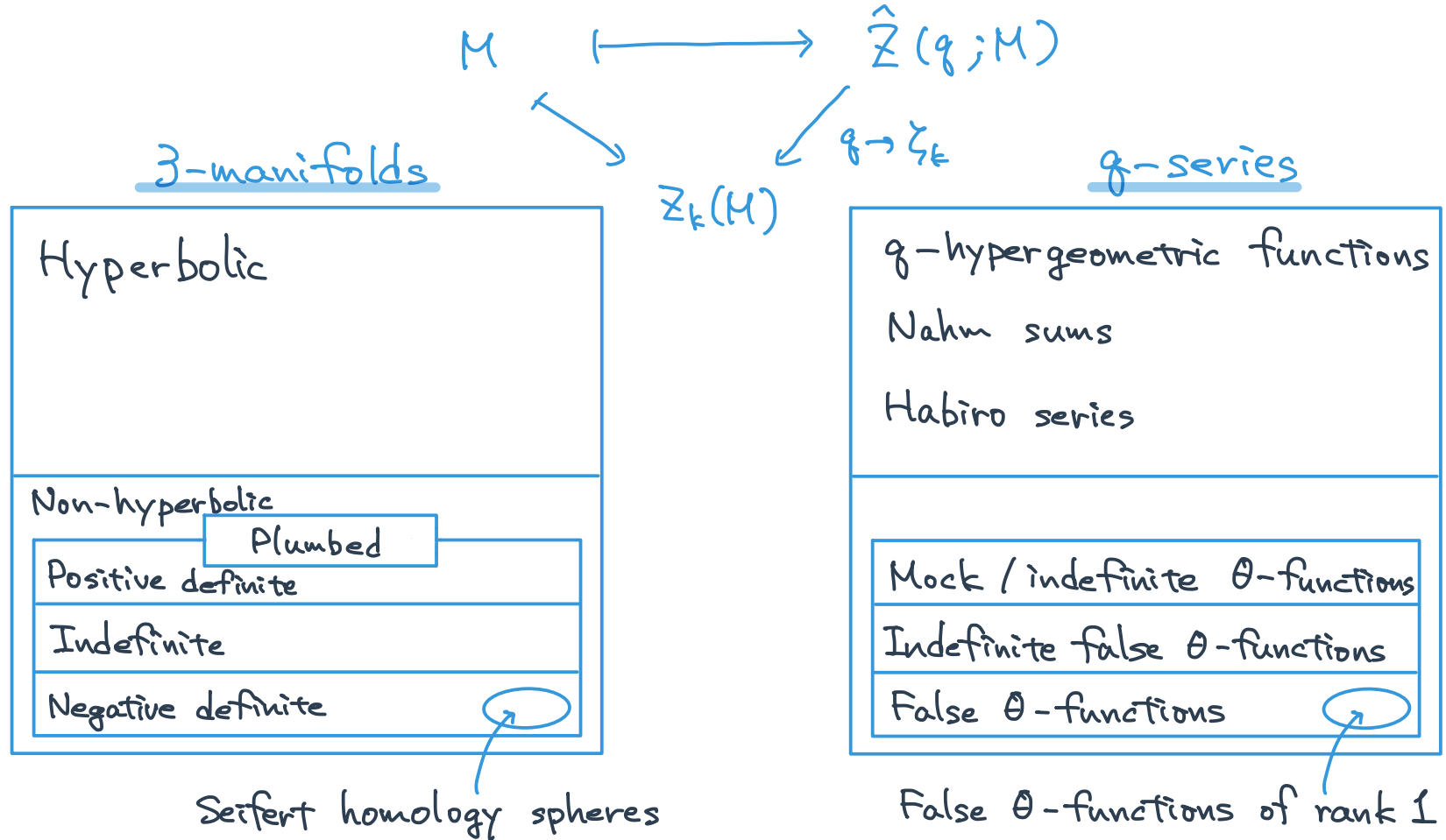
$$\Delta := -\frac{\text{tr } W + |V|}{4}$$

$$\boxed{F_\ell} := \prod_{v \in V} \text{PV} \int_{|z_v|=1} (z_v - z_v^{-1})^{2 - \deg v} \frac{z_v^{-\ell v} dz_v}{2\pi i z_v}$$

- Cauchy principal value

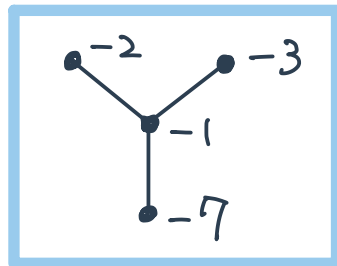
$$\text{PV} \int_{|z|=1} := \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \left(\int_{|z|=1-\varepsilon} + \int_{|z|=1+\varepsilon} \right)$$

● Correspondence between 3-manifolds & q-series



● Example: a Brieskorn homology sphere $\Sigma(2, 3, 7)$

$$\bullet W = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & & \\ 1 & & -3 & \\ 1 & & & -7 \end{pmatrix}, \quad \det W = 1$$



$$\bullet H_1(M, \mathbb{Z}) \cong \mathbb{Z}^4 / W(\mathbb{Z}^4) = 0$$

$$\bullet \text{Spin}^c(M) \cong (\delta + 2\mathbb{Z}^4) / 2W(\mathbb{Z}^4) = (\delta + 2\mathbb{Z}^4) / 2\mathbb{Z}^4, \quad \delta := (3, 1, 1, 1)$$

$$\bullet \text{WRT inv} \quad \mathbb{Z}_k(M) = \frac{-\zeta_{4k}^9}{2(2k)^2(\zeta_{2k} - \zeta_{2k}^{-1})} \sum_{\mu \in (\mathbb{Z} - k\mathbb{Z})^4 / 2k\mathbb{Z}^4} \zeta_{4k}^{\mu W \mu} \frac{\prod_{2 \leq i \leq 4} (\zeta_{2k}^{\mu_i} - \zeta_{2k}^{-\mu_i})}{\zeta_{2k}^{\mu_1} - \zeta_{2k}^{-\mu_1}}$$

$$\bullet \text{GPPV inv} \quad \hat{\mathbb{Z}}_{\delta}(q; M) = q^{\frac{5}{2}} \sum_{m=0}^{\infty} \chi(m) q^{\frac{m^2-1}{168}} = \frac{1}{2} q^{\frac{5}{2}} \sum_{m=-\infty}^{\infty} \text{sgn}(m) \chi(m) q^{\frac{m^2-1}{168}}$$

where $\chi : \mathbb{Z}/84\mathbb{Z} \rightarrow \{0, \pm 1\}$

$$1, -13, -29, 41 \mapsto 1$$

$$-1, 13, 29, -41 \mapsto -1$$

$$\text{otherwise} \mapsto 0$$

rank 1

false theta function!

rank $\in \mathbb{Z}_{>0}$ in general $\leadsto$ Difficult!

- CS inv $\mathcal{S} = \left\{ -\frac{n^2}{168} \bmod \mathbb{Z} \mid n \in (\mathbb{Z}/42\mathbb{Z})^\times \right\}$

- ② Quantum modularity for GPPV inv

admits $\tau \rightarrow -k$
& $k \rightarrow \infty$

$$\hat{Z}_{\mathcal{S}}(-\frac{1}{\tau}) = \tilde{q}^{\frac{5}{2} - \frac{1}{168}} \cdot 16i \sqrt{\frac{84i}{\tau}} \sum_{\theta \in \mathcal{S}} Z_{\theta}^*(q; M) + \Omega(\tau)$$

where $\triangleright q := e^{2\pi i \tau}$, $\tilde{q} := e^{2\pi i \frac{-1}{\tau}}$

rank 1 false theta

$$\triangleright Z_{\theta}^*(q; M) := \sum_{\substack{n \in \mathbb{Z}, \\ \theta \equiv -\frac{n^2}{168} \bmod \mathbb{Z}}} (-1)^n \sin \frac{\pi n}{2} \sin \frac{\pi n}{3} \sin \frac{\pi n}{7} q^{\frac{n^2}{168}}$$

$$\triangleright \Omega(\tau) := \int_0^{\infty + i\epsilon} \tilde{q}^{84 \zeta^2} G(e^{2\pi i \zeta}) d\zeta, \quad G(z) := \frac{(z^{\frac{1}{2}} - z^{-\frac{1}{2}})(z^{\frac{1}{3}} - z^{-\frac{1}{3}})(z^{\frac{1}{7}} - z^{-\frac{1}{7}})}{z - z^{-1}}$$

converges for $\tau = -k$

- ③ Asymptotic expansion $Z_k(M) \sim \sum_{\theta \in \mathcal{S}} e^{2\pi i k \theta} Z_{\theta}^*(k) + \tilde{\varphi}(k)$
 $\in \mathbb{C}[\sqrt{k}]$ $\in \mathbb{C}(\frac{1}{\sqrt{k}})$

§4 Quantum modularity & asymptotics for false theta functions

Def A false theta function of rank r & depth $\leq d$ is

$$\tilde{\Theta}(\tau) := \sum_{m \in \alpha + \mathbb{Z}^r} \text{sgn}(Am) P(m) q^{Q(m)},$$

where $Q : \mathbb{Q}^r \rightarrow \mathbb{Q} : \text{pos def quad form}$, $A \in M_{s,r}(\mathbb{Z}) : \text{rank } d$,
 $\alpha \in \mathbb{Q}^r$, $P(x) \in \mathbb{Q}[x]$, $\text{sgn}(y) := \text{sgn}(y_1) \cdots \text{sgn}(y_s)$, $q := e^{2\pi i \tau}$.

Main Thm 2 (② Quantum modularity for false theta functions)

$$\sqrt{\frac{i}{\tau}}^r \tilde{\Theta}\left(-\frac{1}{\tau}\right) = \sum_{1 \leq i \leq M} \overset{\exists}{\tilde{\Theta}_i}(\tau) \overset{\exists}{\Omega_i}(\tau) + \overset{\exists}{\Omega}(\tau)$$

$\uparrow$ false theta func of rank $\leq r$ & depth $\leq d$
 $\uparrow$ hol func on $\mathbb{C} \setminus \{0\}$

• Previous works

• Rank 1 case

- ▷ Lawrence-Zagier (1999) : for radial limits
- ▷ Creutzig-Milas (2014) : first proof
- ▷ Andersen-Mistegård (2022), Han-Li-Sauzin-Sun (2023),
Creu-Fantini-Goswami-Osbern-Wheeler (2025) :
proof by resurgence theory

• General rank case

- ▷ Creutzig-Milas (2017) : modular transform as integrals
- ▷ Bringmann-Nazaroglu (2019) : modular completion for $P=1$

• Formulation of quantum modularity

- ▷ Zagier (2010), Garoufalidis-Zagier (2023), (2024),
Wheeler's thesis (2023)

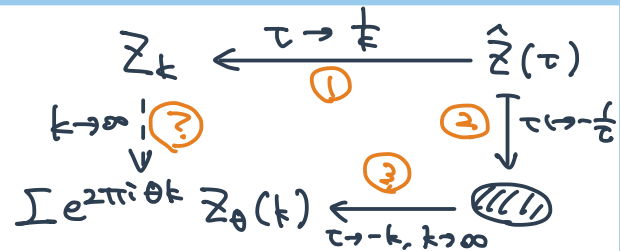
Main Thm 3 (③ Asymptotic expansions)

$$(1) \forall p \in \mathbb{Q}, \quad \tilde{\Theta}(t) \underset{t \rightarrow p}{\sim} \sum_{-\infty \ll j \in \frac{1}{2}\mathbb{Z}} c_j(p) \cdot (t - p)^j.$$

(2) $\rho = \frac{h}{k}$, $h \neq 0$ fixed

$$\Rightarrow c_{\vec{j}}(\rho) \underset{k \rightarrow \infty}{\sim} \sum_{\substack{\theta \in \Xi \mathcal{J} \subset \mathbb{Q}/\mathbb{Z} \\ \text{fin, explicit}}} e^{2\pi i \theta \cdot \frac{-1}{\rho}} \cdot \varphi_{\theta, h, \vec{j}}(k) \cdot \mathcal{O}(k^{-\frac{1}{2}})$$

Applying $\tilde{\mathbb{H}}(\tau)$ as GPPV inv
 $\leadsto$ Thm 1.

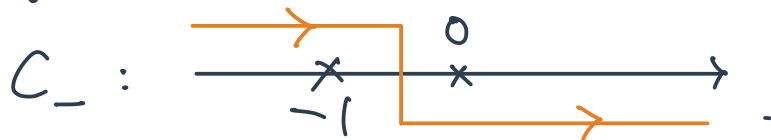


§5 Poisson summation formula with sign

Thm (YM)

$$\sum_{m \in \mathbb{Z}} \operatorname{sgn}(m) f(m) = \sum_{n \in \mathbb{Z}} \operatorname{sgn}(n) \hat{f}(n) + 2 \int_{C_-} \hat{f}(\xi) \frac{d\xi}{1 - e^{2\pi i \xi}},$$

where $\operatorname{sgn}(0) := 1$, $\hat{f}(\xi)$: the Fourier transform of f ,



Rem . $\exists \mathbb{Z}^r$ version. $\sum \int$ terms appear.

• Proof : LHS $\xleftrightarrow{\text{Fourier inverse formula}}$ $=$ $+ 2 \int_{C_-} \hat{f}(\xi) \frac{d\xi}{1 - e^{2\pi i \xi}} \rightarrow \text{RHS}$

Fourier inverse formula

• Same strategy as resurgence : $x \cdots x \cdots x \cdots x \cdots$

● Sketch of a proof of Thm 2

False theta functions ($\supset$ GPPV invariants)



$$\widetilde{\Theta}(\tau) = \sum_{m \in \mathbb{Z}^r} \text{sgn}(m) f(\tau; m)$$

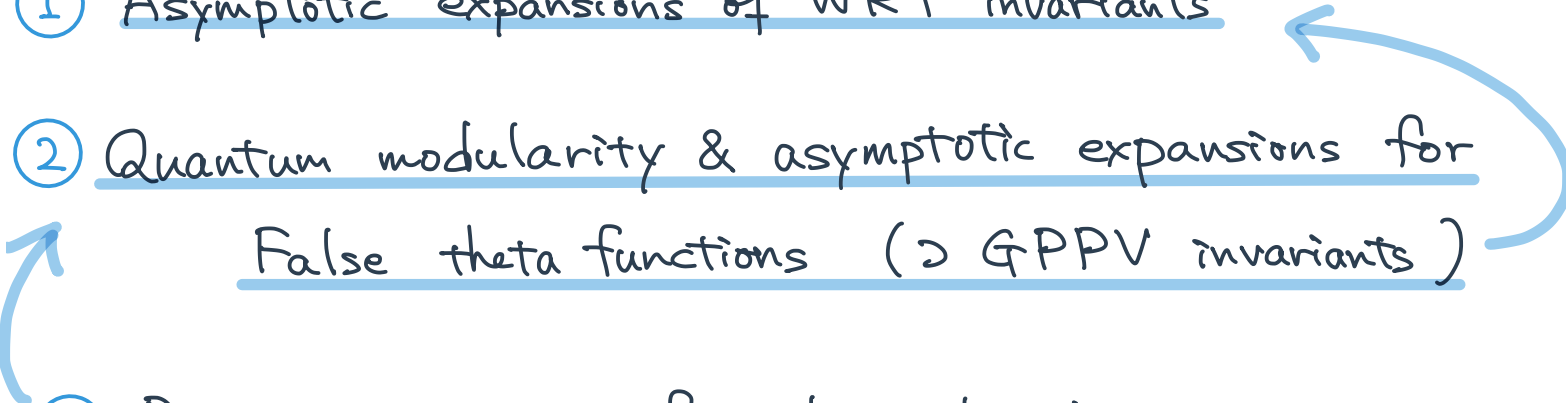
$$= \sum_{n \in \mathbb{Z}^r} \text{sgn}(n) \widehat{f}(\tau; n) + \sum \int_{C_-} + \dots + \int_{C_-^r} \quad \left(\text{Poisson summation formula with sign} \right)$$

$$= \sum_i \widetilde{\Theta}_i(-\tfrac{1}{\tau}) \cdot \tau^{\bullet} + \sum_j \widetilde{\Theta}_j(-\tfrac{1}{\tau}) \underbrace{\Omega_j(\tau)}_{\uparrow} + \dots + \underbrace{\Omega(\tau)}_{\uparrow} \quad \left(\begin{array}{l} \text{complete squares} \\ \text{in Gaussians \& } \\ \text{contour deformation} \end{array} \right)$$

extend holomorphically to $\mathbb{C} \setminus \{0\}$

$\rightsquigarrow$ Quantum modularity !!

Conclusion

- ① Asymptotic expansions of WRT invariants
 - ② Quantum modularity & asymptotic expansions for
False theta functions ($\supset$ GPPV invariants)
 - ③ Poisson summation formula with sign
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Thank you!