Monopole versus center vortex via lens of twisted partition function.

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1. Introduction — Confinement mechanisms via monopoles and/or center vortices —
2. Gauge-invariant criteria for condensation
   - Center-vortex condensation ← T^4-twisted Z.
   - Monopole condensation ← $L(N;1)\times S^1$-twisted Z.
   - Why are they "reasonable"?
3. Center vortex condensation + Mass gap ⇒ Monopole condensation.

1. Introduction

Big question in QCD: Why don't we observe isolated quarks/gluons? (Confinement problem).

pure Yang-Mills (gluon theory)

$$\mathcal{L}=\int d^4x \ \frac{1}{g^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}).$$

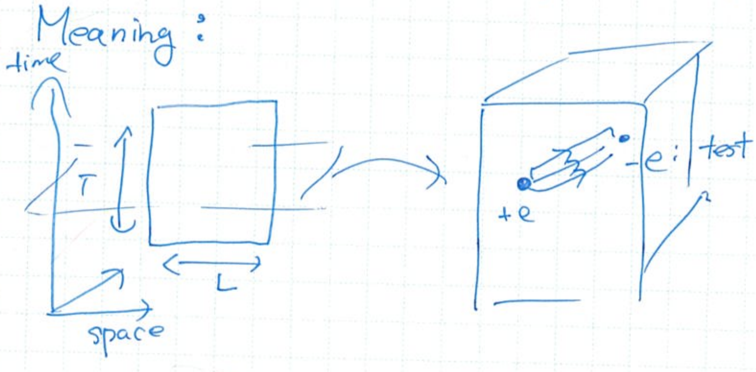
Symmetry : $\mathbb{Z}_N^{(1)}$ symmetry

Order parameter : Wilson loop $W(C)=\frac{1}{N}\mathrm{tr}(\mathcal{P}e^{i\oint a})\xmapsto{\mathbb{Z}_N^{(1)}} e^{\frac{2\pi i}{N}}W(C)$.

Assume nonzero mass gap (= all local correlators show exponential decays), then there are two possible patterns for $\langle W(C)\rangle$:

$$\langle W(C)\rangle \overset{C:\text{large}}{\sim}\begin{cases}\exp(-\sigma\times \mathrm{Area}(C)) & \text{Area law} \iff \mathbb{Z}_N^{(1)} \text{ unbroken}\\ \exp(-\mu\times\mathrm{Length}(C)) & \text{Perimeter law} \iff \mathbb{Z}_N^{(1)}\text{ SSB}\end{cases}.$$

Meaning :



-e: test electric charge.

σ = energy density of confining string.

What causes confinement ?

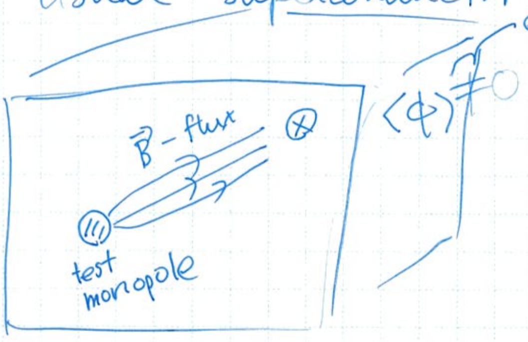
i.e, Why electric flux forms a flux tube ?

Popular scenarios

- Monopole condensation (Nambu, Mandelstam, Polyakov, 't Hooft, ---)
- Center vortex condensation ('t Hooft, Cornwal, Olesen, ---)

× Monopole condensation (dual superconductivity)

In usual superconductor



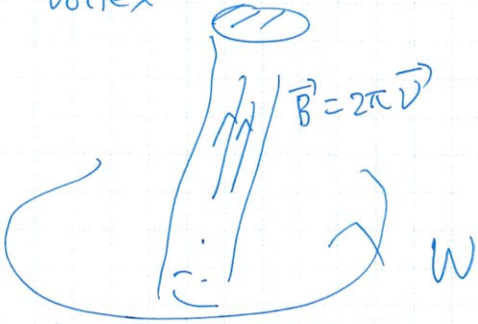
electrically charged scalar condense in vacuum ⇝ Vacuum EoM; $f_{\mu\nu} \doteq \vec{B} = 0$.

Exchange $\vec{E} \leftrightarrow (-1)\vec{B}$

magnetically charged scalar condense

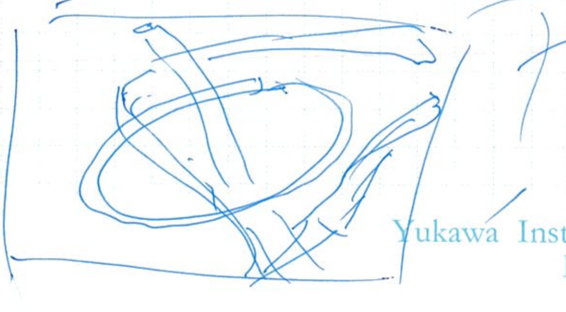
× Vortex condensation (← Analogue of domain-wall condensation of Ising model for $\mathbb{Z}_2$-sym. phase)

Center vortex = vortex soliton w/ $\vec{B} = 2\pi\vec{\nu}$



(In Abelian superconductor, this is Abrikosov vortex).

$$W(C) \sim e^{\frac{2\pi i}{N}}$$



If vortex is randomly distributed, $\langle W(C) \rangle$ has fluctuating phases ⇝ Area law of $\langle W(C) \rangle$.

Both pictures are intuitive & insightful, but ...

In non-Abelian gauge theory, people does not have a "nice" definition / characterization of neither monopoles nor center vertices

that satisfy

- gauge inv.
- locality

(In a gauge-fixed formalism, lattice numerical simulations have a good evidence for their condensation when one chooses a "suitable" gauge)

$$\boxed{\begin{array}{l}\underline{\text{Goal}}:\ \text{Characterize monopole \& vertex condensation} \\ \qquad \text{in a gauge-inv. language using } \mathbb{Z}_N^{(1)} \text{ symmetry.} \\ \quad \text{Then, study their relation.}\end{array}}$$

Idea: 4d $\overset{\mathbb{Z}_N^{(1)}\text{-sym.}}{\wedge}$ QFT w/ mass gap $\xrightarrow{\text{RG flow}}$ 4d $\overset{\mathbb{Z}_N^{(1)}\text{-sym}}{\wedge}$ topological QFT.

$$Z^{(\mathrm{QFT})}_{M_4}[\underset{\substack{\uparrow \\ \text{Background 2-form}}}{B}] \xrightarrow{\mathrm{size}(M_4) \gg \frac{1}{\text{mass gap}}} e^{-\int_{M_4} d^4x\sqrt{g}\,(\underset{\sim}{\Lambda} + \underset{\sim}{\frac{1}{2k}} R + \cdots)} \times \underbrace{Z^{(\mathrm{TQFT})}_{M_4}[B]}_{\text{genuinely low-energy phys. data.}}$$

$$\underset{\nearrow\ \ \nwarrow}{\text{non-universal constants.}}$$

Thus,

$$\frac{Z^{(\mathrm{QFT})}_{M_4}[B]}{Z^{(\mathrm{QFT})}_{M_4}[0]} \rightarrow \frac{Z^{(\mathrm{TQFT})}_{M_4}[B]}{Z^{(\mathrm{TQFT})}_{M_4}[0]}$$

$\mathbb{Z}_N^{(1)}$-twisted partition functions are providing us good probes of the low-energy realization of symmetry!

2. Gauge-inv. criteria for Monopole / Center-Vortex Condensation

Def 1 (Center-vortex condensation)

On T^4, let's introduce B : $\mathbb{Z}_N$ 2-form background gauge field s.t. $\int_{(T^2)_{3,4}} \frac{2\pi}{N} B = \frac{2\pi}{N}$ & others $= 0$.

We define weak center-vortex condensation by

$$\frac{Z_{T^4}[B]}{Z_{T^4}[0]} \xrightarrow{\text{vol.}\to\infty} {}^{\exists}\text{const.} \neq 0 \qquad (\text{weak vortex cond.})$$

— 〃 — strong — 〃 —— by

$$\frac{Z_{T^4}[B]}{Z_{T^4}[0]} \xrightarrow{\text{vol.}\to\infty} 1 .$$

//

Def. 2 (Monopole condensation)

Consider the lens $\times S^1$ space : $M_4 = \overbrace{L(qN;1)}^{\overset{S^3/\mathbb{Z}_{qN}}{\|}} \times S^1$

$\gamma \subset L(qN;1)$ generates $\pi_1(L(qN;1)) = \mathbb{Z}_{qN}$.

We take B s.t. $\int_{\gamma\times S^1} \frac{2\pi}{N} B = \frac{2\pi}{N}$,

which then satisfies $\underset{\substack{\uparrow \\ \text{Bockstein map : } H^2(M_4;\mathbb{Z}_N)\to \text{Tor}\, H^3(M_4;\mathbb{Z})}}{\beta} B \left(= \frac{1}{N} d\hat{B}\right) = q\,\delta(\gamma)$.

We define weak charge-$[q]_{Nq\mathbb{Z}}$ monopole condensation by

$$\frac{Z_{L(qN;1)\times S^1}[B]}{Z_{L(qN;1)\times S^1}[0]} \xrightarrow{\text{vol.}\to\infty} {}^{\exists}\text{const.} \neq 0$$

—— 〃 — strong —— 〃 — by

$$\frac{Z_{L(qN;1)\times S^1}[B]}{Z_{L(qN;1)\times S^1}[0]} \xrightarrow{\text{vol.}\to\infty} 1 .$$

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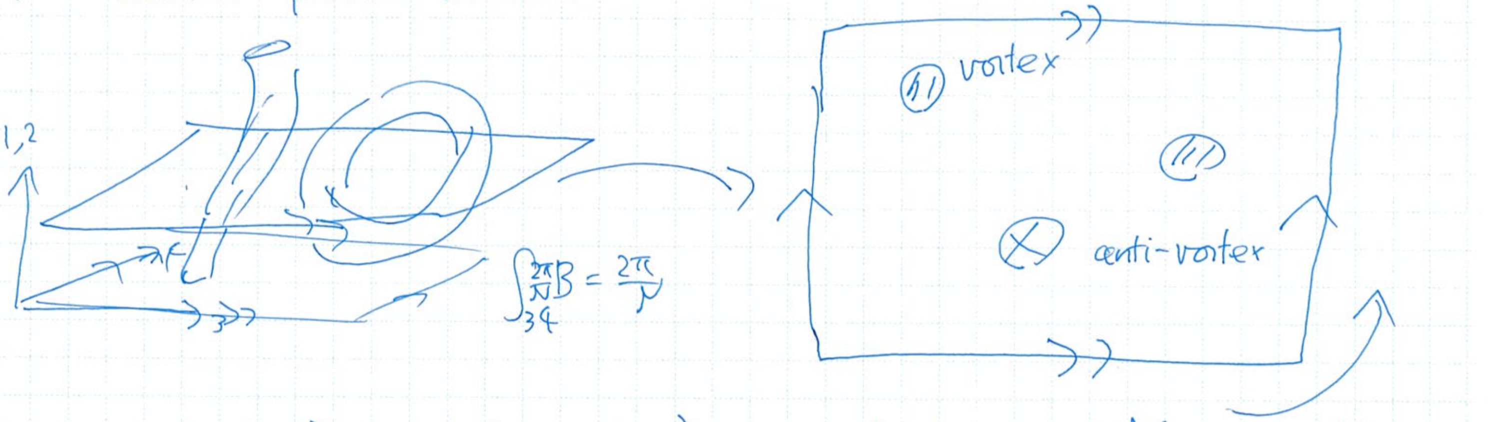
Why these definitions?

Let's compute them for the 4d SU(N) YM + multiple Adj. Higgses.

$\mathbb{Z}_N^{(1)}$ is spontaneously broken.

↳ well-defined monopoles & center vortex.

T^4 - twisted partition function



#(vortex) − #(anti-vortex) = twist mod N. $\mathbb{Z}_N^{(1)}$-twist

⟹ At least one vortex should always cross the $(T^2)_{3,4}$ surface for any x_1, x_2.

$$\Rightarrow \frac{Z_{T^4}[B]}{Z_{T^4}[0]} \sim \exp\left(-\overset{\text{vortex tension}}{T_{\text{vortex}}} \times L_1 L_2\right) \xrightarrow{\text{vol.} \to \infty} 0 .$$

The center-vortex condensation criterion is achieved if $T_{\text{vortex}} = 0$.

⟹ Vortex is proliferated in the vacuum.

$L(qN;1)\times S^1$ – twisted partition function.

Let's consider 4d $SU(N)$ + one Adj. Higgs. : $SU(N) \xrightarrow{\text{Adj}} U(1)^{N-1}$.

$$Z^{\text{Coulomb}}_{M_4}[B] \sim \int \underbrace{\mathcal{D}a}_{\substack{U(1)^{N-1} \\ \text{electric}}} \sum_{\underbrace{n}_{\substack{\text{Dirac} \\ \text{flux}}}} \underbrace{\mathcal{D}\tilde{a}}_{\substack{\text{dual} \\ \text{magnetic}}} \exp\Big[-\frac{1}{g^2}\int |da+2\pi(n+\frac{1}{N}\hat{B})|^2 + i\frac{1}{2\pi}\int d\tilde{a}\wedge(da+2\pi(n+\frac{1}{N}\underbrace{\hat{B}}_{\substack{\mathbb{Z}\text{-valued} \\ \text{lift of } B}}))\Big]$$

(modified Villain variable)

On lens space $\times S^1$,

$$d\hat{B} = N\times q\delta(\gamma)$$

$$\Rightarrow \beta B = \frac{1}{N}d\hat{B} = q\delta(\underset{\nwarrow \pi_1(L(qN;1))}{\gamma})$$

Thus, EoM of $\tilde{a}$ gives

$$dn + \underbrace{\beta B}_{\substack{\shortparallel \\ q\delta(\gamma).}} = 0.$$

$\Rightarrow$ No such n exists as $q\gamma \neq \partial(\text{2-surface})$.

To find the leading non-vanishing contribution, we must take into account monopole excitations :

charge-p monopole on worldline C $\Rightarrow$ $-M^{p}_{\text{mon}}\times \text{Length}(C) + ip\int_C \tilde{a}$.

$\Rightarrow$ EoM of $\tilde{a}$ now gives

$$dn + \underbrace{\beta B}_{q\delta(\gamma).} - \cancel{q}p\,\delta(C) = 0.$$

$\therefore$ Such n exists iff $[q\gamma - Cp] = 0$ in $H_1(L(qN;1)\times S^1;\mathbb{Z}) \simeq \mathbb{Z}_{qN}\times\mathbb{Z}$.

One may set $C \overset{\text{hom.}}{\sim} \gamma$ and $-p+q \in Nq\mathbb{Z}$.

$$\frac{Z_{L(qN;1)\times S^1}[B]}{Z_{L(qN;1)\times S^1}[0]} \sim \exp\Big(-\min_{p\in q+Nq\mathbb{Z}} M^{p}_{\text{mon}} \times \min_{C \overset{\text{hom.}}{\sim}\gamma} \text{Length}(C)\Big).$$

mon.-cond. criterion $\Rightarrow$ some mon. w/ charge $p \in q+Nq\mathbb{Z}$ is massless mon.

3. Theorem

Th'm (Hayashi, YT 2606.17708)
For 4d QFT w/ $\mathbb{Z}_N^{(1)}$-symmetry,
if it has a non-zero mass gap, we find the following relation

Unique vacuum

$\Longrightarrow$ Strong center-vortex condensation

$\Longrightarrow$ Unbroken $\mathbb{Z}_N^{(1)}$ symmetry

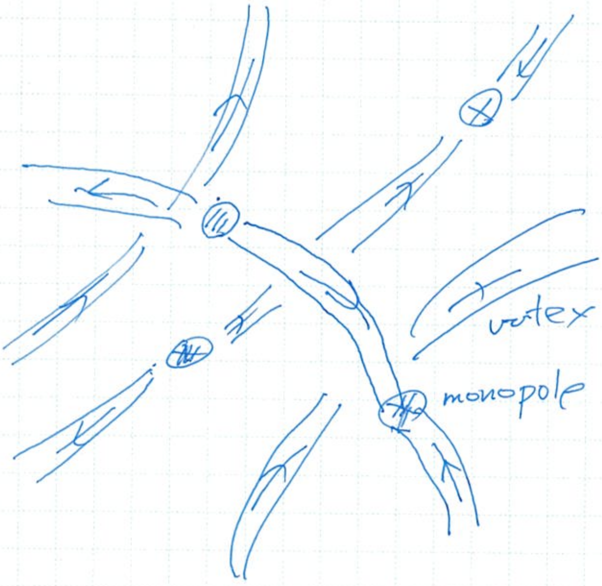
$\Longleftrightarrow$ Strong charge-$[\xi]_{N\xi\mathbb{Z}}$ monopole condensation for some $\xi > 0$.

$\Longrightarrow$ Weak center-vortex condensation

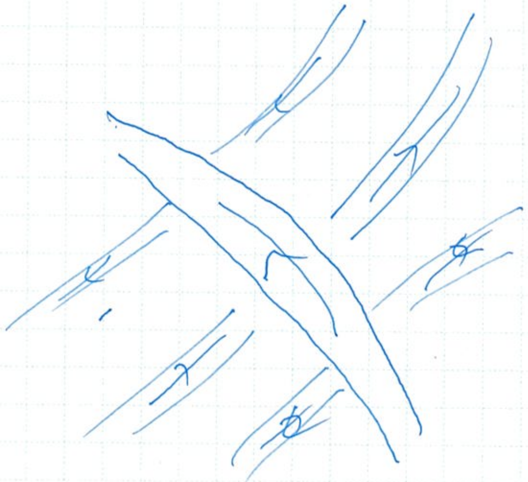
$\Longleftrightarrow$ Weak charge-$[\xi]_{N\xi\mathbb{Z}}$ monopole condensation for some $\xi > 0$ //

In particular,

$$\boxed{\text{center-vortex condensation} + \text{mass gap} \Longrightarrow \text{monopole condensation}}$$



$\Rightarrow$ gapped & confinement



$\Rightarrow$ gapless Coulomb phase.

[cf. M. Nguyen, T. Sulejmanpasic, M. Ünsal '24
J. Giansiracusa, D. Lanners, T. Sulejmanpasic '25].

Weak center-vortex condensation ⇒ Weak monopole condensation

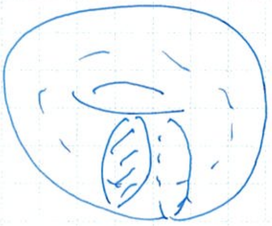
(Proof).

Locality of QFT ⇒ We can cut & glue manifolds to relate partition functions on different manifolds.

For M w/ $\partial M = N$, $Z_M[B] \in \mathcal{H}(N, B_{\partial N}\text{-twisted})$.

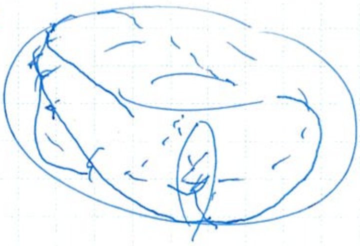
↑ In particular for TQFT, this Hilbert space is finite-dim.

Lens space $L(qN;1)$ can be constructed by gluing two solid tori $(D^2\times S^1)$ via the Dehn twist.



$$\in T^2 = \partial(D^2\times S^1)$$

$$(x_1,x_2) \xmapsto[\text{Dehn}]{} (x_1, x_2+qNx_1) \in T^2$$



$$L(qN;1) = (D^2\times S^1) \cup_{(\mathrm{Dehn})^q} (D^2\times S^1)$$

As the Dehn twist is a diffeo. $T^2 \to T^2$, it gives a unitary operator $\hat{U}_{\mathrm{Dehn}}$.

$$Z_{L(qN;1)\times S^1} = \langle D^2\times S^1 | \hat{U}_{\mathrm{Dehn}}^{q} | D^2\times S^1 \rangle .$$

Since $\exists q$ s.t. $\hat{U}^q_{\mathrm{Dehn}} \approx 1$,

monopole cond. $\Longleftrightarrow$ $|D^2\times S^1, \text{twist}\rangle \neq 0$.

$Z_{T^4}[B] \neq 0$ ensures $|D^2\times S^1, \text{twist}\rangle \neq 0$, which proves the result. //

(Converse is easy).

Strong center vortex cond. $\Longrightarrow$ Unbroken $\mathbb{Z}_N^{(1)}$ sym.

(Proof). Strong vortex condensation $\Longleftrightarrow$ No states on $\mathcal{H}^{\xi}(T^3)$ is charged under $\mathbb{Z}_N^{(1)}$ sym. generator.

If there exists a deconfined $\mathbb{Z}_N^{(1)}$-charged line, one can wrap it along a nontrivial cycle of T^3, which gives a $\mathbb{Z}_N^{(1)}$-charged state on $\mathcal{H}(T^3)$. → Contradiction //

Unbroken $\mathbb{Z}_N^{(1)}$ $\Longrightarrow$ Strong monopole condensation.
for some $\xi > 0$.

(Proof).

Strong charge-$[\xi]_{N\xi\mathbb{Z}}$ monopole condensation

$\Longleftrightarrow$ Take $\Sigma \subset L(\xi \cdot N;1) \times \{pt\} \subset L(\xi N;1) \times S^1$

s.t. $\partial \Sigma = \xi N \gamma$.

$\mathbb{Z}_N^{(1)}$ sym. generator on $\Sigma = 1$ on $\mathcal{H}(L(\xi N;1))$.

Thus, we should prove this is true for some $\xi > 0$
if $\mathbb{Z}_N^{(1)}$ is unbroken & gapped.

※ Important thm for 4d TQFT [Lan, Kong, Wen '18, Johnson-Freyd '20]

4d TQFT w/ "remote detectability"

= 4d discrete 2-group gauge theory

ε (Kapustin, Thorngren '13).

4d discrete gauge theory (+ $\mathbb{Z}_2$ 2-form gauge field)

Let's then take a canonical 4d $\overset{\text{discrete group}}{G}$-gauge theory description & a : G-gauge field.

Assuming unbroken $\mathbb{Z}_N$ symmetry,

B can couple to the G-gauge field a via magnetic coupling

$$\frac{2\pi i}{N}\int_{M_4} B\cup\eta(a) \qquad \left(\eta\in H^2(BG,\mathbb{Z}_N)\right)$$

(↑ example for $G=\mathbb{Z}_N\times\mathbb{Z}_N$ $\frac{2\pi i}{N}\int a_1\cup a_2\cup B$ is a typical example)

and

$$2\pi i\int_{M_4}\langle \Theta B, a\rangle \qquad \left(\Theta\in H^3(B\mathbb{Z}_N,\hat{G})\right)$$

(↑ example for $G=\mathbb{Z}_N$ $\frac{2\pi i}{N}\int a\cup\beta B$ is a typical example)

As we take $B=\delta(\Sigma)$ for $L(\xi N;1)\times S^1$,

we should prove these couplings become trivial for some $\xi>0$.

For this case $B=\delta(\Sigma)$ both magnetic coupling turns out to be Bockstein-type coupling.

$\rightsquigarrow$ $\xi=\gcd\left(\operatorname{lcm}_{g\in G}(\operatorname{ord}(g)), N\right)$ satisfies the condition.

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