

Imaginary pseudo entropy as a temporal-orientation record

Modular covariance and replica interferometer

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Modular Hamiltonian response

My original question around 2022: In non-Hermitian quantum systems, energy is generally complex valued. Is entropy complex-valued? If so, what is physical meaning of imaginary part of entropy? Energy eigenvalues in non-Hermitian systems:

$$E_n = E_n^{\text{Re}} + iE_n^{\text{Im}} \in \mathbb{C}.$$

The partition function naively can then be complex-valued:

$$Z = \sum_n e^{-\beta E_n} \in \mathbb{C}.$$

Entropy naively obtained from this can be complex-valued:

$$S \propto \partial_T(k_B T \log Z) \in \mathbb{C}.$$

The imaginary part of complex entropy may be related with the arrow of time.... More practically, exceptional (PT-transitoin) points in PT-symmetric QM may be detected by this. $\rightarrow$ pseudo entropy (See Kamata, TM, Pazarbasi, Taya (2026) for EWKB framework for this problem.)

Talk plan

Part 1: short-time response

Real-time pseudo entropy and modular-Hamiltonian correlations

- Motivate pseudo entropy from holography and quantum information.
- Define transition matrices and real-time pseudo entropy.
- Derive the short-time response.
- Identify $\text{Im } S_A$ with a modular covariance.
- Check the formula in two-qubit and Ising systems, and extend to PT-symmetric non-Hermitian systems.

Part 2: measurable meaning

Imaginary pseudo entropy encodes temporal orientation

- Show the imaginary pseudo entropy is measurable in an interferometer.
- It encodes temporal orientation in a replica interferometer.
- It also gives a bounded trace-distance meaning in the measurement.
- Coarse graining erases the orientation record, leading to irreversibility.

Imaginary pseudo entropy is a coherent record of transition orientation.

Talk plan

- Motivation: why pseudo entropy is important in holography and quantum information.
- Basic definitions: transition matrix, pseudo entropy, and transition orientation.
- Part 1: short-time formula and modular covariance.
- Part 2: replica interferometer, trace distance, and information loss.
- Examples: two qubits, the transverse-field Ising chain and PT-symmetric non-Hermitian QM.
- Summary and outlook.

Guiding question

Can the imaginary part of pseudo entropy be more than a convention-dependent phase?

Why pseudo entropy? Holographic viewpoint

Holographic entanglement entropy

For a static holographic state, the entanglement entropy of a boundary region A is

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}, \quad \partial\gamma_A = \partial A.$$

Here γ_A is the **minimal surface in the bulk** whose boundary coincides with the boundary of A .

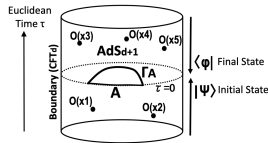
From a state to a transition

Nakata–Takayanagi–Taki–Tamaoka–Wei (2020)

Pseudo entropy extends it from one state to a transition between two states,

$$\mathcal{T}^{\psi|\phi} = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}, \quad \mathcal{T}_A^{\psi|\phi} = \text{Tr}_{\bar{A}} \mathcal{T}^{\psi|\phi}.$$

The states $|\psi\rangle$ and $|\phi\rangle$ can be prepared by different Euclidean path integrals.



Holographic prescription

Different boundary preparations generally lead to a **Euclidean-time-dependent bulk geometry**. In this geometry, pseudo entropy is proposed to be

$$S(\mathcal{T}_A^{\psi|\phi}) = \frac{\text{Area}(\Gamma_A)}{4G_N}$$

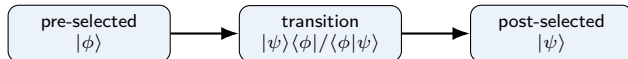
where Γ_A is the **minimal surface in the bulk** anchored on the boundary of A . Thus pseudo entropy extends the geometric interpretation of entanglement entropy from a **state** to a **transition between two states**.

Why pseudo entropy? Quantum-information viewpoint

Post-selection and transition matrices

Nakata et al. (2020)

- Pseudo entropy is defined from a transition matrix, not from a density matrix.
- It gives an entanglement-like quantity for pre- and post-selected processes.
- In simple qubit/random-state settings, it can count effective Bell pairs after post-selection.



Why $\text{Im } S_A$ is subtle

- $\tau_A^{\psi|\phi}$ is generally non-Hermitian.
- $\log \tau_A^{\psi|\phi}$ has branch choices.
- The normalization $\langle\phi|\psi\rangle^{-1}$ can amplify quantities, as in weak values.

Question

Can $\text{Im } S_A$ have a bounded and measurable physical meaning?

From density matrices to transition matrices

Ordinary entanglement entropy

A state gives a positive reduced density matrix,

$$\rho_A = \text{Tr}_{\bar{A}} |\psi\rangle\langle\psi|.$$

Then

$$S_A = -\text{Tr}_A \rho_A \log \rho_A$$

is real.

Pseudo entropy

Nakata et al. (2020)

Two states give a reduced transition matrix,

$$\tau^{\psi|\phi} = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}, \quad \tau_A^{\psi|\phi} = \text{Tr}_{\bar{A}} \tau^{\psi|\phi}.$$

Since $\tau_A^{\psi|\phi}$ is generally non-Hermitian,

$$S_A^{\psi|\phi} = -\text{Tr}_A \tau_A^{\psi|\phi} \log \tau_A^{\psi|\phi}$$

can be complex.

The arrow-of-time tension

- In a closed Hermitian quantum system, the total von Neumann entropy is conserved.
- Irreversibility usually needs an additional step: choosing a subsystem, measuring, coupling to an environment, or coarse graining.
- Forward/backward distinguishability in thermodynamics is usually a statement about probabilities or channels.
- But pseudo entropy is defined at the amplitude level.

Microscopic question

Can pseudo entropy already carry an orientation record before it becomes thermodynamic entropy production?

pseudo entropy $\longrightarrow$ temporal records $\longrightarrow$ loss of them $\longrightarrow$ irreversibility?

A useful distinction

Transition orientation

Exchange the ket and bra boundary states in the transition matrix:

$$F : |\Psi_t\rangle\langle\Psi_0|, \quad B : |\Psi_0\rangle\langle\Psi_t|.$$

For Hermitian unitary dynamics this complex conjugates the reduced transition data.

Physical time reversal

This is stronger. It requires an antiunitary symmetry and a consistent transformation of the Hamiltonian, preparation, and readout.

A nonzero imaginary pseudo entropy in a closed system is not, by itself, temporal irreversibility.

Real-time transition matrix

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ and $|\Psi_t\rangle \equiv |\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$.

Unnormalized and normalized transition data

$$\underbrace{\chi(t) = \langle \Psi_0 | \Psi_t \rangle}_{\text{total overlap}}, \quad \underbrace{T_A^F(t) = \text{Tr}_{\bar{A}} |\Psi_t\rangle \langle \Psi_0|}_{\text{unnormalized reduced transition matrix}},$$
$$\underbrace{\tau_A^F(t) = \frac{T_A^F(t)}{\chi(t)}}_{\text{normalized reduced transition matrix}}.$$

Real-time pseudo entropy

$$S_A^F(t) \equiv S_A(t, 0) = -\text{Tr}_A \tau_A^F(t) \log \tau_A^F(t).$$

At $t = 0$,

$$\tau_A^F(0) = \rho_A = \text{Tr}_{\bar{A}} |\Psi_0\rangle \langle \Psi_0|, \quad S_A^F(0) = -\text{Tr}_A \rho_A \log \rho_A.$$

Orientation-odd part

Forward and backward reduced transition matrices

$$T_A^F(t) = \text{Tr}_{\bar{A}} |\Psi_t\rangle\langle\Psi_0|, \quad T_A^B(t) = \text{Tr}_{\bar{A}} |\Psi_0\rangle\langle\Psi_t| = (T_A^F)^\dagger.$$

$$\tau_A^F = \frac{T_A^F}{\chi}, \quad \tau_A^B = \frac{T_A^B}{\chi^*}, \quad \chi(t) = \langle\Psi_0|\Psi_t\rangle.$$

If the logarithmic branch is followed continuously,

$$S_A^B(t) = S_A(0, t) = S_A^F(t)^*.$$

Therefore the orientation-odd part is

$$\mathcal{A}_A(t) = \frac{1}{2i} \{S_A^F(t) - S_A^B(t)\} = \text{Im } S_A^F(t).$$

Question

Does $\mathcal{A}_A$ has information of temporal orientation?

Modular-Hamiltonian response

Definitions:

$$\rho_A = \text{Tr}_{\bar{A}} |\Psi_0\rangle\langle\Psi_0|, \quad K_A = -\log \rho_A, \quad \langle O \rangle \equiv \langle\Psi_0|O|\Psi_0\rangle.$$

Here, K_A is called **modular Hamiltonian**, which encodes entanglement structure between A and $\bar{A}$. For a trace-preserving variation of the reduced transition matrix,

$$\delta S_A = \text{Tr}_A(\delta\tau_A K_A).$$

Expanding the normalized transition matrix before tracing out the complement,

$$\tau^F(t) = |\Psi_0\rangle\langle\Psi_0| - it(H - \langle H \rangle)|\Psi_0\rangle\langle\Psi_0| + O(t^2),$$

gives

$$S_A^F(t) = S_A(0) - it \langle K_A(H - \langle H \rangle) \rangle + O(t^2).$$

To first order in time, $S_A^F(t)$ is given by the expectation value of K_A times H .

Imaginary part = symmetrized covariance of modular Hamiltonian

For Hermitian H and K_A , define

$$C_A = \langle K_A(H - \langle H \rangle) \rangle, \quad \Delta O \equiv O - \langle O \rangle.$$

The real and imaginary parts separate as

$$\left. \frac{d}{dt} \operatorname{Im} S_A^F(t) \right|_0 = -\frac{1}{2} \langle \{ \Delta K_A, \Delta H \} \rangle \equiv -C_{KH}$$

$$\left. \frac{d}{dt} \operatorname{Re} S_A^F(t) \right|_0 = \frac{1}{2i} \langle [K_A, H] \rangle.$$

The imaginary response is **symmetrized covariance of Hamiltonian and modular Hamiltonian**

Physical meaning

H generates real-time evolution, while $K_A = -\log \rho_A$ encodes the entanglement spectrum of A . The imaginary slope measures their correlation.

Why this is not just a branch effect

- The response is present at infinitesimal real time.
- No eigenvalue has to circle a logarithmic branch point to generate it.
- The slope is fixed by the initial-state correlation function

$$C_{KH} = \frac{1}{2} \langle \{ \Delta K_A, \Delta H \} \rangle, \quad K_A = -\log \rho_A, \quad \Delta O = O - \langle O \rangle.$$

- H evolves the whole state; K_A describes the entanglement structure of the chosen subsystem A .

Message

$\text{Im } S_A^F$ is a state-dependent response of the reduced transition amplitude, not merely a choice of logarithmic branch.

Solvable Schmidt-diagonal model

Assume a Schmidt basis and a diagonal Hamiltonian in the same sectors:

$$|\Phi\rangle = \sum_n \sqrt{p_n} |n\rangle_A |n\rangle_{\bar{A}}, \quad H |n\rangle_A |n\rangle_{\bar{A}} = E_n |n\rangle_A |n\rangle_{\bar{A}}.$$

Overlap and modular Hamiltonian:

$$\chi(t) = \langle \Phi | e^{-iHt} | \Phi \rangle = \sum_m p_m e^{-iE_m t}, \quad K_n = -\log p_n.$$

Eigenvalues of the reduced transition matrix $\tau_A^F(t)$:

$$\lambda_n(t) = \frac{p_n e^{-iE_n t}}{\chi(t)}.$$

With $\vartheta = -it$, $M(\vartheta) = \sum_n p_n e^{\vartheta E_n}$, and $\langle O \rangle_\vartheta \equiv \sum_n \lambda_n(\vartheta) O_n$,

$$\lambda_n(\vartheta) = \frac{p_n e^{\vartheta E_n}}{M(\vartheta)}, \quad S_A(\vartheta) = \langle K \rangle_\vartheta - \vartheta \langle E \rangle_\vartheta + \log M(\vartheta).$$

Complex response, not a second law

For the Schmidt-diagonal class, all averages are over p_n :

$$\text{Cov}(K, E) = \sum_n p_n K_n E_n - \left(\sum_n p_n K_n \right) \left(\sum_n p_n E_n \right).$$

The expansion of pseudo entropy is (where $\kappa(K, E, E)$ is the mixed third cumulant)

$$S_A(t) = S_A(0) - it \text{Cov}(K, E) + \frac{t^2}{2} \{ \text{Var}(E) - \kappa(K, E, E) \} + O(t^3).$$

Thus the imaginary part $\text{Im } S_A(t) = -\text{Cov}(K, E)t + O(t^3)$ is temporal-sign sensitive, while the real part starts at $O(t^2)$ and is not.

Thermal pseudo entropy as a check

Goto–Nozaki–Tamaoka (2021)

For $p_n = e^{-\beta E_n} / Z(\beta)$, $K_n = \beta E_n + \log Z$ and

$$\text{Im } S_{\text{TPE}}(\beta, t) = -\beta \text{Var}_\beta(H)t + O(t^3).$$

Two-qubit check: noncommuting cases

Model definition

$$|\Psi_0\rangle = \sqrt{p}|00\rangle + e^{i\phi}\sqrt{1-p}|11\rangle,$$

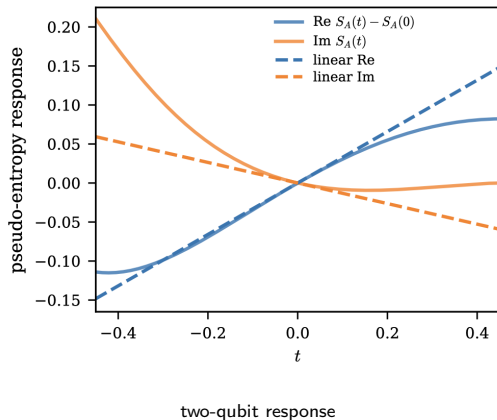
$$H = J\sigma_z \otimes \sigma_z + g\sigma_x \otimes \sigma_x.$$

For $\phi \neq 0$, $\langle [K_A, H] \rangle$ can be nonzero, so there are both the real and imaginary slopes.

Check

Short-time slopes from direct expansion agree with

$$\left. \frac{d}{dt} \text{Im } S_A^F \right|_0 = -C_{KH}, \quad \left. \frac{d}{dt} \text{Re } S_A^F \right|_0 = \frac{1}{2i} \langle [K_A, H] \rangle.$$



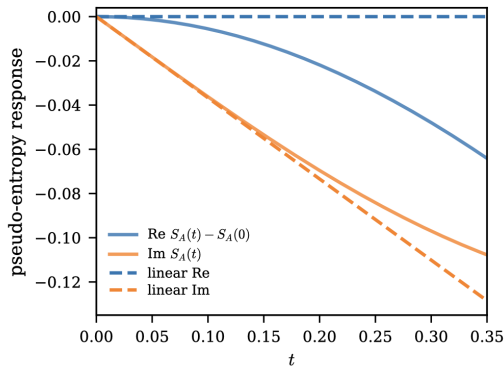
Ising-chain check: many-body cases with quench

Open transverse-field Ising chain

$$H(h) = - \sum_{j=1}^{L-1} \sigma_z^{(j)} \sigma_z^{(j+1)} - h \sum_{j=1}^L \sigma_x^{(j)}.$$

Initial state: ground state of $H(h_i)$. we suddenly changes $h_i \rightarrow h_f$, then time evolution is given by $H(h_f)$. Here, subsystem means half chain A .

- For real initial h_i , $\langle [K_A, H(h_f)] \rangle = 0$.
- Hence the real part has no linear slope.
- The imaginary part follows the modular-covariance slope.



Ising-chain quench

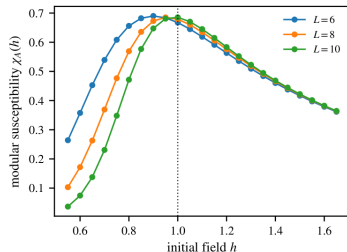
Modular susceptibility near criticality

For an infinitesimal field quench $h \rightarrow h + \delta h$ in the open Ising chain,

$$V = \partial_h H(h) = - \sum_j \sigma_x^{(j)}, \quad \chi_A(h) = \text{Re} \langle K_A (V - \langle V \rangle) \rangle_h.$$

Then

$$\left. \frac{d}{dt} \text{Im} S_A^F(t) \right|_0 = -\delta h \chi_A(h) + O(\delta h^2).$$



- we call $\chi_A(h)$ **modular susceptibility**.
- It correlates the perturbation $V = \partial_h H$ with $K_A = -\log \rho_A$.
- It peaks near the critical value $h \simeq J$.
- It probes how the entanglement structure reacts to a Hamiltonian deformation.

Non-Hermitian extension

For a non-Hermitian Hamiltonian, use right and left states,

$$H|R_n\rangle = E_n|R_n\rangle, \quad \langle L_n|H = E_n\langle L_n|.$$

A one-sided transition matrix is

$$|R(t)\rangle = e^{-iHt}|R\rangle, \quad \tau^{RL}(t, 0) = \frac{|R(t)\rangle\langle L|}{\langle L|R(t)\rangle}, \quad \langle O\rangle_{RL} \equiv \langle L|O|R\rangle.$$

With $S_A^{RL}(t, 0) = -\text{Tr}\tau^{RL}(t, 0) \log \tau^{RL}(t, 0)$ and $K_A^{RL} = -\log \rho_A^{RL}$,

$$S_A^{RL}(t, 0) = S_A^{RL}(0) - it \langle K_A^{RL}(H - \langle H\rangle_{RL}) \rangle_{RL} + O(t^2).$$

Caution

For genuinely non-Hermitian dynamics this is a biorthogonal modular correlator, not a covariance of two ordinary Hermitian observables.

PT-symmetric two-level illustration

$$H_{PT} = g\sigma_x + i\gamma\sigma_z, \quad E_{\pm} = \pm\Delta, \quad \Delta = \sqrt{g^2 - \gamma^2}.$$

With $s = \beta + it$,

$$S_{PT}(s) = \log(2 \cosh s\Delta) - s\Delta \tanh(s\Delta).$$

For $|\gamma| < |g|$, Δ is real, then

$$\text{Im} S_{PT}(\beta, t) = -\beta\Delta^2 \text{sech}^2(\beta\Delta) t + O(t^3).$$

For $|\gamma| > |g|$, writing $\Delta = i\kappa$, one finds

$$S_{PT}(s) = \log(2 \cos \kappa s) + \kappa s \tan(\kappa s).$$

Zeros of $\cos \kappa s$ generate logarithmic singularities

$$\kappa s = \frac{\pi}{2} + \pi n, \quad n \in \mathbb{Z}.$$

Unbroken PT

$|\gamma| < |g|$, so $\Delta \in \mathbb{R}$. The entropy has a hyperbolic, thermal-like structure.

Broken PT

$|\gamma| > |g|$, so $\Delta = i\kappa$. Hyperbolic functions become trigonometric, and branch sensitivity becomes explicit. It shows that the analytic structure of complex entropy changes across the exceptional point.

Summary of Part 1

- ① Real-time pseudo entropy has a universal first-order expansion.
- ② Its imaginary part is controlled by a covariance of modular and total Hamiltonians.
- ③ Its real linear response is controlled by their commutator expectation value.
- ④ The formula works in both simple, many-body examples and non-Hermitian systems.
- ⑤ The result identifies a coherent temporal-orientation response.

Next step

The covariance explains how the imaginary part is generated. We now ask how it can be read out as a bounded physical signal.

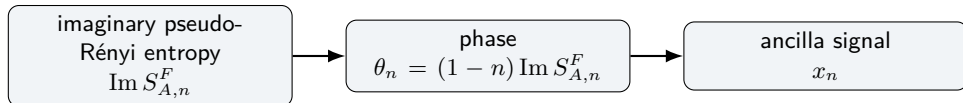
Part 2: from imaginary entropy to a measured phase

Question after Part 1

Part 1 explained how the imaginary part is generated at short times:

$$\partial_t \operatorname{Im} S_A^F(t)|_0 = -C_{KH}.$$

Now we ask how the same kind of imaginary quantity can be **read out** at integer replica number.



Message of Part 2

The imaginary pseudo-Rényi entropy is a phase. A calibrated replica interferometer converts this phase into an ancilla Y signal whose sign distinguishes the two transition orientations.

Step 1: transition data before replicas

Full-system overlap

$$\chi(t) = \langle \Psi_0 | \Psi_t \rangle = |\chi(t)| e^{i\alpha(t)}.$$

- $\chi(t)$ is the transition amplitude of the **full system**.
- $\alpha(t) = \arg \chi(t)$ is its phase.

Reduced transition matrices on subsystem A

$$T_A^F(t) = \text{Tr}_{\bar{A}} |\Psi_t\rangle \langle \Psi_0|, \quad \tau_A^F(t) = \frac{T_A^F(t)}{\chi(t)}.$$

- T_A^F : unnormalized reduced transition matrix.
- τ_A^F : normalized reduced transition matrix used in pseudo entropy.

Step 2: replica moments G_n^F and Z_n^F

Unnormalized replica moment

$$G_n^F(t) = \text{Tr}_A [(T_A^F(t))^n].$$

- G_n^F is a **moment** of the unnormalized transition matrix T_A^F .
- In the interferometer, the same G_n^F is realized as a many-copy branch overlap.

Normalized pseudo-Rényi moment

$$Z_n^F(t) = \text{Tr}_A [(\tau_A^F(t))^n] = \frac{G_n^F(t)}{\chi(t)^n}.$$

- Z_n^F is the moment of the normalized matrix τ_A^F .
- The factor $\chi(t)^{-n}$ is the normalization overlap.

Step 3: pseudo-Rényi entropy and phase

Pseudo-Rényi entropy

$$S_{A,n}^F(t) = \frac{1}{1-n} \log Z_n^F(t).$$

This is the replica version of pseudo entropy for the reduced transition matrix τ_A^F .

Pseudo-Rényi phase

$$\theta_n(t) = \arg Z_n^F(t) = (1-n) \operatorname{Im} S_{A,n}^F(t).$$

Equivalently,

$$\theta_n(t) = \arg G_n^F(t) - n \arg \chi(t).$$

The quantity to be measured is the calibrated phase θ_n .

Why calibration is needed

Unnormalized versus normalized moments

The interferometer naturally measures the unnormalized replica moment G_n^F . But the pseudo-Rényi entropy is defined from the normalized moment

$$Z_n^F = \frac{G_n^F}{\chi^n}.$$

Therefore the desired phase is

$$\theta_n = \arg Z_n^F = \arg G_n^F - n \arg \chi.$$

Calibration

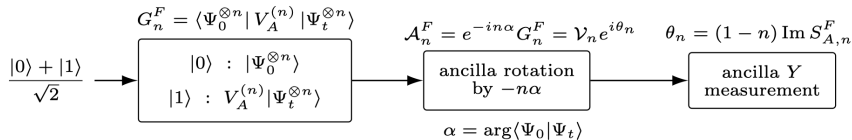
A one-copy interferometer measures

$$\alpha = \arg \chi.$$

Then rotating the ancilla by $-n\alpha$ removes the trivial normalization phase from χ^n .

Replica interferometer measures the moment G_n^F

(a) Calibrated replica interferometer



Replica SWAP/cyclic-permutation measurement

$V_A^{(n)}$: cyclic permutation of the subsystem A among n copies.

$$G_n^F = \langle \Psi_0^{\otimes n} | V_A^{(n)} | \Psi_t^{\otimes n} \rangle = \text{Tr}_A[(T_A^F)^n].$$

Thus the circuit measures the **unnormalized replica moment** G_n^F .

Calibrated ancilla amplitude

After the $-n\alpha$ ancilla rotation

$$\mathcal{A}_n^F = e^{-in\alpha} G_n^F = \mathcal{V}_n e^{i\theta_n}.$$

Phase

$$\theta_n = \arg Z_n^F = (1 - n) \operatorname{Im} S_{A,n}^F.$$

This is the imaginary pseudo-Rényi entropy written as a measurable phase.

Visibility

$$\mathcal{V}_n = |G_n^F|.$$

This is the length of the ancilla Bloch vector. It affects the signal size.

Transition orientation reverses the phase

Forward and backward orientations

$$F : |\Psi_t\rangle\langle\Psi_0|, \quad B : |\Psi_0\rangle\langle\Psi_t|.$$

For Hermitian unitary dynamics,

$$T_A^B = (T_A^F)^\dagger, \quad Z_n^B = (Z_n^F)^*.$$

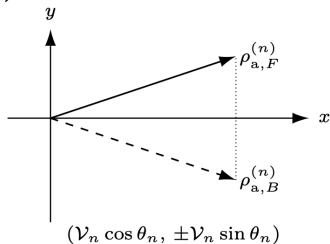
Effect on the phase

$$\theta_n^B = -\theta_n^F.$$

The imaginary pseudo-Rényi phase is therefore **orientation odd**.

Ancilla Bloch vector: where the phase appears

(b) Ancilla Bloch vectors



$$\mathcal{D}_n = \mathcal{V}_n |\sin \theta_n| \rightarrow P_{\text{succ}}^* = \frac{1}{2}(1 + \mathcal{D}_n)$$

After tracing out the replica registers

$$\rho_{a,F}^{(n)} = \frac{1}{2} \{I + \mathcal{V}_n \cos \theta_n \sigma_x + \mathcal{V}_n \sin \theta_n \sigma_y\},$$

$$\rho_{a,B}^{(n)} = \frac{1}{2} \{I + \mathcal{V}_n \cos \theta_n \sigma_x - \mathcal{V}_n \sin \theta_n \sigma_y\}.$$

same x component, opposite y component

$$x_n \equiv \mathcal{V}_n \sin \theta_n = \mathcal{V}_n \sin\{(1 - n) \operatorname{Im} S_{A,n}^F\}.$$

It has record of transition time orientation.

Meaning

x_n is the signed Y component of the calibrated ancilla Bloch vector:

$$\langle \sigma_y \rangle_F = +x_n, \quad \langle \sigma_y \rangle_B = -x_n.$$

Why this matters

The sign of x_n is the experimentally readable record of transition orientation:

$$F \leftrightarrow B : \quad \theta_n \rightarrow -\theta_n, \quad x_n \rightarrow -x_n.$$

Ancilla Y measurement

Measurement outcomes

Measure the ancilla in the σ_y basis. Let the outcome be $y = +1$ or -1 . Then

$$p_F(y) = \frac{1 + yx_n}{2}, \quad p_B(y) = \frac{1 - yx_n}{2}.$$

Interpretation

The two orientations give opposite biases in the same Y measurement.

F : positive x_n favors $y = +1$,

B : positive x_n favors $y = -1$.

Thus the phase θ_n has become a directly measured orientation signal.

Trace distance and optimal readout

Distinguishability of the two ancilla outputs

$$D_n = \frac{1}{2} \|\rho_{a,F}^{(n)} - \rho_{a,B}^{(n)}\|_1 = |x_n|.$$

Using $x_n = \mathcal{V}_n \sin \theta_n$,

$$D_n = \mathcal{V}_n |\sin \theta_n| = \mathcal{V}_n |\sin\{(1-n) \operatorname{Im} S_{A,n}^F\}|.$$

Best one-shot success probability

For equal prior probabilities of F and B ,

$$P_{\text{succ}}^* = \frac{1}{2}(1 + D_n).$$

Finite-replica dynamics: phase generation

Definitions

$$\rho_A = \text{Tr}_{\bar{A}} |\Psi_0\rangle\langle\Psi_0|, \quad P_n = \text{Tr}_A \rho_A^n, \quad \langle O \rangle_0 = \langle \Psi_0 | O | \Psi_0 \rangle.$$

Initial growth of the phase

$$\dot{\theta}_n(0) = -n \left[\frac{1}{2} \frac{\langle \{ \rho_A^{n-1}, H \} \rangle_0}{P_n} - \langle H \rangle_0 \right].$$

This is the finite-replica response that generates the imaginary pseudo-Rényi phase.

Finite-replica dynamics: signal generation

Initial growth of the measured signal

At $t = 0$, $\theta_n(0) = 0$ and $\mathcal{V}_n(0) = P_n$. Therefore

$$\dot{x}_n(0) = -n \left[\frac{1}{2} \langle \{ \rho_A^{n-1}, H \} \rangle_0 - P_n \langle H \rangle_0 \right].$$

$n = 2$: SWAP covariance

For $n = 2$, $V_A^{(2)}$ is the subsystem SWAP between two copies and $H_\Sigma = H^{(1)} + H^{(2)}$:

$$\dot{x}_2(0) = -\frac{1}{2} \langle \{ \Delta V_A^{(2)}, \Delta H_\Sigma \} \rangle_{\Psi_0^{\otimes 2}}.$$

This is accessible with replica observables, without constructing $K_A = -\log \rho_A$.

Von Neumann limit: bridge back to Part 1

Finite-replica side

The interferometer measures

$$\theta_n = (1 - n) \operatorname{Im} S_{A,n}^F, \quad x_n = \mathcal{V}_n \sin \theta_n, \quad D_n = |x_n|.$$

$n \rightarrow 1$ side

If the continuation to $n = 1$ is smooth,

$$\left. \frac{d}{dt} \operatorname{Im} S_A^F(t) \right|_0 = -\frac{1}{2} \langle \{ \Delta K_A, \Delta H \} \rangle_0 \equiv -C_{KH}, \quad K_A = -\log \rho_A.$$

$$\lim_{n \rightarrow 1} \lim_{t \rightarrow 0} \frac{D_n(t)}{|n-1| |t|} = |C_{KH}|.$$

Two-qubit example: model and signal

Model

$$|\Psi_0\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle, \quad H = J\sigma_z \otimes \sigma_z + g\sigma_x \otimes \sigma_x.$$

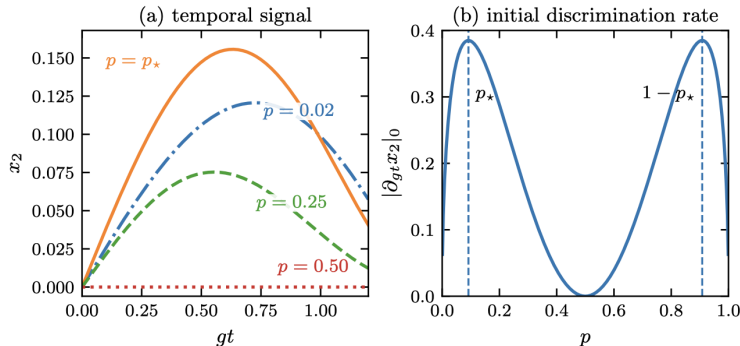
$n = 2$ signed signal

$$x_2(t) = \mathcal{V}_2 \sin \theta_2 = \frac{2r\delta^2 c^3 s}{1 - \delta^2 s^2}.$$

$$r = \sqrt{p(1-p)}, \quad \delta = 2p - 1, \quad c = \cos gt, \quad s = \sin gt.$$

The $g\sigma_x \otimes \sigma_x$ term mixes the two Schmidt sectors and generates the phase signal.

Two-qubit example: where the signal is largest



Reading the plots

- x_2 is the signed ancilla Y signal.
- Product states $p = 0, 1$: no Schmidt-sector coherence.
- Maximally entangled state $p = 1/2$: flat modular spectrum.
- Largest initial rate at $\delta^2 = 2/3$, i.e. $p_* \simeq 0.092$ and $1 - p_* \simeq 0.908$.

Ising-chain example: definitions

Transverse-field Ising chain

$$H(h) = - \sum_{j=1}^{L-1} \sigma_z^{(j)} \sigma_z^{(j+1)} - h \sum_{j=1}^L \sigma_x^{(j)}, \quad A = \text{half chain.}$$

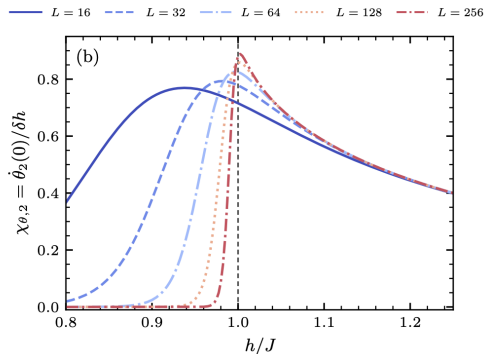
Small field quench and susceptibility

For $h \rightarrow h + \delta h$, define two susceptibilities

$$\chi_{\theta,2}(h, L) = \frac{\dot{\theta}_2(0)}{\delta h}$$

$$\chi_{D,2}(h, L) = \frac{|\dot{x}_2(0)|}{|\delta h|} = P_2 |\chi_{\theta,2}|, \quad P_2 = \text{Tr } \rho_A^2.$$

Ising-chain example: phase response near criticality

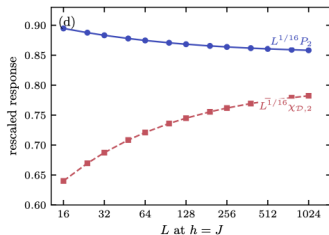
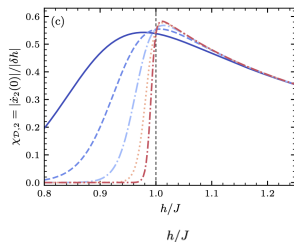


Susceptibility

- $\chi_{\theta,2}$ is the initial generation rate of the phase θ_2 .
- The peak approaches the critical field $h = J$ as L grows.
- This is the finite-replica analogue of the modular susceptibility in Part 1.

Ising-chain example: measured response

— $L = 16$ - - $L = 32$ - · - $L = 64$ ····· $L = 128$ - - - $L = 256$



Susceptibility

$$\chi_{D,2} = P_2 |\chi_{\theta,2}|.$$

This is the same phase response after conversion to the directly measured ancilla signal x_2 .

Visibility factor

At the critical point, boundary CFT gives

$$P_2 = \text{Tr} \rho_A^2 \propto L^{-1/16}$$

for the critical Ising half chain. The suppression is weak.

Coarse graining: what can be lost?

Orientation information before coarse graining

$$I_n^{\text{arrow}} = D(\rho_{a,F}^{(n)} \| \rho_{a,B}^{(n)}), \quad D(\rho \| \sigma) = \text{Tr } \rho (\log \rho - \log \sigma).$$

I_n^{arrow} is quantum relative entropy between the two ancilla outputs.

Loss under a common channel

Let any map Λ (detector noise, dephasing, discarding an environment, etc) act on both outputs. Then

$$\Delta_{\Lambda} I_n^{\text{arrow}} = I_n^{\text{arrow}} - D(\Lambda(\rho_{a,F}^{(n)}) \| \Lambda(\rho_{a,B}^{(n)})) \geq 0.$$

This follows from monotonicity of relative entropy. It is temporal-orientation information discarded by the chosen coarse graining.

Example of coarse graining: dephasing

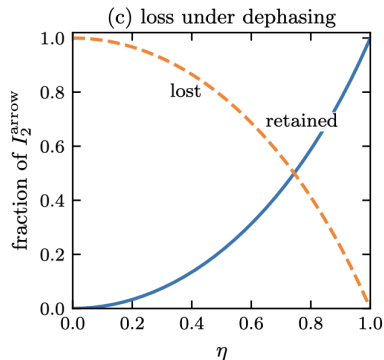
Phase-damping channel

$$\Lambda_\eta(\sigma_x) = \eta\sigma_x, \quad \Lambda_\eta(\sigma_y) = \eta\sigma_y, \quad \Lambda_\eta(\sigma_z) = \sigma_z, \quad 0 \leq \eta \leq 1.$$

$\eta = 1$: non-dephasing; $\eta = 0$: full dephasing.

Retained and lost fractions for two-qubit example

$$\text{retained} = \frac{I_{2,\eta}^{\text{arrow}}}{I_{2,1}^{\text{arrow}}}, \quad \text{lost} = 1 - \frac{I_{2,\eta}^{\text{arrow}}}{I_{2,1}^{\text{arrow}}}.$$



Coherent record versus entropy production

What the interferometer measures

It measures a coherent distinction between transition orientations:

$$F \leftrightarrow B : \quad \theta_n \rightarrow -\theta_n, \quad x_n \rightarrow -x_n, \quad D_n = |x_n|.$$

What it does not automatically measure

D_n and I_n^{arrow} can be nonzero in a closed, reversible unitary process. They are not thermodynamic entropy production by themselves.

Where irreversibility can enter

A noninvertible coarse graining can discard part of the record:

$$\Delta_\Lambda I_n^{\text{arrow}} \geq 0.$$

Overall structure of Part 2

Definitions and readout

$$\begin{aligned} G_n^F = \text{Tr}_A[(T_A^F)^n] &\longrightarrow Z_n^F = \frac{G_n^F}{\chi^n} \longrightarrow \theta_n = (1-n) \text{Im } S_{A,n}^F \\ \theta_n &\longrightarrow x_n = \mathcal{V}_n \sin \theta_n \longrightarrow D_n = |x_n|. \end{aligned}$$

Physics

- Generation: $\dot{\theta}_n(0)$ and $\dot{x}_n(0)$ are finite-replica covariance responses.
- Orientation: $F \leftrightarrow B$ sends $\theta_n \rightarrow -\theta_n$ and $x_n \rightarrow -x_n$.
- Coarse graining: a channel can erase part of the record, measured by $\Delta_\Lambda I_n^{\text{arrow}} \geq 0$.

The phase θ_n , i.e. the imaginary pseudo-Rényi entropy records time orientation and is detectable in calibrated interferometer. The information can be lost by coarse graining, leading to irreversibility of time!

What has been established

- 1 The imaginary part is not merely a logarithmic branch artifact.
- 2 At short times it is governed by modular covariance.
- 3 At integer replica number, the phase becomes physical through the signed ancilla signal x_n and the trace distance D_n .
- 4 The operational quantity is transition-orientation distinguishability, not automatically entropy production.
- 5 Coarse graining can remove the orientation record, where the irreversibility emerges!

Point

Imaginary pseudo entropy records a coherent distinction between transition orientations. It becomes connected to irreversibility only after a physical coarse graining is specified.

- Can the $n \rightarrow 1$ continuation be made operational beyond the short-time slope?
- What is the field-theoretic interpretation of the finite-replica covariance $\dot{x}_n(0)$?
- Can the interferometric phase signal detect Fisher-zero approaches or dynamical quantum phase transitions?
- How does the construction behave in holographic or timelike-entanglement settings?
- What is the best non-Hermitian extension for experiments: eigenvalue pseudo entropy, SVD entropy (Parzygnat et al. 2023), or a biorthogonal interferometer?

Imaginary pseudo entropy is an amplitude-level temporal-orientation phase.

- Its initial generation is controlled by modular covariance.
- Its finite-replica content is visibility-dressed orientation distinguishability.
- Its loss under a common coarse graining is monotonic and recoverability-controlled.

Unitary dynamics creates the record; replica interference reads it; noninvertible coarse graining can erase it, leading to time irreversibility.