SCALES 1st General Meeting - Superfluid Condensates in Astrophysics and Laboratory Experiments
June 23, 2026

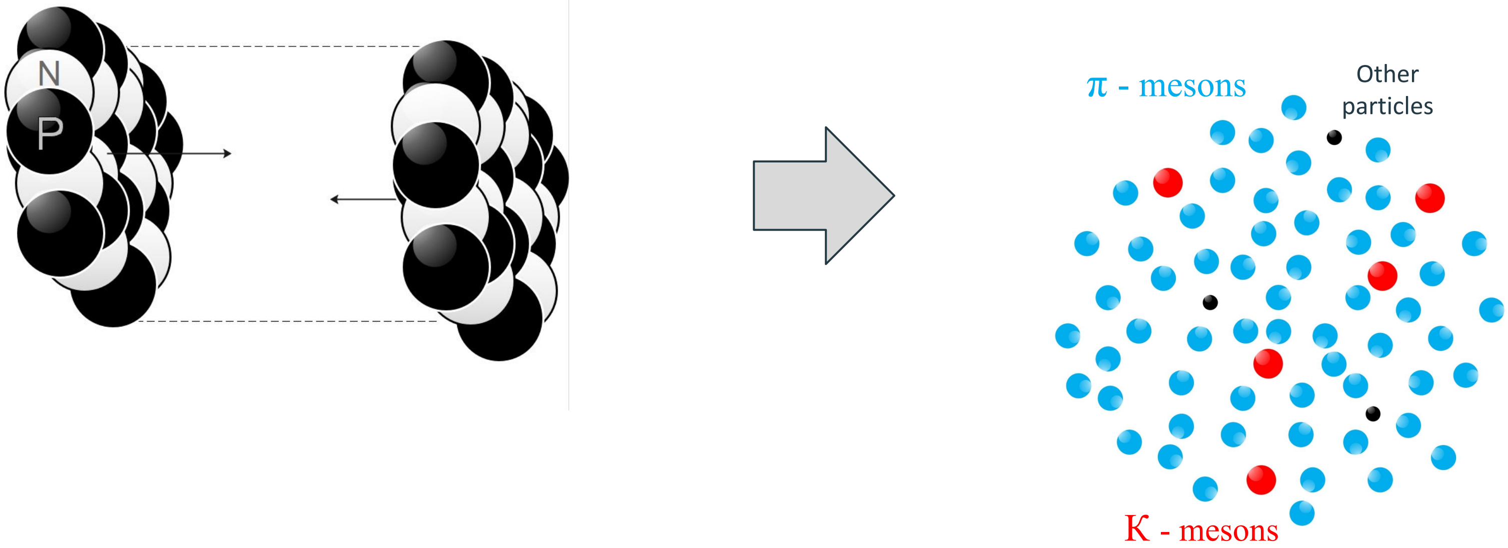
BEC phase transitions in a system of interacting relativistic bosonic particles and antiparticles at finite temperatures

Speaker: Denys Zhuravel

In collaboration with **Dmytro Anchishkin**, **Volodymyr Gnatovskyy**, **Vladyslav Karpenko**

Bogolyubov Institute for Theoretical Physics

High energy collisions



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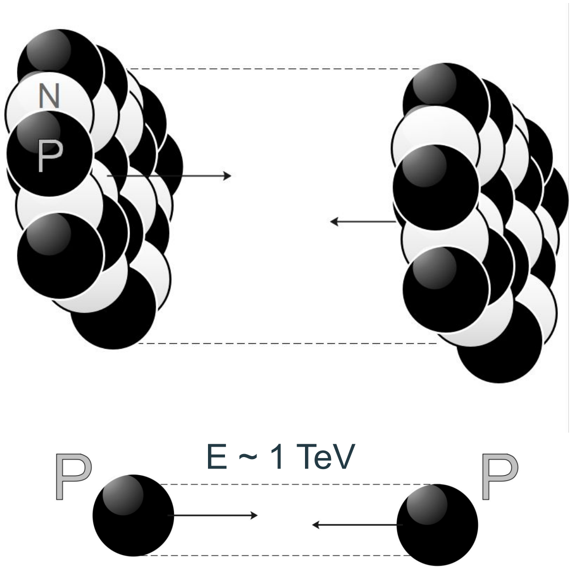

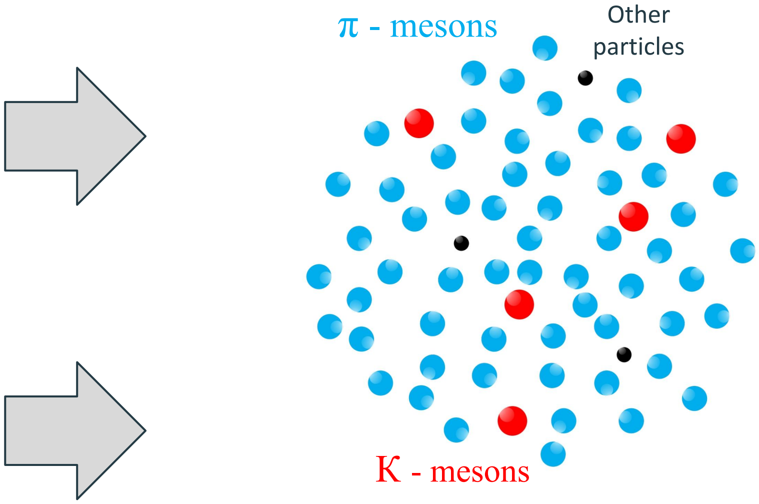

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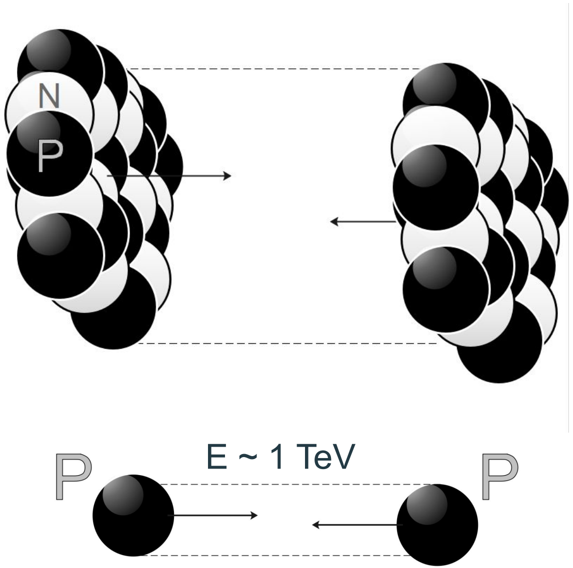

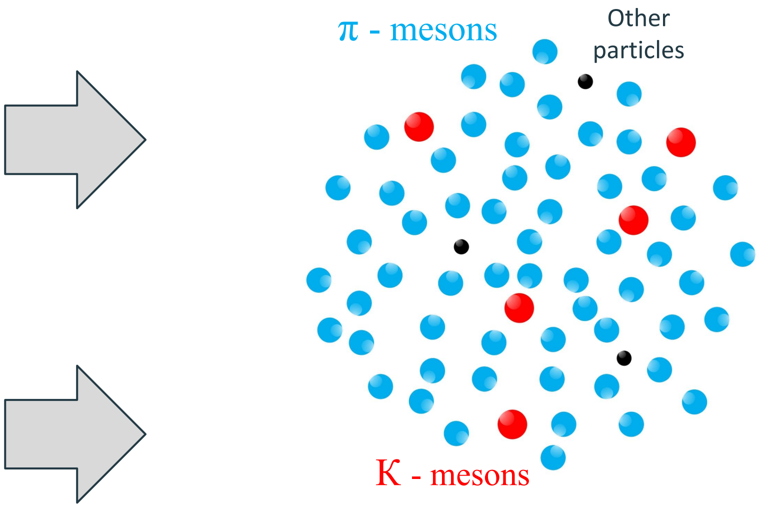

$N \sim 10^2 - 10^4$

Probability < 1%

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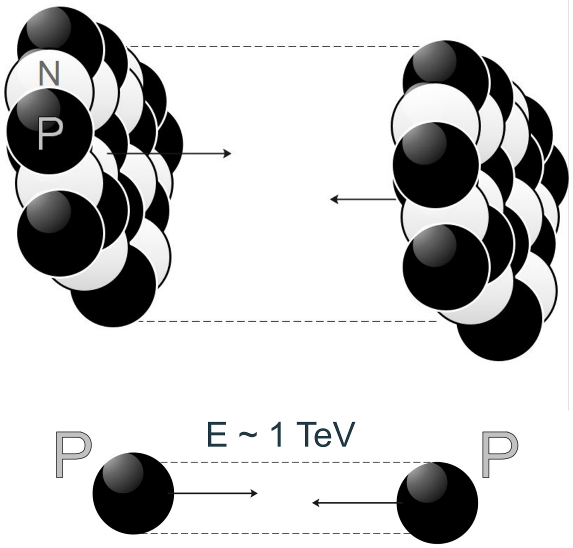
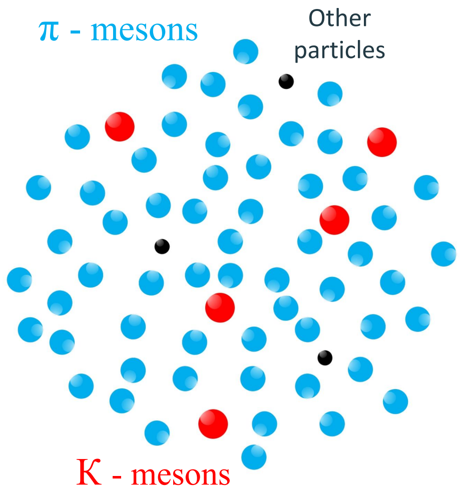


Task: Study of the conditions for the occurrence of Bose condensate in the pion-antipion system with interaction



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Historical remarks and Bose-Einstein distribution

Bose-Einstein distribution function:

$$f_{\mathrm{BE}}\left(\varepsilon_j, \mu\right) = \frac{1}{\exp\left[\frac{\varepsilon_j - \mu}{T}\right] - 1}$$

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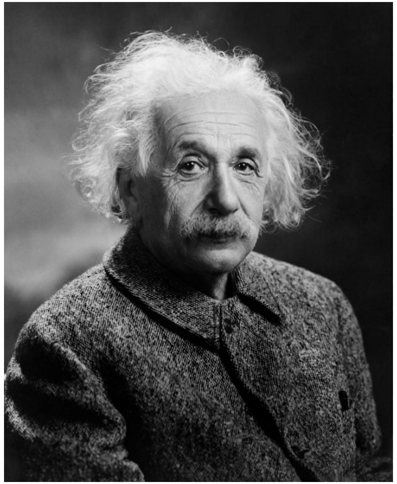

Albert Einstein

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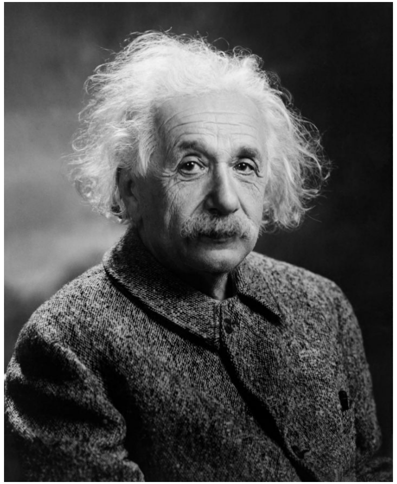

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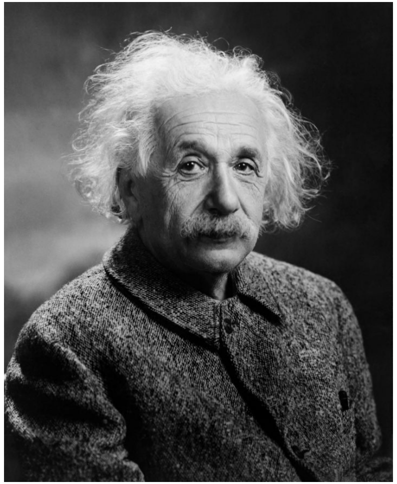

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$$E = \sum_s \frac{8\pi h \nu^{s3}}{c^3} V \frac{1}{e^{\frac{h\nu^s}{kT}} - 1} \mathrm{d}\nu^s$$

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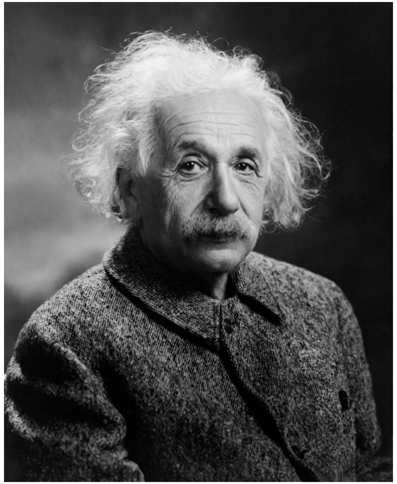

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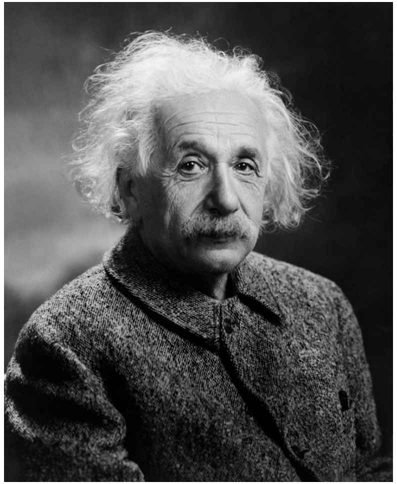

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1st paper: Extended Bose's statistical method from photons to massive Bose particles.
2nd paper: Introduced ideas of bosons condensation

When does Bose-Einstein condensation (BEC) occur?

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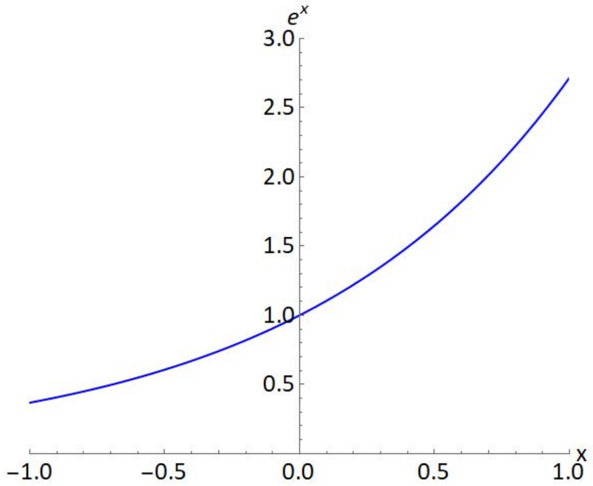

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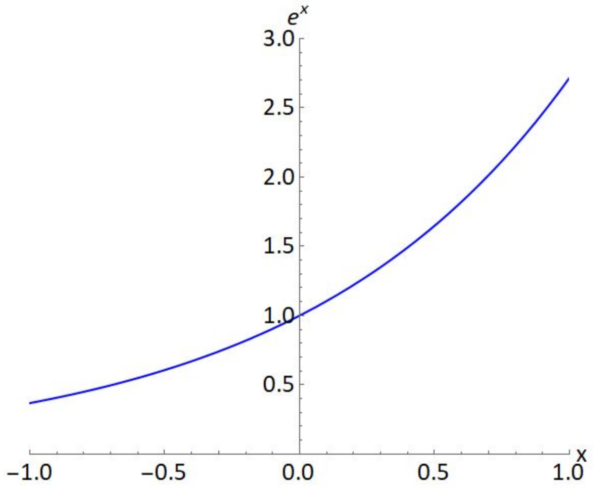

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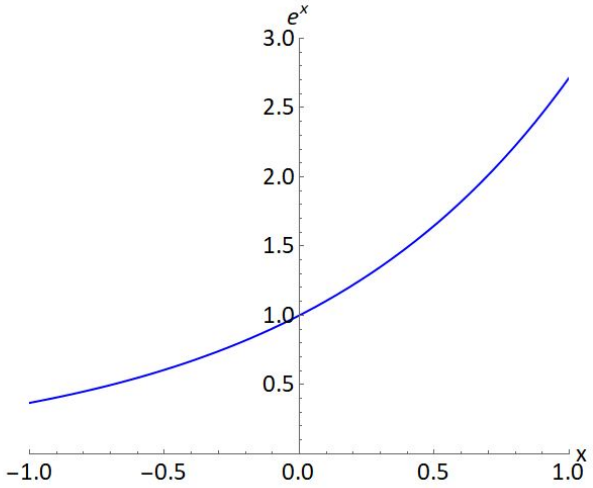

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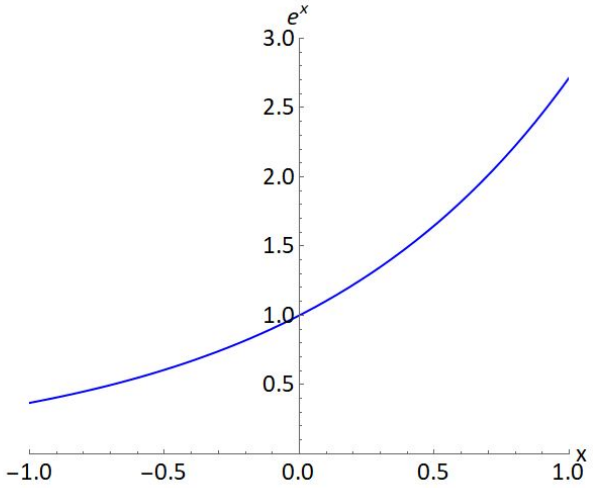
Energy levels of system:

$$\varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \ldots$$

Ground state



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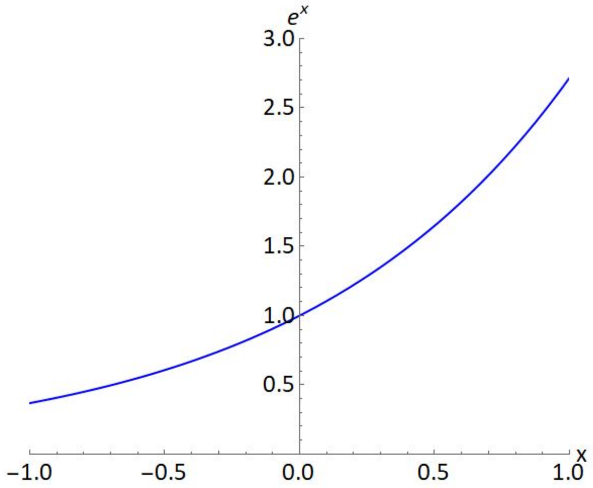

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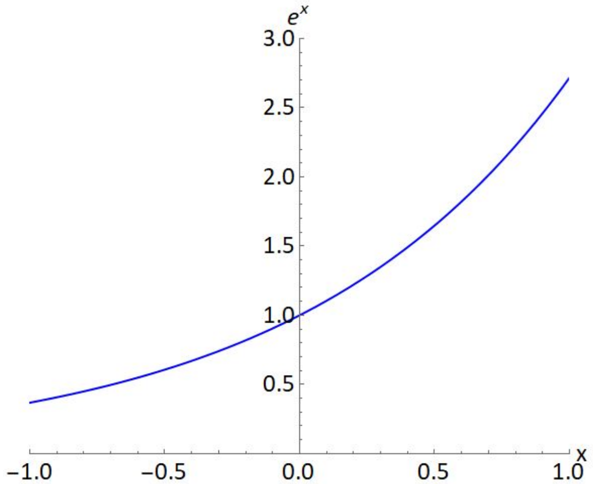
$$\varepsilon_0 = \mu$$



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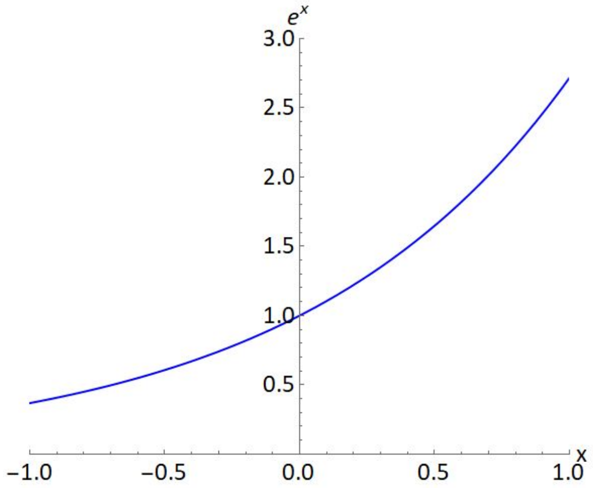
Condition for the onset of Bose-Einstein condensation



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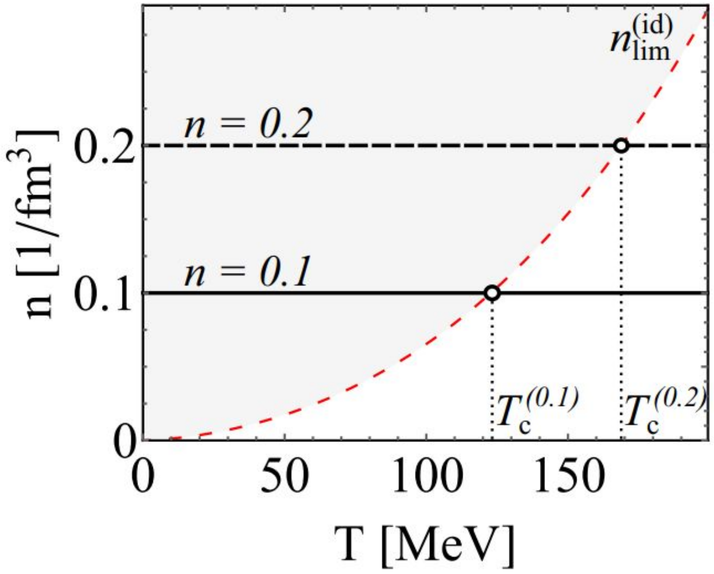
Problem!

1-component ideal gas



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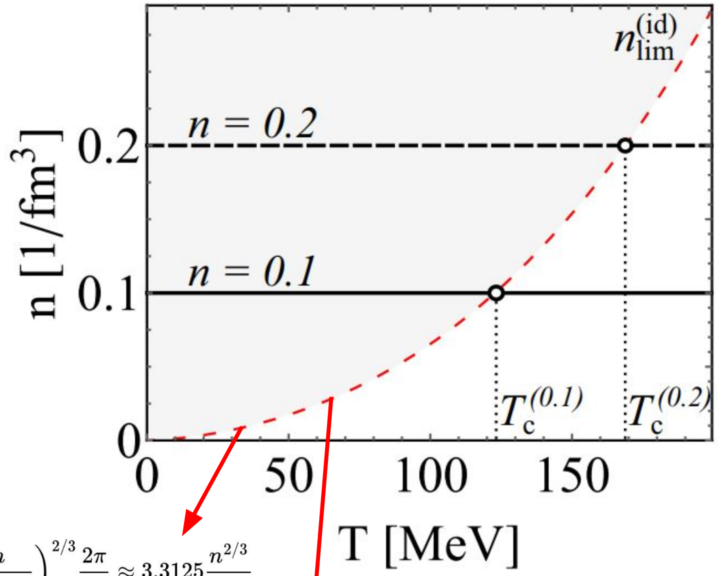

$$T_c = \left(\frac{n}{\zeta(3/2)}\right)^{2/3} \frac{2\pi}{m} \approx 3.3125 \frac{n^{2/3}}{m}$$

Classical system

$$n_{\mathrm{lim}}^{(\mathrm{id})}(T) = g \int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{1}{\exp\left[\frac{\sqrt{m^2+k^2}-m}{T}\right] - 1}$$

Relativistic system

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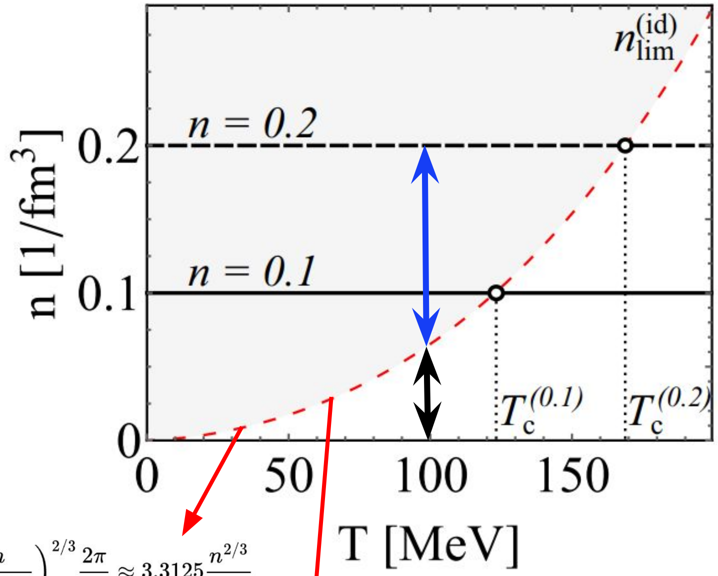

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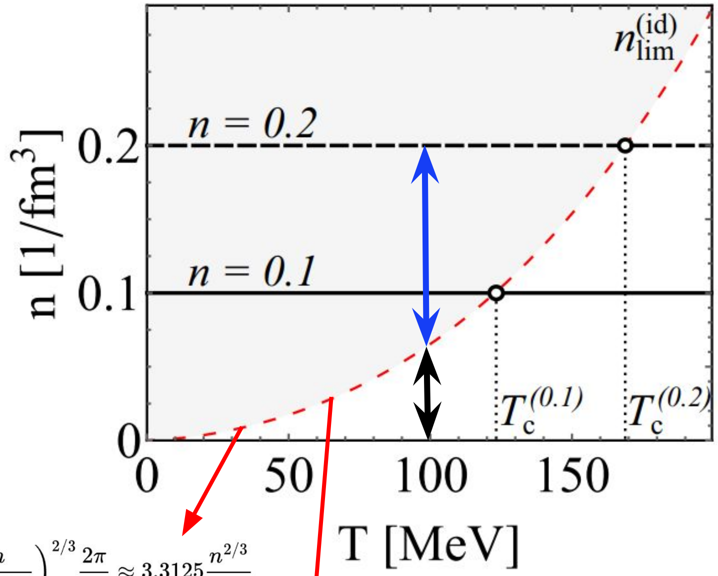

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When $T < T_c$:

$$n = n_{\text{cond}} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{\exp\left[\frac{\varepsilon_k - \mu}{T}\right] - 1}\Bigg|_{\mu=\varepsilon_0}$$

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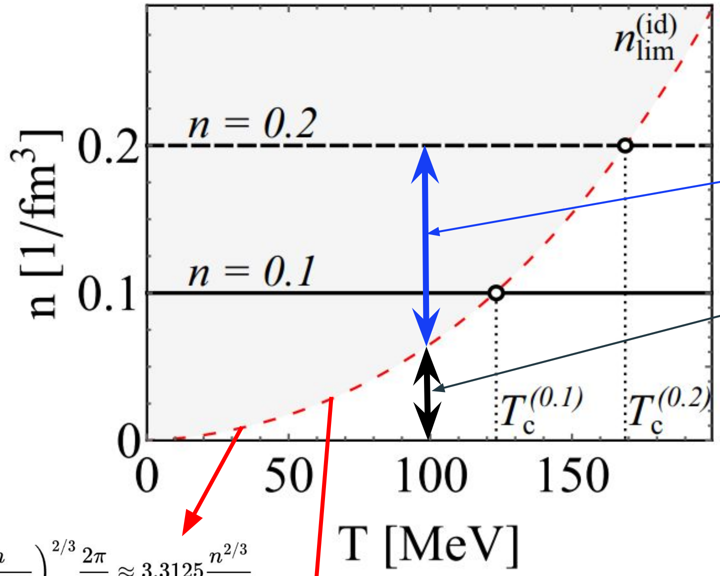

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Relativistic system

How to describe 2-component system with interaction?

Scalar field model[3,4]

Thermodynamic mean-field model[1-3]

Starting point

The Lagrangian density:

$$\mathcal{L}(x) = \partial_\mu \phi^+(x)\, \partial^\mu \phi(x) - m^2\, \phi^+(x)\phi(x) + \mathcal{L}_{\text{int}}(x)$$

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

Self-consistent set of equations

Parameterization of interaction

[1] D. Anchishkin and V. Vovchenko, J. Phys. G 42, 105102 (2015).
[2] D. Anchishkin, I. Mishustin, and H. Stoecker, J. Phys. G 46, 035002 (2019).
[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).
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Parameterization of interaction

$$m = 140\,\mathrm{MeV}$$

$$\mathcal{L}_{\rm int}(\sigma) = \frac{a}{2}\sigma^2 - \frac{b}{3}\sigma^3, \qquad \omega - \mu_I = 0$$

$$\kappa = \frac{a}{2m\sqrt{b}}.$$

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$$f_{\rm BE}(E, \mu) \,=\, \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$$

Parameterization of interaction

$$U(n) = -An + Bn^2, \qquad E - \mu = 0$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

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Starting point

The Lagrangian density:

$$\mathcal{L}(x) = \partial_\mu \phi^+(x)\, \partial^\mu \phi(x) - m^2\, \phi^+(x)\phi(x) + \mathcal{L}_{\text{int}}(x)$$

Self-consistent set of equations

$$\sigma = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k(\sigma)} \left\{ f_{\text{BE}}[\omega_k(\sigma), \mu_I] + f_{\text{BE}}[\omega_k(\sigma), -\mu_I] \right\},$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left\{ f_{\text{BE}}[\omega_k(\sigma), \mu_I] - f_{\text{BE}}[\omega_k(\sigma), -\mu_I] \right\}.$$

Fixed (n_I)

$$\omega_k(\sigma) = \sqrt{M(\sigma)^2 + \boldsymbol{k}^2}, \quad M^2(\hat{\sigma}) = m^2 - \frac{\partial \mathcal{L}_{\text{int}}(\hat{\sigma})}{\partial \hat{\sigma}}$$

$$f_{BE}(\omega, \mu_I) = \left\{ \exp\left[\frac{\omega - \mu_I}{T}\right] - 1 \right\}^{-1}, \quad \hat{\sigma}(x) = \phi^+(x)\phi(x)$$

Parameterization of interaction

$$m = 140\,\text{MeV}$$

$$\mathcal{L}_{\text{int}}(\sigma) = \frac{a}{2}\sigma^2 - \frac{b}{3}\sigma^3, \qquad \omega - \mu_I = 0$$

$$\kappa = \frac{a}{2m\sqrt{b}}.$$

Interplay between a and b
Parameter b - fixed

Thermodynamic mean-field model[1-3]

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

$$n = \int \frac{d^3k}{(2\pi)^3} \left[f_{\text{BE}}(E(k,n), \mu_I) + f_{\text{BE}}(E(k,n), -\mu_I) \right],$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left[f_{\text{BE}}(E(k,n), \mu_I) - f_{\text{BE}}(E(k,n), -\mu_I) \right].$$

Fixed (n_I)

$$E(k,n) = \sqrt{m^2 + \boldsymbol{k}^2} + U(n)$$

$$f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$$

$$U(n) = -An + Bn^2, \qquad E - \mu = 0$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

Interplay between A and B
Parameter B - fixed

Scalar field model[3,4]

Starting point

The Lagrangian density:

$$\mathcal{L}(x) = \partial_\mu \phi^+(x)\, \partial^\mu \phi(x) - m^2\, \phi^+(x)\phi(x) + \mathcal{L}_{\text{int}}(x)$$

Self-consistent set of equations

$$\sigma = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k(\sigma)} \left\{ f_{\text{BE}}[\omega_k(\sigma), \mu_I] + f_{\text{BE}}[\omega_k(\sigma), -\mu_I] \right\},$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left\{ f_{\text{BE}}[\omega_k(\sigma), \mu_I] - f_{\text{BE}}[\omega_k(\sigma), -\mu_I] \right\}.$$

Fixed (n_I)

$$\omega_k(\sigma) = \sqrt{M(\sigma)^2 + k^2}, \quad M^2(\hat\sigma) = m^2 - \frac{\partial \mathcal{L}_{\text{int}}(\hat\sigma)}{\partial \hat\sigma}$$

$$f_{BE}(\omega, \mu_I) = \left\{ \exp\left[\frac{\omega - \mu_I}{T}\right] - 1 \right\}^{-1}, \quad \widehat{\sigma}(x) = \phi^+(x)\phi(x)$$

Parameterization of interaction

$$m = 140\,\text{MeV}$$

$$\mathcal{L}_{\text{int}}(\sigma) = \frac{a}{2}\sigma^2 - \frac{b}{3}\sigma^3, \qquad \omega - \mu_I = 0$$

$$\kappa = \frac{a}{2m\sqrt{b}}.$$

Interplay between a and b
Parameter b - fixed

Thermodynamic mean-field model[1-3]

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

$\kappa < 1$ - weak attraction regime

$\kappa = 1$ - critical attraction regime

$\kappa > 1$ - strong(over-critical) attraction regime

$$U(n) = -An + Bn^2, \qquad E - \mu = 0$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

Interplay between A and B
Parameter B - fixed

Ideal 2-component relativistic gas

Mean field model

$$U(n) = 0$$

No interaction

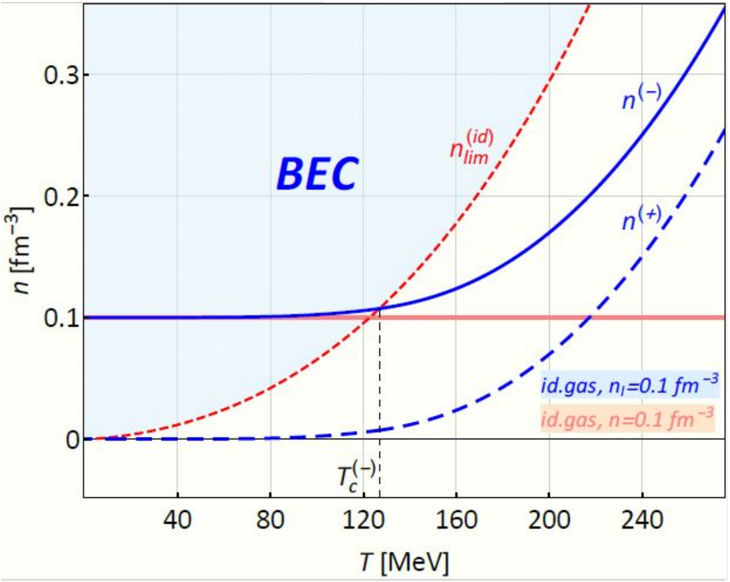


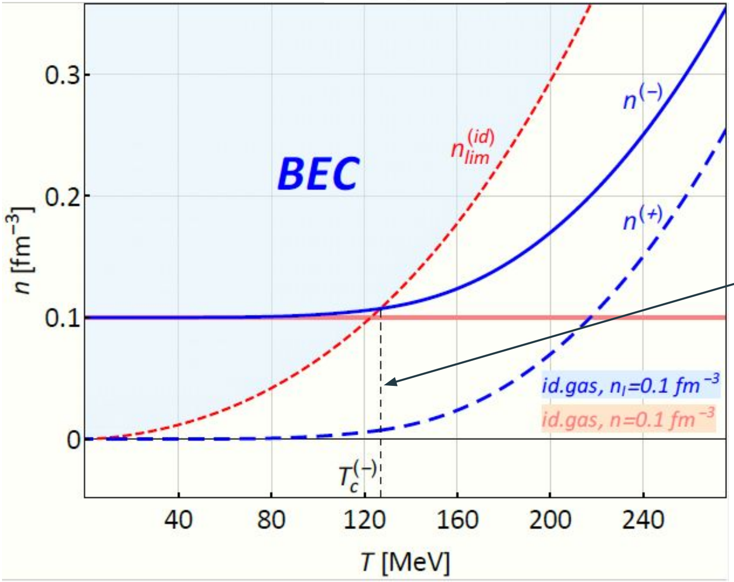
Figure 1. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the ideal π^+-π^- pion gas. The total isospin density is kept constant, n_I = 0.1 fm^{-3}.

Ideal 2-component relativistic gas

Mean field model

No interaction

$$U(n) = 0$$



2nd order phase transition

Figure 1. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the ideal π^+-π^- pion gas. The total isospin density is kept constant, $n_I = 0.1\ fm^{-3}$.

Ideal 2-component relativistic gas

Scalar model

No interaction

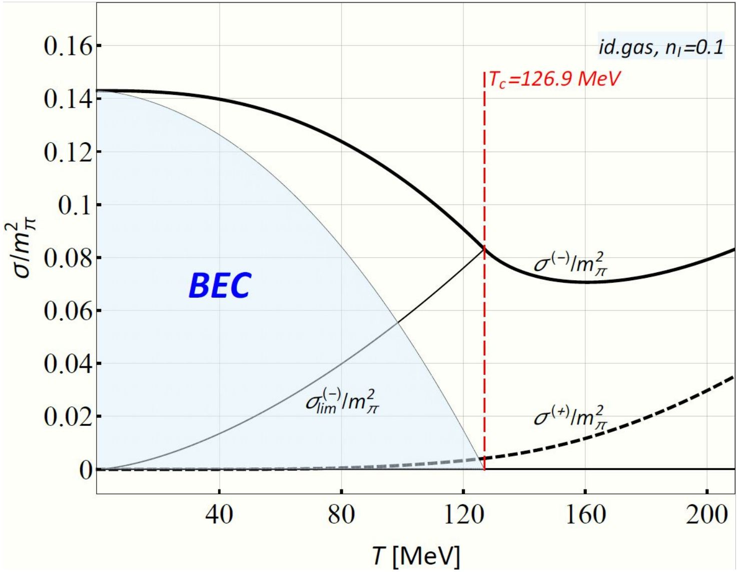


Figure 2. Dependence of scalar density $\sigma^{(+)}/m^2$ and $\sigma^{(-)}/m^2$ on temperature for ideal gas of π^+-π^- pions. Isospin density is considered to be fixed, $n_I = 0.1\ fm^{-3}$.

Ideal 2-component relativistic gas

No interaction

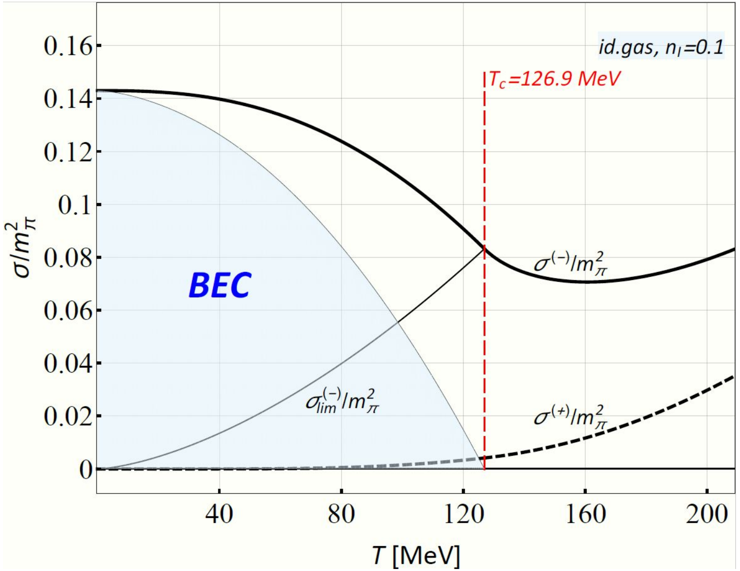


Figure 2. Dependence of scalar density $\sigma^{(+)}/m^2$ and $\sigma^{(-)}/m^2$ on temperature for ideal gas of π^+-π^- pions. Isospin density is considered to be fixed, $n_I = 0.1$ fm^{-3}.

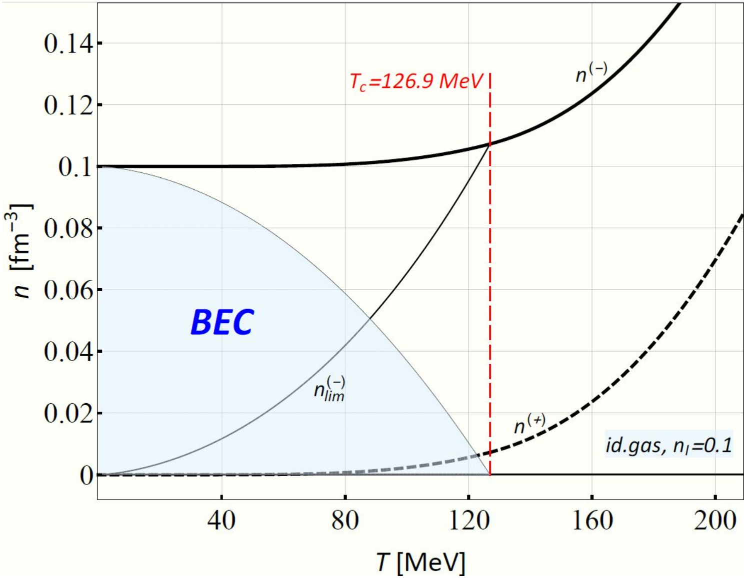


Figure 3. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for ideal gas of π^+-π^- pions within the framework of the scalar field approach. Isospin density is considered to be fixed, $n_I = 0.1$ fm^{-3}.

Ideal 2-component relativistic gas

No interaction

2nd order phase transition

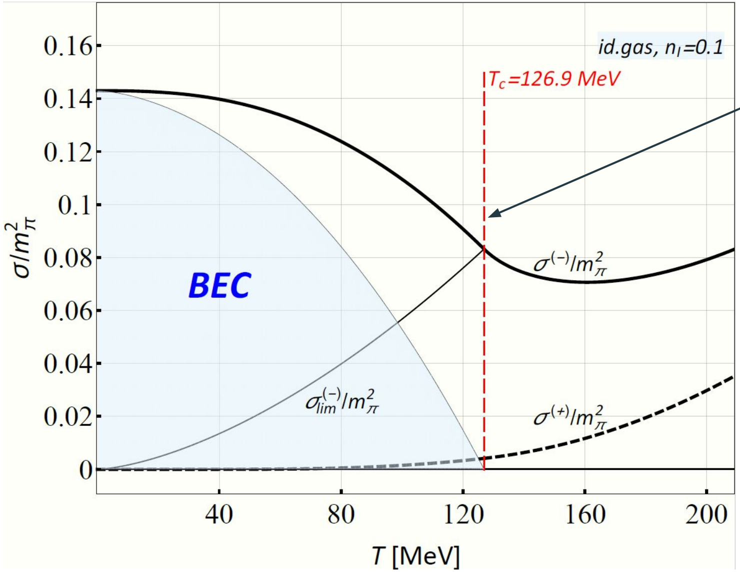


Figure 2. Dependence of scalar density $\sigma^{(+)}/m^2$ and $\sigma^{(-)}/m^2$ on temperature for ideal gas of π^+-π^- pions. Isospin density is considered to be fixed, $n_I = 0.1$ fm^{-3}.

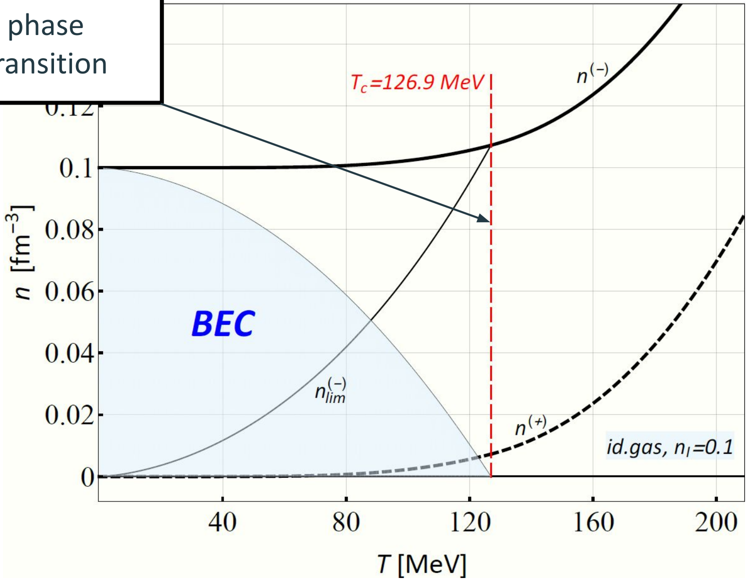


Figure 3. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for ideal gas of π^+-π^- pions within the framework of the scalar field approach. Isospin density is considered to be fixed, $n_I = 0.1$ fm^{-3}.

No interaction

Thermodynamic approach

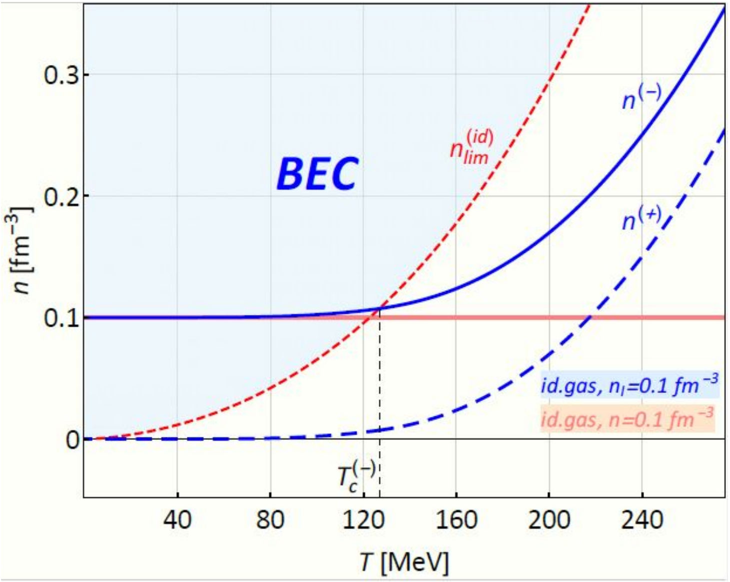


Figure 1. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for ideal gas of π^+-π^- pions within the framework of the thermodynamic mean-field approach. Isospin density is considered to be fixed, n_I = 0.1 fm^{-3}.

Field approach

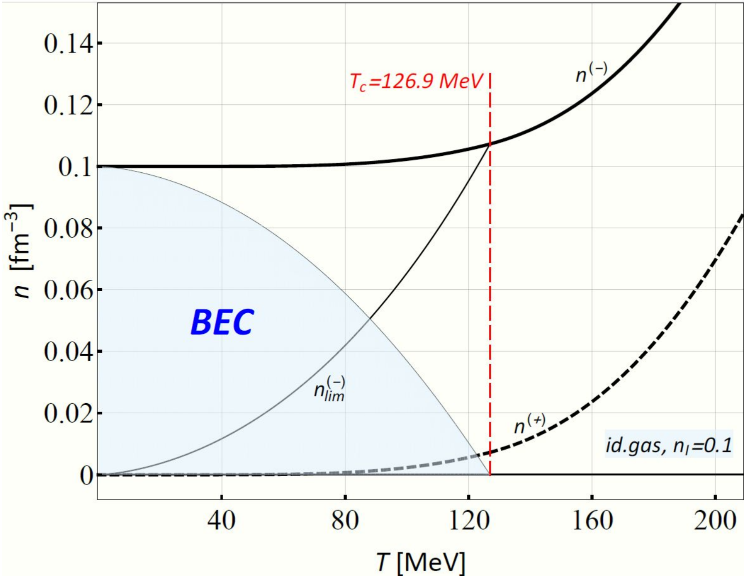


Figure 3. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for ideal gas of π^+-π^- pions within the framework of the scalar field approach. Isospin density is considered to be fixed, n_I = 0.1 fm^{-3}.

Thermodynamic approach

Field approach

No interaction

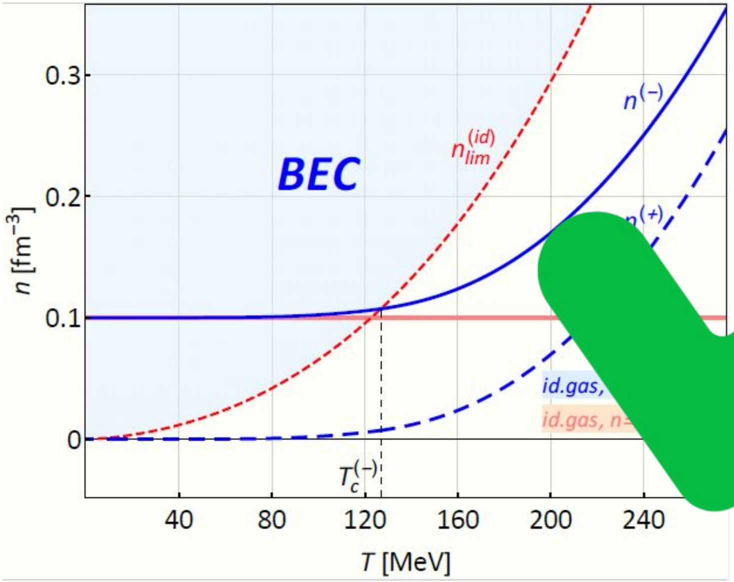


Figure 1. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for ideal gas of π^{+}-π^{-} pions within the framework of the thermodynamic mean-field approach. Isospin density is considered to be fixed, n_I = 0.1 fm^{-3}.

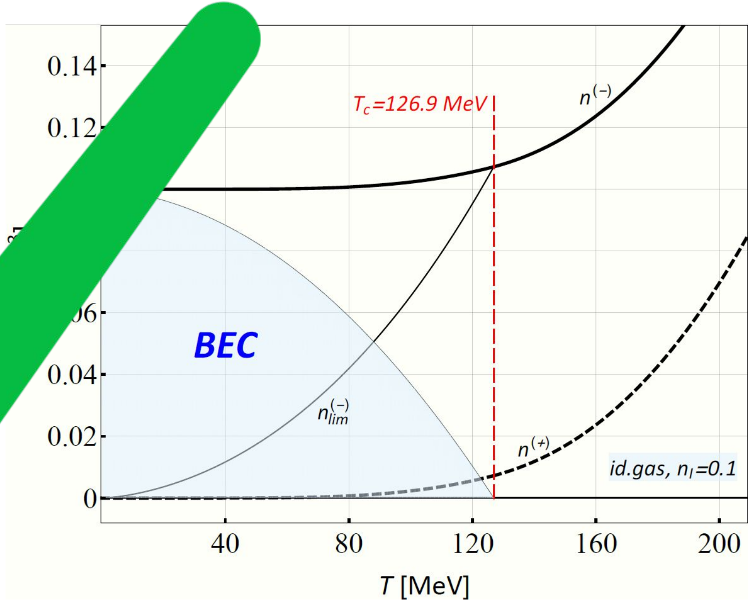


Figure 3. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for ideal gas of π^{+}-π^{-} pions within the framework of the scalar field approach. Isospin density is considered to be fixed, n_I = 0.1 fm^{-3}.

Let's turn the attraction ON!

Type 1 phase transition (2nd order ph. tr.)

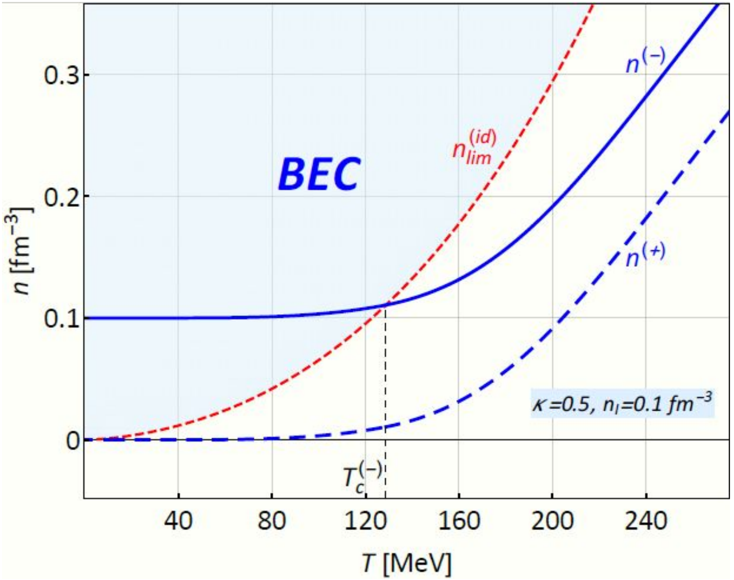


Figure 4. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas in the mean-field model. The total isospin density is kept constant, $n_I = 0.1\ fm^{-3}$, and the attraction parameter is **κ = 0.5**.

Weak attraction regime
$\kappa = 0.5$

Type 1 phase transition (2nd order ph. tr.)

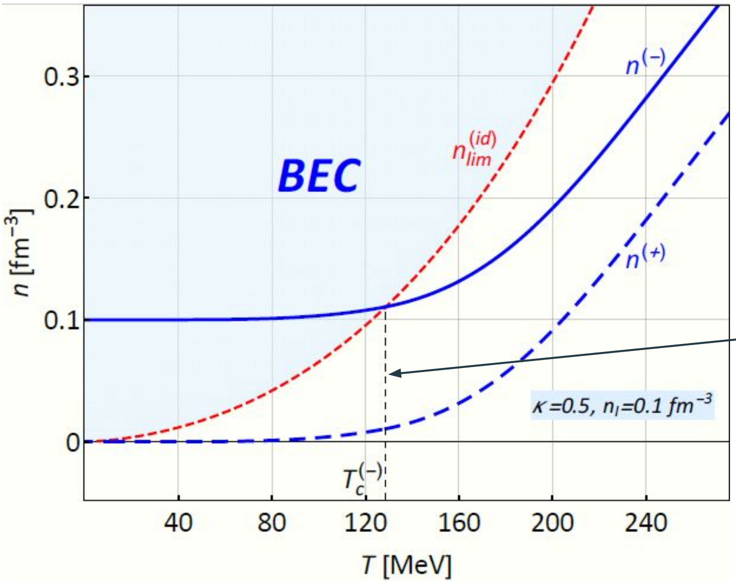
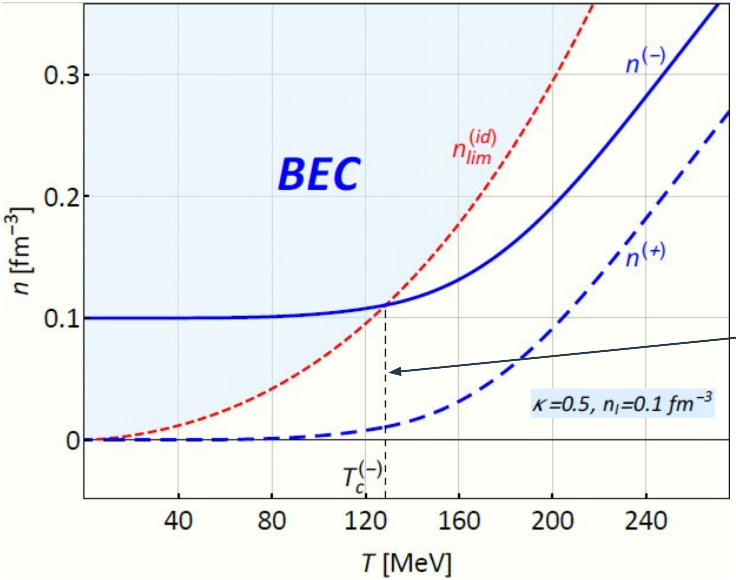


Figure 4. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas in the mean-field model. The total isospin density is kept constant, $n_I = 0.1$ fm^{-3}, and the attraction parameter is **$\kappa = 0.5$**.

Phase transition of the 2nd order

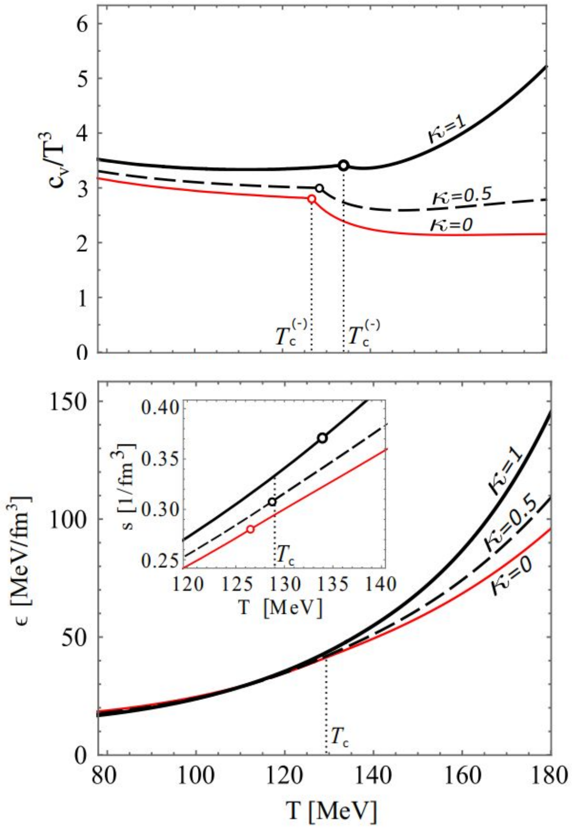
Weak attraction regime
$\kappa = 0.5$

Type 1 phase transition (2nd order ph. tr.)

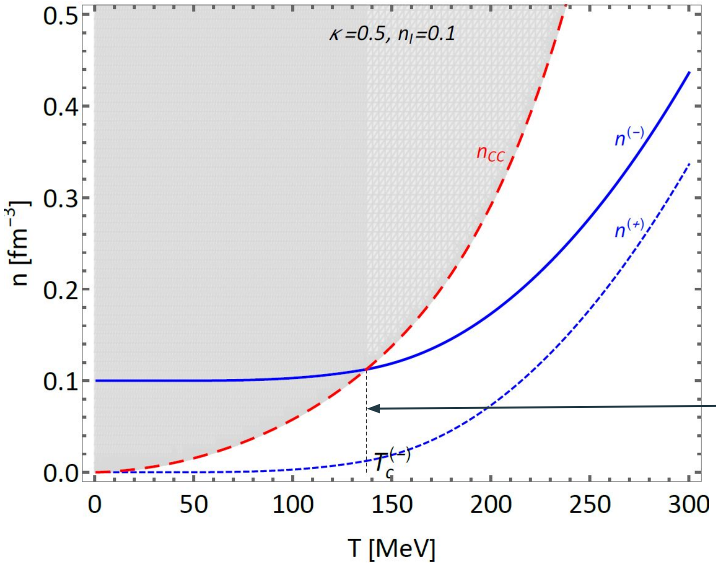


Phase transition of the 2nd order

Figure 4. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^{+}-π^{-} pion gas in the mean-field model. The total isospin density is kept constant, n_I = 0.1 fm^{-3} , and the attraction parameter is **κ = 0.5**.



Type 1 phase transition (2nd order ph. tr.)



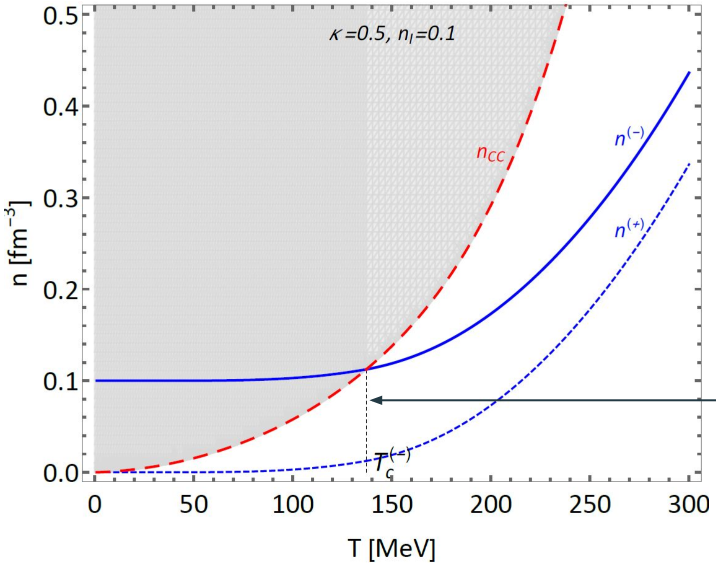
2nd order phase transition

Weak attraction regime
$\kappa = 0.5$

Figure 5. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for π^{+}-π^{-} pion gas with interaction within the framework of the scalar field approach. Isospin density is considered to be fixed, $n_I = 0.1$ fm^{-3}, the attraction parameter **κ = 0.5**.

Type 1 phase transition (2nd order ph. tr.)

Field approach



2nd order phase transition

Thermodynamic approach

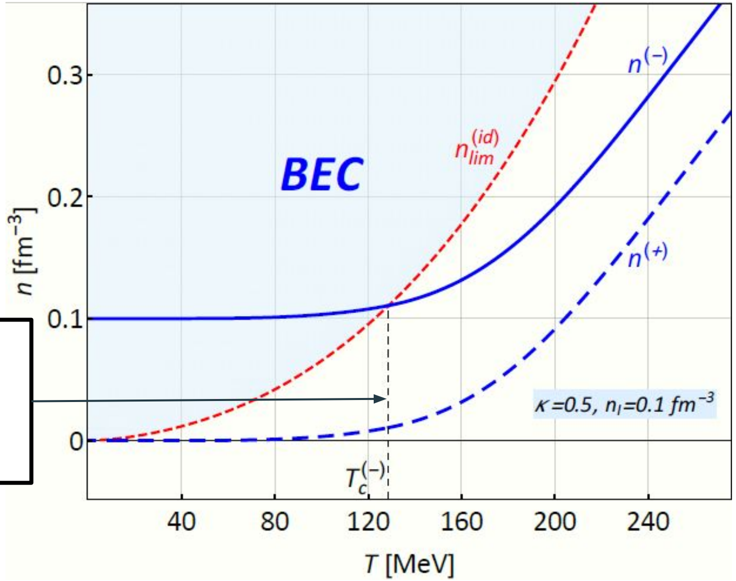


Figure 5. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for π^+-π^- pion gas with interaction within the framework of the scalar field approach. Isospin density is considered to be fixed, $n_I = 0.1\ fm^{-3}$, the attraction parameter **κ = 0.5**.

Figure 4. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas in the mean-field model. The total isospin density is kept constant, $n_I = 0.1\ fm^{-3}$, and the attraction parameter is **κ = 0.5**.

Type 1 phase transition (2nd order ph. tr.)

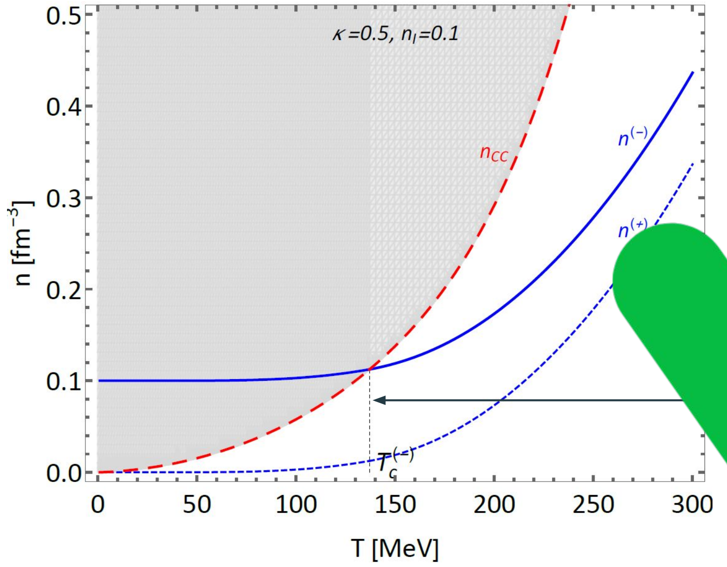


Figure 5. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for π^+-π^- pion gas with interaction within the framework of the scalar field approach. Isospin density is considered to be fixed, $n_I = 0.1\ fm^{-3}$, the attraction parameter **κ = 0.5**.

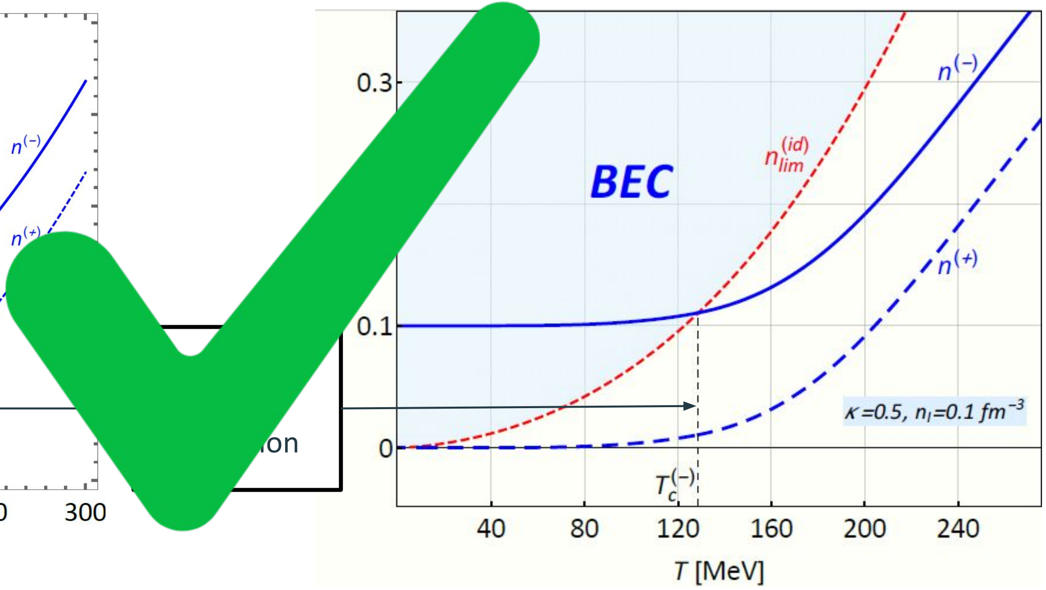


Figure 4. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas in the mean-field model. The total isospin density is kept constant, $n_I = 0.1\ fm^{-3}$, and the attraction parameter is **κ = 0.5**.

What will happen if we increase attraction parameter κ in such system?

Type 2 phase transition (2nd order ph. tr.)

Only particles π⁻ are depicted

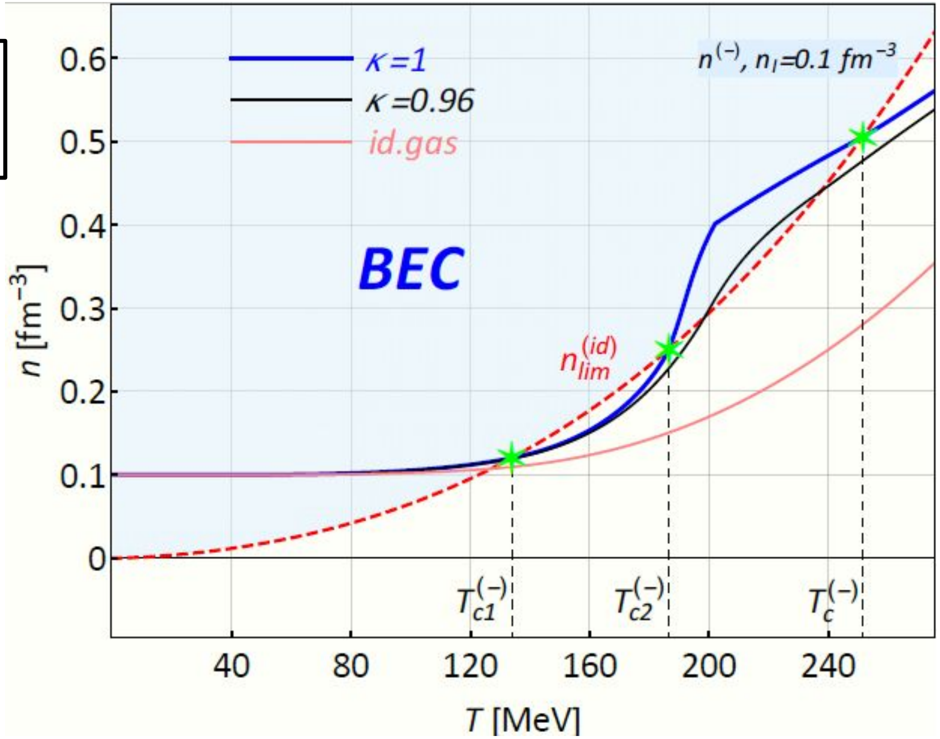
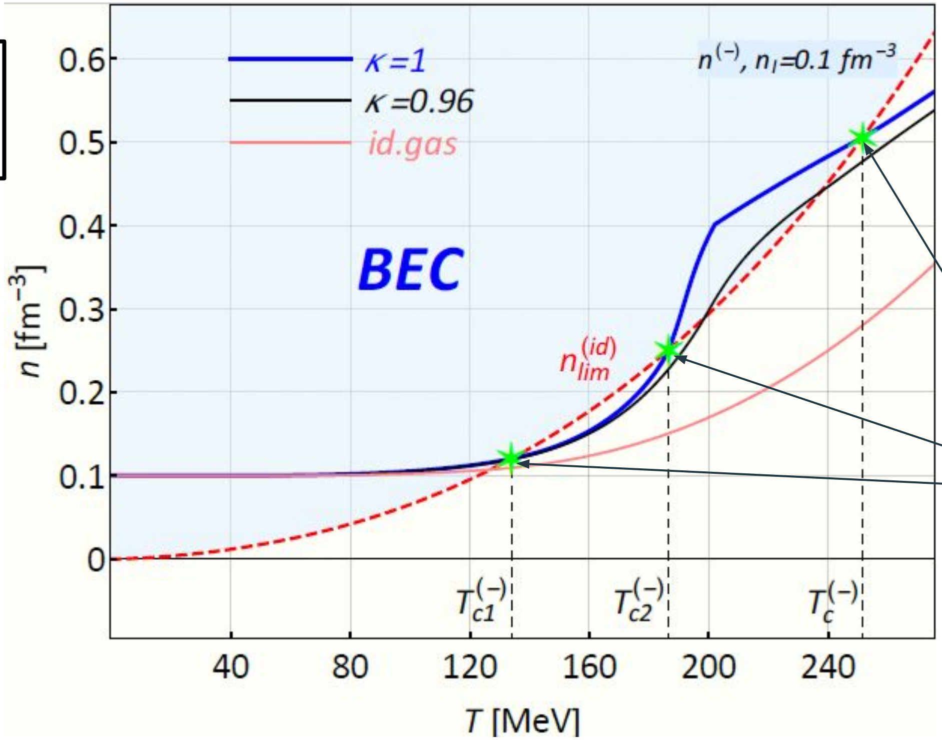


Figure 6. The particle density $n^{(-)}$ versus temperature for the interacting π^{+}-π^{-} pion gas, the attraction parameter is **κ = 1**.

Type 2 phase transition (2nd order ph. tr.)

Only particles π^- are depicted



2nd order phase transitions

Figure 6. The particle density $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas, the attraction parameter is **κ = 1**.

Type 3 phase transition (2nd order ph. tr.)

Only antiparticles π^+ are depicted

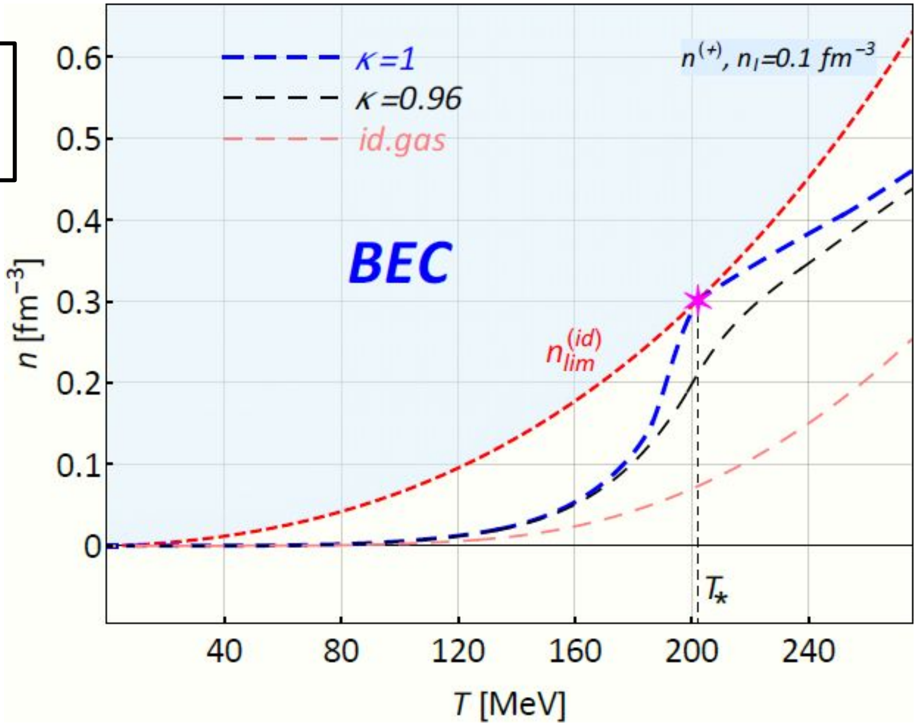
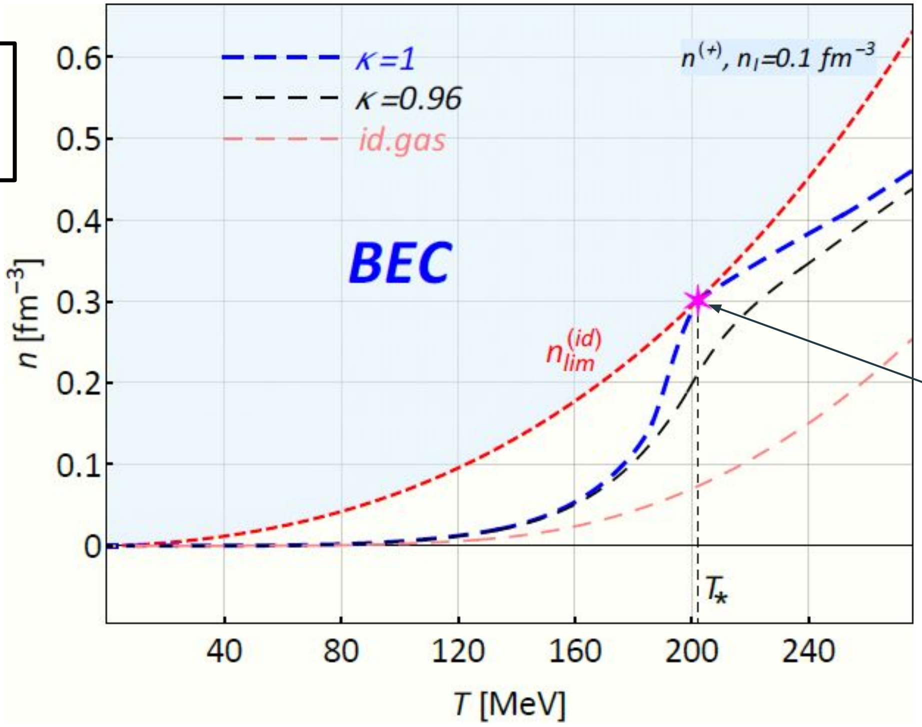


Figure 7. The particle density $n^{(+)}$ versus temperature for the interacting π^+-π^- pion gas, the attraction parameter is **κ = 1**.

Type 3 phase transition (2nd order ph. tr.)

Only antiparticles π^+ are depicted



2nd order point-like phase transition

Figure 7. The particle density $n^{(+)}$ versus temperature for the interacting π^+-π^- pion gas, the attraction parameter is **κ = 1**.

Type 3 phase transition (2nd order ph. tr.)

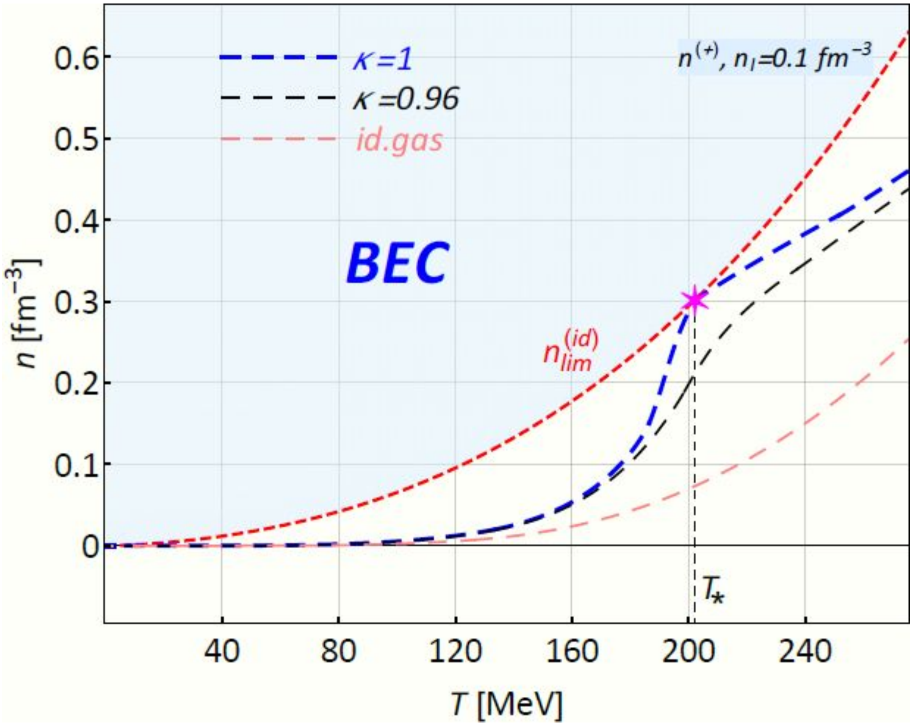
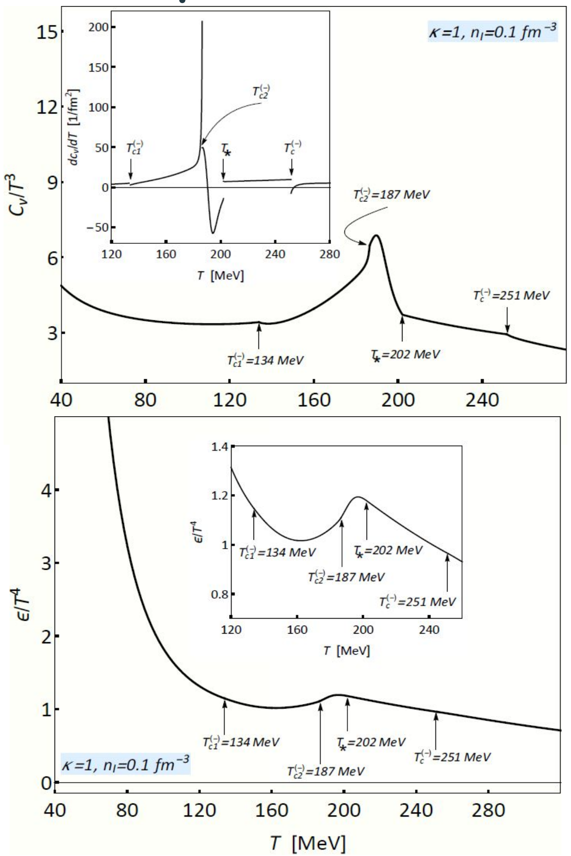


Figure 7. The particle density $n^{(+)}$ versus temperature for the interacting π^+-π^- pion gas, the attraction parameter is **κ = 1**.



Type 2,3 phase transition (2nd order ph. tr.)[3]

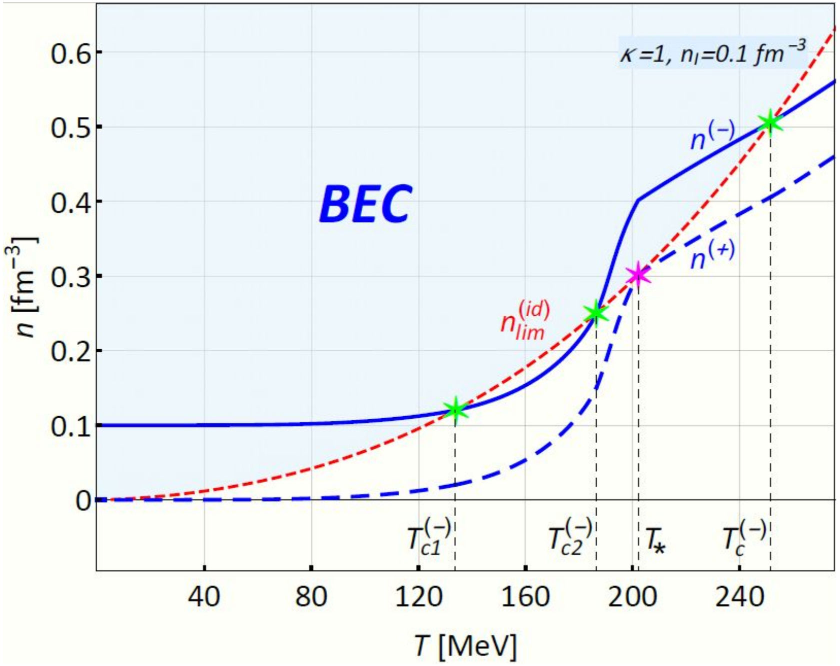


Figure 8. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^{+}-π^{-} pion gas. The attraction parameter is **κ = 1**.

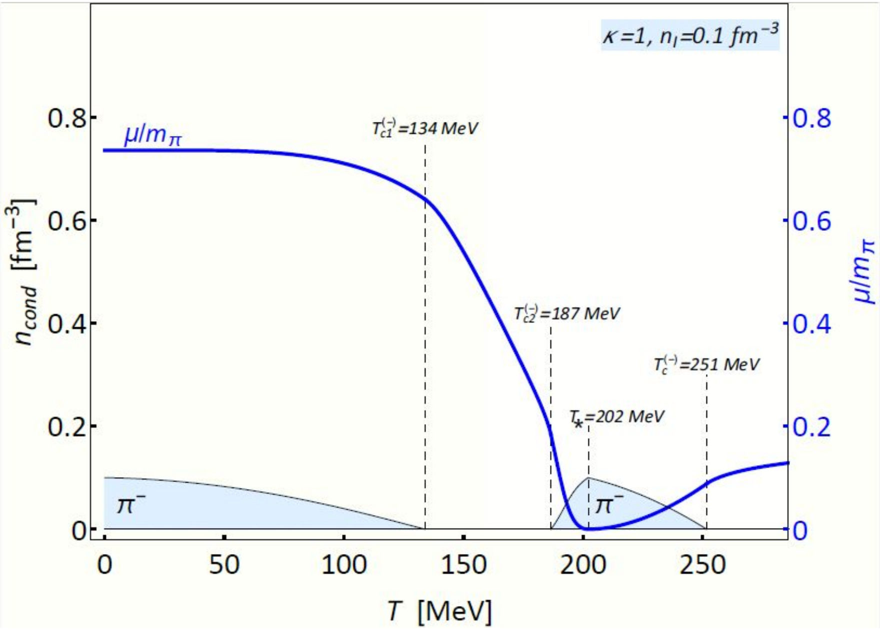


Figure 9. The condensate particle-number densities n_{cond} versus temperature. The attraction parameter is κ = 1.

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).

Point-like phase transition in scalar model

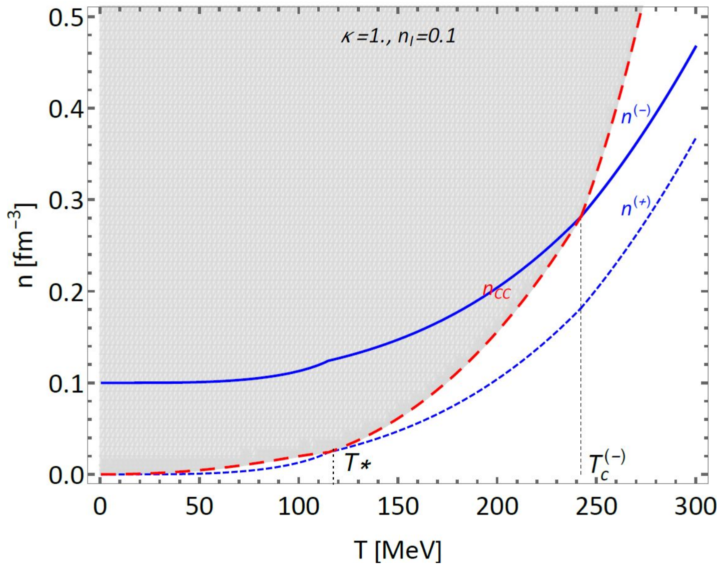
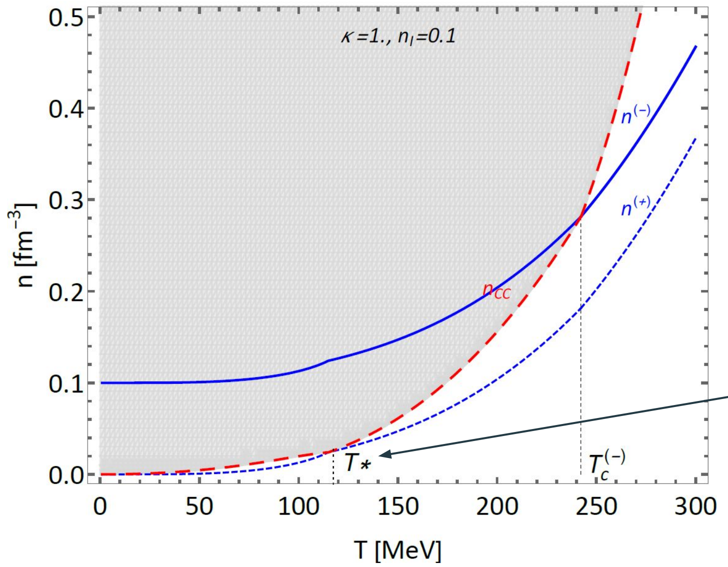


Figure 10. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for π^{+}-π^{-} pion gas with interaction within the framework of the scalar field approach. Isospin density is considered to be fixed, $n_I = 0.1\ fm^{-3}$, the attraction parameter **κ = 1**.

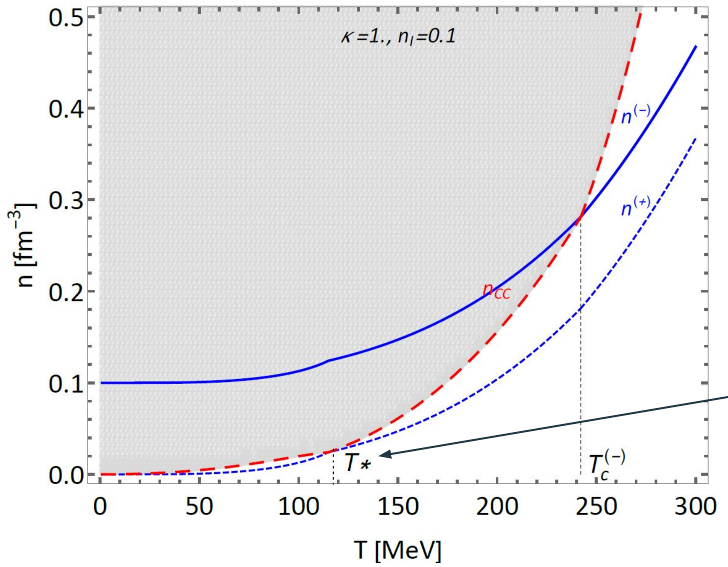
Point-like phase transition in scalar model



Point-like phase transition of the 2nd order?

Figure 10. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for π^{+}-π^{-} pion gas with interaction within the framework of the scalar field approach. Isospin density is considered to be fixed, $n_I = 0.1$ fm^{-3}, the attraction parameter **κ = 1**.

Point-like phase transition in scalar model



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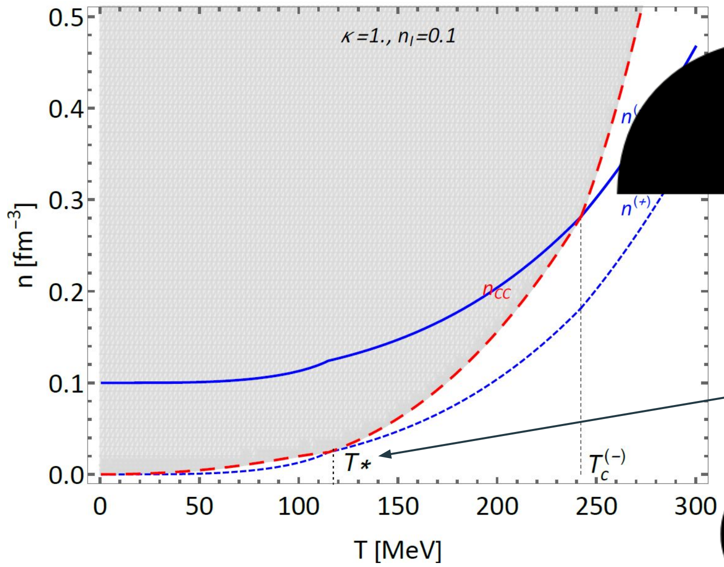
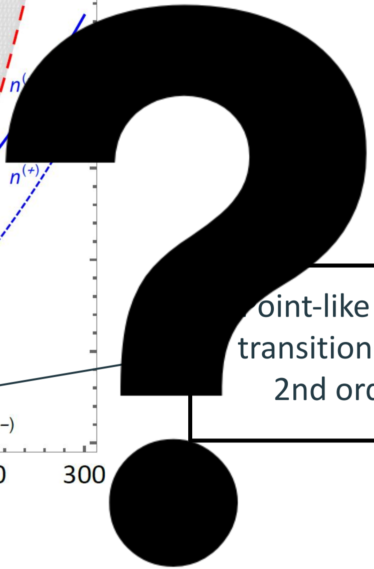
Point-like phase transition of the 2nd order?

Technical difficulties with describing the condensate phase within the scalar field approach

Point-like phase transition in scalar model





Technical difficulties with describing the condensate phase within the scalar field approach

oint-like phase transition of the 2nd order?

Figure 10. Dependence of particle density $n^{(+)}$, $n^{(-)}$ on temperature for π^{+}-π^{-} pion gas with interaction within the framework of the scalar field approach. Isospin density is considered to be fixed, $n_I = 0.1\ fm^{-3}$, the attraction parameter **κ = 1**.

To be continued…

What if κ is in over-critical regime?

$\kappa > 1$

Type 4 phase transition (1st order ph. tr.)[3,5]

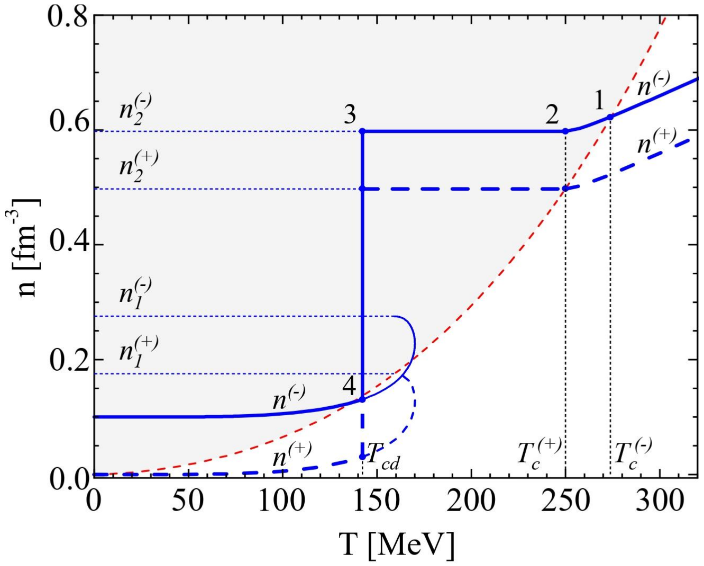
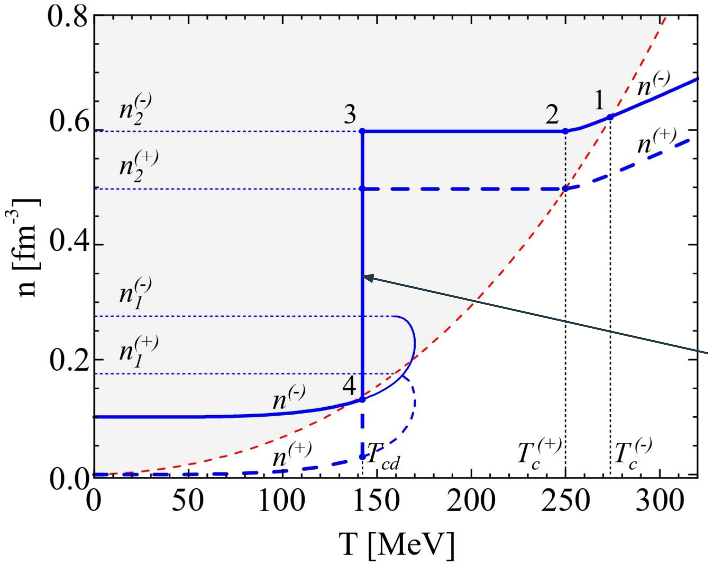


Figure 11. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas. The attraction parameter is **κ = 1.1**.

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).
[5] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).

Type 4 phase transition (1st order ph. tr.)[3,5]



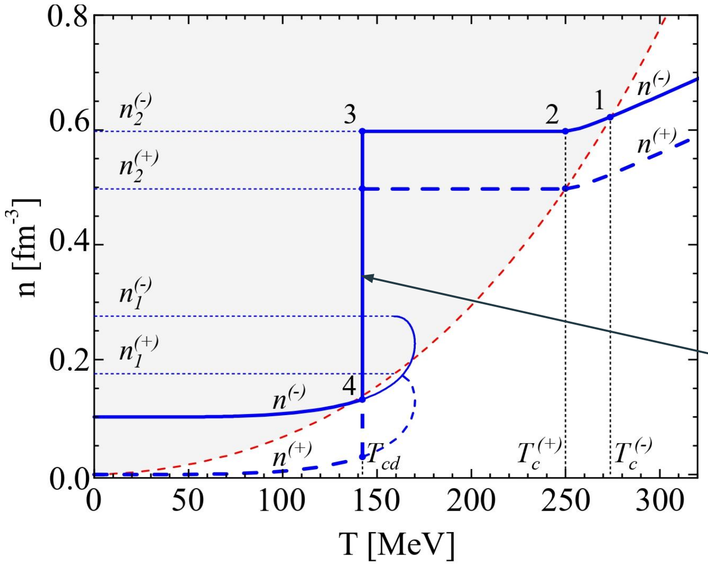
1st order phase transition

(Both components are in the condensate phase)

Figure 11. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas. The attraction parameter is **κ = 1.1**.

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).
[5] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).

Type 4 phase transition (1st order ph. tr.)[3,5]



1st order phase transition

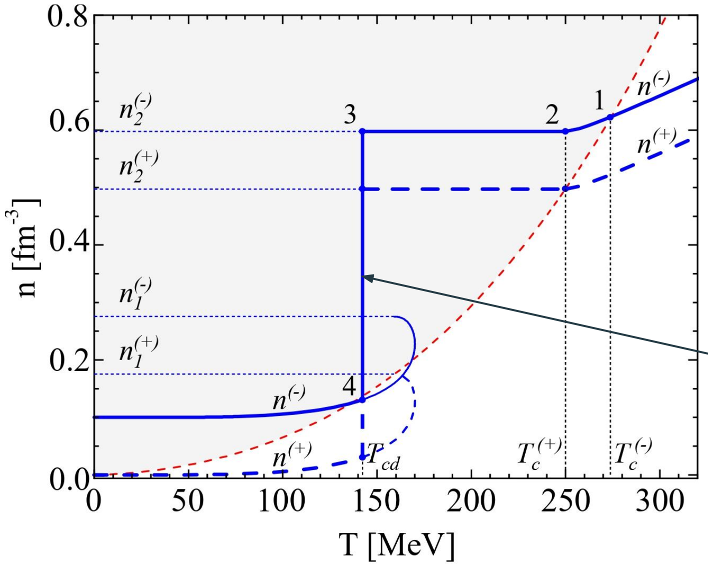
(Both components are in the condensate phase)

Figure 11. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas. The attraction parameter is **κ = 1.1**.

$$n_1 = \sqrt{\frac{m}{B}}\left(\kappa - \sqrt{\kappa^2 - 1}\right), \quad n_2 = \sqrt{\frac{m}{B}}\left(\kappa + \sqrt{\kappa^2 - 1}\right),$$

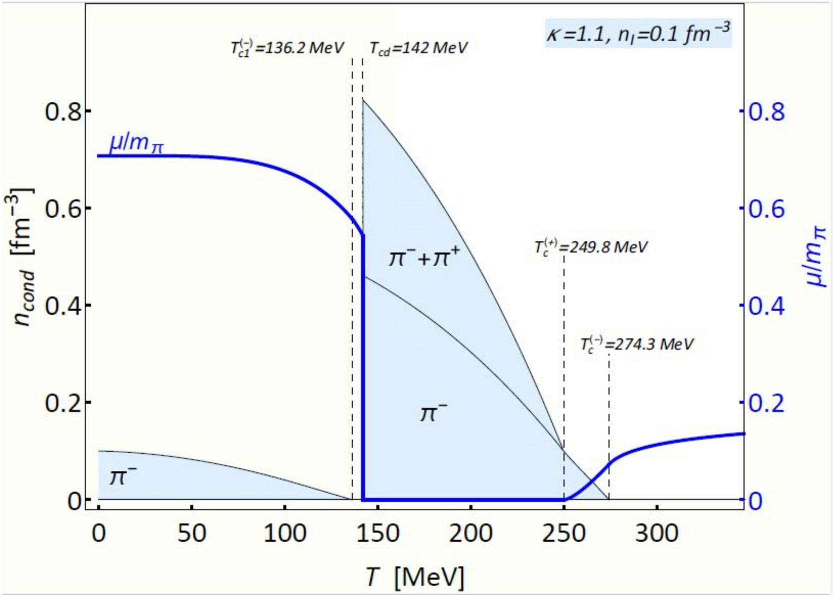
[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).
[5] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).

Type 4 phase transition (1st order ph. tr.)[3,5]



1st order phase transition

(Both components are in the condensate phase)



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$$n_1 = \sqrt{\frac{m}{B}}\left(\kappa - \sqrt{\kappa^2 - 1}\right), \quad n_2 = \sqrt{\frac{m}{B}}\left(\kappa + \sqrt{\kappa^2 - 1}\right),$$

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).
[5] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).

Comparison with lattice data

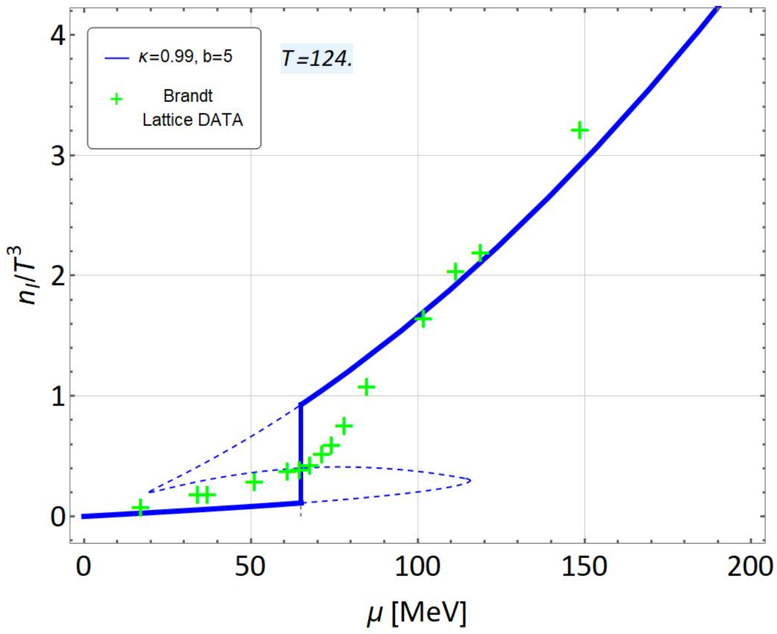


Figure 12. Dependence of normalized isospin density n_I/T^3 on chemical potential for π^+-π^- pions sytem within field approach showed with blue line. Lattice data for π-mesons configurations at T=124 MeV from [6].

[6] B. B. Brandt, G. Endrodi, S. Schmalzbauer, EPJ Web Conf. Volume 175 (2018).

Conclusions

- The mean-field thermodynamic model and the scalar field model were used to describe a 2-component system of interacting mesons (bosons) at high temperatures in the presence of condensate.
- 4 types of phase transitions in the system with repulsion and attraction were introduced: **type 1** - one component single phase transition (2nd order), **type 2** - one component multiple phase transition (2nd order), **type 3** - second component point-like phase transition (2nd order), **type 4** - both components phase transition (1st order).
- Parameters of interaction within scalar field approach were fitted to up-to-date lattice data. Attraction parameter k in this case is close to its critical value.

Thank you for your attention!

Over-critical regime with $n_I = 0$

Термодинамічна модель середнього поля

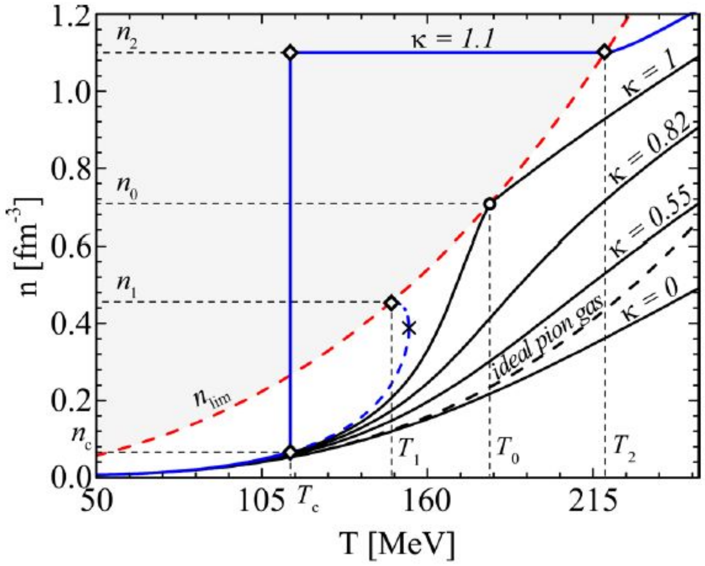


Figure 18. Залежність густин частинок n від температури для взаємодіючого газу π⁺-π⁻ піонів в рамках термодинамічної моделі середнього поля. Густина ізоспіну вважається сталою, **n_I=0**.

Модель скалярного поля

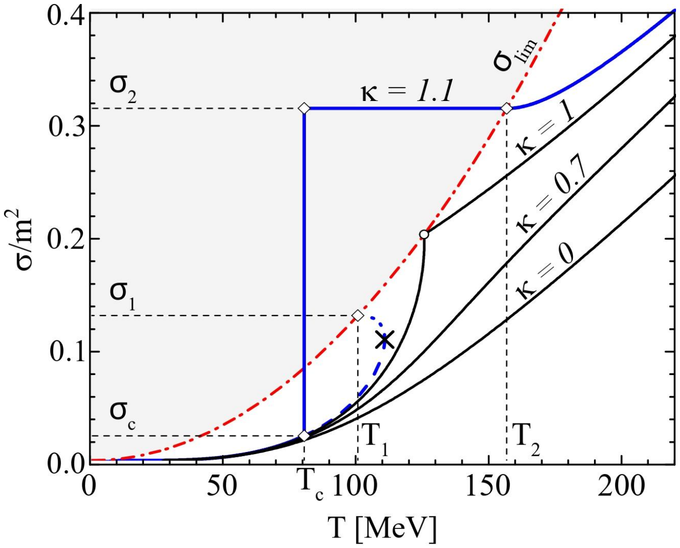
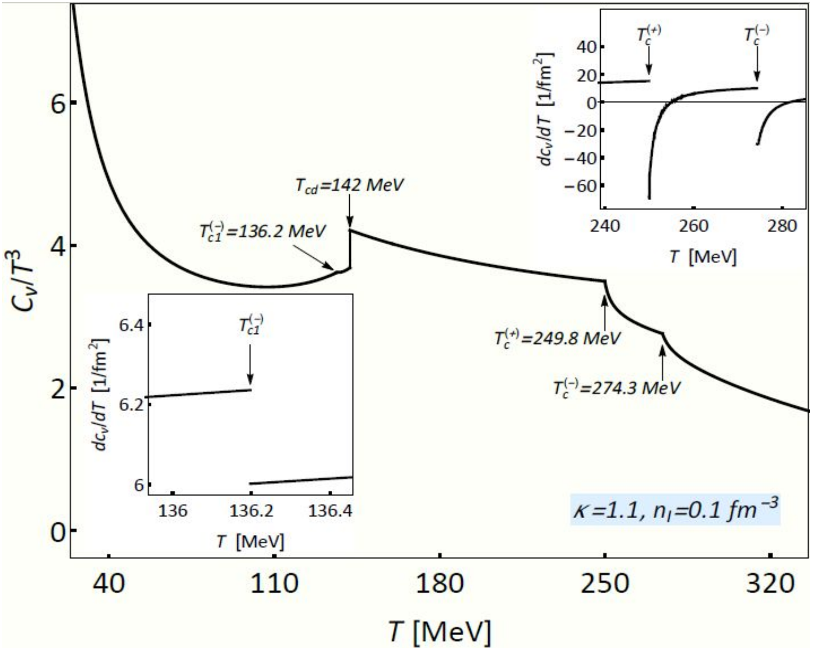
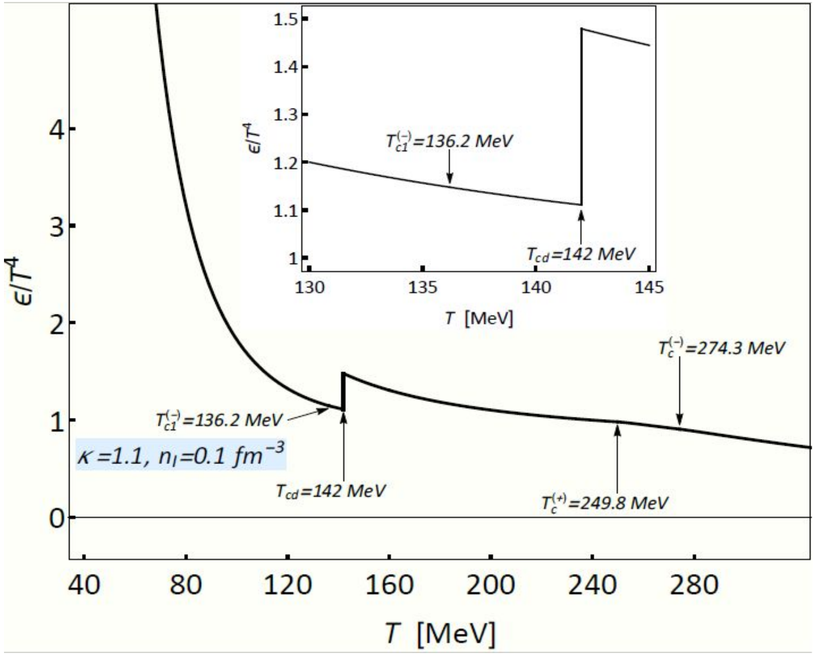


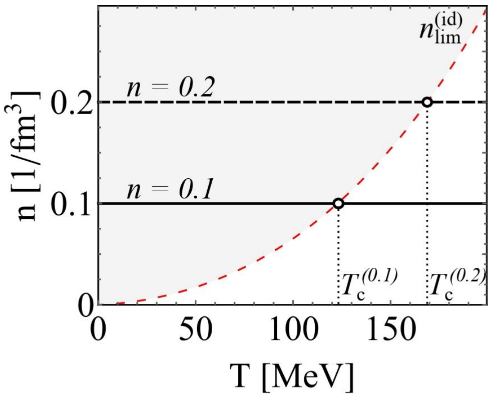
Figure 19. Залежність скалярної густини σ /m² від температури для газу π⁺-π⁻ піонів із взаємодією. Густина ізоспіну вважається сталою, **n_I = 0**.

Type 4 phase transition (1st order ph. tr.)[3,4]
Proof

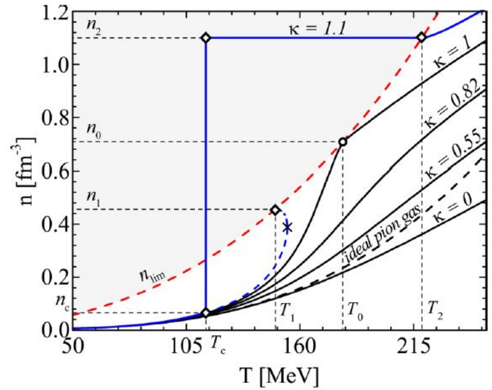




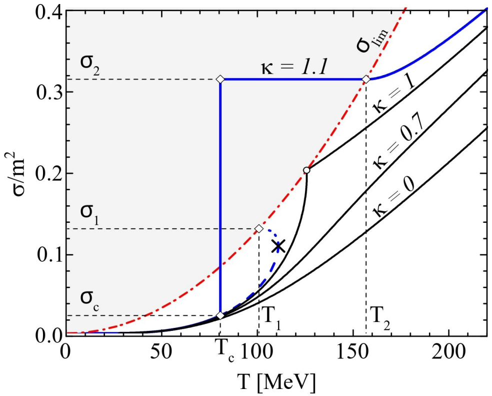
An ideal relativistic gas



Thermodynamic mean-field model



Scalar field model



Thermodynamic mean-field model

$$n = \int \frac{d^3k}{(2\pi)^3} \left[f_{\rm BE}\big(E(k,n),\mu_I\big) + f_{\rm BE}\big(E(k,n),-\mu_I\big) \right] ,$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left[f_{\rm BE}\big(E(k,n),\mu_I\big) - f_{\rm BE}\big(E(k,n),-\mu_I\big) \right] .$$

$$E(k,n) = \sqrt{m^2 + \boldsymbol{k}^2} + U(n)$$

$$f_{\rm BE}(E,\mu) = \left\{ \exp\left[\frac{E-\mu}{T}\right] - 1 \right\}^{-1}$$

Scalar field model

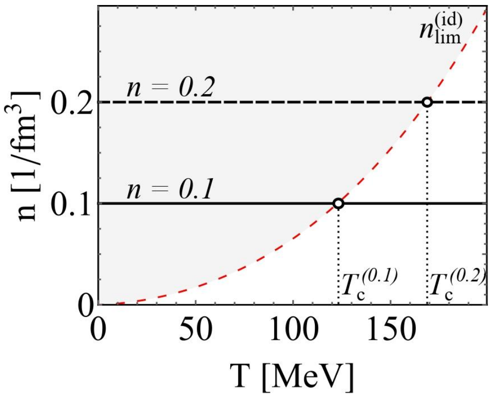
$$\sigma = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k(\sigma)} \left\{ f_{\rm BE}[\omega_k(\sigma),\mu_I] + f_{\rm BE}[\omega_k(\sigma),-\mu_I] \right\} ,$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left\{ f_{\rm BE}[\omega_k(\sigma),\mu_I] - f_{\rm BE}[\omega_k(\sigma),-\mu_I] \right\} .$$

$$\omega_k(\sigma) = \sqrt{M(\sigma)^2 + \boldsymbol{k}^2}$$

$$f_{\rm BE}(\omega,\mu_I) = \frac{1}{e^{(\omega-\mu_I)/T} - 1}$$

Ideal relativistic gas



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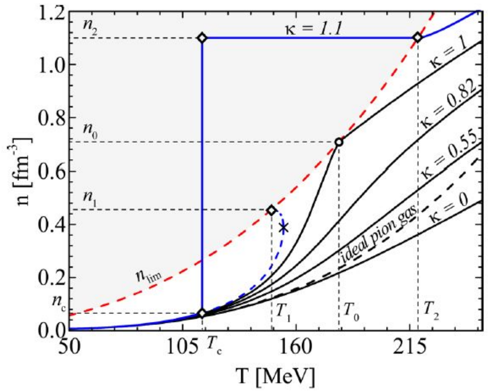
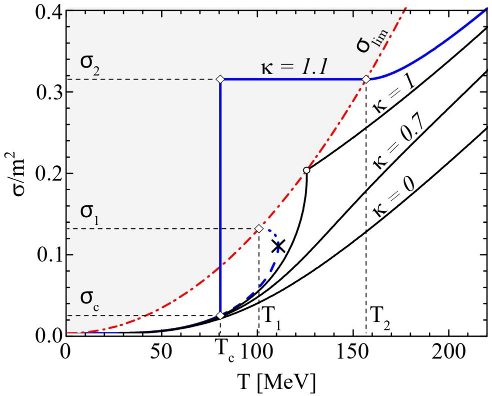


Scalar field model



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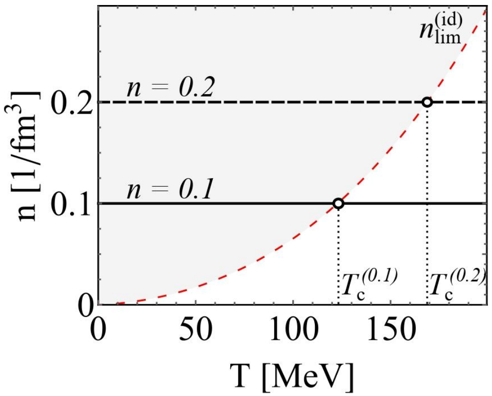
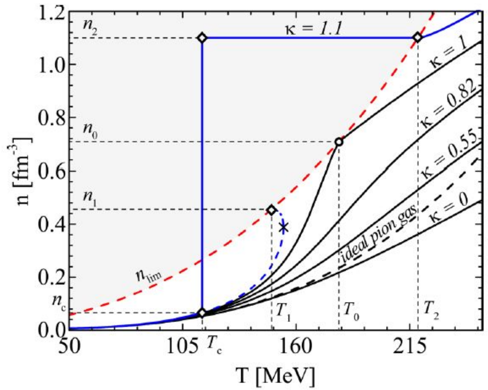
$$\omega_k(\sigma) = \sqrt{M(\sigma)^2 + \boldsymbol{k}^2} \quad , \quad M^2(\sigma) \equiv m^2 + U(\sigma)$$

$$f_{\rm BE}(\omega,\mu_I) = \frac{1}{e^{(\omega-\mu_I)/T} - 1}$$

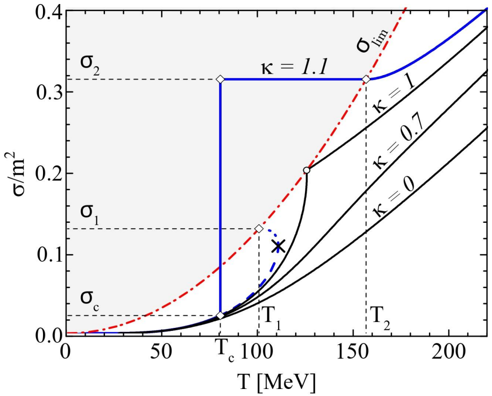
Ideal relativistic gas



Thermodynamic mean-field model



Scalar field model



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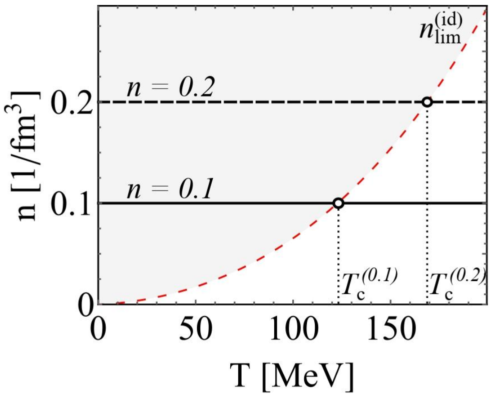
Ideal relativistic gas



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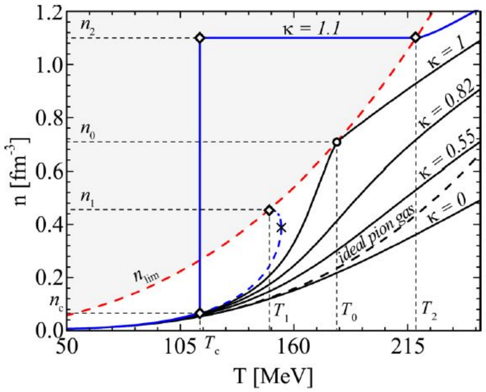
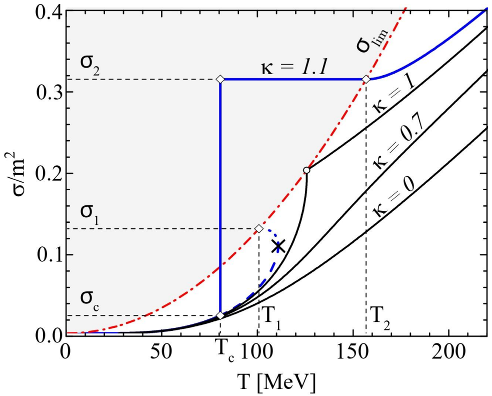


Scalar field model



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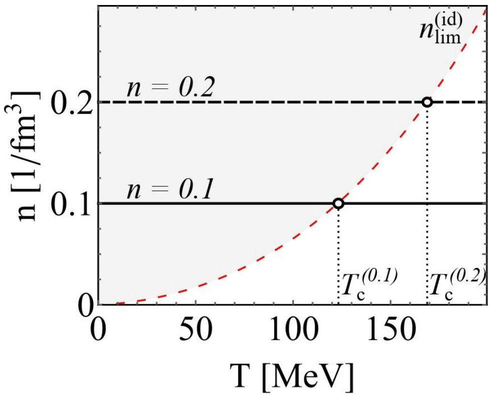
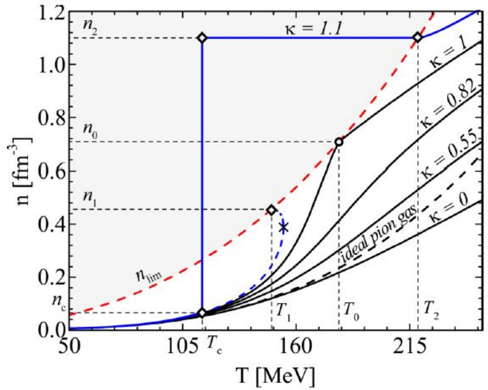
$$n^{(-)} = \int \frac{d^3k}{(2\pi)^3} f_{\bf BE}\left[\omega_k(\sigma),\mu_I\right], \quad n^{(+)} = \int \frac{d^3k}{(2\pi)^3} f_{\bf BE}\left[\omega_k(\sigma),-\mu_I\right]$$

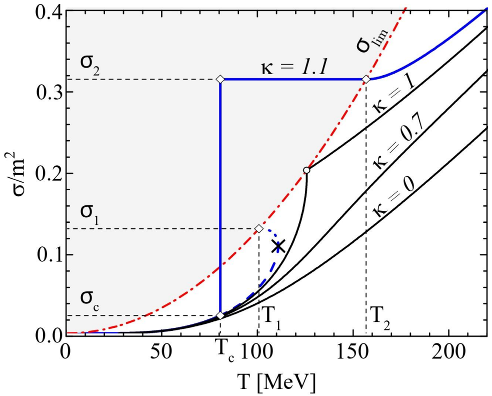
Ideal relativistic gas



Thermodynamic mean-field model



Scalar field model



Thermodynamic mean-field model

$$n = \int \frac{d^3k}{(2\pi)^3} \left[f_{\rm BE}(E(k,n),\mu_I) + f_{\rm BE}(E(k,n),-\mu_I) \right] ,$$

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$$f_{\rm BE}(E,\mu) = \left\{ \exp\left[\frac{E-\mu}{T}\right] - 1 \right\}^{-1}$$

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$$\sigma = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k(\sigma)} \left\{ f_{\rm BE}[\omega_k(\sigma),\mu_I] + f_{\rm BE}[\omega_k(\sigma),-\mu_I] \right\} ,$$

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$$n = \int \frac{d^3k}{(2\pi)^3} \left\{ f_{\rm BE}[\omega_k(\sigma),\mu_I] + f_{\rm BE}[\omega_k(\sigma),-\mu_I] \right\}$$

Thermodynamic mean-field model[1-3]

Starting point

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\rm int}(n, T),$$

Self-consistent set of equations

$$n = \int \frac{d^3k}{(2\pi)^3} \left[f_{\rm BE}(E(k,n), \mu_I) + f_{\rm BE}(E(k,n), -\mu_I) \right],$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left[f_{\rm BE}(E(k,n), \mu_I) - f_{\rm BE}(E(k,n), -\mu_I) \right].$$

Fixed

$$E(k,n) = \sqrt{m^2 + \boldsymbol{k}^2} + U(n)$$

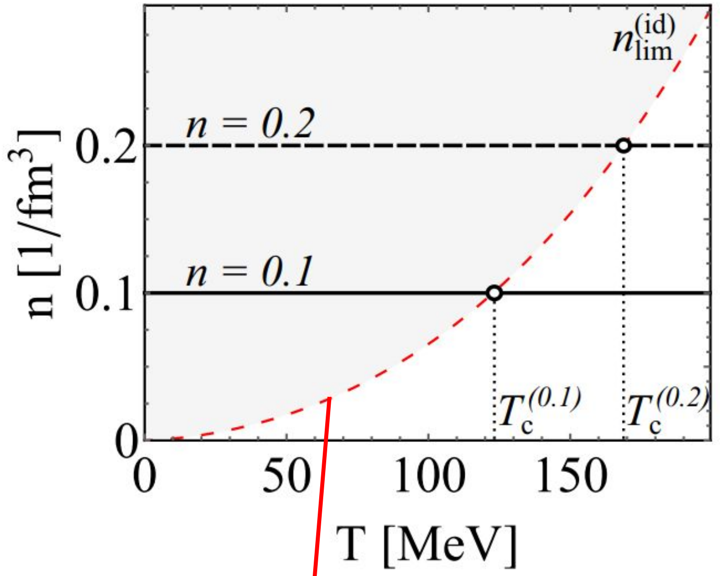
$$f_{\rm BE}(E, \mu) = \left\{ \exp\left[\frac{E-\mu}{T}\right] - 1 \right\}^{-1}$$

Parameterization of the mean-field

$$U(n) = -An + Bn^2, \quad n_1 = \sqrt{\frac{m}{B}}\left(\kappa - \sqrt{\kappa^2 - 1}\right), \quad n_2 = \sqrt{\frac{m}{B}}\left(\kappa + \sqrt{\kappa^2 - 1}\right),$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

Interplay between A and B
Parameter B - fixed



$$n_{\rm lim}^{\rm (id)}(T) = g \int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{1}{\exp\left[\frac{\sqrt{m^2+k^2}-m}{T}\right] - 1}$$

[1] D. Anchishkin and V. Vovchenko, J. Phys. G 42, 105102 (2015).
[2] D. Anchishkin, I. Mishustin, and H. Stoecker, J. Phys. G 46, 035002 (2019).
[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).

Introduction