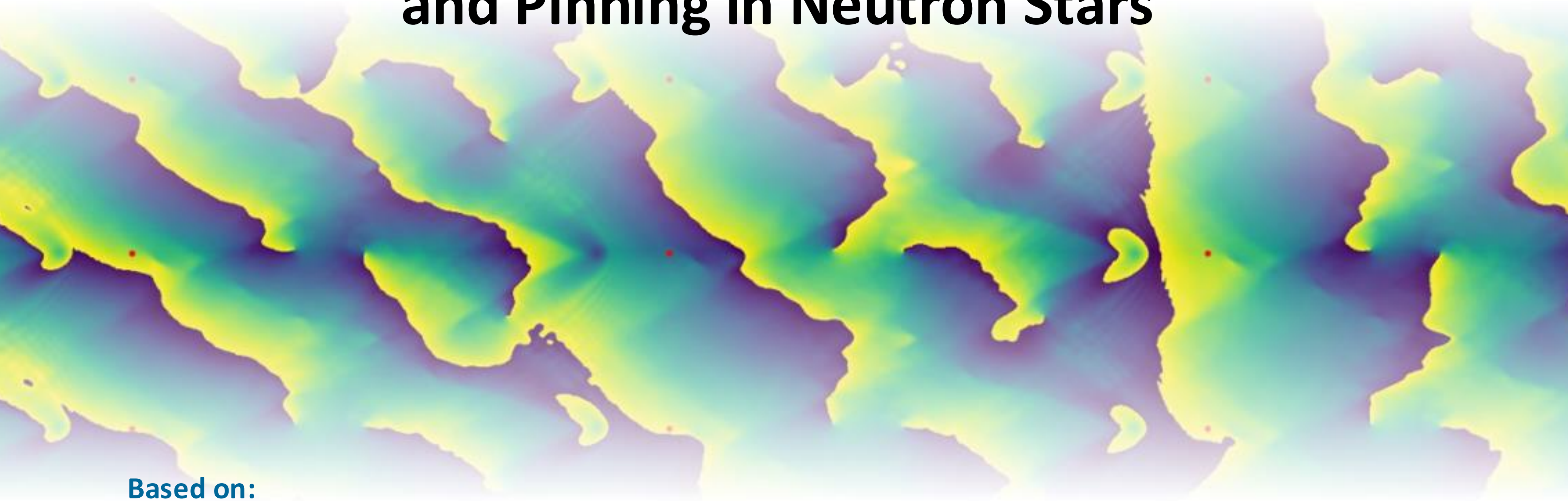
Bouquets of Topological Defects and Pinning in Neutron Stars



Based on:

A. Karekkat, G. Montefusco, M. Antonelli (in progress)

F. Magistrelli, M. Antonelli, PRA 113, 013307 (2026)

GINTONIC repo: https://github.com/Magistrelli/gintonic

Marco Antonelli

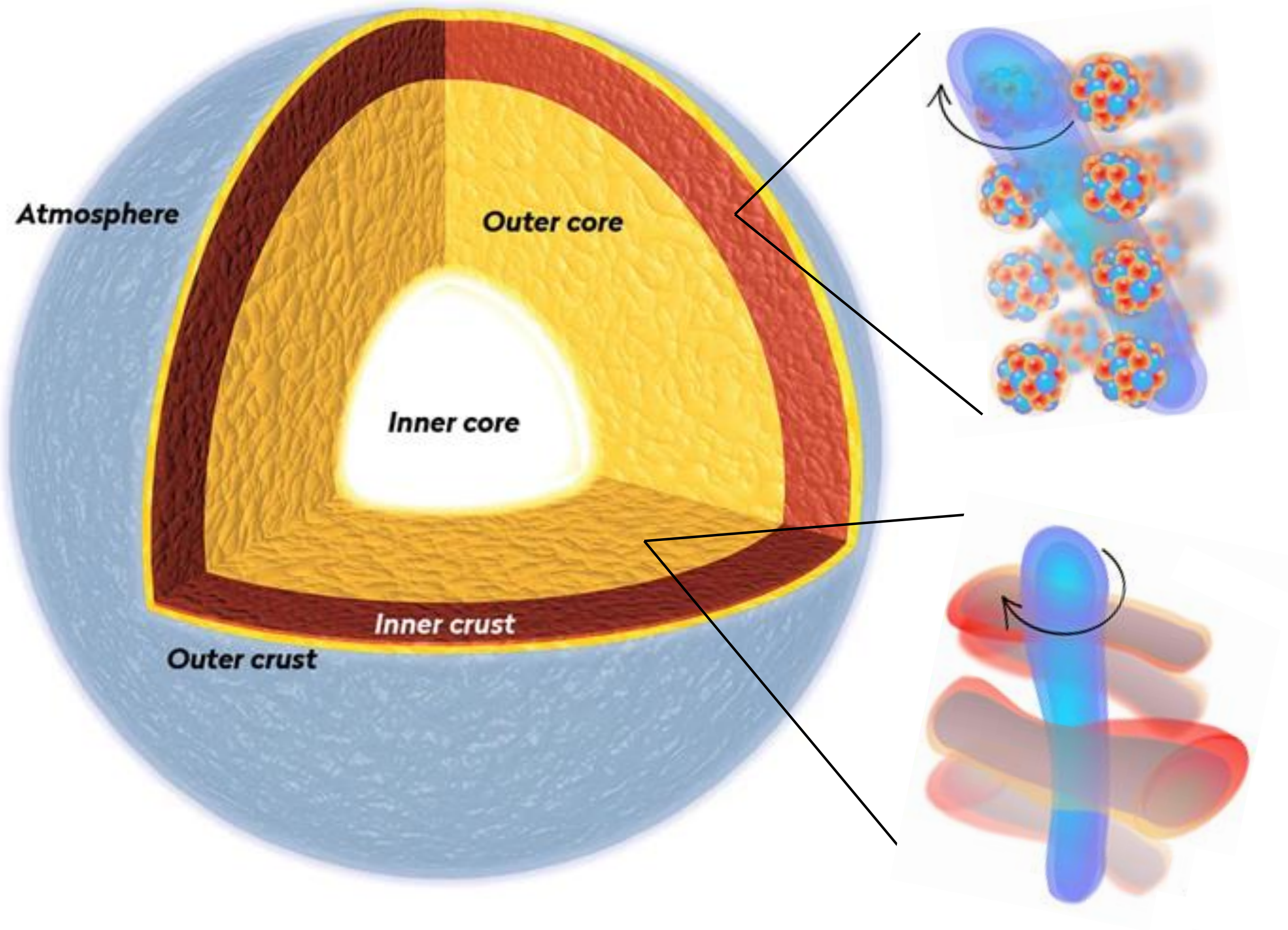
CNRS – LPC Caen

antonelli@lpccaen.in2p3.fr

June 2026 – University of Coimbra

Institut national de physique nucléaire et de physique des particules

Motivation: Topological Defects in Neutron Stars



<u>Inner crust</u>

Vortices in the **neutron superfluid** interact with the crustal lattice

→ 2D toy-model (GINTONIC)

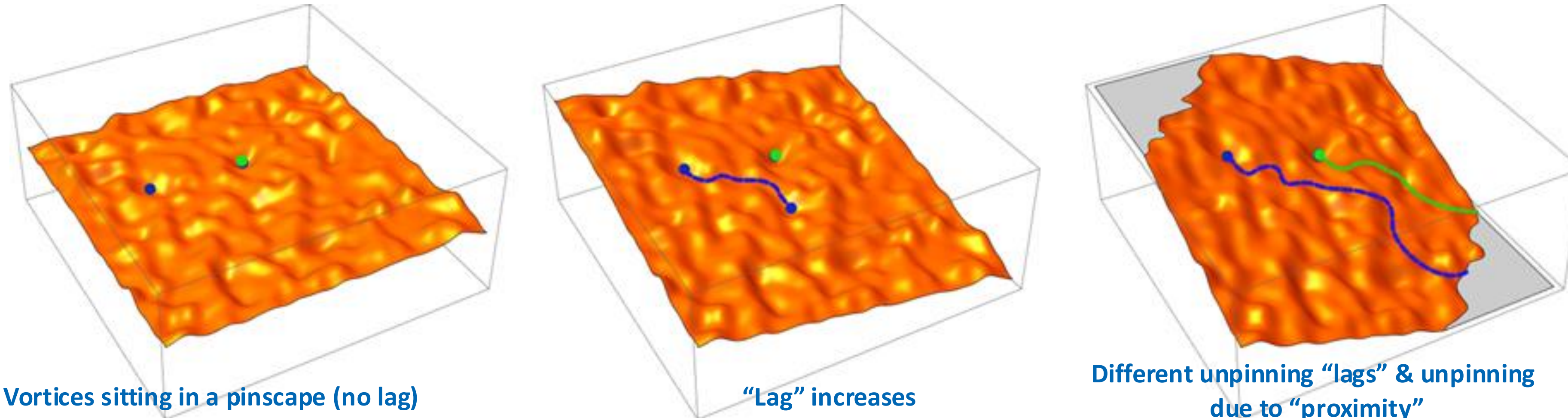
<u>Outer core</u>

Vortices in the **neutron superfluid** interact with the ***flux-tubes*** in the **proton superconductor**

→ 2D toy-model (work in progess by *Adarsh Karekkat*, PhD @LPC Caen)

Fluid elements in the bulk limit:
- total non-zero net vorticity
- no hard walls / confining potential
- possibility of sustaining homogeneous persistent currents

From vortex dynamics to hydrodynamics



Vortices sitting in a pinscape (no lag)

"Lag" increases
→ effective pinscape tilts

Different unpinning "lags" & unpinning due to "proximity"
→ possible avalanches

We can not take into account each vortex (~10^{16} in a pulsar): need to **average** over **many vortices** in the **bulk**

Macroscopic scale: Euler-like equations + **mutual friction**

$$\rho_n D_t \mathbf{v}_n + \ldots = \mathbf{F}_{MF}$$
$$\rho_p D_t \mathbf{v}_p + \ldots = -\mathbf{F}_{MF}$$

The **dynamics of vortices** in a fluid Element gives the form and strength of the macroscopic **"mutual friction"**

$$\mathbf{F}_{MF} = -\kappa\, n_v\, \hat{\boldsymbol{\kappa}} \times (\langle \dot{\boldsymbol{x}} \rangle - \boldsymbol{v}_{np})$$

"lag"

Average vorticity

Average vortex velocity (normal frame)

Antonelli, Montoli, Pizzochero arXiv:2301.12769 (2023)

From vortex motion to mutual friction

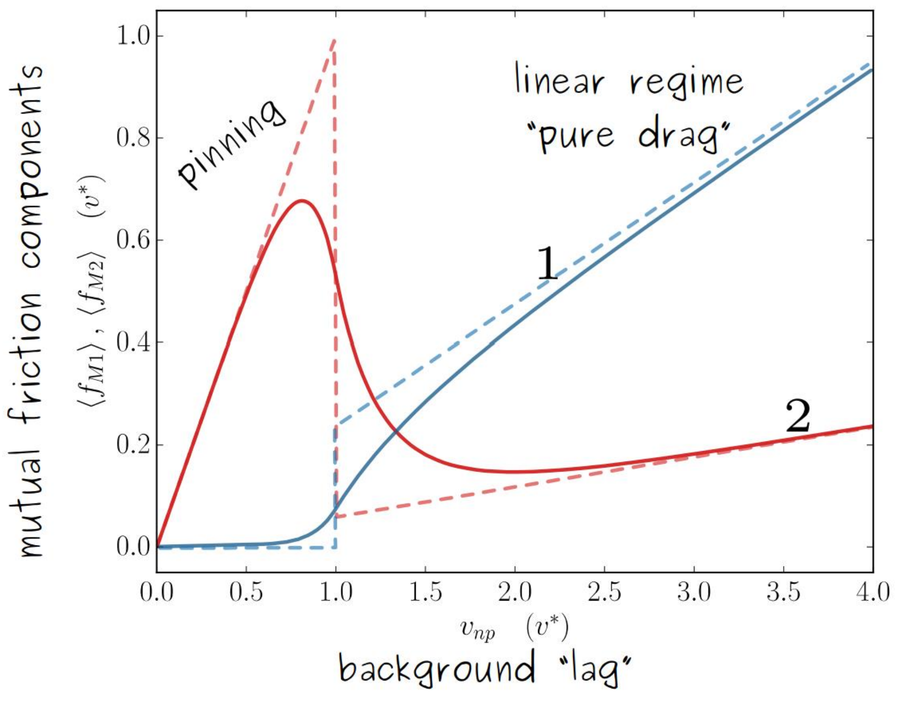
- Fix a background "lag" (velocity of superfluid neutrons in the pinscape's frame)

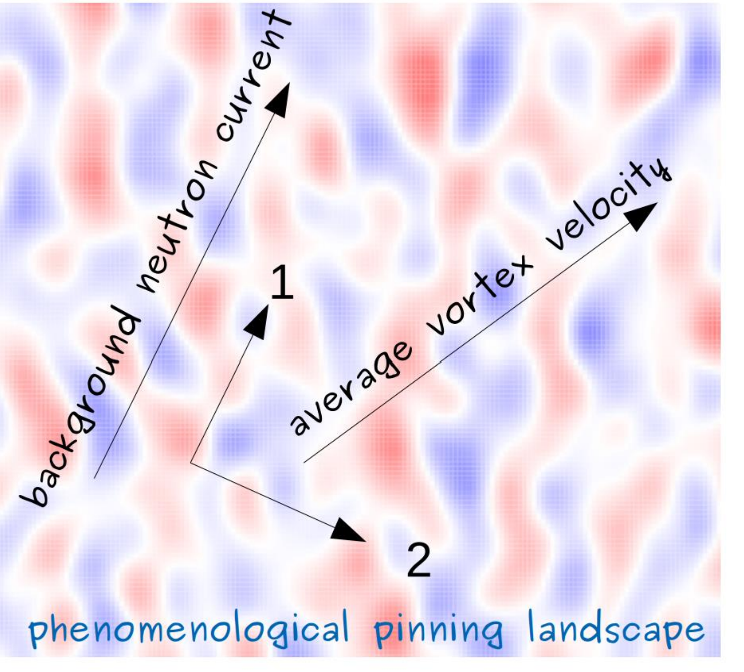
- Assign random position of a vortex in the pinning landscape and solve the trajectory:

$$\hat{\boldsymbol{\kappa}} \times (\dot{\boldsymbol{x}}(t) - \boldsymbol{v}_{np}) - \mathcal{R}\,\dot{\boldsymbol{x}}(t) + \boldsymbol{f} = 0$$

- Repeat many times and find the average vortex velocity for the given "lag".

- The mutual friction is given by: $\boldsymbol{F}_n = -\kappa\, n_v\, \hat{\boldsymbol{\kappa}} \times (\langle \dot{\boldsymbol{x}} \rangle - \boldsymbol{v}_{np})$





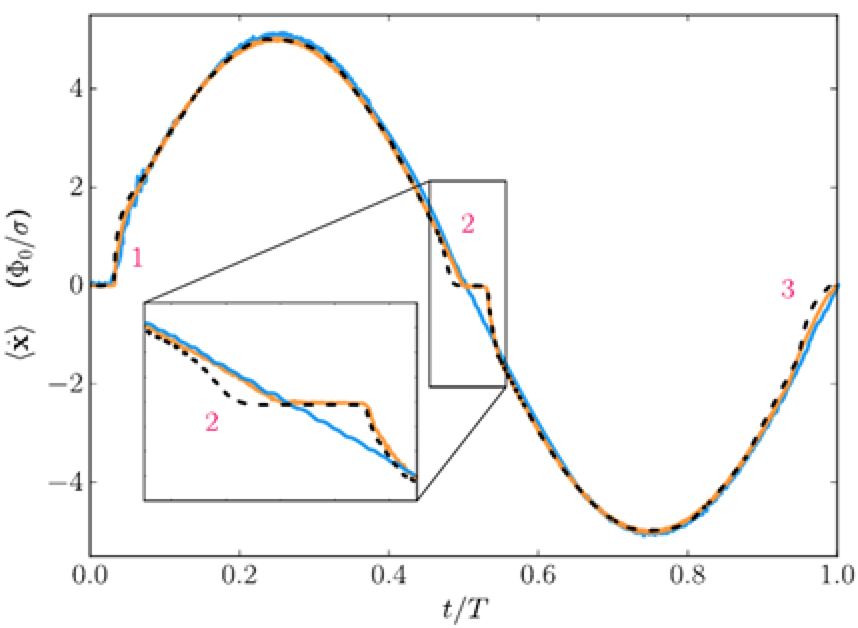
Antonelli & Haskell, MNRAS (2020)

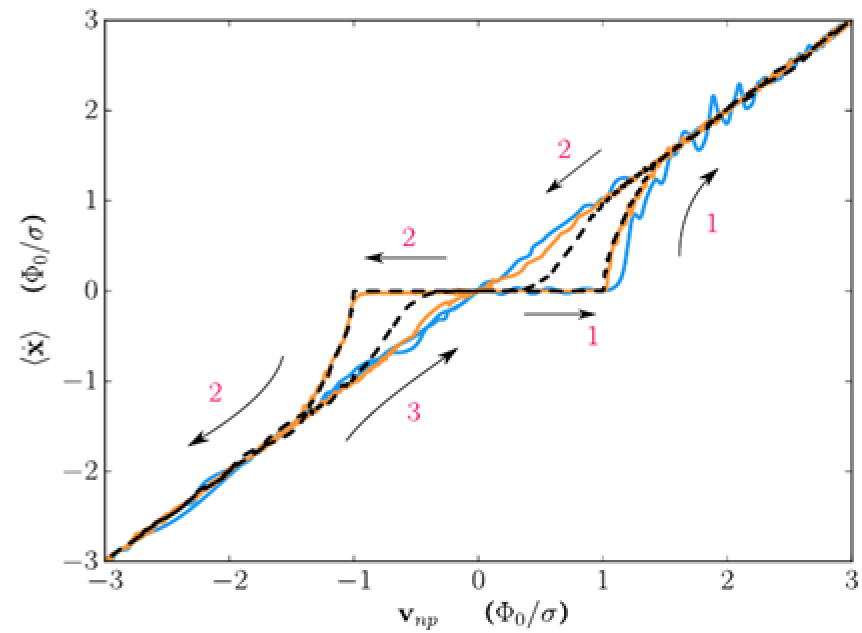
Beyond hydrodynamics: hysteresis

Rate–dependent hysteresis: lag between an input and an output that disappears if the input is varied more slowly. If the input is reduced to zero, the output continues to respond for a finite time.

Instantaneous drop to null lag → vortex velocity drops to zero immediately (NO pinscape)
Instantaneous drop to null lag → vortex velocity relaxes to zero (with pinscape)

Rate–independent hysteresis → vortex-vortex interactions





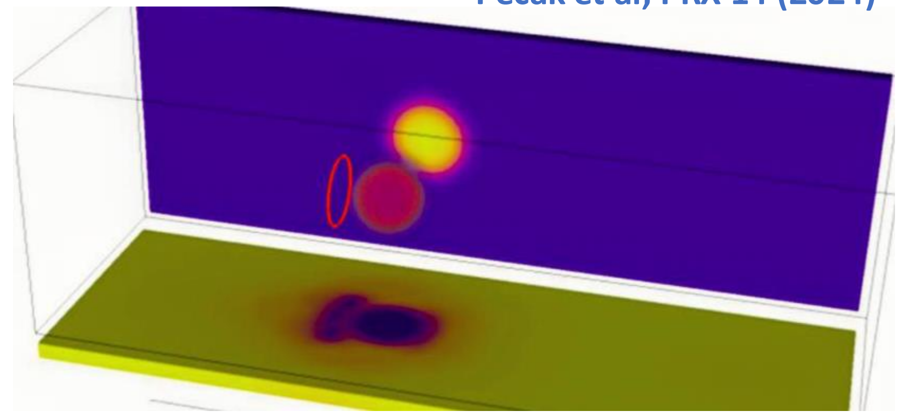
Upgrade to 3D (vortex lines): possible way to enhance hysteresis

Link & Levin, ApJ 2022

Antonelli & Haskell, MNRAS (2020)

Two kinds of vortex simulations

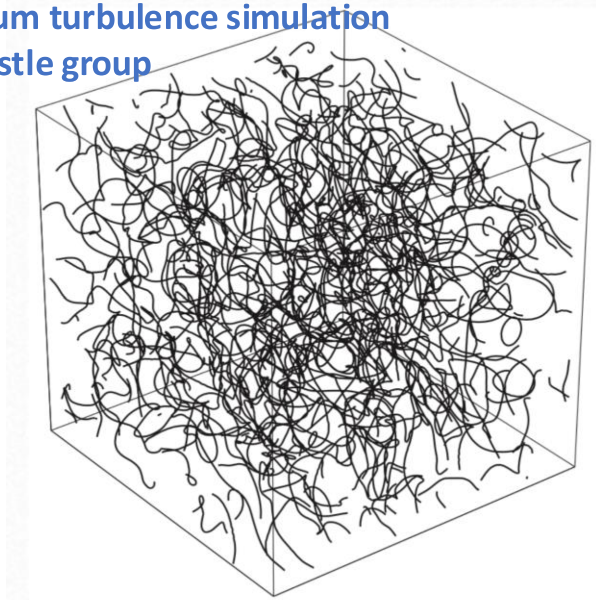
Pecak et al, PRX 14 (2024)

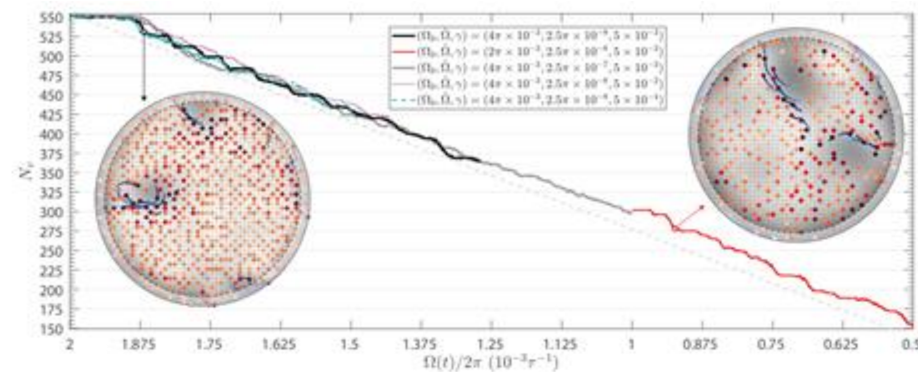


“Fluid element” → PBC

- No boundaries (possible homogeneous persistent currents)
- Zero total vorticity (each vortex is a loop)

Quantum turbulence simulation
Newcastle group

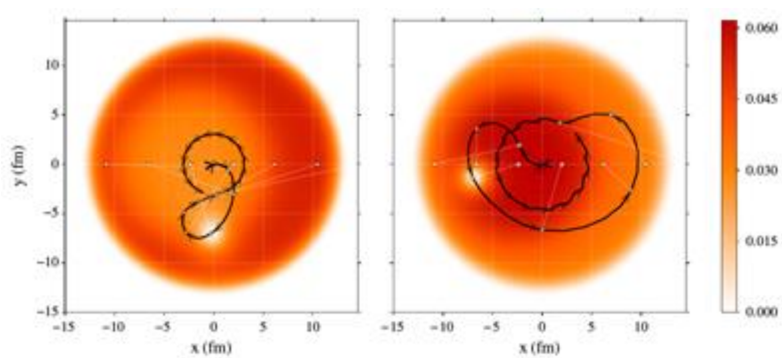




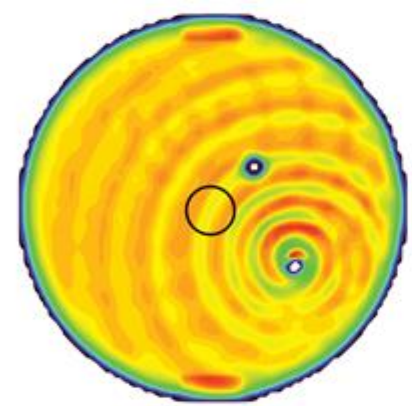
Liu et al,
ApJ Lett 2025

“Whole system”

- Walls / trap
- Possible finite vorticity
- Possible global rotation



Bulgac et al. PRL 2013



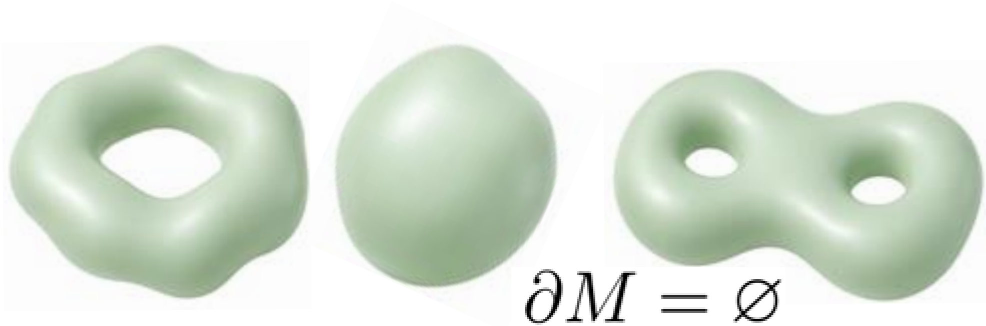
Warszawski et al.
MNRAS 2012

Topological obstruction

Two **incompatible** requests:

→ total non-zero net vorticity (non-zero net number of vortices in the domain)
→ computational domain with no boundaries (finite-size proxy for the bulk limit)

Take a surface without boundaries and a 1-form u field on it



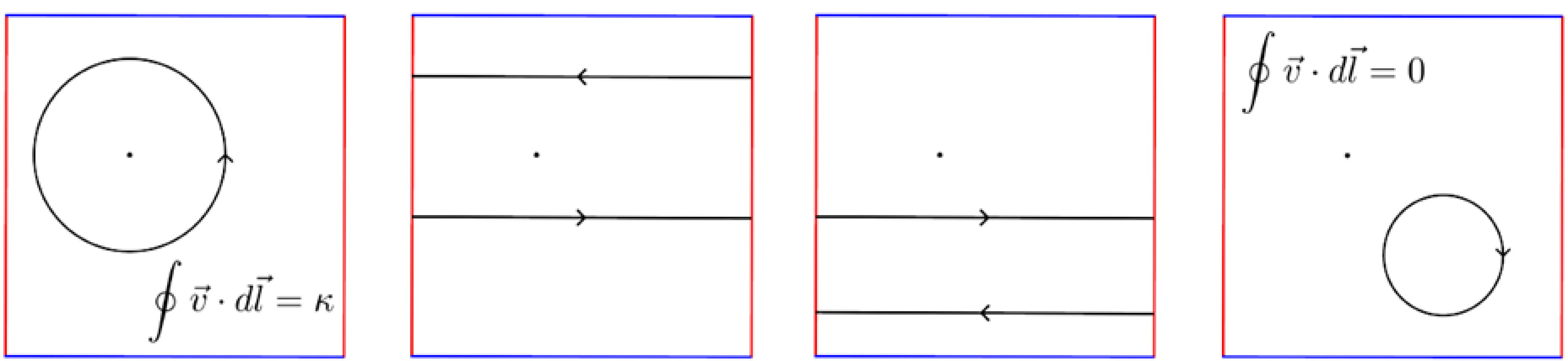
$$\partial M = \varnothing$$

Stokes theorem

$$\int_M du = \int_{\partial M} u = 0$$

Vorticity 2-form (for du) — **Velocity 1-form** (for u)

Our specific case: the impossibility of a quantised vortex on a flat torus



Quasi-Periodic Boundary Conditions

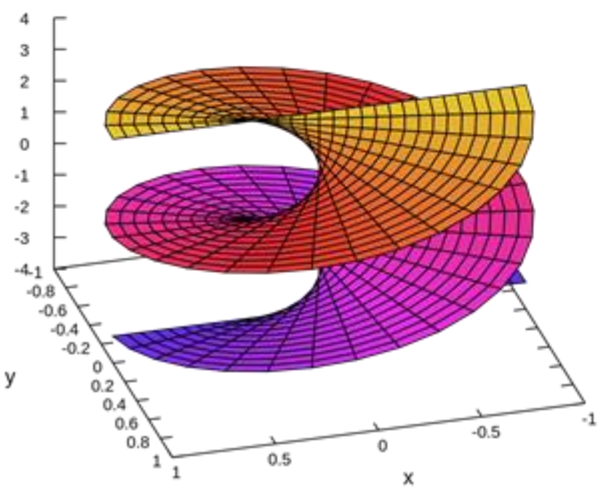
Tkachenko, Sov. Phys. JETP 22, 1282 (1966)
Wood et al, PRB 100, 024505 (2019)
Doran & Billam, PRE 102, 033309 (2020)

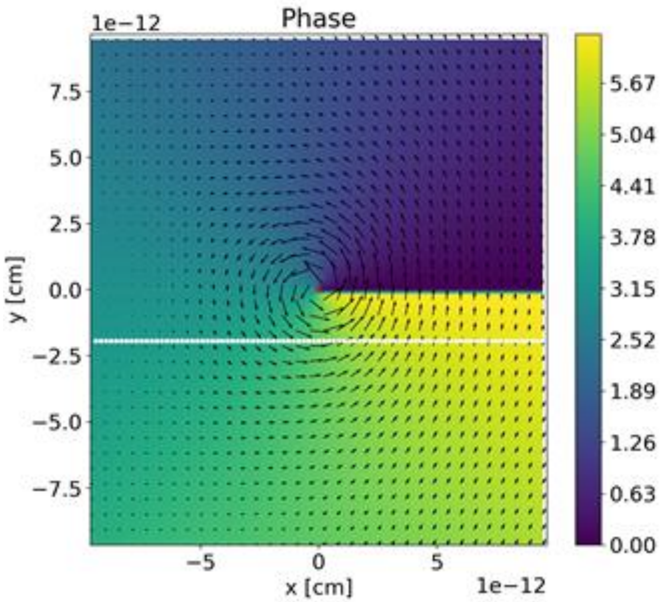
Complex order parameter or collective wave function

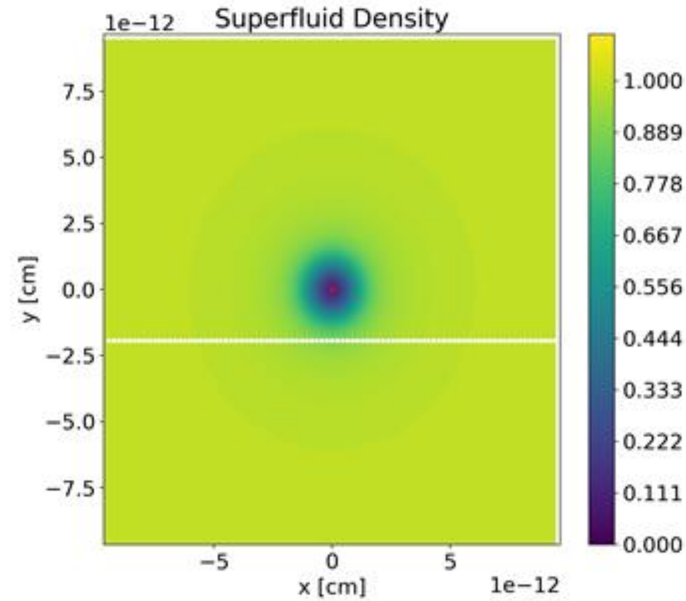
$$\psi(\vec{x},t) = \sqrt{n(\vec{x},t)}e^{i\theta(\vec{x},t)}$$

density field — phase field

Phase field for a single-vortex configuration







QPBC on the **phase** field:

$$\theta(\vec{x}+\vec{L}) = \theta(\vec{x}) + \frac{\pi}{V_0}(\vec{N}_v \times \vec{L})\cdot\vec{x} + c \mod 2\pi$$

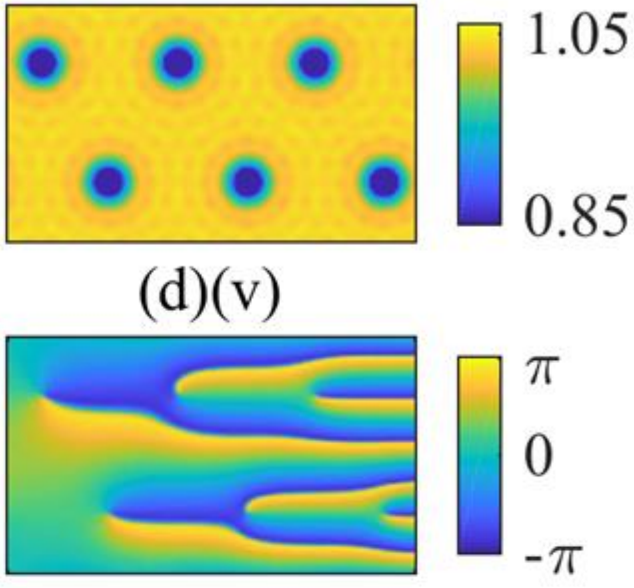
PBC on the **density** field:

$$n(\vec{x}+\vec{L},t) = n(\vec{x},t)$$

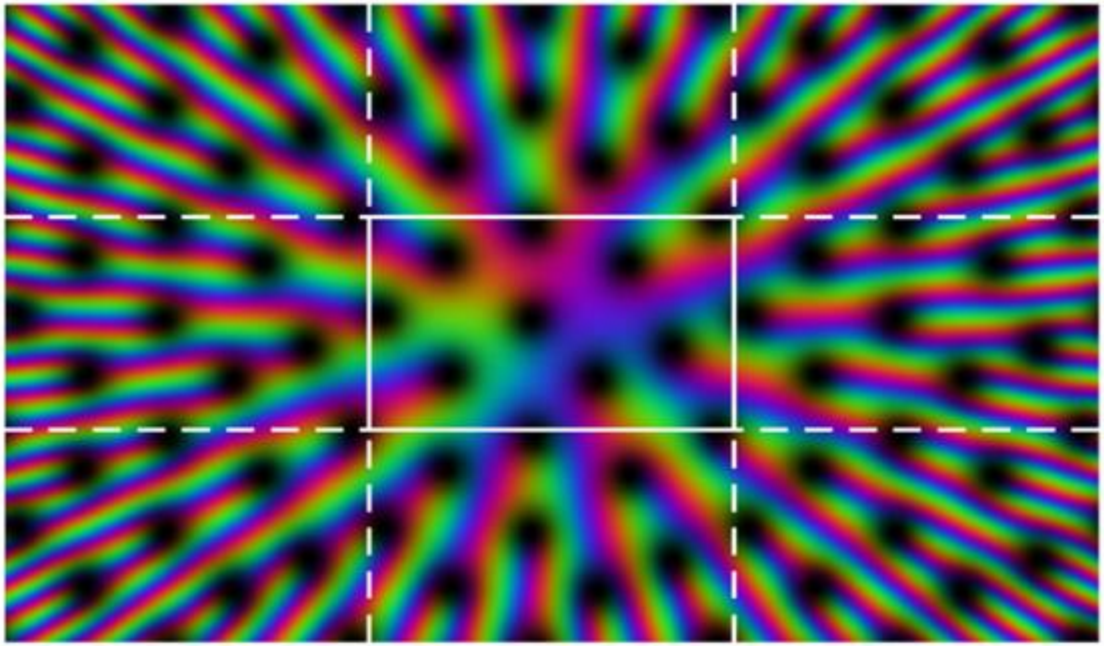
Vortex lattices

Natural application of QPBC is the study of vortex lattices!

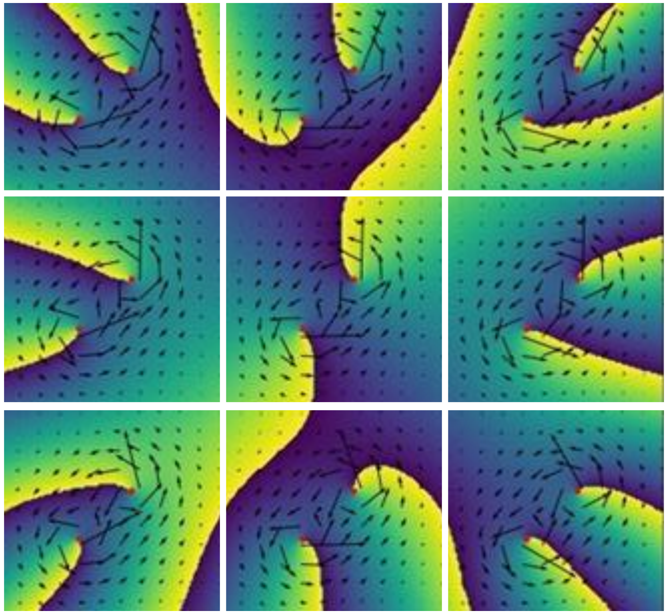
Tkachenko, Stability of vortex lattices, JETP, 1966
Mingarelli et al, J. Phys. Cond. Matt. (2016)
Mingarelli et al, PRA (2018)



Doran & Billam, PRE 2024



Wood et al, PRB 2019



Full information is all in the “central” cell, then it can be uniquely propagated to get the phase field everywhere.

2 vortices pinned in a cell (GINTONIC)

GINTONIC code

https://github.com/Magistrelli/gintonic

GPE equation
in code units

$$(i-\gamma)\,\partial_t\psi = \left[\frac{|\psi|^2}{n_\infty} + \frac{V(\boldsymbol{x},t)}{\mu} - 1 - \frac{\nabla^2}{2} + i\boldsymbol{\Omega}\cdot(\boldsymbol{x}\times\boldsymbol{\nabla})\right]\psi$$

length	time	density	speed
$\xi = \hbar/\sqrt{m\mu}$	$\tau = \hbar/\mu$	$n_\infty = \mu/g$	$c_1 = \sqrt{\mu/m}$

$$2\boldsymbol{\Omega} = \frac{h}{m}\frac{\boldsymbol{N}_v}{V_0}$$

Collective wavefunction decomposition (real – imaginary) and QPBC:

$$\psi = R + iI$$

$$R(\boldsymbol{x}+\boldsymbol{L},t) = R(\boldsymbol{x},t)A(\boldsymbol{x},t) - I(\boldsymbol{x},t)B(\boldsymbol{x},t)$$

$$I(\boldsymbol{x}+\boldsymbol{L},t) = I(\boldsymbol{x},t)A(\boldsymbol{x},t) + R(\boldsymbol{x},t)B(\boldsymbol{x},t)$$

$$\theta(\boldsymbol{x}+\boldsymbol{L}) = \theta(\boldsymbol{x}) + \frac{\pi}{V_0}(\boldsymbol{N}_v\times\boldsymbol{L})\cdot\boldsymbol{x} + c \mod 2\pi$$

$$A(\boldsymbol{x},t) = \cos\left[\theta(t)\big|_{\boldsymbol{x}}^{\boldsymbol{x}+\boldsymbol{L}}\right]$$

$$B(\boldsymbol{x},t) = \sin\left[\theta(t)\big|_{\boldsymbol{x}}^{\boldsymbol{x}+\boldsymbol{L}}\right]$$

GINTONIC code

https://github.com/Magistrelli/gintonic

→ Documentation: present (hopefully clear enough)
→ C++ backbone & Python scripts for plotting / analysis
→ Parameters: cell geometry, resolution, initial config. (vortex positions, solitons), ext. potential
→ Phenomenological dissipation (if zero, #particles conserved very well)
→ Total vorticity (net number of vortices): **always conserved exactly**
→ If you try to use it, please give us feedback :)

README   MIT license

GINTONIC - Gross-pitaevskii INtegrator on a TOrus with Non-zero vortICity

Main developer: Fabio Magistrelli - fabio.magistrelli@uni-jena.de
Co-developer: Marco Antonelli - antonelli@lpccaen.in2p3.fr

Description

The code integrates the Gross-Pitaevskii equation (GPE) to simulate the dynamics of a superfluid made of spinless particles and immersed in an external background potential. A set of Quasi-Periodic Boundary Conditions (QPBC) is employed to study the bulk properties of the superfluid, allowing for systems with non-zero net vorticity to be represented on a torus.

Reference paper (ref [1] in the code): "Dynamics of quantized vortices under quasi-periodic boundary conditions" arXiv:2509.15298

arXiv-2509.15298 Latest
on Sep 22, 2025

Packages

No packages published

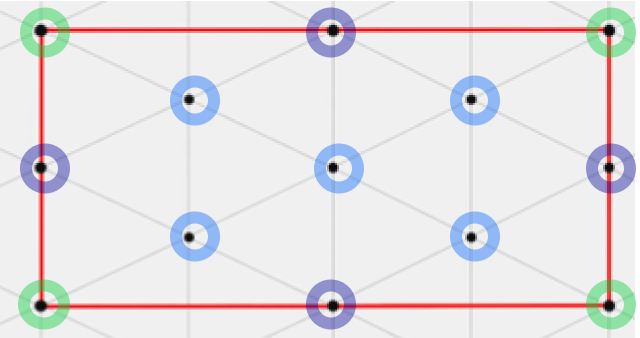
Contributors 1

Magistrelli Fabio Magistrelli

Languages

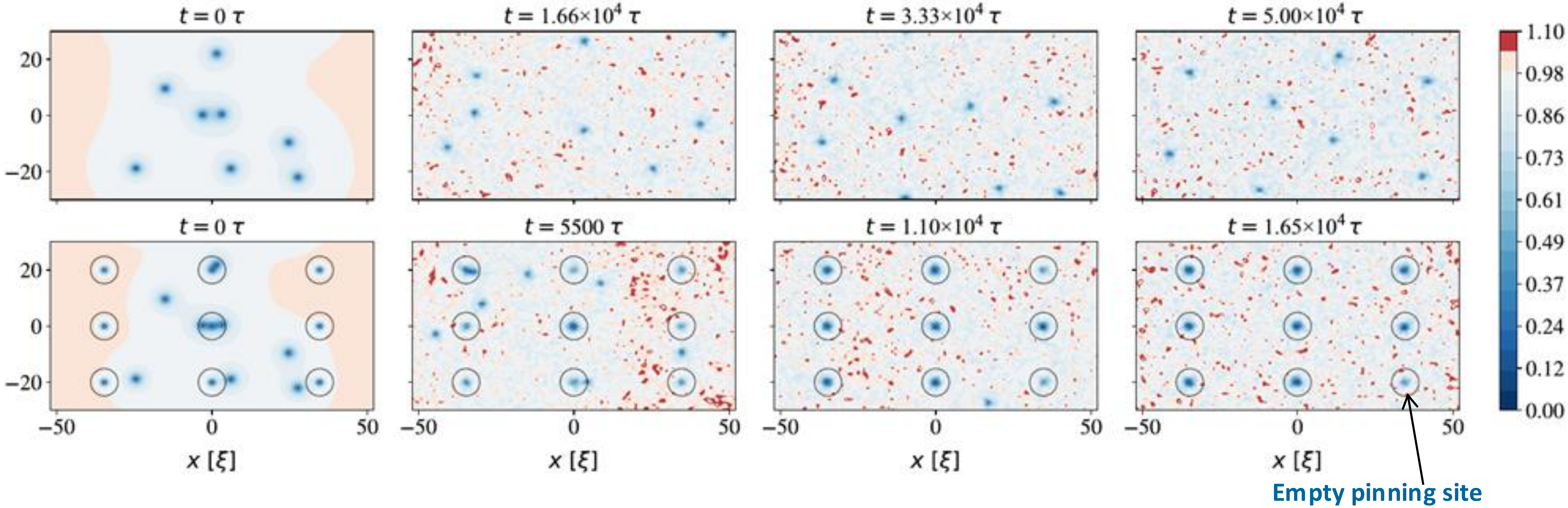
- **C++** 75.7%
- **Python** 16.1%
- **Shell** 4.1%
- **C** 3.9%
- **Makefile** 0.2%

GINTONIC: lattice formation



Numerical dissipation, 8 vortices, cell sides: $60\sqrt{3}\,\xi \times 60\,\xi$

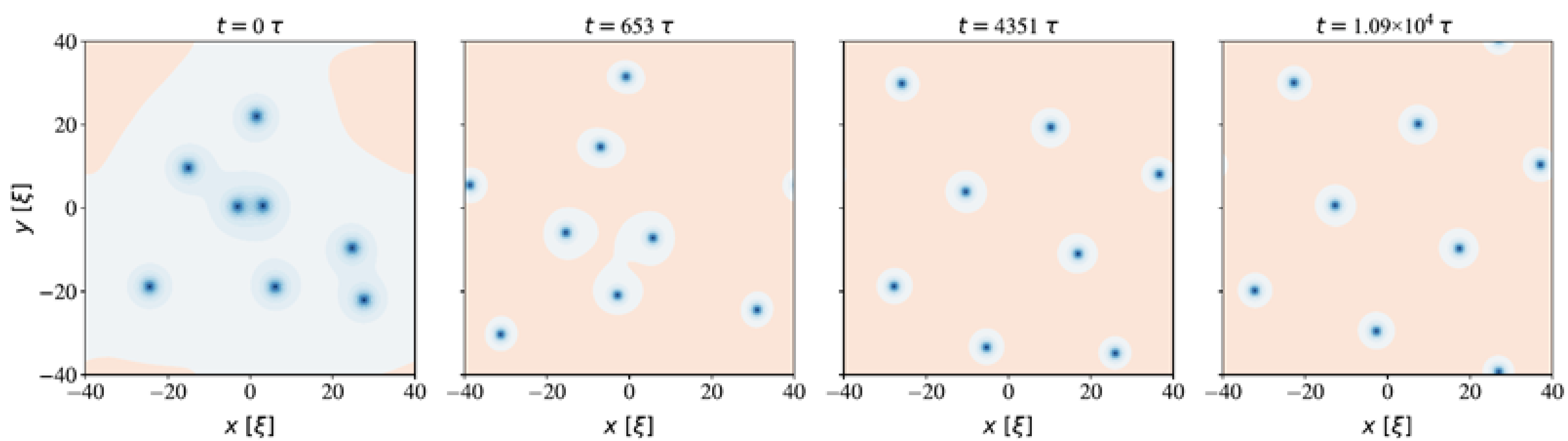
$\gamma = 0 \qquad \delta l = \xi$



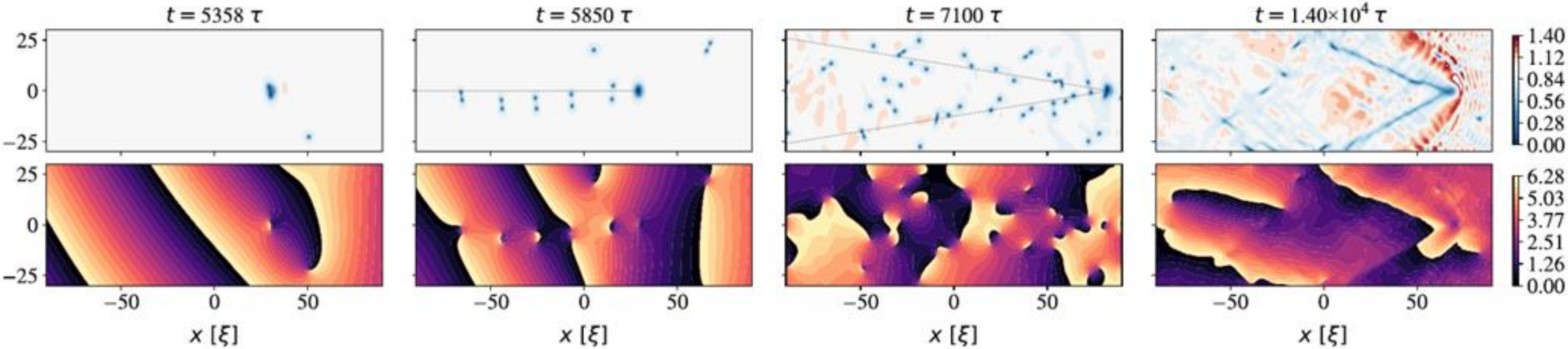
Complex time, 8 vortices, cell aspect ratio (1 : 1) → frustrated lattice (metastable)

$\gamma = 0.03$

$\delta l \simeq 0.3$



GINTONIC: unpinning and nucleation

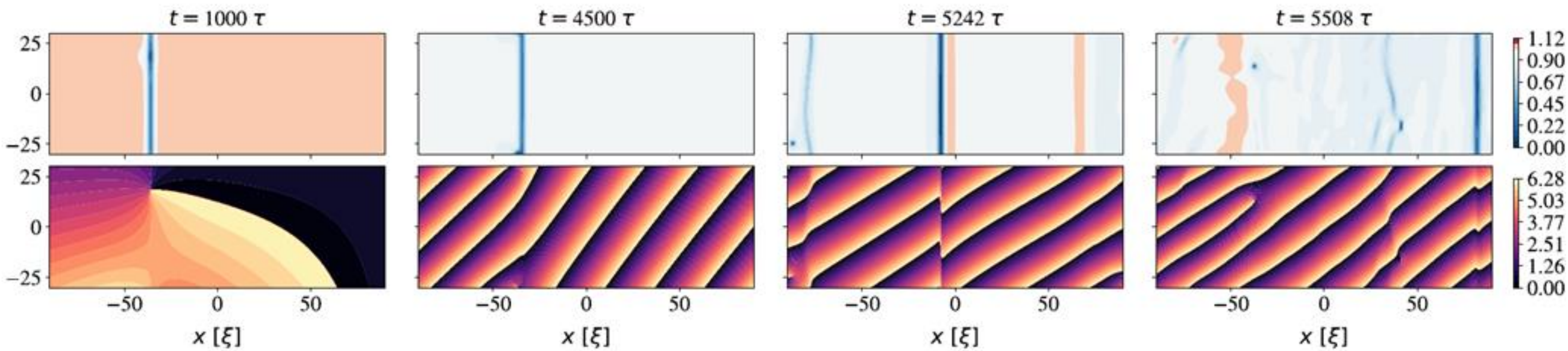


Nucleation of the first vortex pair

Single-branched Kármán vortex street

Double-branched Kármán vortex street

Sonic boom (supersonic impurity)



Rod accelerates (from left to right)...

...the vortex (positive polarization) moves upwards and unpins

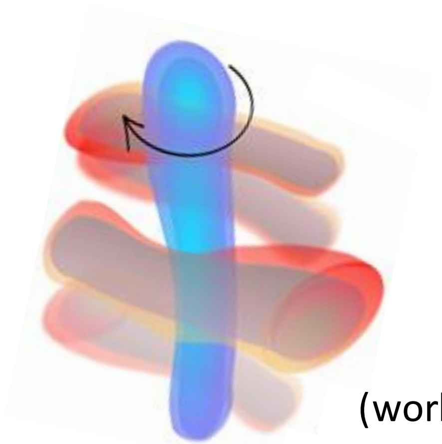
...first encounter vortex-density depletion

Sound decay (vortex pair nucleation)

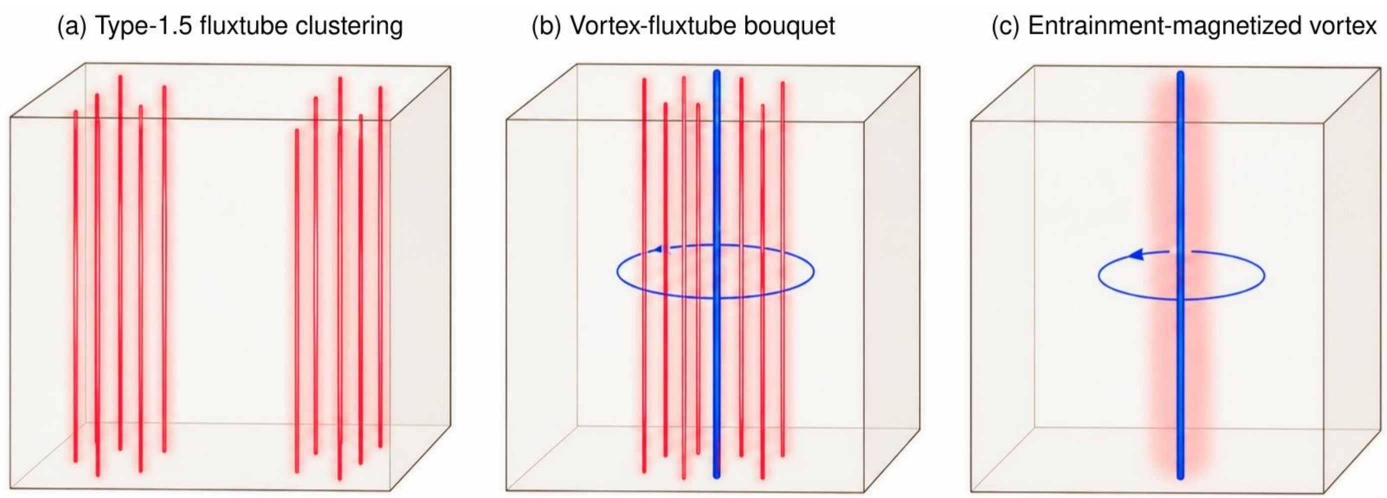
Outer core: superconductor-superfluid mixture

Fluid element in the bulk of a neutron star outer core.
Two kinds of topological defects:
- Fluxtubes in the proton superconductor
- Vortices in the neutron superfluid

How do they arrange?



(work in progess by *Adarsh Karekkat*, PhD @LPC Caen)



Alpar et al., ApJ (1984)
Alford & Good, PRB (2008)
Drummond et al., MNRAS (2017)
Sedrakian et al., MNRAS (1997)
Wood & Graber, Univ. (2022)
Thong et al. MNRAS (2023)

Outer core: superconductor-superfluid mixture

$$F[\psi_p, \psi_n, \mathbf{A}] \propto \int \mathcal{F} dx dy$$

"cost" functional to be minimised

$$\mathcal{F} = \frac{g_{pp}}{2}\left(|\psi_p|^2 - \frac{n_p}{2}\right)^2 + \frac{g_{nn}}{2}\left(|\psi_n|^2 - \frac{n_n}{2}\right)^2 + g_{pn}\left(|\psi_p|^2 - \frac{n_p}{2}\right)\left(|\psi_n|^2 - \frac{n_n}{2}\right) + \frac{\hbar^2}{4m_u}\left|\left(\nabla - \frac{2ie}{\hbar c}\boldsymbol{A}\right)\psi_p\right|^2 + \frac{\hbar^2}{4m_u}|\nabla\psi_n|^2 + \frac{1}{8\pi}|\nabla \times \boldsymbol{A}|^2 + \tilde{g}\frac{\hbar^2}{4m_u}\left[\psi_p^*\psi_n^*(\nabla\psi_n)\left(\nabla - \frac{2ie}{\hbar c}\boldsymbol{A}\right)\psi_p + \psi_p\psi_n^*(\nabla\psi_n)\left(\nabla + \frac{2ie}{\hbar c}\boldsymbol{A}\right)\psi_p^* + \mathbf{c.c.}\right]$$

Energy density for stationary configurations

Alpar et al., ApJ (1984)
Alford & Good, PRB (2008)
Drummond et al., MNRAS (2017)
Sedrakian et al., MNRAS (1997)
Wood & Graber, Univ. (2022)
Thong et al. MNRAS (2023)

Outer core: superconductor-superfluid mixture

$$F[\psi_p, \psi_n, \mathbf{A}] \propto \int \mathcal{F} dx dy$$

"cost" functional to be minimised

$$\mathcal{F} = \frac{1}{2}(1 - |\psi_p|^2)^2 + \frac{R^2}{2\epsilon}(1 - |\psi_n|^2)^2 +$$

Mexican hats

$$+ \frac{\alpha}{\epsilon}(1 - |\psi_p|^2)(1 - |\psi_n|^2) +$$

Amplitude-amplitude interaction

$$+ |(\nabla - i\mathbf{A})\psi_p|^2 + \frac{1}{\epsilon}|\nabla\psi_n|^2 + k^2|\nabla \times \mathbf{A}|^2$$

"free" terms

$$+ \frac{\tilde{g}}{\epsilon}\left(\nabla|\psi_p|^2 \cdot \nabla|\psi_n|^2\right)$$

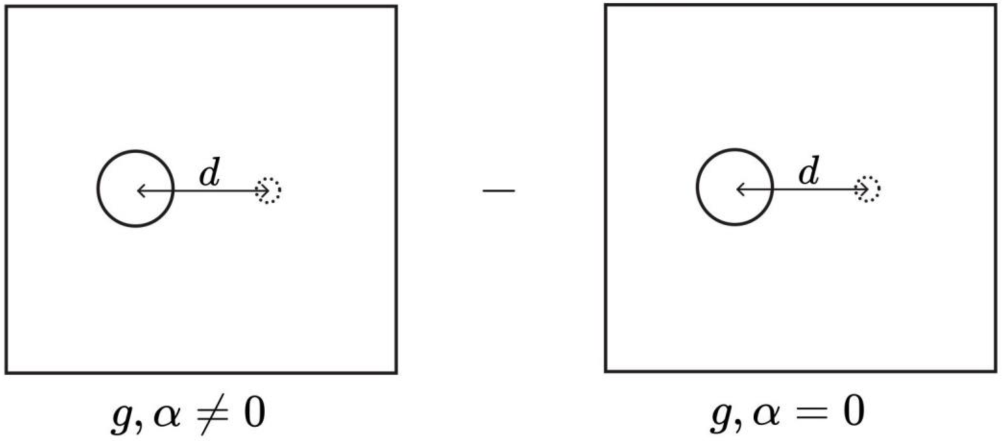
Gradient-gradient interaction

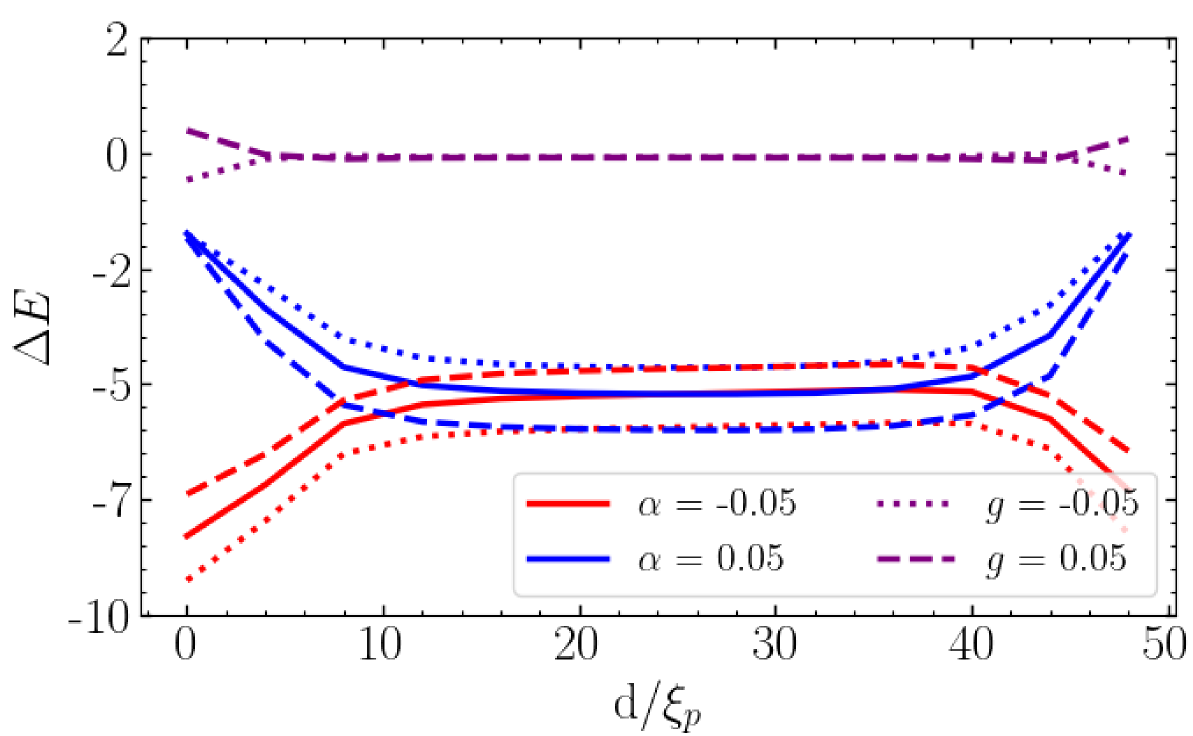
Rescaling done per:
Wood and Graber, Univ. (2022)
Alford and Good, PRB (2008)

$$\kappa \equiv \frac{\lambda}{\xi_p}, \; R \equiv \frac{\xi_p}{\xi_n}, \; \epsilon \equiv \frac{n_p}{n_n}, \; \alpha \equiv \frac{g_{pn}}{g_{pp}}$$

Outer core: vortex-fluxtube pinning

Numerical experiment: impose QPBC for 1 **fluxtube** and 1 **vortex** and constrain the position of the phase singularities, then minimise the energy functional

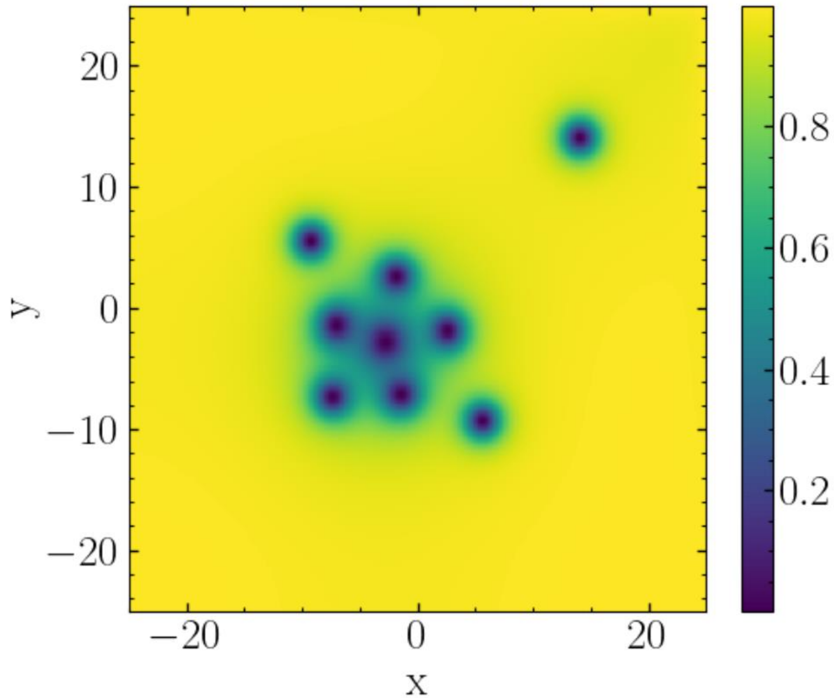




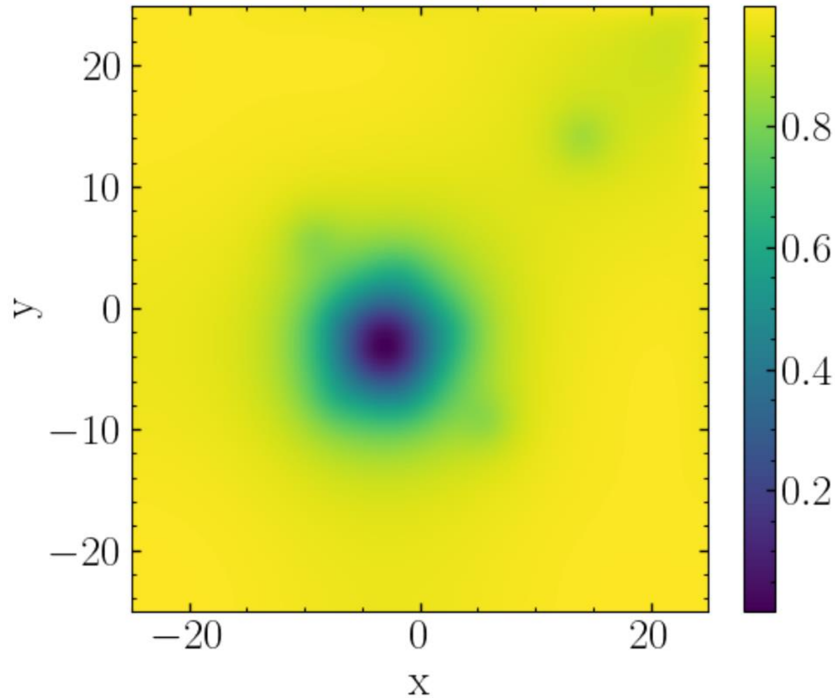
For α < 0 the energy increases as you separate the vortex from the fluxtube (pinning)

Outer core: *bouquet* of fluxtubes around a vortex

Minimize the energy functional with QPBC for 1 vortex and 9 fluxtubes ($\alpha = -0.05$)



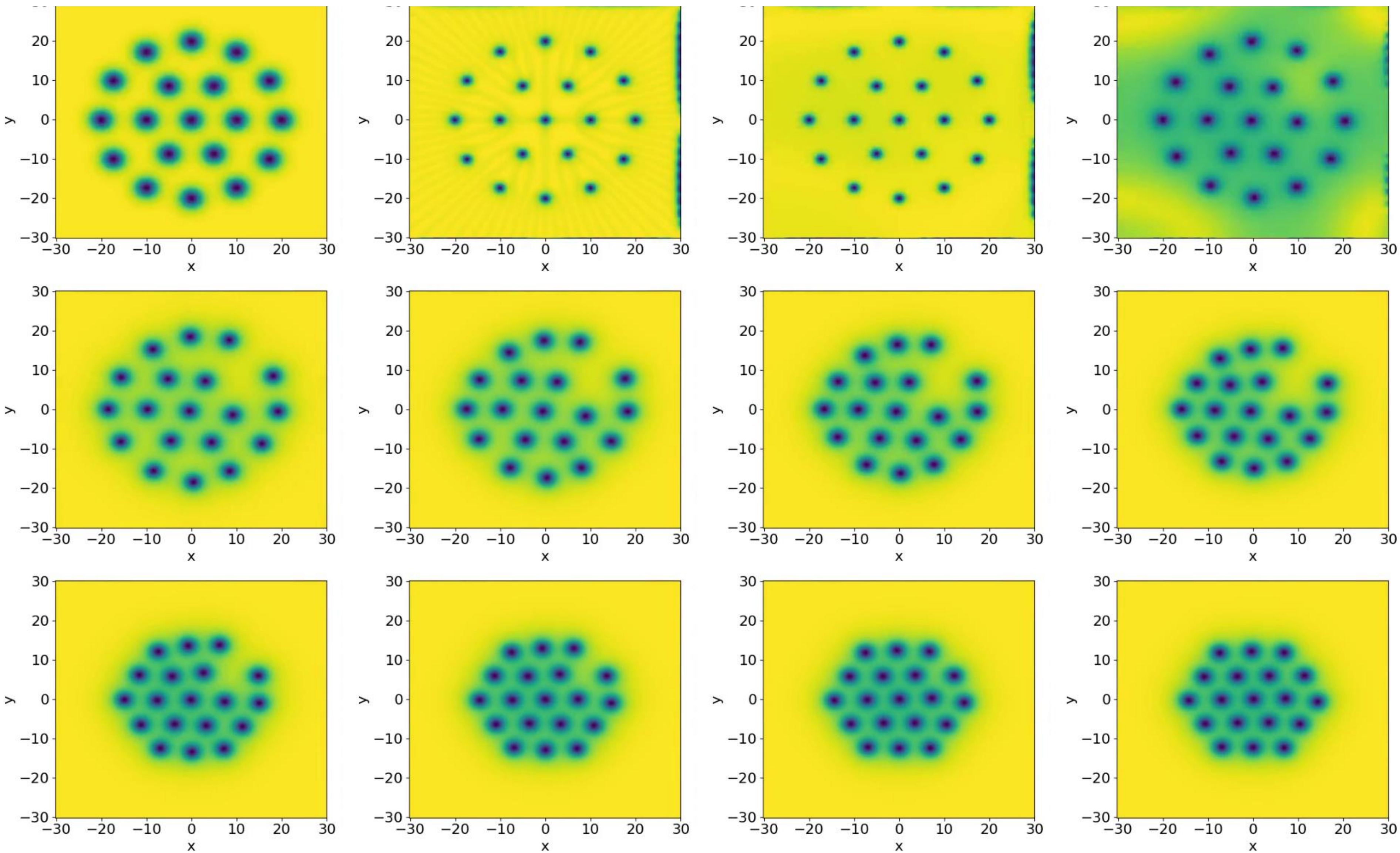
Proton condensate amplitude



Neutron condensate amplitude

Outer core: type 1.5 superconductivity

Minimize the energy functional with QPBC for 19 fluxtubes ($\alpha = 0.1$)



Final considerations

Motivation for QPBC:

→ need to simulate fluid elements with persistent supercurrents and finite vorticity

Possible use: every time you need a boundary-less domain!

→ revision of classical numerical experiments (pinning, Karman vortex street, vortex nucleation)

→ diagnostic tool: alternative to hard walls to check if spurious effects are present in the simulation

→ GPE upgrade of point-vortex simulations to extract mutual friction

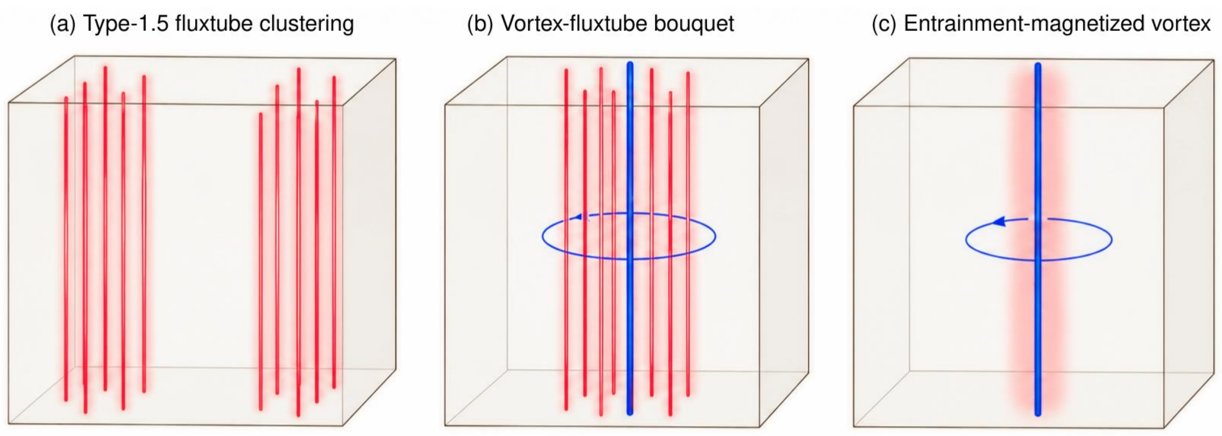
→ natual tool for: vortex lattice formation & type 1.5 superconductivity, vortex-fluxtube bouquets

→ natural tool for unpinning experiments (dragging a pinning site indefinitely across the torus)

Caveat: "no boundaries" does not mean "no finite size effects"

→ the rectangular domain is finite: QPBC do not guarantee "true" bulk limit

Ongoing work: adding entrainment



GINTONIC code

https://github.com/Magistrelli/gintonic

→ 2D rectangular Cartesian grid with QPBC
→ Code units: usual GPE dimensionless units (e.g., space ~ “healing length”)
→ Real and imaginary parts are evolved in time with a 3rd-order Adams–Bashforth linear multistep
→ Spatial derivatives: 5-point stencils
→ Diagnostic: alternative 4th-order explicit Runge–Kutta (time), 3-point stencils (space)
→ Numerical dissipation sets in when “grid spacing” ~ “healing length”

README   MIT license

GINTONIC - Gross-pitaevskii INtegrator on a TOrus with Non-zero vortICity

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arXiv-2509.15298 Latest
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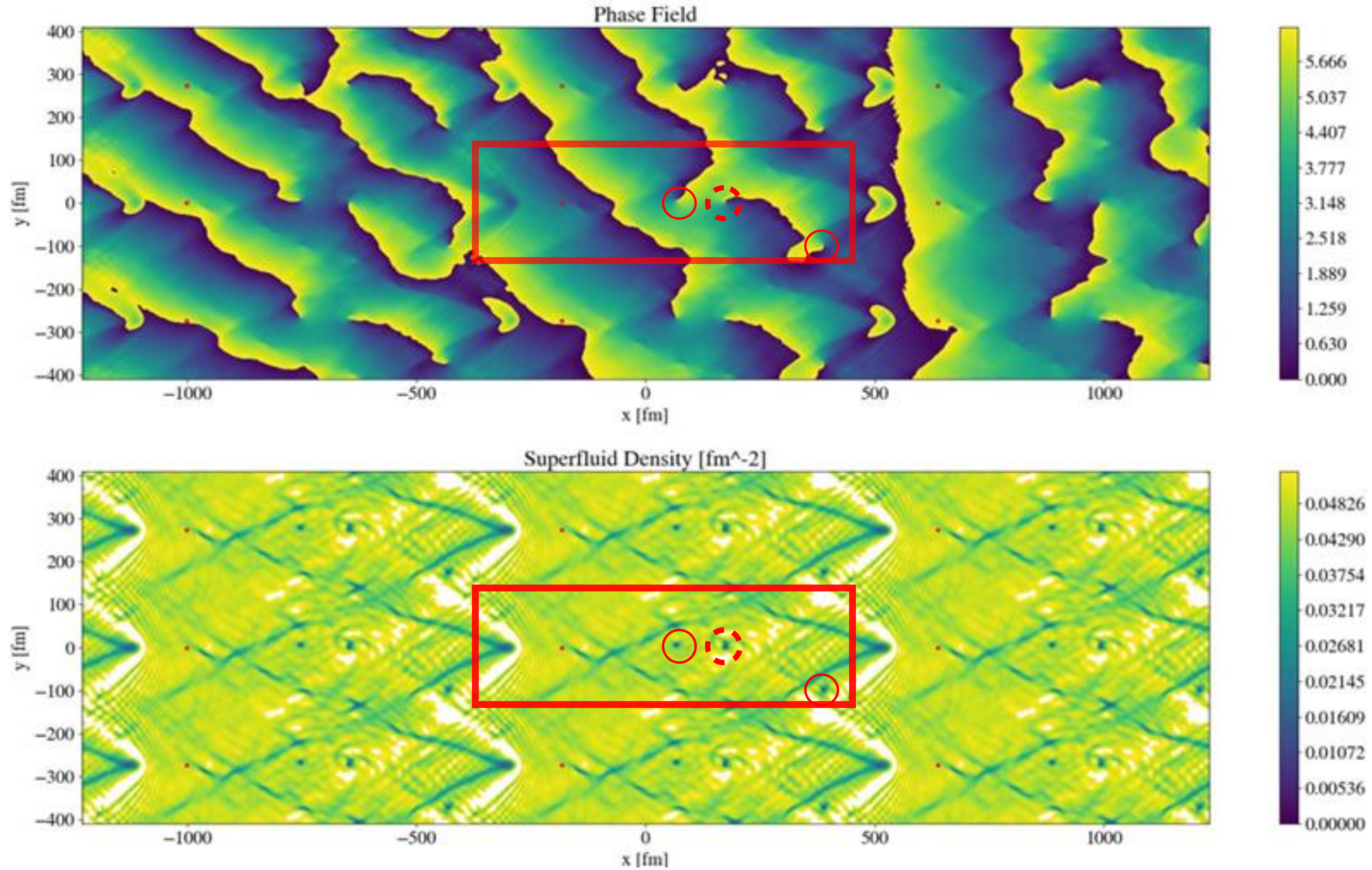
- **Makefile** 0.2%

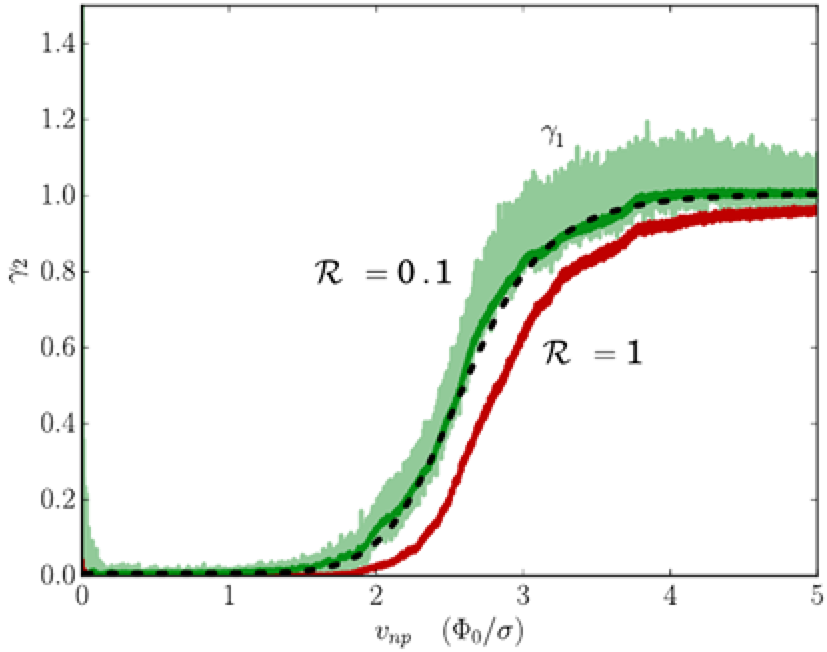
GINTONIC: sonic boom

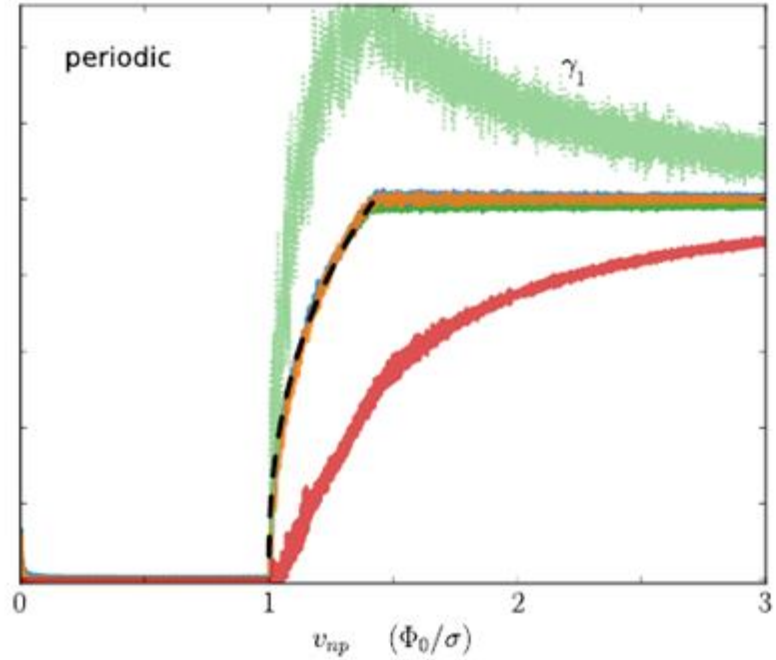


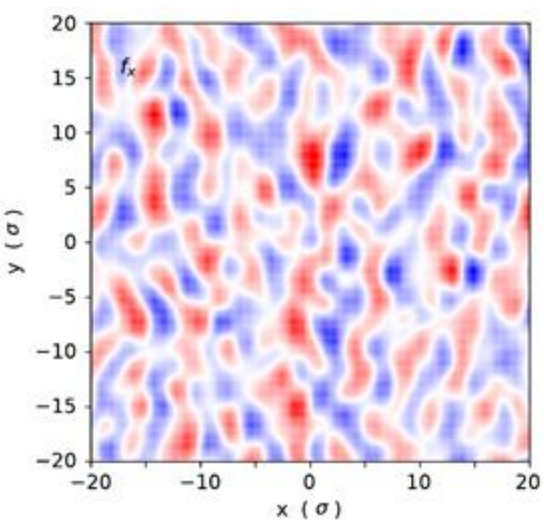
Stress test for the code: in regimes where dissipation is important, the result is only as physical as the "gamma prescription" (or numerical dissipation)

→ lack of explicit *condensate/thermal component* transfusion

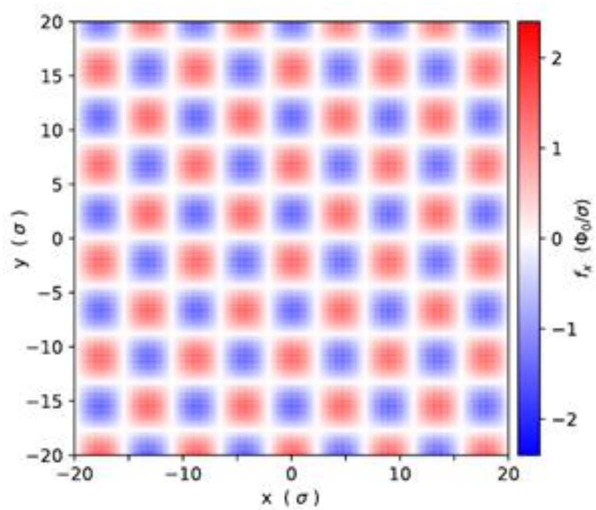
Unpinning & creep: effect of the pinscape







Random
pinning landscape



Periodic
Pinning landscape

$$\gamma_{1,2} = \frac{1}{v_{1,2}^{fr} N_v} \sum_i \hat{e}_{1,2}^i \cdot \langle \dot{\boldsymbol{x}} \rangle$$

Antonelli & Haskell, MNRAS (2020)

Vortex Lattice Ansatz

Tkachenko, JETP 1966, Wood et al, PRB 2019, Doran & Billam PRE 2020

Initial condition (no background flow):

$$\hat{\psi}(\vec{x}, t=0) = f_{\rm bg}(\vec{x}) \prod_{k=1}^{N} f_P(|\vec{x} - \vec{x}_k|) \exp\{i s_k \theta_J(\vec{x}|\vec{x}_k)\}$$

Initial phase: $$\theta_J(\vec{x}|\vec{x}_v) = \frac{x(y - 2y_v)}{\lambda_{2,y}} + \mathrm{Arg}\left[\Theta_1\left(x - x_v + i(y - y_v)\,,\, \tau\right)\right]$$

$$\pi\tau \equiv \lambda_{2,x} + i\,\lambda_{2,y}$$

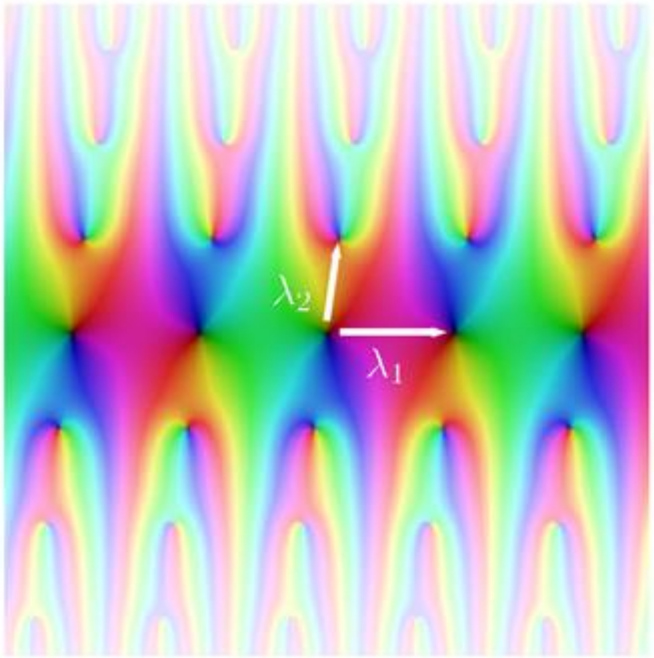
Jacobi Theta Function:

$$\Theta_1(z, q) \equiv 2\sum_{n=0}^{\infty}(-1)^n\, q^{\left(n+\frac{1}{2}\right)^2} \sin\left[(2n+1)z\right]$$

$$z \equiv x + iy \qquad q = e^{i\pi\tau}$$

Initial density:

$$|\psi(\vec{x})| = f_P(|\vec{x} - \vec{x}_v|) \equiv \left[\left(\frac{r^2}{2s^2 + r^2}\right) n_\infty\right]^{1/2}$$



Point vortex orbits ("attractive" pinning potential)

$$\hat{\boldsymbol{\kappa}} \times (\dot{\boldsymbol{x}}(t) - \boldsymbol{v}_{np}) - \mathcal{R}\,\dot{\boldsymbol{x}}(t) + \boldsymbol{f} \;=\; 0$$

Magnus force — Drag — Pinning

$$\mathcal{R} = \tan\theta_d$$

Dissipation angle

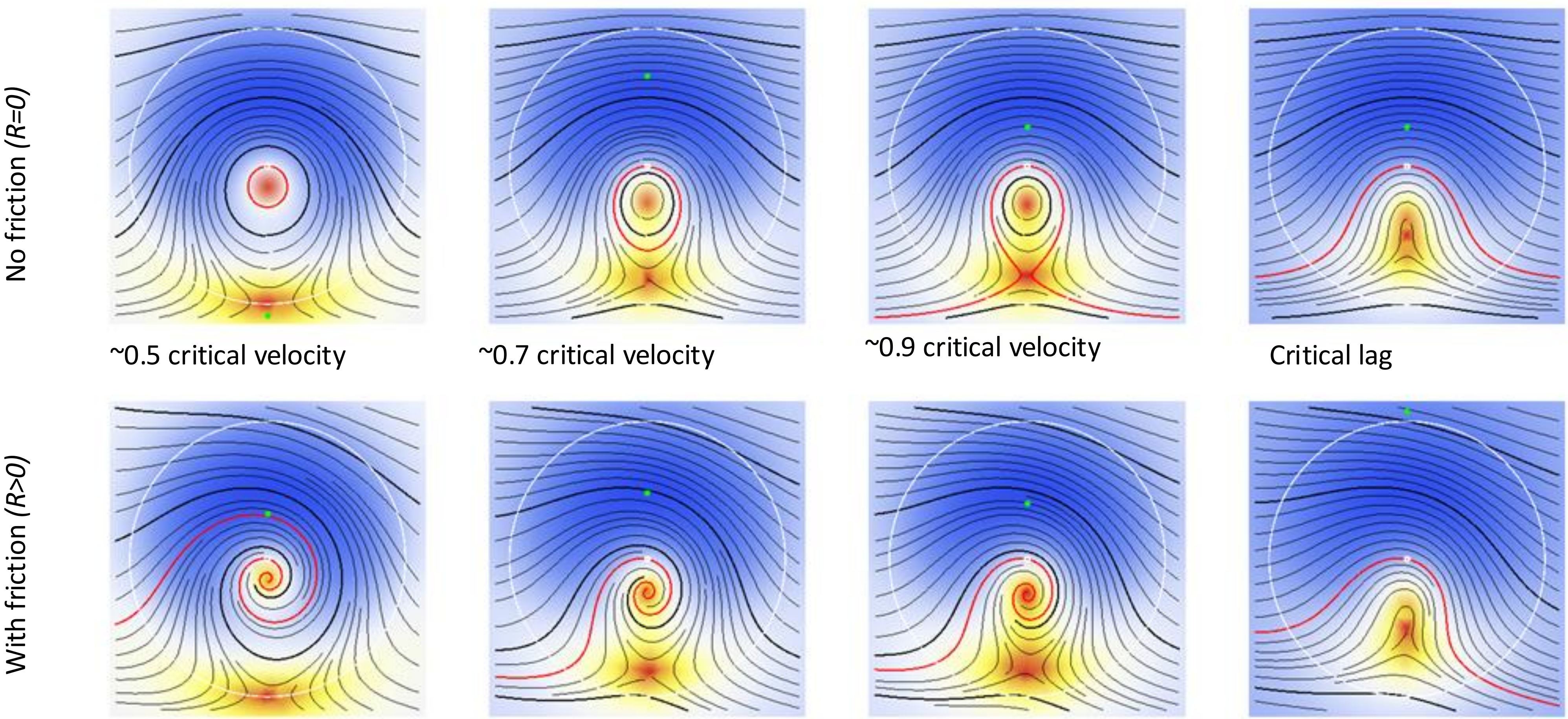
Orbits of a point vortex interacting with an "attractive" effective potential

$$\dot{\boldsymbol{x}} \;=\; \cos\theta_d \, R_{\theta_d}^{-1} \, (\boldsymbol{v}_{np} + R_{\pi/2}\boldsymbol{f})$$

$$\boldsymbol{f} \;=\; -\nabla\Phi$$

Negative

$$\Phi(\boldsymbol{x}) = \Phi_0 \; e^{-|\boldsymbol{x}-\boldsymbol{r}_a|^2/2\sigma^2}$$



Point vortex orbits ("repulsive" pinning potential)

$$\hat{\boldsymbol{\kappa}} \times (\dot{\boldsymbol{x}}(t) - \boldsymbol{v}_{np}) - \mathcal{R}\,\dot{\boldsymbol{x}}(t) + \boldsymbol{f} = 0$$

Magnus force — Drag — Pinning

$$\mathcal{R} = \tan\theta_d$$

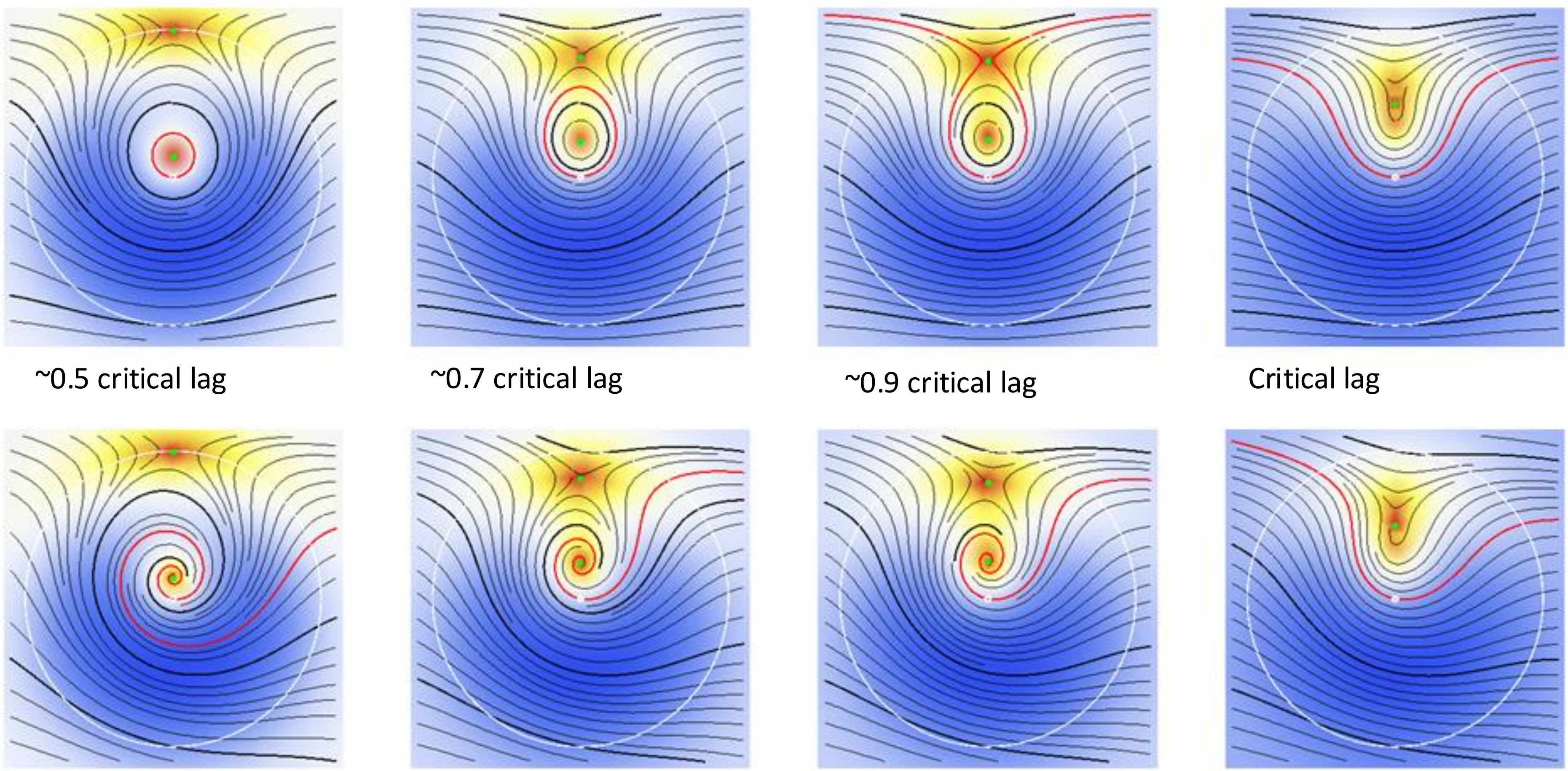
Dissipation angle

Orbits of a point vortex interacting with a "repulsive" effective potential

$$\dot{\boldsymbol{x}} = \cos\theta_d\, R_{\theta_d}^{-1}\,(\boldsymbol{v}_{np} + R_{\pi/2}\boldsymbol{f}) \qquad \boldsymbol{f} = -\nabla\Phi \qquad \Phi(\boldsymbol{x}) = \Phi_0\; e^{-|\boldsymbol{x}-\boldsymbol{r}_a|^2/2\sigma^2}$$

Positive

No friction (*R=0*)



~0.5 critical lag — ~0.7 critical lag — ~0.9 critical lag — Critical lag

With friction (*R>0*)

ArXiv: 2007.11720 (NS, non rel)
2012.10288 (NS, He relativistic)
2301.12769 (NS, non-rel, review)

PEVM (meso) → HVBK (macro)

- **Mesoscopic** scale: PEVM (“phenomenological equation for vortex motion”)
  Just find the *most general* form that is *linear in the velocities* (e.g. arXiv: *2301.12769*, *2007.11720*)

$$M^N\,(\boldsymbol{v}_L-\boldsymbol{v}_N)+M^S\,(\boldsymbol{v}_L-\boldsymbol{v}_S)+\boldsymbol{f}=0$$

“Magnus” *“super drag (?)”*

$$M^S_{ij}=K_{ij}-\xi_\perp\ \perp_{ij}$$

$$M^N_{ij}=\mathcal{R}_\times\,K_{ij}-\mathcal{R}_\perp\,\perp_{ij}$$

“Iordanskii” *“normal drag”*

$$K_{ij}=\epsilon_{iaj}\hat{\kappa}^a$$ *Vorticity “dual”*

$$\perp_{ij}=\delta_{ij}-\hat{\kappa}_i\hat{\kappa}_j=-K_{ia}K_{aj}$$ *orthogonal projector*

- Normal form of the PEVM:

$$\dot{\boldsymbol{x}}=\boldsymbol{v}_L-\boldsymbol{v}_N$$ *vortex velocity*

$$\boldsymbol{v}_{SN}=\boldsymbol{v}_S-\boldsymbol{v}_N$$ *velocity “lag”*

velocities in the normal frame

$$B_\perp=(1+\mathcal{R}_\times+\xi_\perp\mathcal{R}_\perp+\xi_\perp^2)/D$$
$$B_\times=(\mathcal{R}_\perp-\mathcal{R}_\times\xi_\perp)/D$$
$$C_\perp=(\mathcal{R}_\perp+\xi_\perp)/D$$
$$C_\times=(1+\mathcal{R}_\times)/D$$
$$D=(1+\mathcal{R}_\times)^2+(\xi_\perp+\mathcal{R}_\perp)^2$$

$$\dot{\boldsymbol{x}}=(M^N+M^S)^{-1}\,(M^S\boldsymbol{v}_{SN}-\boldsymbol{f})$$
$$=(B_\perp-B_\times\,K)\boldsymbol{v}_{SN}+(C_\perp+C_\times\,K)\boldsymbol{f}$$

ed” Iordanskii)

$$\dot{\boldsymbol{x}}=\frac{1}{1+\mathcal{R}_\perp^2}\boldsymbol{v}_{SN}-\frac{\mathcal{R}_\perp}{1+\mathcal{R}_\perp^2}\hat{\kappa}\times\boldsymbol{v}_{SN}$$

Always 2 parameters: degeneracy

PEVM (meso) → HVBK (macro)

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$$M^N\,(\boldsymbol{v}_L - \boldsymbol{v}_N) + M^S\,(\boldsymbol{v}_L - \boldsymbol{v}_S) + \boldsymbol{f} = 0$$

"Magnus" *"super drag (?)"*

$$M^S_{ij} = K_{ij} - \xi_\perp \perp_{ij}$$

$$M^N_{ij} = \mathcal{R}_\times K_{ij} - \mathcal{R}_\perp \perp_{ij}$$

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$$K_{ij} = \epsilon_{iaj}\hat{\kappa}^a$$ *Vorticity "dual"*

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- Normal form of the PEVM:

$$\dot{\boldsymbol{x}} = \boldsymbol{v}_L - \boldsymbol{v}_N$$ *vortex velocity*

$$\boldsymbol{v}_{SN} = \boldsymbol{v}_S - \boldsymbol{v}_N$$ *velocity "lag"*

velocities in the normal frame

$$B_\perp = (1 + \mathcal{R}_\times + \xi_\perp\mathcal{R}_\perp + \xi_\perp^2)/D$$
$$B_\times = (\mathcal{R}_\perp - \mathcal{R}_\times\xi_\perp)/D$$
$$C_\perp = (\mathcal{R}_\perp + \xi_\perp)/D$$
$$C_\times = (1 + \mathcal{R}_\times)/D$$
$$D = (1 + \mathcal{R}_\times)^2 + (\xi_\perp + \mathcal{R}_\perp)^2$$

$$\dot{\boldsymbol{x}} = (M^N + M^S)^{-1}\,(M^S\boldsymbol{v}_{SN} - \boldsymbol{f})$$
$$= (B_\perp - B_\times K)\boldsymbol{v}_{SN} + (C_\perp + C_\times K)\boldsymbol{f}$$

Relevant part if no pinning

… for the coarse … straight array case, 2D).
… k into "Magnus" and multiply by vortex density

Mutual friction in the simplest "2D" case: again always 2 parameters, whatever the PEVM: degeneracy!

ArXiv: 2007.11720 (NS, non rel)
2012.10288 (NS, He relativistic)
2301.12769 (NS, non-rel, review)

Iordanskii "controversy"

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"Magnus" "super drag (?)"

$$M^S_{ij} = K_{ij} - \xi_\perp \perp_{ij}$$

$$M^N_{ij} = \mathcal{R}_\times K_{ij} - \mathcal{R}_\perp \perp_{ij}$$

"Iordanskii" "normal drag"

$$K_{ij} = \epsilon_{iaj} \hat{\kappa}^a$$ Vorticity "dual"

$$\perp_{ij} = \delta_{ij} - \hat{\kappa}_i \hat{\kappa}_j = -K_{ia} K_{aj}$$ orthogonal projector

- Iordanskii "controversy": the value of ?

→ Is it possible to deduce it from "micro"? $\mathcal{R}_\times$n / Thouless incompatible results, e.g. see arXiv 2012.10288)
→ Is there a "universal" expression in terms of circulation quantum and "fraction" (like for Magnus)?
→ Is it possible to have insight from experiments? (caveat! Degeneracy "problem", e.g. see arXiv 2007.11720)

→ May degeneracy

$$\dot{\boldsymbol{x}} = (B_\perp - B_\times \, K) \boldsymbol{v}_{SN}$$

$$\boldsymbol{v}_L - \boldsymbol{v}_N = \cos\theta_d R^{-1}_{\theta_d} \boldsymbol{v}_{SN}$$

$$\tan\theta_d = \frac{B_\times}{B_\perp} = \frac{\mathcal{R}_\perp - \mathcal{R}_\times \xi_\perp}{1 + \mathcal{R}_\times + \xi_\perp \mathcal{R}_\perp + \xi_\perp^2}$$

Iordanskii force (inner crust)

ArXiv: 2007.11720 (NS, non rel)
2012.10288 (NS, He relativistic)
2301.12769 (NS, non-rel, review)

ArXiv:2012.10288 – Model for rel. hydro HVBK-style for super neutrons in the **inner crust**:
→ Based on 2 fluids (no relativistic electrons, no heat): normal (lattice) & superfluid
→ Not all neutrons are unbounded: what goes in the "super" component?
Better: *some neutrons are locked to the the protons in nuclei*. This gives rise to an ambiguity in the definition

→ Whatever the choice, physical quantities must be "chemical gauge" independent!

Conclusions in ArXiv:2012.10288:

→ To ensure **chemical gauge covariance** (i.e. physics does not change for different chemical choices) of the hydro ec

→ Different results present in the literature regarding the "controversy" around the presence of the Iordanskii force in He may pertain to **two (opposite) dynamical regimes** of the fluid system: revised version of **Wexler's "experiment"**