Turbulence Modelling in Magnetised Neutron Stars

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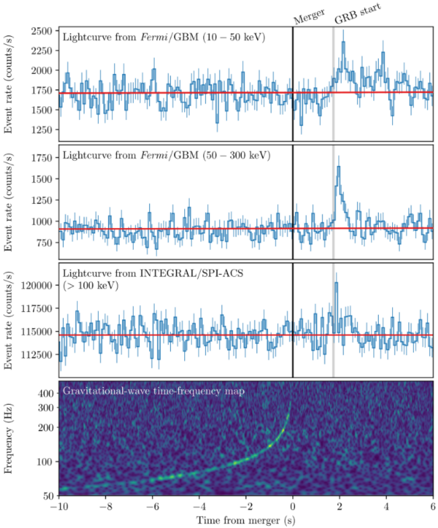
BNS mergers

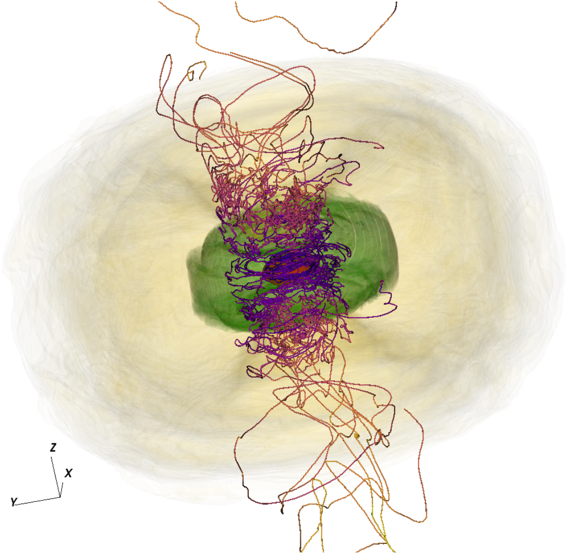
- GW170817 first observation of BNS merger
- Multimessenger observation of GW and coincident Gamma Ray Burst, followed by long term EM kilonova.
- Evolution driven by all four forces of nature in extreme regimes.
- Connecting fundamental physics to multimessenger observables requires numerical simulations that can capture all these effects.



[Abbott + '17]

Magnetic fields in BNS

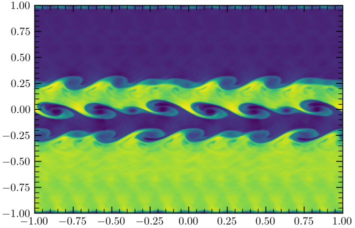
- During a BNS merger magnetic and fluid instabilities amplify B field and trigger turbulent flow.
- Large scale B field forms, triggers jet launching, and associated gamma ray burst.
- Magnetic field in remnant drives ejecta winds, impacts ejected material, associated nucleosynthesis
- Magnetic stresses transport angular momentum, impact collapse time of remnant and associated GW emmission.

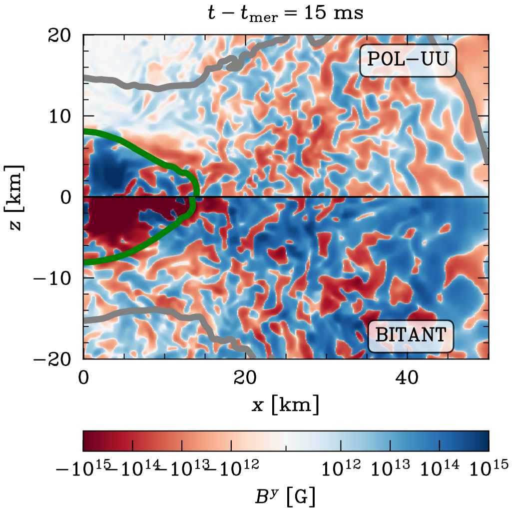


[Gutierrez (WC) + '25]

Magnetic field instabilities

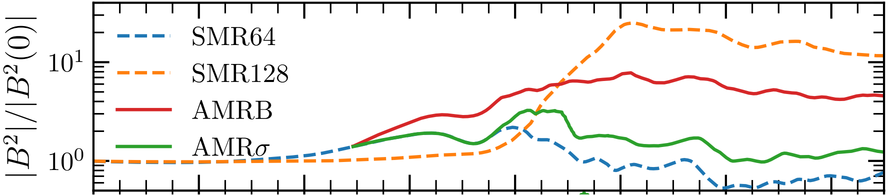
- How does the B field amplify during BNS merger / post-merger?
- At merger Kelvin-Helmholtz instability. Fields $\sim 10^{12}$ may amplify up to $\sim 10^{16}$ [Kiuchi + '26] , as B field undergoes turbulent cascade to small scales.
- In post-merger disk/remnant star MRI, Taylor Instability.
- Can power dynamo processes: $\alpha - \Omega$, Taylor-Spruit; and generate effective viscosity





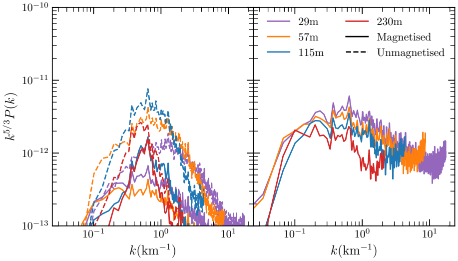
[Gutierrez (WC)+ '25]

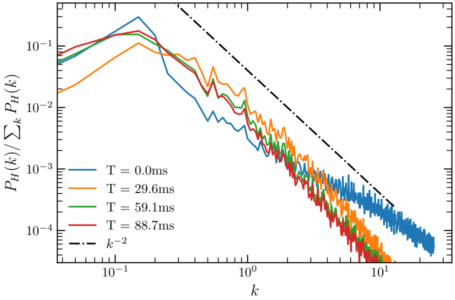
- Want to perform global simulations of BNS mergers. Dynamical GR, perfect fluid with nuclear EOS, ideal MHD, neutrino transport. Expensive simulations.
- Numerical simulation limited in its ability to capture amplification through Kelvin-Helmholtz instability by resolution. $\sigma \propto 1/\Delta x$
- How to model turbulence in numerical simulations? Cannot span the relevant range of length scales from viscous length $\mathcal{O}(1\mathrm{cm})$ to e.g. GW wavezone $\mathcal{O}(1000\mathrm{km})$.



[WC+ '23]

Numerics

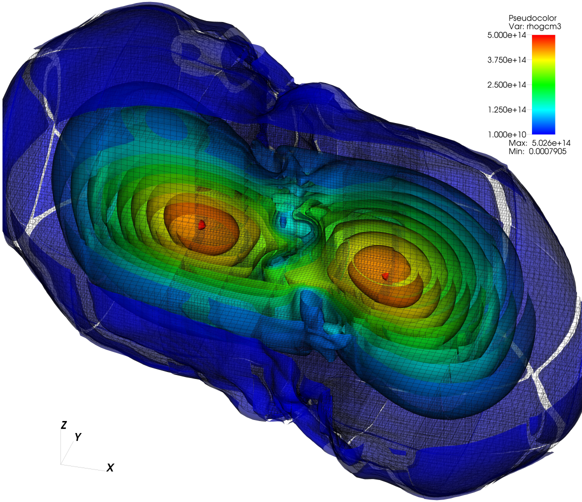




- Modelling strategies: Learn what we can from few VHR sim - [WC+ '25] for single star, 29m, e.g. [Kiuchi+ '26] BNS, 6m.
- Runs incredibly costly e.g. 28,000 GPU node hours ; 5×10^8 CPU hours.
- Cannot explore parameter space with this approach. Use **Large Eddy Simulations + Subgrid Model** to perform low resolution simulations with model that incorparates back reaction of small scale physics on large scale evolution

Large Eddy Simulations

- In numerical simulations we discretise continuous eqs and evolve volume averaged variables on cells size Δx.
- Lose information on scale $\ell < \Delta x$.
- Instead ask : what is the continuous equation that a volume averaged or "filtered" fluid obeys?
- Gives an **exact** equation for the evolution of the volume averaged variable.
- What's the catch? Equations are not closed. Must prescribe a model for the behaviour on the subgrid scale "subgrid model".
- Conservation of particle number $\nabla_\mu(\rho u^\mu) = 0$
- Conservation of energy-momentum $\nabla_\mu(T^{\mu\nu}) = 0$
- Maxwells Eq $\nabla_\mu {}^*F^{\mu\nu} = 0$
- Give evolution equations for (D, S^i, τ, B^i), (mass, momentum and energy densities and mag. field in Eulerian frame)



Concrete example: B in Newtonian Physics

- Induction equation in ideal MHD

$$\partial_t B^k + \partial_i (B^k v^i - B^i v^k) = 0 \quad (1)$$

- What equation does this obey when filtered on scales $> \Delta x$?
- Filtering operator:

$$\overline{f}(x,t) = \int_{x'_{\rm min}}^{x'_{\rm max}} G(x-x') f(x',t) dx' \quad (2)$$

$$G(x-x') = \frac{1}{\Delta x} H\left(\frac{\Delta x}{2} - |x-x'|\right) \quad (3)$$

- Filtered equation:

$$\partial_t \overline{B^k} + \partial_i (\overline{B^k v^i} - \overline{B^i v^k}) = 0 \quad (4)$$

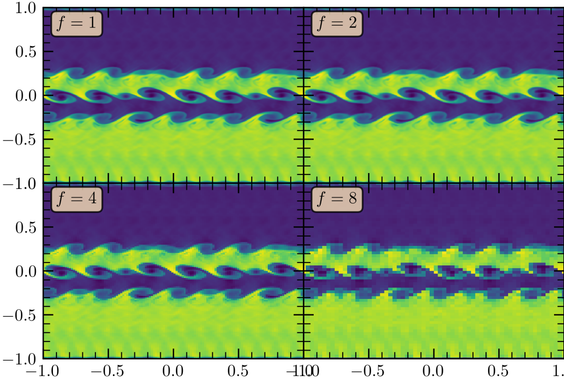
Concrete example: B in Newtonian Physics

$$\partial_t \overline{B^k} + \partial_i (\overline{B^k v^i} - \overline{B^i v^k}) = 0 \tag{5}$$

- Problem: what is $\overline{B^k v^i}$? We know $\overline{B^k}$, $\overline{v^i}$, but filtration and non linear terms do not commute. Would need to know true continuum solution, take product, then filter.
- Traditionally we ignore this (Implicit LES).
- To capture the back reaction of small scale physics on the large scale filtered physics, we construct a subgrid model for these terms to close the system of Eqs.
- Various closure schemes exist Smagorinsky [Smagorinsky '63, Radice '17] , gradient model [Vigano+ '19, Carrasco+ '20, Palenzuela+ '21] , MINIT model [Miravet-Tenes+ '22,'23]

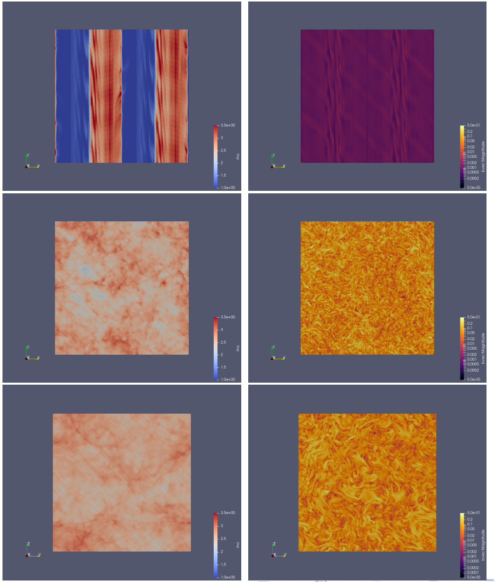
Subgrid models

- Data driven approach: Machine Learning [Rosofsky, Huerta '19]
- Perform high resolution simulation of turbulent physics. Treat this as the "true" "continuum" solution.
- Calculate product terms like $B^i v^k$, as well as individual terms like B^i, v^i.
- Filter everything down to low resolution $B^i v^k \rightarrow \overline{B^i v^k}, B^i \rightarrow \overline{B^i}, v^i \rightarrow \overline{v^i}$.
- Train a neural network on the relationship between inputs: $\overline{B^i}, \overline{v^i}$ (evolved variables in our simulations) and outputs $\overline{B^i v^k}$ (the unknown subgrid terms)
- Perform a low res simulation and evaluate the neural network to add subgrid contribution.
- Low res simulation should capture the physics of the high res simulation used for training, at fraction of the cost.



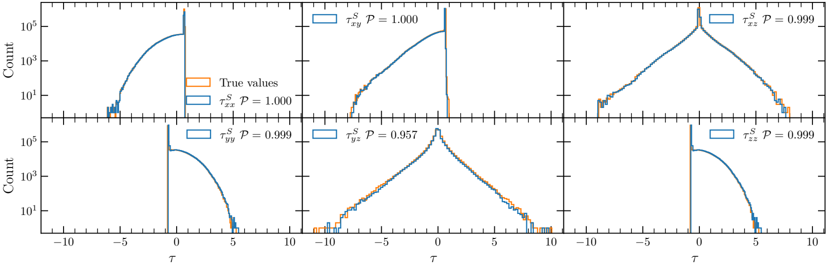
Kelvin-Helmholtz Simulation

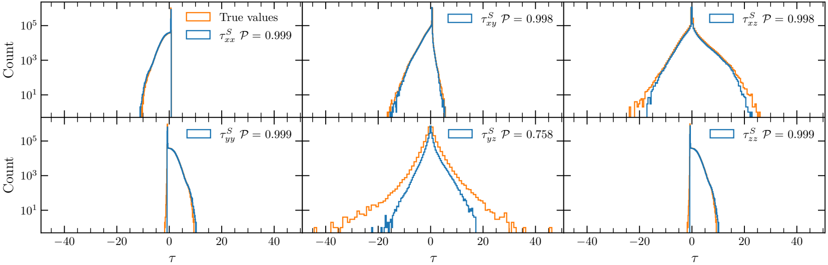
- We use an idealised model, local Kelvin-Helmholtz instability in 3D, special relativity.
- We run 3 resolutions (LR, SR, HR)
- Filter the SR data by a factor 2 $\rightarrow$ LR
- Filter the HR data by a factor 4 $\rightarrow$ LR
- 2 sets of models `f2, f4`
- Code used GR-Athena++, modern NR code capable of solving BNS mergers in dynmical spacetime with GRMHD and M1 neutrino transport. [Daszuta (WC) + 21, WC+ '23, Daszuta (WC)+ '26]



A priori testing

- Given a slice of data not in the training set can our model predict the correct subgrid tensors?

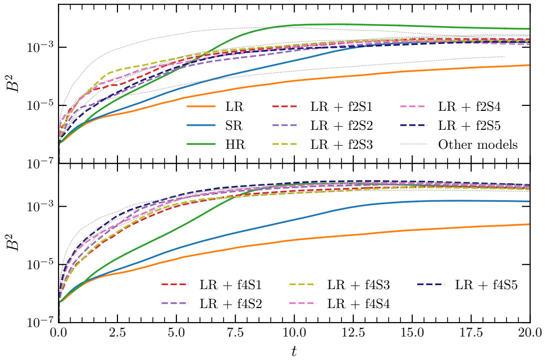




A posteriori testing

- We generate models with different hyperparameters and select those that give the target amplification at the end of the simulation.
- All models are designed to be small: all faster than the equivalent higher res run.

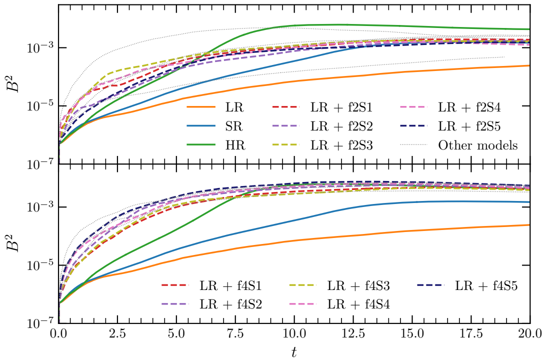
Name	$B^2(t=20)$	Speed up to target
LR	2.43×10^{-4}	-
SR	1.48×10^{-3}	-
HR	4.31×10^{-3}	-
f2S1	1.88×10^{-3}	4.17
f2S2	1.75×10^{-3}	2.80
f2S3	1.62×10^{-3}	4.63
f2S4	1.26×10^{-3}	4.02
f2S5	1.52×10^{-3}	3.94
f4S1	4.02×10^{-3}	40.5
f4S2	4.75×10^{-3}	43.5
f4S3	3.81×10^{-3}	33.3
f4S4	4.46×10^{-3}	38.6
f4S5	5.46×10^{-3}	28.4



A posteriori testing: growth rate

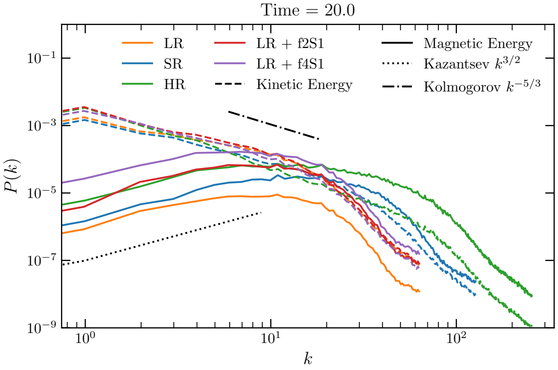
- Final B matches, but what about growth rate?
- Growth rate at finite resolution $B^2 \propto \exp(\gamma/\Delta x)$
- Can estimate γ from "no-subgrid runs" and effective $\Delta x = L/N_{\text{eff}}$.

Name	$N_{\text{mesh,eff}}$
LR	128
SR	256
HR	512
f2S1	270
f2S2	288
f2S3	159
f2S4	175
f2S5	285
f4S1	508
f4S2	541
f4S3	528
f4S4	428
f4S5	409



A posteriori testing: spectra

- How do the structures of the magnetic and kinetic energies compare?
- More energy injected on longer wavelengths.
- Match to expected Kolmogorov and Kazantsev scaling for dynamo process.
- Robust kinetic energy modelling even though model selection criterion based on magnetic fields.



Conclusion

- First machine learning subgrid model for relativistic MHD implemented
- Can be used to incorporate HR physics accurately in LR runs, for factor ~ 44 speedup.
- Can be applied to BNS simulations capturing magnetic field amplification from KHI.
- Do we need separate models for different physics, e.g. MRI vs KHI?
- Gauge invariance of model when moving to full GR?

- Model the relationship between inputs and outputs with a sequence of linear transformations, with a simple non-linear activation function between each:

$$X_i^{(\ell)} = g(\tilde{X}_i^{(\ell-1)}) \tag{6}$$

$$\tilde{X}_i^{(\ell-1)} = W_{ij} X_j^{(\ell-1)} + b_j \tag{7}$$

- where e.g. $g(x) = max(0, x)$ and W, b are parameters to be tuned, through stochastic gradient descent.
- Training process tunes these parameters by minimising the error of the model on training dataset.