



Useful things to know about accelerators – part I

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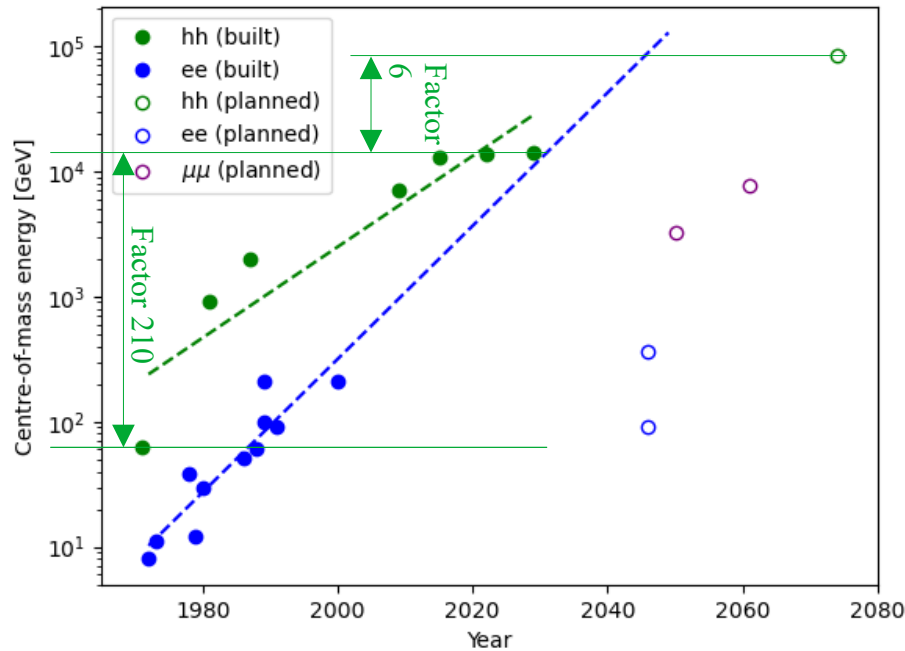
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Accelerators - A Window on Nature

- Particle accelerators provide the source for most high energy physics experiments
 - Provide high luminosity, high energy beams for colliders
 - Provide high brightness beams for secondary particle production
 - Also key technology for life sciences, engineering, chemistry
- How do they work?
 - How can we get to high energy?
 - How can we keep the beam in the accelerator?
 - How can we get to high luminosity?
- What are the main HEP facilities in the world today?
- What might HEP facilities look like in the future?

Particle Colliders



- Particle collider energy grew exponentially in 20th century
- E.g. proton collider energy grew by factor **210** in **last 50 years**
- Proton collider energy set to grow by factor **6.3** in **next 50 years**
- End of the era of exponential growth?
 - Where are the challenges
 - What are the alternatives



Accelerator Components

- Most accelerators share similar components
- Main components of an accelerator
 - Bending – dipoles
 - Focussing – quadrupoles
 - Acceleration - RF cavities
- Also
 - Vacuum
 - Diagnostics
 - Targets for secondary particle production
- First Lecture: Derive basic theory of accelerator physics
- Second Lecture: Discuss accelerator equipment and techniques

Lorentz force law

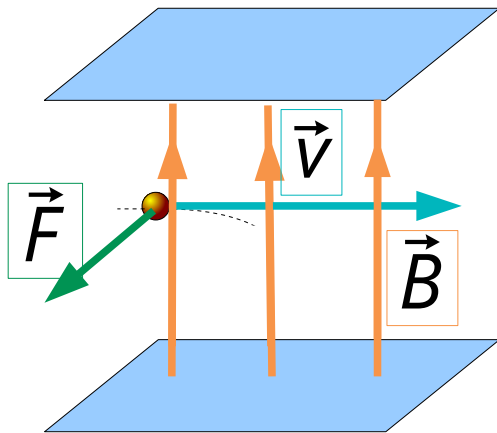
- Fundamental equation for particles moving through fields

The diagram shows the Lorentz force law equation $\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$ with five labels in boxes above it: 'Force' (green), 'Charge' (cyan), 'Velocity' (red), 'Magnetic Field' (purple), and 'Electric Field' (orange). Colored arrows point from each label to its corresponding variable in the equation: Force to \vec{F} , Charge to q , Velocity to \vec{v} , Magnetic Field to \vec{B} , and Electric Field to \vec{E} . The equation is labeled '(eq. 1)' on the right.

- **Magnetic force** is perpendicular to velocity
 - Magnetic field conserves energy
- **Electric force** is weaker by factor velocity
 - Magnets are better for bending and focussing

Magnetic Rigidity and Bending

- Simplest magnet - "dipole"
 - Uniform magnetic field perpendicular to beam direction



Lorentz force (eq. 1) + centripetal motion:

$$qvB = \frac{pv}{\rho}$$

Radius

Rearranging:

$$B\rho = \frac{p}{q}$$

Magnetic Rigidity

- Constant force \rightarrow constant curvature \rightarrow circular motion
- Magnetic rigidity parameterises momentum
- Charge-to-mass ratio important when accelerating multiple particle species

Worked example – LHC

- If we wanted to accelerate, say, 7 TeV/c particles, what bending radius is required?
- Maximum dipole field around 8.3 T

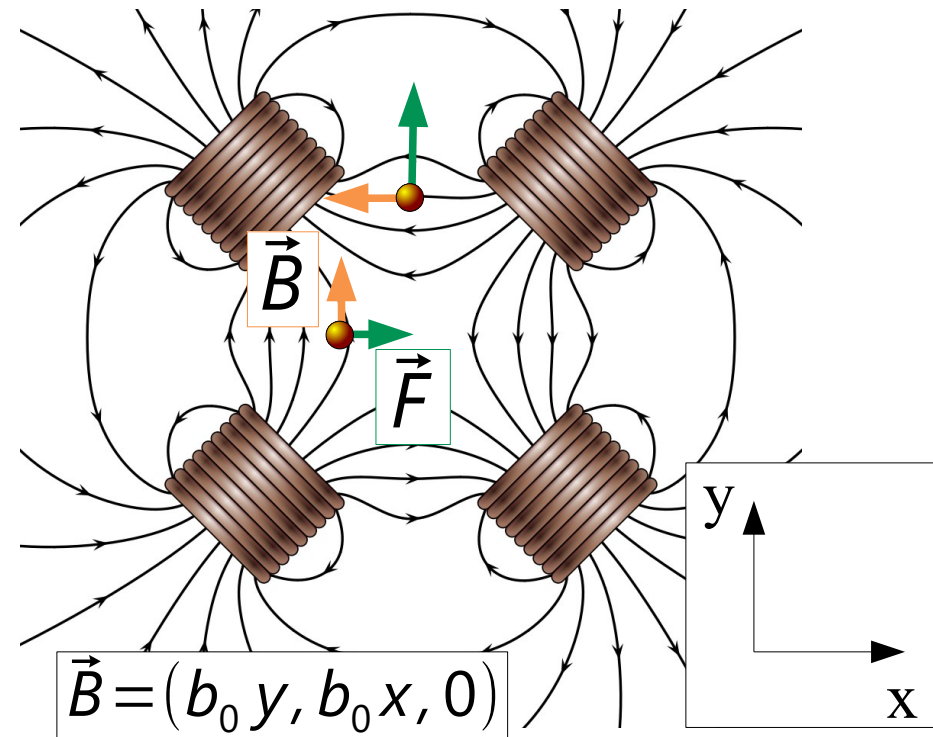
$$B \rho = \frac{p}{q}$$

$$\rho = \frac{p}{qB} = \frac{7e12}{8.3 \times 3e8} = 2800 \text{ m}$$

- Nb: LHC radius \sim 4.1 km
 - Need space for detectors, etc

Quadrupole magnets

- If we only had bending magnets, particles would soon be lost from the accelerator
- Need to keep the particles in the accelerator using focussing elements
 - Usually use quadrupoles
- Field stronger away from beam centre
 - Like a spring or pendulum
 - Simple harmonic motion
- "F" quad focuses in x and defocuses in y
- "D" quad focuses in y and defocuses in x
- Overall focussing by alternating "F" and "D"
 - Just reverse the field



Quadrupole field – horizontal (1)

- For a particle moving near to the z-axis

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$

$$\vec{B} = (b_0 y, b_0 x, 0)$$

- Considering only p_x for now

$$\frac{dp_x}{dt} = q \frac{dz}{dt} B_y$$

$$\vec{v} \times \vec{B} = \begin{pmatrix} v_z B_y \\ -v_z B_x \\ 0 \end{pmatrix}$$

- Use the chain rule

$$\frac{dp_x}{dt} = \frac{dp_x}{dz} \frac{dz}{dt}$$

- Combining these equations:

$$\frac{dp_x}{dz} = q b_0 x$$

Quadrupole field – horizontal (2)

$$\frac{dp_x}{dz} = qb_0 x \quad \text{😊}$$

- Definition of x-component of momentum

$$p_x = m \gamma v_x = m \gamma \frac{dz}{dt} \frac{dx}{dz} = p_z \frac{dx}{dz}$$

- Substitute this definition into 😊 gives

$$p_z \frac{d^2 x}{dz^2} = qb_0 x$$

- Rearrange and wrap up constant terms in focussing strength k

$$\frac{d^2 x}{dz^2} - k x = 0$$

Quadrupole field – vertical

- Lorentz force law with quadrupole field definition

$$\frac{dp_y}{dt} = -q b_0 v_z y$$

- Use chain rule and eliminate v_z

$$p_z \frac{d^2 y}{dz^2} = -q b_0 y$$

- Rearrange and wrap up constant terms in defocussing strength k

$$\frac{d^2 y}{dz^2} + k y = 0$$

Solutions

- Motion is governed by

$$\frac{d^2 x}{dz^2} - k x = 0 \qquad \frac{d^2 y}{dz^2} + k y = 0$$

- This is simple harmonic motion – solutions are of form

$$x = x_0 \cos(\sqrt{k} z) + \frac{dx_0}{dz} \frac{1}{\sqrt{k}} \sin(\sqrt{k} z)$$

- Taking derivative

$$\frac{dx}{dz} = -x_0 \sqrt{k} \sin(\sqrt{k} z) + \frac{dx_0}{dz} \cos(\sqrt{k} z)$$

For y

$$y = y_0 \cosh(\sqrt{k} z) + \frac{dy_0}{dz} \frac{1}{\sqrt{k}} \sinh(\sqrt{k} z)$$

$$\frac{dy}{dz} = y_0 \sqrt{k} \sinh(\sqrt{k} z) + \frac{dy_0}{dz} \cosh(\sqrt{k} z)$$

Transfer Matrix

- Just thinking about x , the particles move according to

$$x_1 = x_0 \cos(\sqrt{k} z) + \frac{dx_0}{dz} \sin(\sqrt{k} z)$$

$$\frac{dx_1}{dz} = -x_0 \sqrt{k} \sin(\sqrt{k} z) + \frac{dx_0}{dz} \sqrt{k} \cos(\sqrt{k} z)$$

- We can rewrite this as a matrix

$$\begin{pmatrix} x_1 \\ \frac{dx_1}{dz} \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k} z) & \frac{1}{\sqrt{k}} \sin(\sqrt{k} z) \\ -\sqrt{k} \sin(\sqrt{k} z) & \cos(\sqrt{k} z) \end{pmatrix} \begin{pmatrix} x_0 \\ \frac{dx_0}{dz} \end{pmatrix}$$

- This matrix is known as the quadrupole's **transfer matrix**

$$\underline{u}_1 = \mathbf{M}_{01} \underline{u}_0$$



Questions



Questions

- Exercise – what is the transfer matrix for a drift space, that is a region with no fields at all?
 - What is the force acting on the particle?
 - What is $x(z)$ in terms of dx_0/dz and x_0
 - What is dx/dz in terms of dx_0/dz
 - Now write that as a matrix

Questions

- Exercise – what is the transfer matrix for a drift space?

- What is the force acting on the particle?

- No force

- What is $x(z)$ in terms of dx_0/dz and x_0

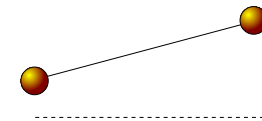
$$x = x_0 + \frac{dx_0}{dz} z$$

- What is dx/dz in terms of dx_0/dz

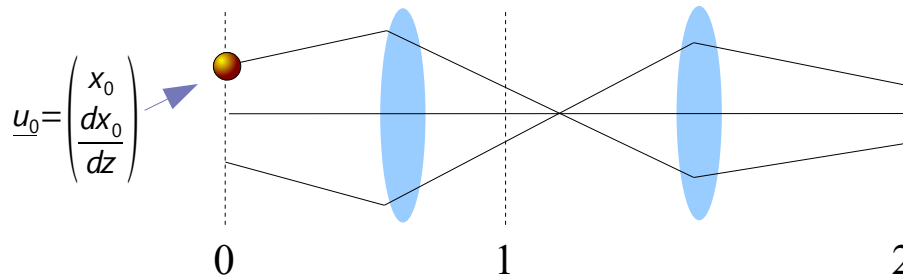
$$\frac{dx}{dz} = \frac{dx_0}{dz}$$

- Now write that as a matrix

$$\begin{pmatrix} x \\ \frac{dx}{dz} \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \frac{dx_0}{dz} \end{pmatrix}$$



Transfer Lines



- Transfer matrix defines transport through a region
- Transfer matrices can be combined by multiplication
- Say we have transfer matrices like:

$$\underline{u}_1 = \mathbf{M}_{01} \underline{u}_0$$

$$\underline{u}_2 = \mathbf{M}_{12} \underline{u}_1$$

- Then

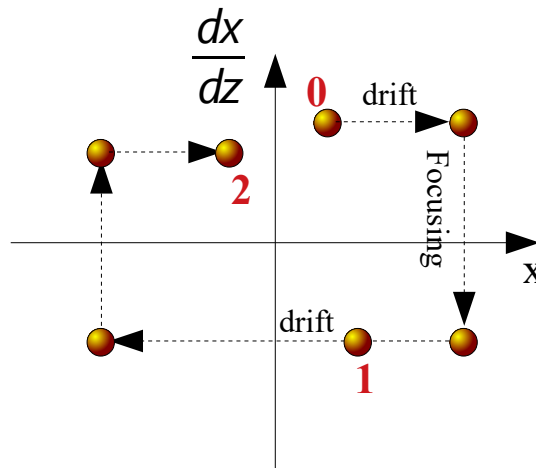
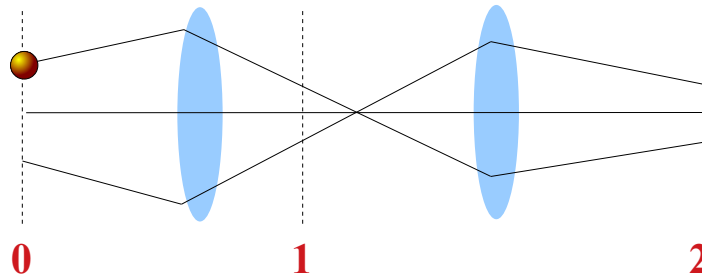
$$\underline{u}_2 = \mathbf{M}_{12} \mathbf{M}_{01} \underline{u}_0$$

- i.e. we can define a combined transfer matrix like

$$\mathbf{M}_{02} = \mathbf{M}_{12} \mathbf{M}_{01}$$

Phase space

- Another instructive way to look at beam optics is by considering the phase space



$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$

- There is a general rule for what transfer matrices are allowed by equations of motion
 - "Symplectic condition"
- Formally a matrix \mathbf{M} is *symplectic* if it satisfies

$$\mathbf{M}^{\text{T}} \mathbf{S} \mathbf{M} = \mathbf{I}$$

transpose Identity matrix

- Where

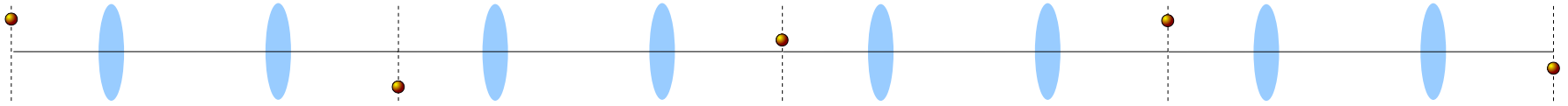
$$\mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- It can be shown that any *symplectic* matrix \mathbf{M} can be written as

$$\mathbf{M} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu$$

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \gamma\beta - \alpha^2 = 1 \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Periodic Lattices



- Following n identical cells or turns in a ring with one-turn matrix \mathbf{M}

$$\underline{u}_n = \mathbf{M}^n \underline{u}_0$$

- Rewrite

$$\mathbf{M} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu$$

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \gamma\beta - \alpha^2 = 1 \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

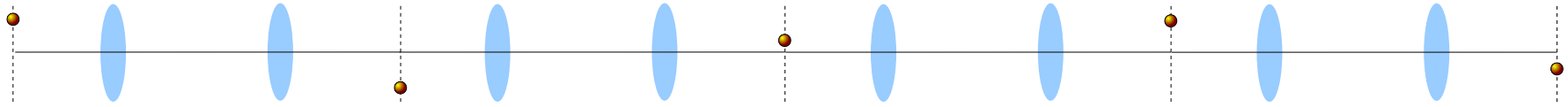
- So

$$\mathbf{J}^2 = -\mathbf{I}$$

- And

$$\mathbf{M}^n = \mathbf{I} \cos(n\mu) + \mathbf{J} \sin(n\mu)$$

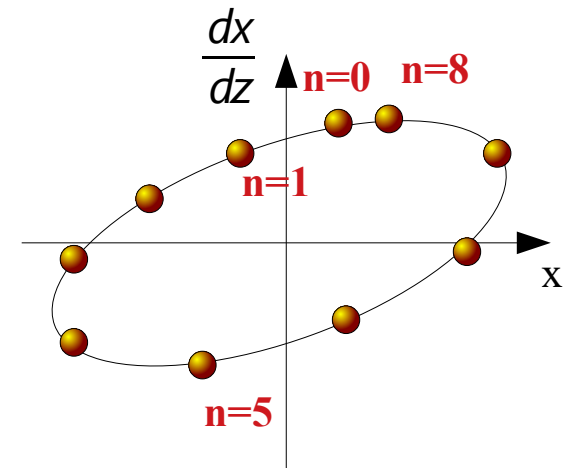
Periodic Lattices



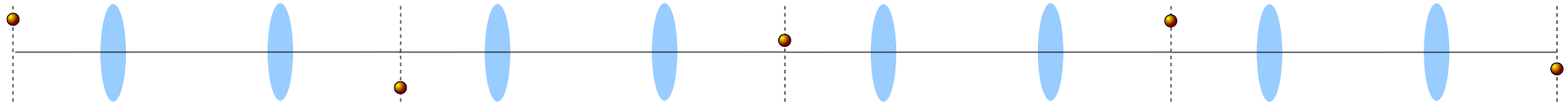
- What does this mean?

$$\mathbf{M}^n = \mathbf{I} \cos(n\mu) + \mathbf{J} \sin(n\mu)$$

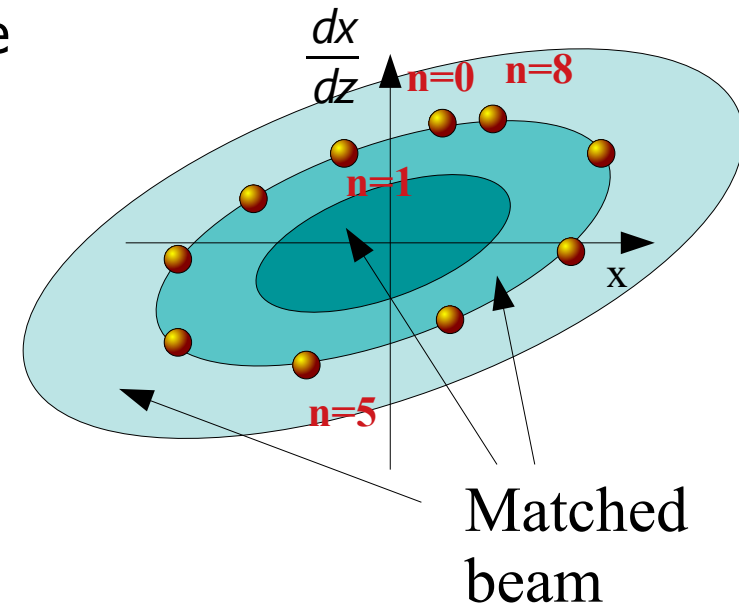
- Particles move around an ellipse in phase space if $\text{Trace}(\mathbf{M}) < 2$
- μ is the “phase advance”
 - Sometimes use “tune” ... $2\pi \nu = \mu$
- α , β and γ are “Twiss parameters”
 - Tell us the alignment of the ellipse
- Each particle sits on ellipse area ε - the particle’s amplitude



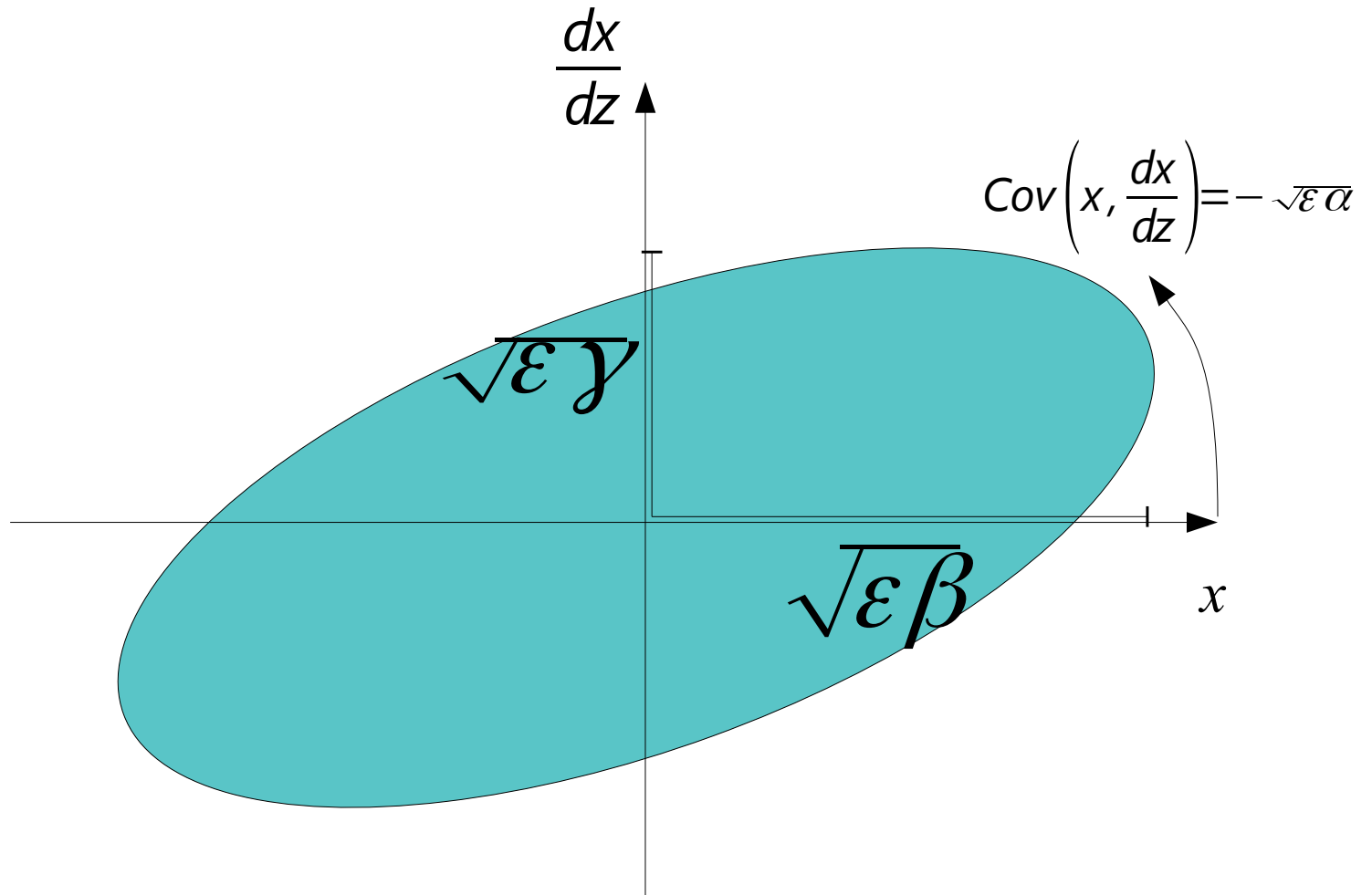
Periodic Lattices and beams



- Beam is composed of many particles
 - Particles occupy a region in phase space
- “Emittance” is area occupied by the entire beam
- Sometimes classify “RMS emittance”
 - Area occupied by ellipse 1 RMS distance from beam centre
- Low emittance is crucial for
 - High luminosity
 - Low losses



Beam ellipse





Questions



Questions

- What is behaviour of particles in phase space if
 - $\text{Trace}(M) < 2$
 - $\text{Trace}(M) = 2$
 - $\text{Trace}(M) > 2$



Questions

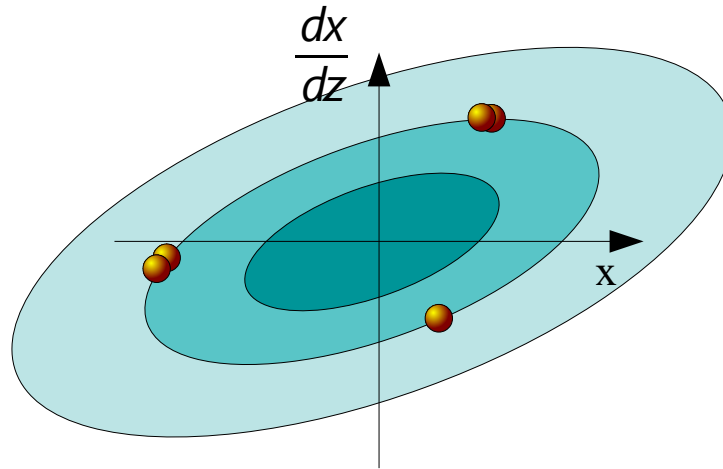
- What is behaviour of particles in phase space if
 - $\text{Trace}(M) < 2$
 - Motion is an ellipse
 - $\text{Trace}(M) = 2$
 - $x \rightarrow +/- x$
 - $\text{Trace}(M) > 2$
 - Motion is a hyperbola



Emittance Growth

- Ideally emittance is conserved, but this is not always the case
- Long list of effects that can cause emittance growth
 - Beam mismatch
 - Scattering off residual gas
 - Scattering off particles in the same beam
 - Scattering off particles in other beams (e.g. in collider)
 - Space charge
 - Resonances

Resonances



- Reminder:-
 - Tune ν is number of SHM oscillations per turn
 - Phase advance $\mu=2\pi\nu$ is "angle" advanced per turn
- The beam does not behave well when

$$\nu_x = l + \frac{m}{n}$$

integer

- Beam passes through the same field region every n^{th} turn
 - Imperfections in the field get amplified
- Resonance

Resonances

- Can see poor performance for $\nu_x = 4 + \frac{1}{3} = 4.33$
- Only a very small area in phase space is transmitted
- In fact, a 2D phenomenon in (ν_x, ν_y)

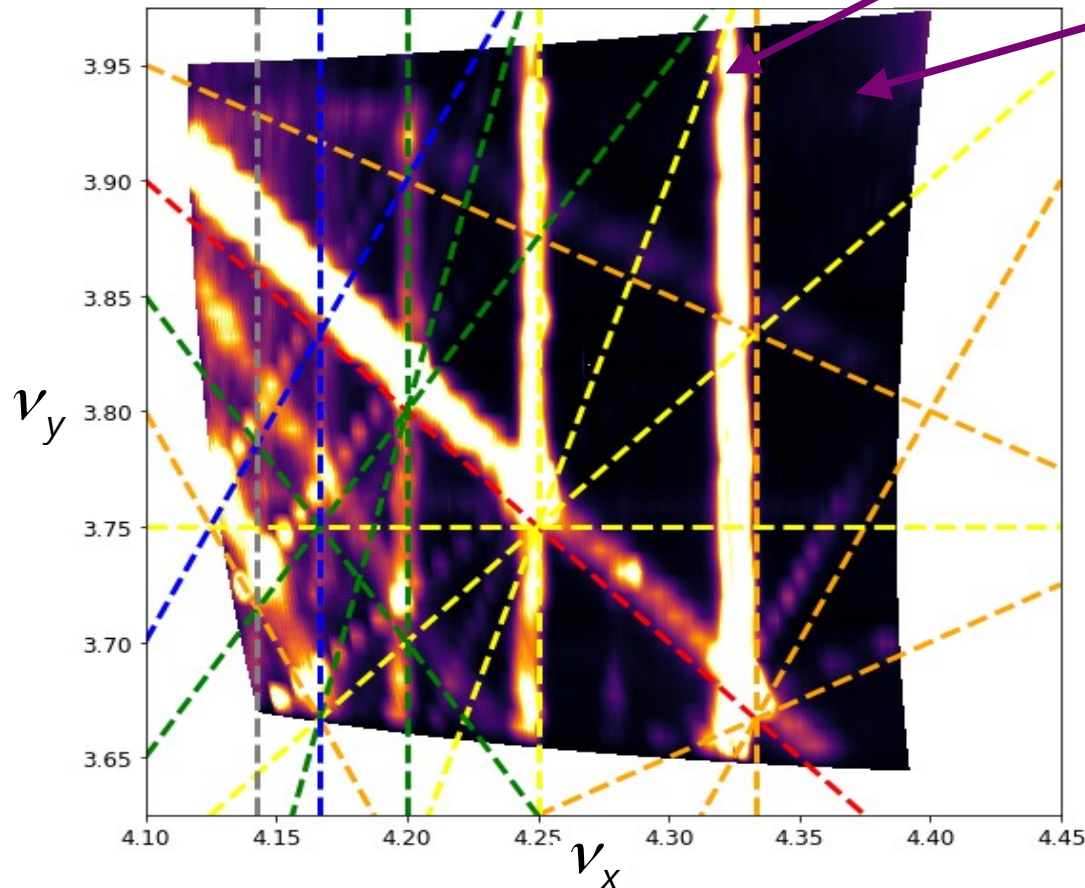
Bad transmission

(hot)

Good transmission

(cold)

ISIS Synchrotron (measured)

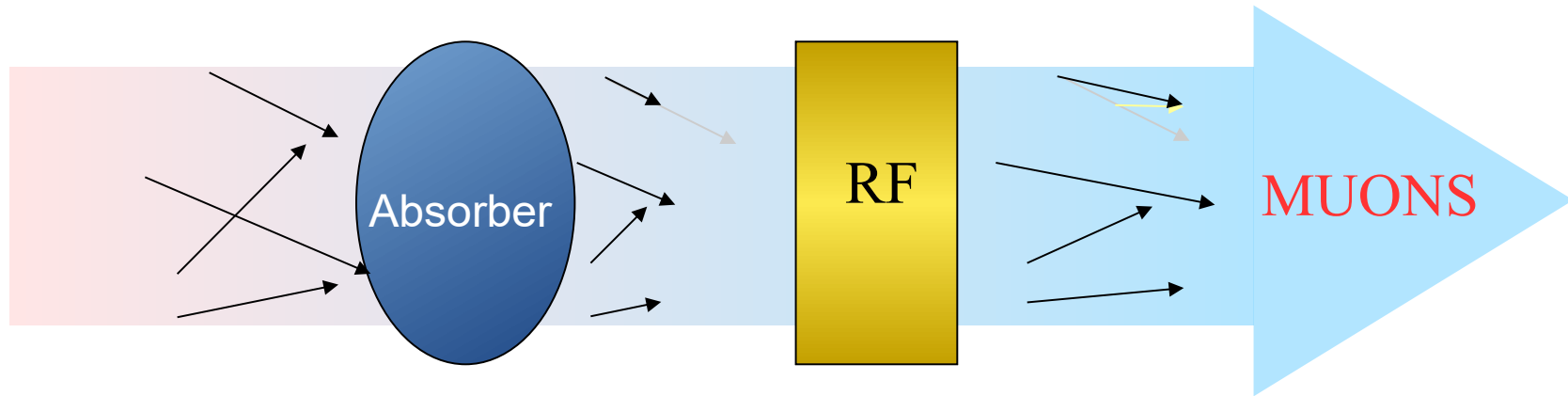




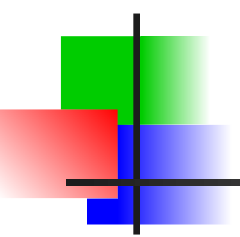
Emittance Reduction (Cooling)

- Several techniques to reduce emittance
 - Synchrotron radiation cooling
 - Stochastic cooling
 - Laser cooling
 - Electron cooling
 - Ionisation cooling
- Fundamental principle is to remove “heat” from the beam using a neighbouring heat sink
 - Comoving electron beam → electron cooling
 - Comoving laser → laser cooling
 - Emission of synchrotron radiation
 - Photon emission caused by (principally) electrons bending in magnetic field

E.g. Ionisation Cooling



- Beam loses energy in absorbing material
 - Absorber removes momentum in all directions
 - RF cavity replaces momentum only in longitudinal direction
 - End up with beam that is more straight
- Multiple Coulomb scattering from nucleus ruins the effect
 - Mitigate with tight focussing
 - Mitigate with low-Z materials
 - Equilibrium emittance where MCS completely cancels the cooling



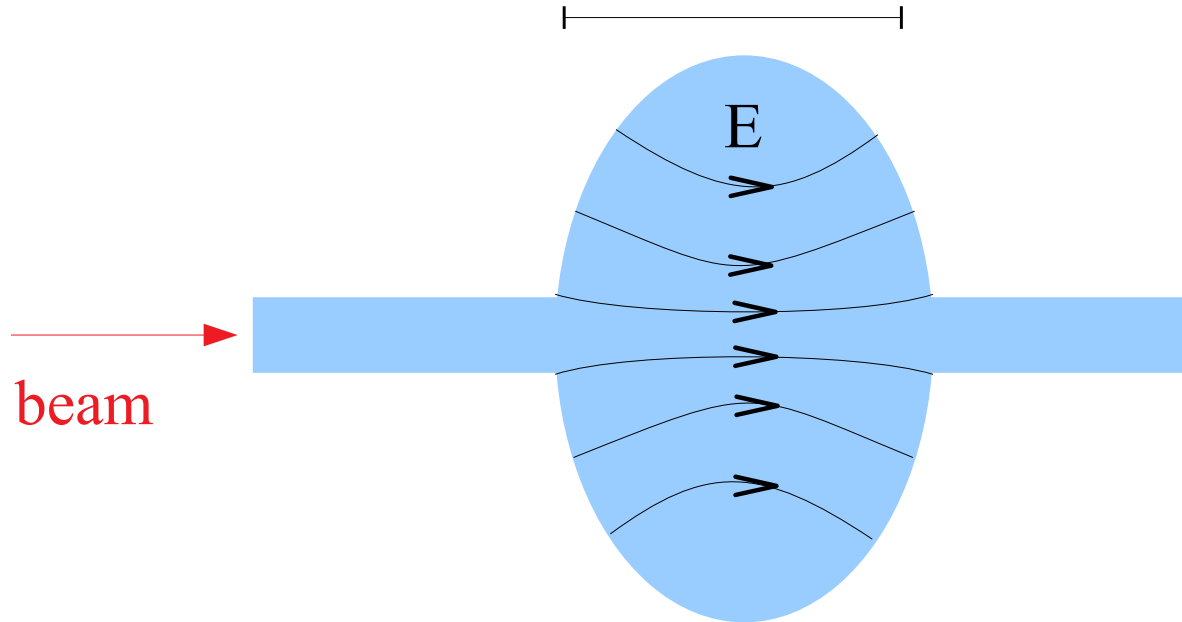
Longitudinal Dynamics and Acceleration



Longitudinal Dynamics

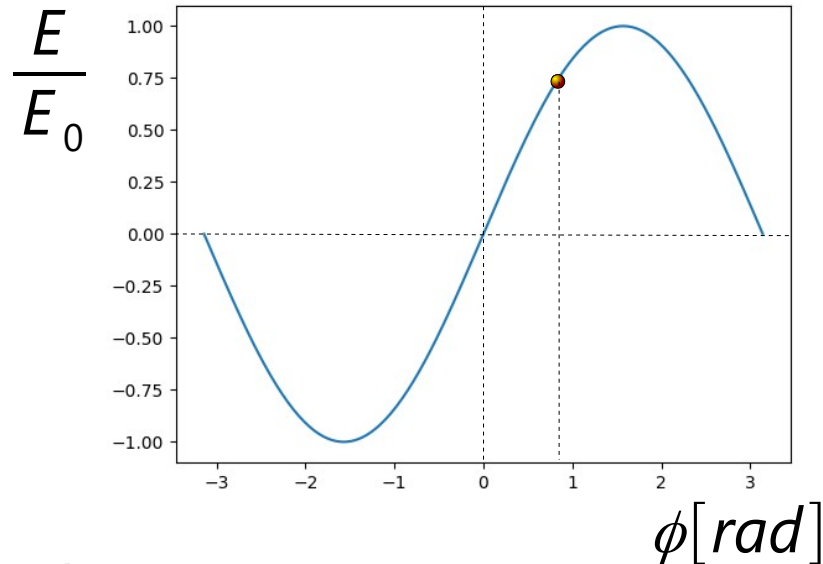
- So much for transverse motion (i.e. x and y planes)
- What about energy and acceleration?
- Electrostatic acceleration limited by breakdown potential
 - Change in energy is given by voltage differential
 - High voltage differentials cause breakdown (sparks)
 - Practically limits electrostatic acceleration to few MeV
- To accelerate beyond MeV require oscillating electric field
- RF Cavities

RF cavity field



- RF cavity holds a resonating EM wave
- Recall Lorentz force law
$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$
- Force is in direction of motion - energy changes!

RF cavity field



- In RF cavity

$$\vec{E} = E_0 \sin(\omega t + \phi) \hat{z}$$

- Energy change of particle crossing at ϕ

$$\delta W = \int F dz = q g E_0 \sin(\phi)$$

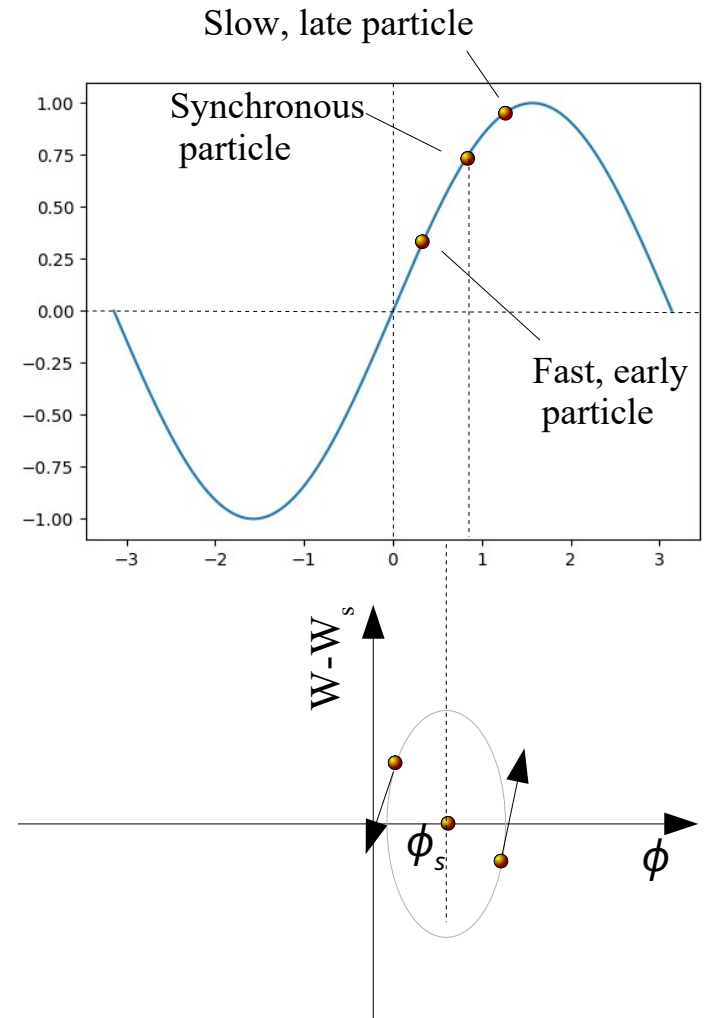
- g is the gap length
- Assumes gap is short so electric field doesn't change much
 - For longer gaps, can introduce an effective gap length $g T$
 - T is the "transit time factor" → reduces the effective gap length

Phase stability

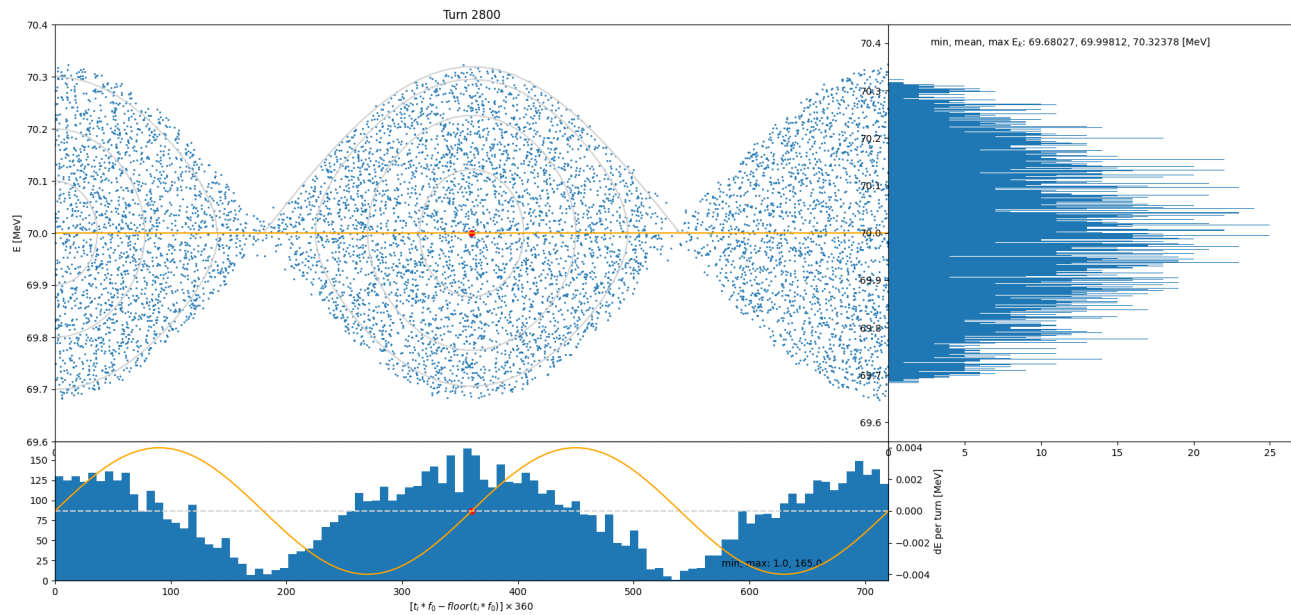
- Phase cavities so that a "synchronous" particle always crosses at phase ϕ_s
- Particle crossing at phase ϕ relative to synchronous particle

$$\delta W = qgT E_0 \sin(\phi + \phi_s)$$

- Particle arriving early
 - Fast
 - t negative
 - Gets smaller energy kick
 - Ends up relatively slower
- Particle arriving late
 - Slow
 - t positive
 - Gets bigger energy kick
 - Ends up relatively faster
- Phase stability!



RF Bucket



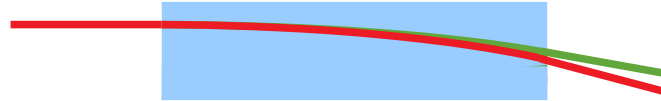
- RF Bucket – region where phase is stable
- Harmonic number – number of RF oscillations per ring turn



Dealing with momentum spread

- Momentum spread introduces a few effects
 - Dispersion
 - Chromaticity
 - Momentum compaction
- Dispersion:
 - Off-momentum particles follow a different trajectory
- Momentum compaction (rings):
 - Different path length yields different time of flight
- Chromaticity:
 - Off-momentum particles get a different focussing strength

Dispersion



- Recall the definition of magnetic rigidity

$$B \rho = \frac{p}{q}$$

- Particles having different momentum (p) get different radius of curvature

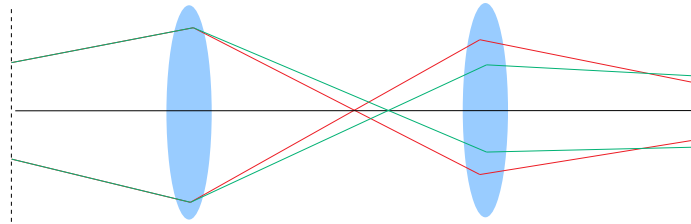
- Introduce dispersion D

$$D = p \frac{dx}{dp}$$

- Which is another optical function that we must make periodic

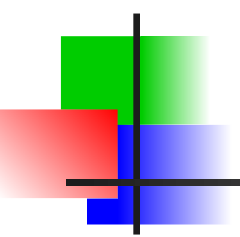
Chromaticity

- Chromaticity arises because quadrupoles focus differently for different momenta



$$k = q \frac{b_0}{p}$$

- This often limits the degree of focussing at a collision point
 - Limits luminosity
- Can deliberately enhance/reduce chromaticity by
 - Introduce a dispersion
 - Using a magnet with variable focussing strength across the aperture - "sextupole"



Questions

Review

- Dipoles are used to bend a beam – rigidity is $B \rho = \frac{p}{q}$
- Quadrupoles are used to focus a beam: $k = q \frac{b_0}{p}$
- Beam in each of x and y can be characterised by 3 Twiss parameters and an emittance
- Lattices can be characterised by a phase advance
- RF cavities are used to accelerate the beam
- Introducing momentum spread, one can also define a dispersion (and its derivative with respect to z)

Finally... luminosity

- Luminosity defines the number of interactions in a collider per unit time for a given cross section
- Luminosity will increase if
 - Beam is narrower
 - Current is higher

The diagram illustrates the luminosity formula $\tilde{L} = \frac{N_1 N_2 f N_b}{4 \pi \sigma_x \sigma_y}$. Each variable in the formula is enclosed in a colored box, and a corresponding text box with a matching border is connected to it by a line. The boxes are: a green box for N_1 and N_2 labeled 'Number of particles in each bunch'; a blue box for f labeled 'Revolution frequency'; a red box for N_b labeled 'Number of bunches'; and an orange box for σ_x and σ_y labeled 'Width of Each bunch'.

$$\tilde{L} = \frac{N_1 N_2 f N_b}{4 \pi \sigma_x \sigma_y}$$



What dictates luminosity?

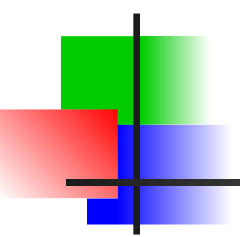
$$\tilde{L} = \frac{N_1 N_2 f N_b}{4 \pi \sigma_x \sigma_y}$$

- Typically
 - Number of particles → space charge
 - Revolution frequency → ring circumference
 - Number of bunches → RF frequency
 - Beam width → $\sqrt{\varepsilon\beta}$
 - Emittance (cooling?)
 - Twiss beta (final focus and chromaticity)



Next lecture...

- Accelerator equipment
- Types of accelerator
- Current facilities
- Future facilities



Backup

Transverse Space Charge 1

- Consider a circular beam of radius a having uniform density

$$\rho(r) = q \frac{I}{\beta_{rel} c \pi a^2} \quad r < a$$

- Quote field around a cylinder of charge/current

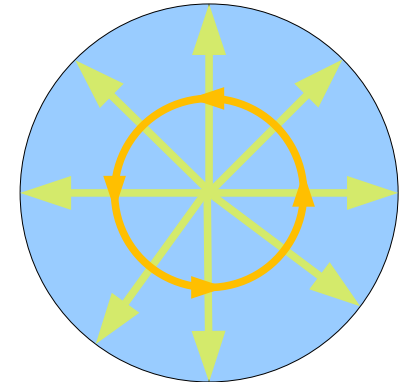
$$E(r) = \frac{1}{2\pi\epsilon_0} \frac{I}{\beta_{rel} c} \frac{r}{a^2}$$

$$B_\phi(r) = \frac{1}{2\pi\epsilon_0} \frac{I}{c^2} \frac{r}{a^2}$$

- Apply Lorentz force law

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$

$$F_r = q\vec{v} \times \vec{B} + q\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{r}{a^2} \left(\frac{I}{\beta_{rel} c} - \frac{I}{\beta_{rel} c} \beta_{rel}^2 \right) = \frac{1}{2\pi\epsilon_0} \frac{r}{a^2} \left(\frac{I}{\gamma^2 \beta_{rel} c} \right)$$



Transverse Space Charge 2

- Force is defocusing

$$\frac{d^2 x}{dz^2} - (k - K_{sc})x = 0 \quad \text{with} \quad K_{sc} = \frac{1}{2\pi\epsilon_0} \frac{1}{a^2} \left(\frac{I}{\gamma_{rel}^3 \beta_{rel}^2 c} \right)$$

- Treat SC as a perturbation

$$\mathbf{M}_p = \mathbf{M} \mathbf{M}_{sc}$$

$$\mathbf{M} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu$$

$$\mathbf{M}_{sc} = \begin{pmatrix} 1 & 0 \\ -K_{sc} & 1 \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

- Change of beam size (β)

- Change of phase advance**

- Drive the beam onto resonances \rightarrow ruin the acceptance

- Phase advance \rightarrow look at Trace of \mathbf{M}_p

$$\text{Tr}(\mathbf{M}_p) = 2 \cos(\mu) + \alpha \sin(\mu) - \alpha \sin(\mu) + \beta K \sin(\mu)$$

Transverse Space Charge 3

- Consider just the $trace(\mathbf{M}_p)$

$$Tr(\mathbf{M}_p) = 2 \cos(\mu) + \beta K \sin(\mu)$$

- Consider compound angle formula

$$\cos(\mu + \delta\mu) = \cos(\mu) \cos(\delta\mu) + \sin(\mu) \sin(\delta\mu)$$

$$\cos(\mu + \delta\mu) \simeq \cos(\mu) + \sin(\mu) \sin(\delta\mu)$$

- Looking at the tune

$$\delta\nu = \frac{\delta\mu}{2\pi} = \frac{\beta K}{4\pi}$$

$$K_{sc} = \frac{1}{2\pi\epsilon_0} \frac{1}{a^2} \left(\frac{I}{\gamma_{rel}^3 \beta_{rel}^2 c} \right)$$

$$\delta\nu = \frac{r_0 N}{2\pi\epsilon\beta_{rel}^2 \gamma_{rel}^3}$$

$$\sigma(x) = \sqrt{\beta\epsilon}$$

Transverse Space Charge 3

