Perturbative and colorful lectures on **Strong Interactions**

lecture 2/3

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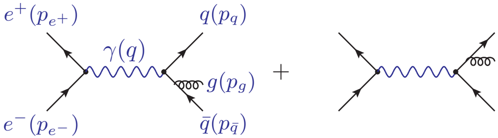
1. A few words about QCD
   1. running of α_S
   2. asymptotic freedom and confinement
2. quarks and gluons from e^+e^- annihilation
   1. quark electric charges
   2. quark and gluon spins
   3. QCD gauge parameters
3. Structure of hadrons
   1. elastic scattering
   2. deep inelastic scattering
   3. proton structure functions
   4. DGLAP evolution equation
4. hadron - hadron interactions
   1. jet production
   2. Drell-Yan process

First pQCD correction

- up to now, what we saw was driven by EW ME + hadronisation
- additional jets are due to pQCD effects: gluon radiation from the quark lines



$e^+e^- \to q\bar{q}$: $\overline{|\mathcal{M}|}^2 = 8(4\pi)^2 \alpha^2 N_c\, Q_i^2 \dfrac{(p_{e^+}\cdot p_q)(p_{e^-}\cdot p_{\bar{q}}) + (p_{e^+}\cdot p_{\bar{q}})(p_{e^-}\cdot p_q)}{q^4}$

$e^+e^- \to q\bar{q}g$: $\overline{|\mathcal{M}|}^2 = 8(4\pi)^3 \alpha^2 \alpha_S\, C_F N_c\, Q_i^2 \dfrac{(p_{e^+}\cdot p_q)(p_{e^-}\cdot p_{\bar{q}}) + (p_{e^+}\cdot p_{\bar{q}})(p_{e^-}\cdot p_q)}{(p_{e^+}\cdot p_{e^-})(p_q\cdot p_g)(p_{\bar{q}}\cdot p_g)}$

1) couplings: $\alpha^2 \to \alpha^2\alpha_S$ - first QCD correction

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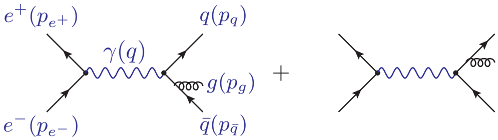

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2) factors: $N_c \to C_F N_c$

C_F is the color combinatory for one gluon radiation from a quark with a given color, $C_F = 4/3$ (coming from the λ^a)

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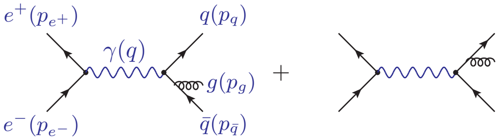

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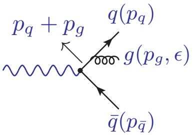
3) kinematics:

- note that: $q^4 = 4(p_{e^+}\cdot p_{e^-})(p_q\cdot p_{\bar{q}})$
- so, the kin effect is: $(p_q\cdot p_g) \to (p_q\cdot p_g)(p_{\bar{q}}\cdot p_g)$ - where does it come from ?

- adding a quark propagator in the ME calculation
- neglecting masses and in soft gluon approximation:



$$\sim \frac{(p_q + p_g) + m_q}{(p_q + p_g)^2 - m_q^2} \not{\epsilon} \overset{m_q=0}{\simeq} \frac{p_q + p_g}{2(p_q \cdot p_g)} \not{\epsilon} \overset{|p_g| \ll |p_q|}{\simeq} \frac{1}{2} \frac{\epsilon \cdot p_q}{p_q \cdot p_g}$$

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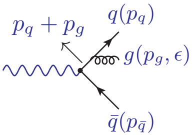

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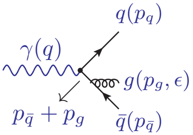

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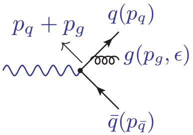

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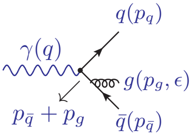

$$\sim \frac{1}{2} \frac{\epsilon \cdot p_{\bar{q}}}{p_{\bar{q}} \cdot p_g}$$

- putting them together and after sum over the gluon polarisation states (only 2 tranverse states):

$$\mathrm{d}\sigma_{q\bar{q}g} = \frac{\alpha_S}{2\pi} C_F \left[\frac{p_{\bar{q}}}{p_{\bar{q}} \cdot p_g} - \frac{p_q}{p_q \cdot p_g} \right]^2 \frac{\mathrm{d}^3 p_g}{(2\pi)^3 p_g^0} \mathrm{d}\sigma_{qq}$$

- note : the sign difference comes from the orientation of $\vec{\epsilon}$

$$\cdots \left[\frac{p_{\bar{q}}}{p_{\bar{q}} \cdot p_g} - \frac{p_q}{p_q \cdot p_g} \right]^2 \mathrm{d}\sigma_{qq} \simeq \cdots 2 \frac{p_{\bar{q}} \cdot p_q}{(p_{\bar{q}} \cdot p_g)(p_q \cdot p_g)} \mathrm{d}\sigma_{qq}$$

- with the term from $\mathrm{d}\sigma_{qq}$:

$$1/q^4$$

$$2 \frac{p_{\bar{q}} \cdot p_q}{(p_{\bar{q}} \cdot p_g)(p_q \cdot p_g)} \frac{1}{4(p_{e^+} \cdot p_{e^-})(p_q \cdot p_{\bar{q}})} = \frac{1}{2(p_{e^+} \cdot p_{e^-})(p_{\bar{q}} \cdot p_g)(p_q \cdot p_g)}$$

$\Rightarrow$ we find back:

$e^+ e^- \to q\bar{q}g$: $$\overline{|\mathcal{M}|}^2 = 8(4\pi)^3 \alpha^2 \alpha_S C_F N_c Q_i^2 \frac{(p_{e^+} \cdot p_q)(p_{e^-} \cdot p_{\bar{q}}) + (p_{e^+} \cdot p_{\bar{q}})(p_{e^-} \cdot p_q)}{(p_{e^+} \cdot p_{e^-})(p_q \cdot p_g)(p_{\bar{q}} \cdot p_g)}$$

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⚠ it presents singularities

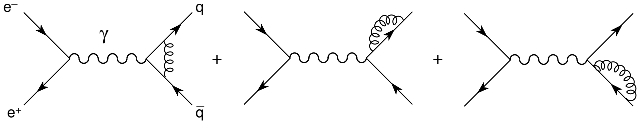
if $p_q \cdot p_g \to 0$ and/or $p_{\bar{q}} \cdot p_g \to 0$

two singularities

$$p_q \cdot p_g = E_q E_g - \vec{p}_q \cdot \vec{p}_g = E_q E_g - |\vec{p}_q||\vec{p}_g| \cos\theta_{qg}$$
$$\simeq E_q E_g (1 - \cos\theta_{qg})$$

$\Rightarrow$ singularities for $p_q \cdot p_g \to 0$ (same for $\bar{q}g$):
- $E_g \to 0$ (soft gluon limit)
- $\theta_{qg} \to 0$ (collinear limit)

- they correspond to a double pole (when both limits occur at the same time) and a single pole.
- these IR poles are exactly cancelled by the virtual corrections



comparison to data

$$d\sigma_{q\bar{q}g} = \frac{\alpha_S}{2\pi} C_F \left[\frac{p_{\bar{q}}}{p_{\bar{q}} \cdot p_g} - \frac{p_q}{p_q \cdot p_g} \right]^2 \frac{d^3 p_g}{(2\pi)^3 p_g^0} \, d\sigma_{qq}$$

- the cross section has 2 physical degrees of freedom :

  $+9$ free variables $(\mathrm{d^3p_q}, \mathrm{d^3p_{\bar{q}}}, \mathrm{d^3p_g})$
  -4 relations ($E, \vec{P}$ conservation)
  -3 independent (non relevant here - unpolaratised case) Euler angles
  $= 2$

- we choose : $x_q = 2E_q/\sqrt{s}$ $\quad x_{\bar{q}} = 2E_{\bar{q}}/\sqrt{s}$ $\quad x_g = 2E_g/\sqrt{s}$

  related by $x_q + x_{\bar{q}} + x_g = 2$ and $1 - x_i = \frac{1}{2} x_j x_k (1 - \cos\theta_{jk})$

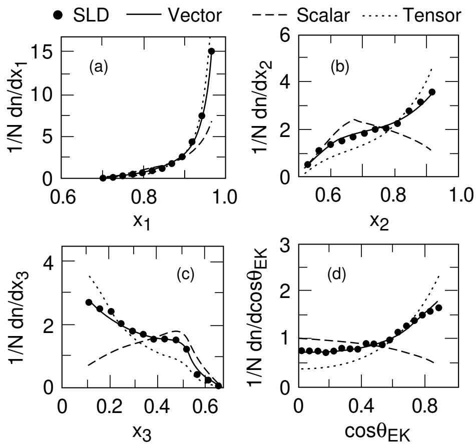
$$\frac{d^2\sigma_{q\bar{q}g}}{dx_q dx_{\bar{q}}} = \frac{\alpha_s}{2\pi} C_F \frac{x_q^2 + x_{\bar{q}}^2}{(1 - x_q)(1 - x_{\bar{q}})} \sigma_{q\bar{q}}$$

- 2 sigularities correspond to $x_q \to 1$ and $x_{\bar{q}} \to 1$

- in data we don't know which is q jet, ...
- sort the 3 jets by decreasing energy :
    - $E_1 > E_2 > E_3$
    - define: $x_i \;=\; 2E_i/\sqrt{s}$

- jet 1: $q/\bar{q}$ jet almost unaffected ($x_1 \to 1$ pole)
- jet 2: $q/\bar{q}$ jet with significant energy loss + g jet such that $E_{q/\bar{q}} > E_g > E_{q/\bar{q}}$
- jet 3: mainy the gluon jet (falling distribution) + $q/\bar{q}$ jet of the above case

- $\cos\theta_{EK} = \frac{\sin\theta_2 - \sin\theta_3}{\sin\theta_1}$



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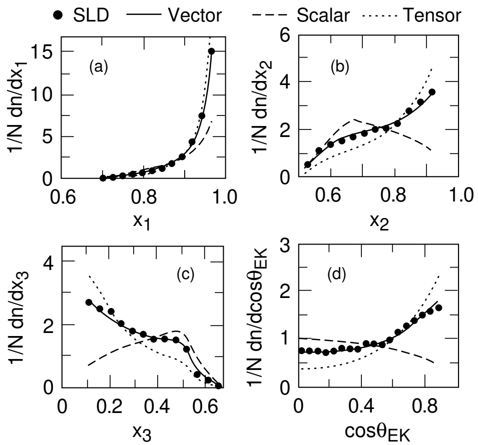

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gluons are spin 1 !

pure vectorial current (γ^μ)

collinear approximation

$$\frac{d^2\sigma_{q\bar{q}g}}{dx_q dx_{\bar{q}}} = \frac{\alpha_s}{2\pi} C_F \frac{x_q^2 + x_{\bar{q}}^2}{(1-x_q)(1-x_{\bar{q}})} \sigma_{q\bar{q}}$$

- let's express the cross section as a function of $z = x_g$ and θ_{qg}
- in the limit of small angles θ_{qg} (such as $1 - \cos\theta \simeq \theta^2/2$), on finds:

$$\mathrm{d}^2\sigma_{q\bar{q}g} = \sigma_{q\bar{q}}\, C_F \frac{\alpha_s}{2\pi} \left[1 + (1-z)^2\right] \frac{\mathrm{d}z}{z} \frac{\mathrm{d}\theta^2}{\theta^2}$$

- the singularity $\theta \to 0$ is avoided by the non zero quark masses
- dead cone effect: radiations are supressed in a cone $\theta < \theta_{min}$
- we integrate over the angle, and keep only the leading term, corresponding to $\theta_{min} = m_q/E_e$ we get the a famous Leading Log approximation (LL):

$$\mathrm{d}\sigma_{q\bar{q}g} = \sigma_{q\bar{q}}\, C_F \frac{\alpha_s}{2\pi} \left[1 + (1-z)^2\right] \ln \frac{E_e^2}{m_q^2} \frac{\mathrm{d}z}{z}$$

- this approximation is used in many Monte Carlo simulation for the parton shower

Jet multiplicity

- let's come back to $e^+e^- \to q\bar{q}g$ - the 3 jet rate defined as $f_3 = \sigma_{3jet}/\sigma_{tot}$ can be computed

$$\sigma_{tot}\; f_3 \;=\; \int d\sigma_{q\bar{q}g} \;=\; C_F \frac{\alpha_S}{2\pi} \int_{2y_{cut}}^{1-y_{cut}} \frac{dx_q}{1-x_q} \int_{1+y_{cut}-x_q}^{1-y_{cut}} \frac{dx_{\bar{q}}(x_q^2+x_{\bar{q}}^2)}{1-x_{\bar{q}}}$$

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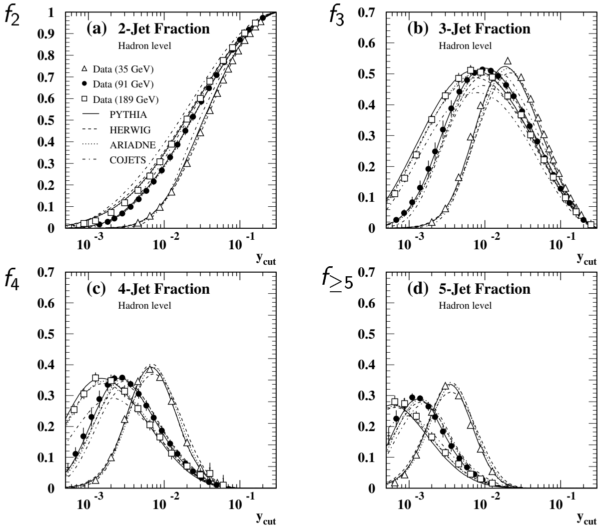
$$= C_F \frac{\alpha_S}{2\pi} \left(4 Li_2\left(\frac{y_{cut}}{1-y_{cut}}\right) + (3-6y_{cut}) \log\left(\frac{y_{cut}}{1-2y_{cut}}\right) + 2\log^2\left(\frac{y_{cut}}{1-y_{cut}}\right) \right.$$

$$\left. -6y_{cut} - \frac{9}{2} y_{cut}^2 - \frac{\pi^2}{3} + \frac{5}{2} \right)$$

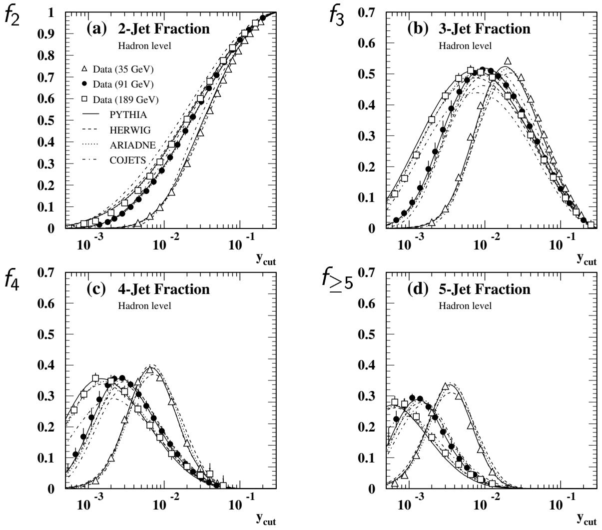


$f_i = \sigma_{i\,jets} / \sigma_{tot}$

- $N_{jet} \geq 2$
- if $y_{cut} \nearrow \Rightarrow N_{jet} \searrow$

(because each jet includes a larger fraction of s)

- depends on s ! - why ?



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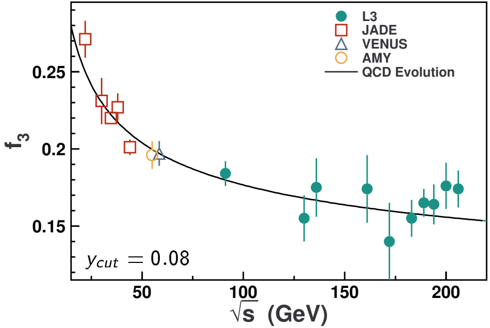
$\Rightarrow$ way to measure α_S!

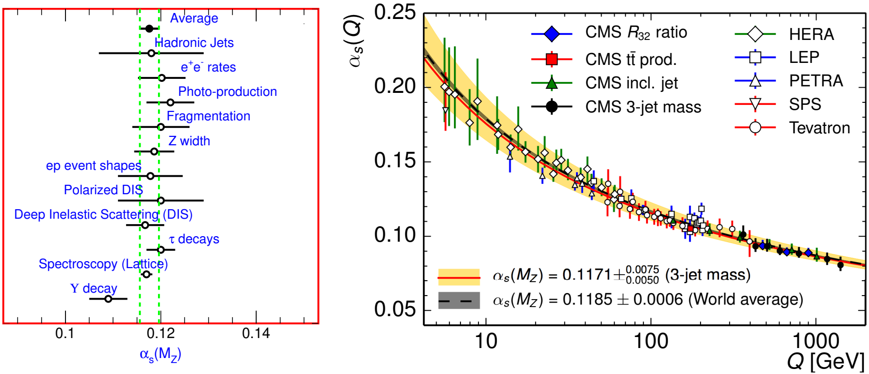
- sensitivity to α_S $\nearrow$ with N_{jet} $\nearrow$ but stat $\searrow$



- energy scale dependence in good agreement with the QCD predicted evolution

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu_R^2)}{1 + \frac{23}{6\pi}\alpha_s(\mu_R^2)\ln\frac{Q^2}{\mu_R^2}}.$$

Comparison of worldwide α_S measurements

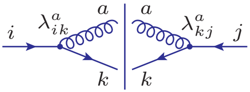


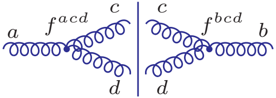
great success

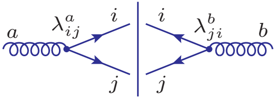
- precision at ‰ level
- it has been measured that α_S is independent of the quark flavor.

Test of QCD gauge structure

- what are the free parameters of the different QCD couplings ?







$$(1)\ \sum_a \left(\lambda^a_{ik}\lambda^a_{kj}\right) = 4\,C_F\,\delta_{ij} \qquad C_F = T_F\frac{N^2-1}{N} = \frac{4}{3}$$

$$(2)\ \sum_{cd} f^{acd} f^{bcd} = C_A\,\delta_{ab} \qquad C_A = 2\,T_F\,N = 3$$

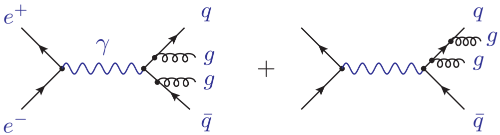
$$(3)\ \mathrm{Tr}\left(\lambda^a\lambda^b\right) = 4\,T_F\,\delta^{ab} \qquad T_F = \frac{1}{2}$$

- to confirm that QCD has the correct gauge structure

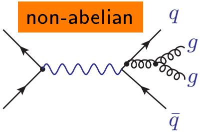
$\Rightarrow$ to access the gauge parameters C_F, C_A and T_F, we need to study 4-jet events.

4-jet events

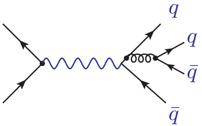
$\mathcal{A}$:



$\mathcal{B}$:

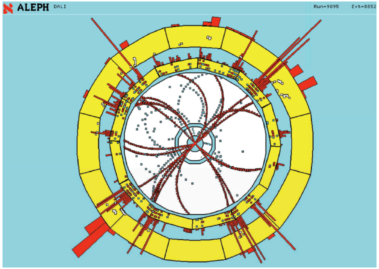


$\mathcal{C}$:



$$d\sigma = \left(\frac{\alpha_S}{2\pi}\right)^2 \left[C_F^2\,\mathcal{A} + C_F\,C_A\,\mathcal{B} + C_F\,T_F\,n_f\,\mathcal{C}\right]$$

- $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are kinematic functions corresponding to the diagrams



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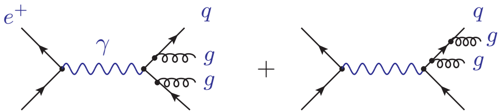
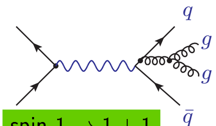


spin $1/2 \rightarrow 1/2 + 1$

$\mathcal{B}$:



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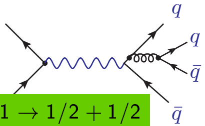

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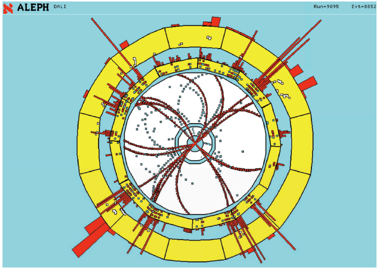
- $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are kinematic functions corresponding to the diagrams
- constrains on the gauge parameters can be obtain from angular distributions



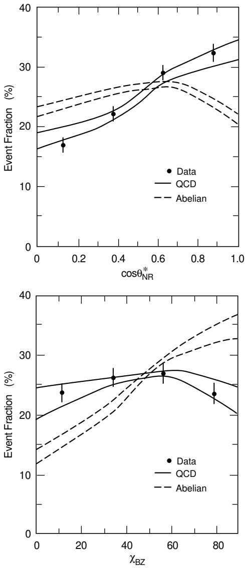
4-jet events

$$E_1 > E_2 > E_3 > E_4$$

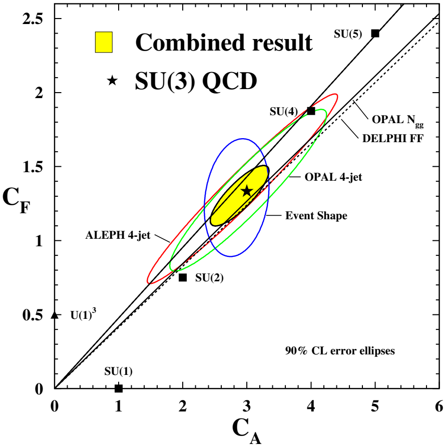
$$\cos\Theta_{NR^*} = \left| \frac{(\vec{p}_1 - \vec{p}_2)\cdot(\vec{p}_3 - \vec{p}_4)}{|\vec{p}_1 - \vec{p}_2|\,|\vec{p}_3 - \vec{p}_4|} \right|$$

$$\cos\chi_{\mathrm{BZ}} = \left| \frac{(\vec{p}_1 \times \vec{p}_2)\cdot(\vec{p}_3 \times \vec{p}_4)}{|\vec{p}_1 \times \vec{p}_2|\,|\vec{p}_3 \times \vec{p}_4|} \right|$$

$\Rightarrow$ confirms the need a the non-abelian term



Constrains on gauge parameters



- combination of 4-jets angle measurements with other measurement
- abelian $U(1)$ excluded by 12 sigmas
- only $SU(3)$ is compatible !

- N_{gg}: limit from fraction of gluon jets
- FF: limit coming from Fragmentation Functions

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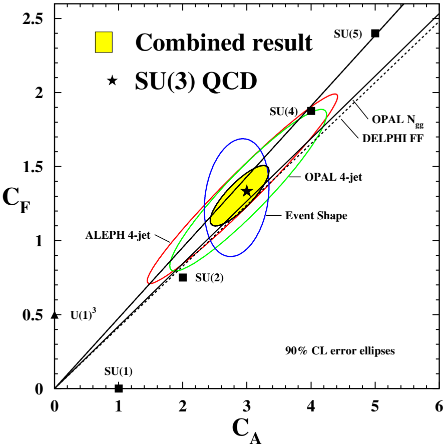

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wonderful success

section conclusions

- ✓ quark electric charges
- ✓ quark spin 1/2
- ✓ gluon spin 1
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complete success !

$\Rightarrow$ QCD is a (the ?) good theory to describe strong interactions

$\Rightarrow$ can it be used to study the hadron structure (next section)

$\Rightarrow$ is it portable in hadron-hadron interactions ? (next-to-next section)

- Part 3 -

Structure of hadrons

Structure of the proton

- proton in the quark model: **2 up quarks, 1 down quark.**

- the picture seems consistent: up-charge $= +\frac{2}{3}$; down charge $= -\frac{1}{3}$

$$2 \times \frac{2}{3} \; - \; 1 \times \frac{1}{3} = +1$$

- but is this **right?** Is it **all?**

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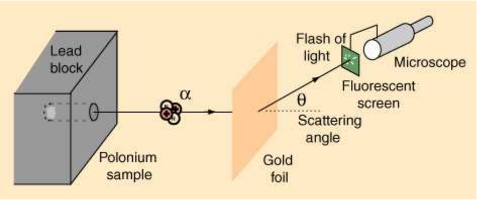
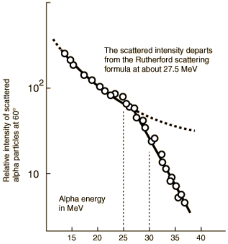
- but is this **right?** Is it **all?**

  - do we see interactions between them or are they free particles?
  - the colour field (gluons) between the quarks fluctuates in $q\bar{q}$ pairs
    $\Rightarrow$ **gluons and sea quarks**
    they carry a small fraction of the proton momentum
    $\Rightarrow$ **small x physics**
    i.e. the study of a colour field inside the proton.
  - How can we study that? $\Rightarrow$ **scattering experiments**

- 1909: Geiger and Marsden and Rutherford : Rutherford scattering - Atomic nucleus



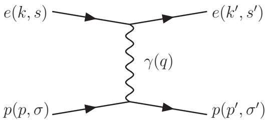


$$\langle r_{nucleus} \rangle < 10^{-14}\, m$$

$\Rightarrow$ let see use the same principle but using an elementary particle (e^-) scattered off a p

$$e^- + p \to e^- + p$$

Mott Scattering: point-like spin 1/2 off point-like spin 1/2



$$\mathcal{M} = \langle k', s' | J_e^\mu | k, s\rangle \frac{g_{\mu\nu}}{q^2} \langle p', \sigma' | J_p^\nu | p, \sigma\rangle$$

$$J_e^\mu = -ie\,\bar{u}(k', s')\,\gamma^\mu\, u(k, s)$$

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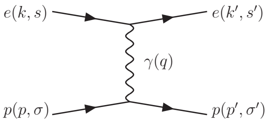

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$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \sum_{s,s',\sigma,\sigma'} |\mathcal{M}|^2 = \frac{e^4}{q^4} L^{\mu\nu}\, W_{\mu\nu}$$

$$L^{\mu\nu} = \frac{1}{2} \sum_{s,s'} J_e^\mu J_e^\nu$$

$$= 2\left(k'^\mu k^\nu + k^\mu k'^\nu - (k \cdot k' + m_e^2) g^{\mu\nu}\right)$$

$$W^{\mu\nu} = 2\left(p'^\mu p^\nu + p^\mu p'^\nu - (p \cdot p' + m_p^2) g^{\mu\nu}\right)$$

Using $p' = p + k - k'$ and $q^2 = -2\,k \cdot k'$ $(m_e \ll k, k')$ the tensor product gives:

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left[\frac{-q^2}{2}(k - k') \cdot p + 2(k \cdot p)(k' \cdot p) + \frac{m_p^2 q^2}{2}\right]$$

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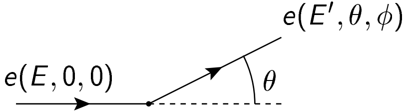
in the lab frame (fixed target - $p = (m_p, 0, 0, 0)$)



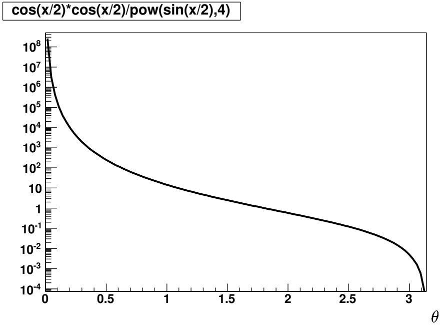
$p\cdot k = m_p\,E,\qquad p\cdot k' = m_p\,E'$
$k\cdot k' = E\,E'\,(1-\cos\theta)$
and $E - E' = -\frac{q^2}{2m_p}$

$$\overline{|\mathcal{M}|^2} = \frac{16e^4}{q^4}EE'm_p^2\left(\cos^2(\theta/2) - \frac{q^2}{2m_p^2}\sin^2(\theta/2)\right)$$

$$\Rightarrow \boxed{\frac{d^2\sigma}{dE'\,d\Omega} = \frac{4\alpha^2}{Q^4}E'^2\left[\cos^2(\theta/2) - \frac{q^2}{2m_p^2}\sin^2(\theta/2)\right]\delta\left(E - E' + \frac{q^2}{2m_p^2}\right)}$$

Note 1: the first term corresponds to the classic calculation called the Mott scattering cross section:

$$\frac{d\sigma_{Mott}}{d\Omega} = \frac{4\alpha^2}{Q^4} E'^2 \cos^2(\theta/2) = \frac{\alpha^2}{4E^2} \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}$$



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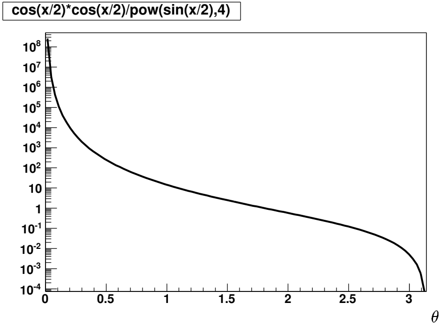


the second term vanishes for $Q^2 \to 0$, i.e. real photons (purely transverse).

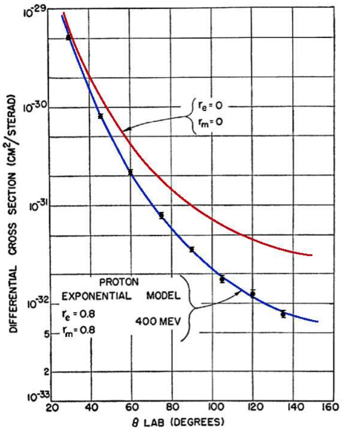
Note 2: in the ultra relativistic limit, using Mandelstam variables:

$$\begin{aligned} s &= (k+p)^2 = 2\,p\cdot k, \\ t &= (k-k')^2 = q^2 = -2\,k\cdot k', \\ u &= (k-p')^2 = (k'-p)^2 = -2\,p\cdot k', \end{aligned}$$

$$\overline{|\mathcal{M}|^2} = 2e^4\,\frac{s^2+u^2}{t^2}$$

Form Factors: elastic point-like spin 1/2 off hadronic target

1953 Hofstadter et al. (SLAC)



$$e^- + N \rightarrow e^- + N$$

Deviation w.r.t point-like scattering

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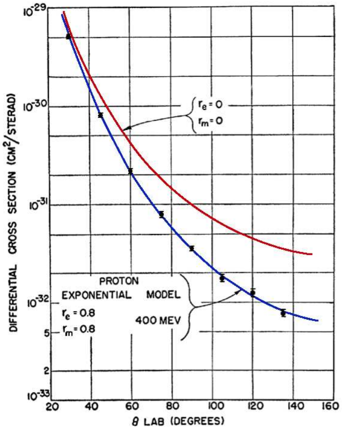

$$e^- + N \to e^- + N$$

Deviation w.r.t point-like scattering

Experimentalist way to introduce the FF:

$$\frac{d\sigma}{d\Omega_e} = \frac{d\sigma_{Mott}}{d\Omega_e} \times \left[G_E(Q)^2 + \frac{\tau}{\epsilon} G_M(Q)^2\right] \frac{1}{1+\tau}$$

$\tau = \frac{Q^2}{4mp^2}$ and ϵ : virtual photon polarisation

$\frac{1}{1+\tau} = \frac{Ee}{Ee'}$: recoil factor

$$Q^2 = -q^2 = -(k' - k)^2 \simeq 2E\,E'(1 - \cos\theta)$$

G_E: Fourier transform of the target electric charge distribution (resp. G_M: distribution of the magnetisation current densities)

$$G_E(Q) = \int \mathrm{d}^3 r\, \rho(r)\, e^{i\mathbf{r}\cdot\mathbf{Q}} \qquad \text{with} \qquad G_E(0) = 1$$

for small $|q|$

$$G_E(Q) = \int \left(1 + iQ \cdot r - \frac{(Q \cdot r)^2}{2} + \dots\right) \rho(r)\, \mathrm{d}^3 r$$

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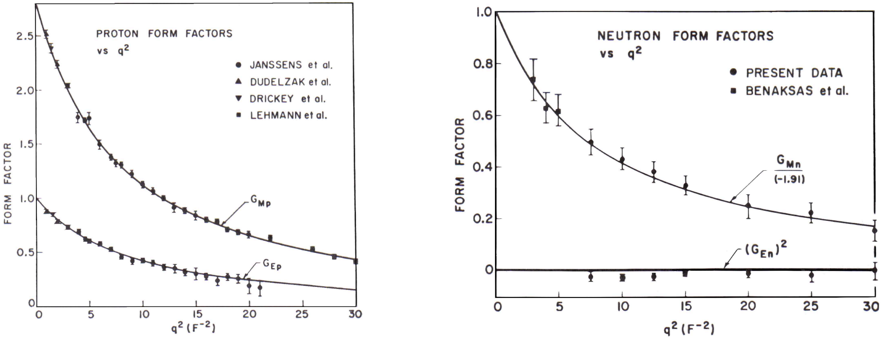
Average squared radius :

$$\langle r^2 \rangle = \int r^2 \rho(r)\, \mathrm{d}^3 r = 4\pi \int r^2 \rho(r)\, r^2\, \mathrm{d}r$$

Assuming spherical symmetry $\Rightarrow \rho(r) = \rho(-r)$:

$$G_E(Q) \simeq 1 + 0 - \frac{Q^2}{6} \langle r^2 \rangle + \ldots .$$

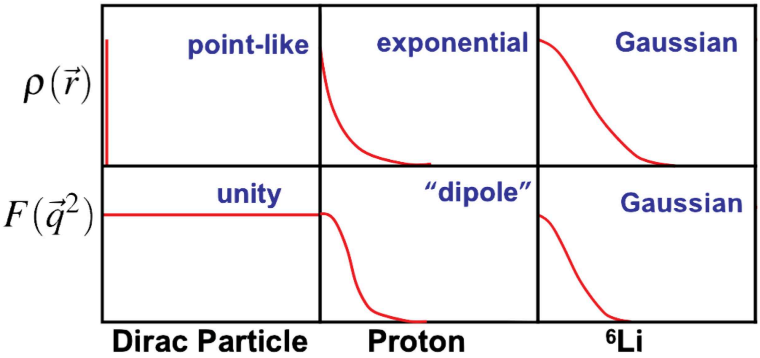
$$\langle r^2 \rangle^{1/2} \simeq 0.74 \pm 0.24\ 10^{-15}\ \mathrm{m}$$



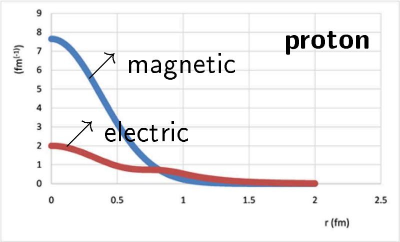
Fitted with a dipole shape:

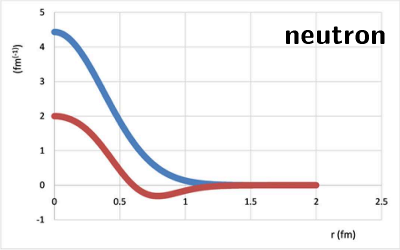
$$|G_D(Q)| \sim \frac{1}{(1+Q^2/\mu_0^2)^2}$$

with $\mu_0^2 \simeq 0.71\ \text{GeV}^2$


$\rho(\vec{r})$
$F(\vec{q}^2)$
point-like
exponential
Gaussian
unity
"dipole"
Gaussian
Dirac Particle
Proton
^{6}Li

Example of recent work: Kurai [2020]





transverse charge distributions as a function of the impact parameter b

$\Rightarrow$ negative charge densities at the neutron center

$\Rightarrow$ contradicts the negative pion cloud surrounding a positively charged core.