Primordial Black Holes and Stochastic Inflation: Noise terms can be important

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1. Brief overview - standard cosmological model and inflation.
2. What are and why PBHs ?
3. Enhancing the spectrum of primordial fluctuations on small scales.
4. Ultra Slow Roll Inflation
5. The Stochastic inflation formalism and the importance of noise
6. Realistic slow roll into USR type behaviour

Subirfest 2023 - Oxford - Sept 12, 2023

A few preliminaries

We live in a large old universe, at least 13.8bn years old, which is expanding, in fact accelerating

It is described on large scales by the homogeneous and isotropic FLRW metric

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\ \left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]}$$

Where a(t) is the scale factor of the universe allowing us to describe it's expansion via the Hubble parameter

$$\boxed{H = \frac{\dot{a}}{a} > 0}$$

Growth of structure is from gravitational amplification of initial instabilities associated with tiny initial homogeneities

The universe appears to be accelerating today, as it may have been in the very earliest moments - although at dramatically different rates.

$$\boxed{\frac{\ddot{a}}{a} > 0}$$

Acceleration in the early universe is known as Inflation and driven by the potential energy of the inflaton
Acceleration today is not called anything in particular and we don't know what is driving it either - we call it dark energy, but it could be a potential energy, modified kinetic energy, modified gravity or a cosmological constant !

Standard model of cosmology - Flat ΛCDM model

Provides excellent description of the large scale evolution of our Universe from about 1 sec to 13.8 bn years

Based on a number of key assumptions - what is the universe made of ?

Standard model of particle physics

Dark matter made of cold non-relativistic particles

Dark Energy is a Cosmological constant Λ

More key assumptions - initial conditions :

Expanding initial conditions (what banged ?)

Homogeneous and isotropic on large scales

Universe is spatially flat

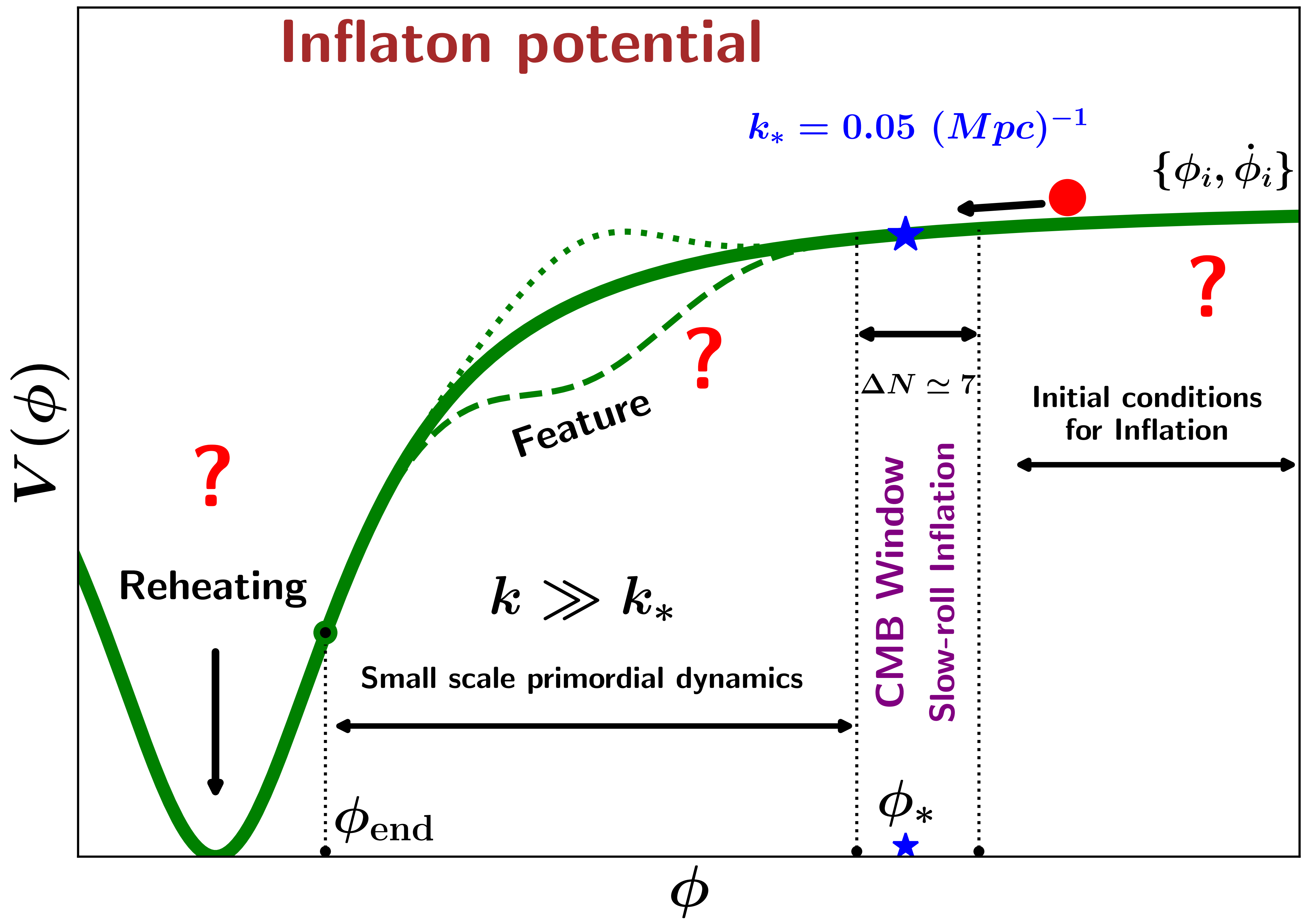
Almost scale invariant, nearly Gaussian and adiabatic initial density fluctuations

Led to the idea behind the Inflationary Universe = a short period of accelerated expansion of space between the GUT and EWK era in which

$$a_{end} > e^{60} a_{ini}$$

Setting the initial conditions for the Hot Big Bang period

Inflation - brief recap



$$S[g_{\mu\nu}, \phi] = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{m_p^2}{2} R - \frac{1}{2} \partial_\mu \phi \, \partial_\nu \phi \, g^{\mu\nu} - V(\phi) \right]$$

Einstein’s equations assuming scalar field dominates the energy density

$$H^2 \equiv \frac{1}{3m_p^2} \rho_\phi = \frac{1}{3m_p^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right]$$

$$\dot{H} \equiv \frac{\ddot{a}}{a} - H^2 = -\frac{1}{2m_p^2} \dot{\phi}^2 ,$$

$$\ddot{\phi} + 3 H \dot{\phi} + V_{,\phi}(\phi) = 0 .$$

Inflation can occur when potential dominated $\dot{\phi}^2 < V(\phi)$

When $\dot{\phi}^2 \ll V(\phi)$ with nearly flat potential dominating we obtain nearly exponential expansion at the background level

$$a \sim e^{Ht}$$

Inflation - produces the initial seeds for structure to grow through Quantum Fluctuations

Action for gravity plus inflaton

$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{m_p^2}{2} R - \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\, g^{\mu\nu} - V(\phi) + ... \right)$$

Metric including fluctuations

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\left(e^{2\Psi(t,\vec{x})}\,\delta_{ij} + h_{ij}(t,\vec{x})\right)\mathrm{d}x^i\mathrm{d}x^j\right]}$$

When mass of inflaton is small compared to Hubble rate : m<<H

Comoving curvature perturbation exists -
will become density and temperature fluctuations

$$\boxed{-\zeta(t,\vec{x}) = \Psi + \frac{H}{\dot{\phi}}\frac{\delta\phi}{m_p}}$$

Tensor perturbations which will become relic gravitational waves

$$h_{ij}(t,\vec{x})$$

Inflation - allows us to predict the form of the fluctuations for a given model

In particular during slow roll inflation, where the potential is flat enough and dominates the energy density

We have $\dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll V'(\phi)$

We quantify the power spectrum and deviations from scale invariance in terms of slow roll parameters $\Rightarrow \boxed{\epsilon_H, \; |\eta_H| \ll 1}$ where $\boxed{\epsilon_H = \frac{\dot{\phi}^2}{2m_p^2 H^2}, \quad \eta_H = \frac{-\ddot{\phi}}{H\dot{\phi}}}$

The Power Spectrum for scalar and tensor fluctuations on large scales

$$\mathcal{P}_\zeta = \frac{1}{8\pi^2}\left(\frac{H}{m_p}\right)^2 \frac{1}{\epsilon_H} = A_S \left(\frac{k}{k_*}\right)^{n_S} \qquad \mathcal{P}_\mathcal{T} = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2 = A_\mathcal{T}\left(\frac{k}{k_*}\right)^{n_T}$$

Slow roll predictions: $n_S = -4\epsilon_H + 2\eta_H, \quad n_\tau = -2\epsilon_H, \quad r \equiv \frac{A_\tau}{A_S} = 16\epsilon_{H*}$

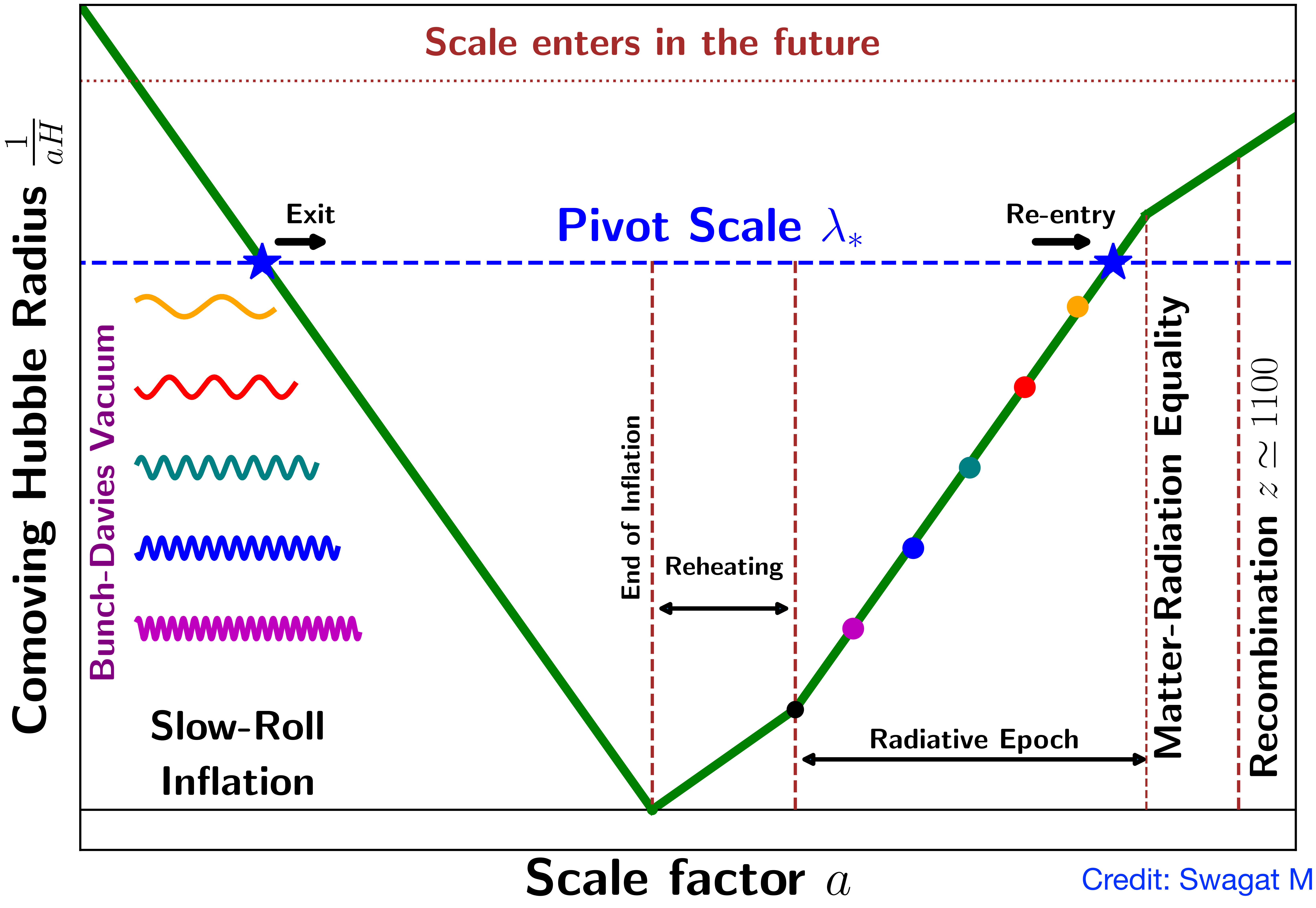
CMB observations: $A_s = 2.1 \times 10^{-9}$ $\qquad A_\mathcal{T} \leq 3.6\% \, A_S$

Scalar Spectral index: Red tilt $\boxed{n_S \simeq -0.033}$

Tensor spectra index: $\boxed{|n_\mathcal{T}| \leq 0.0045}$

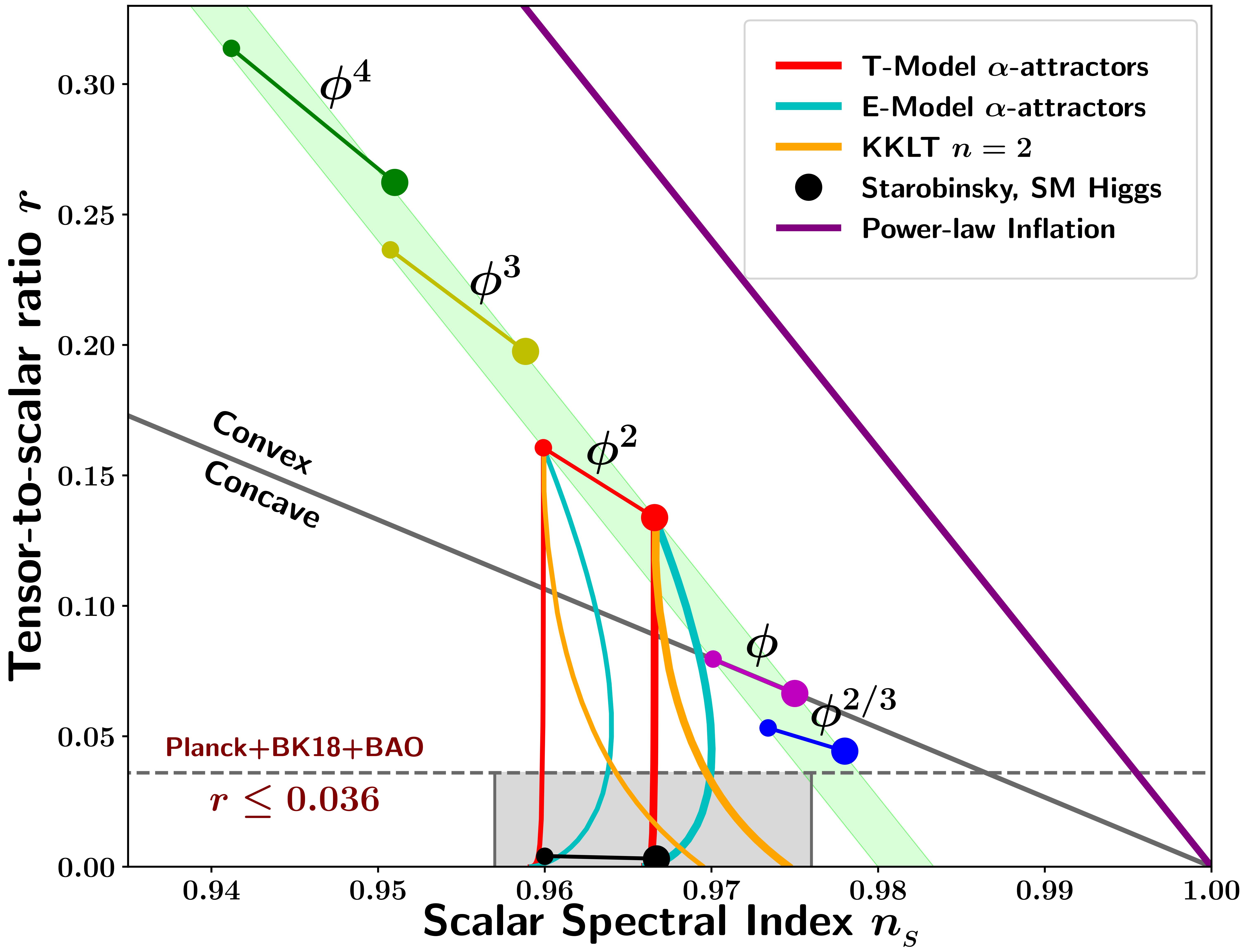
Prediction is nearly scale invariant and are very small on large scales

Pictorially may help

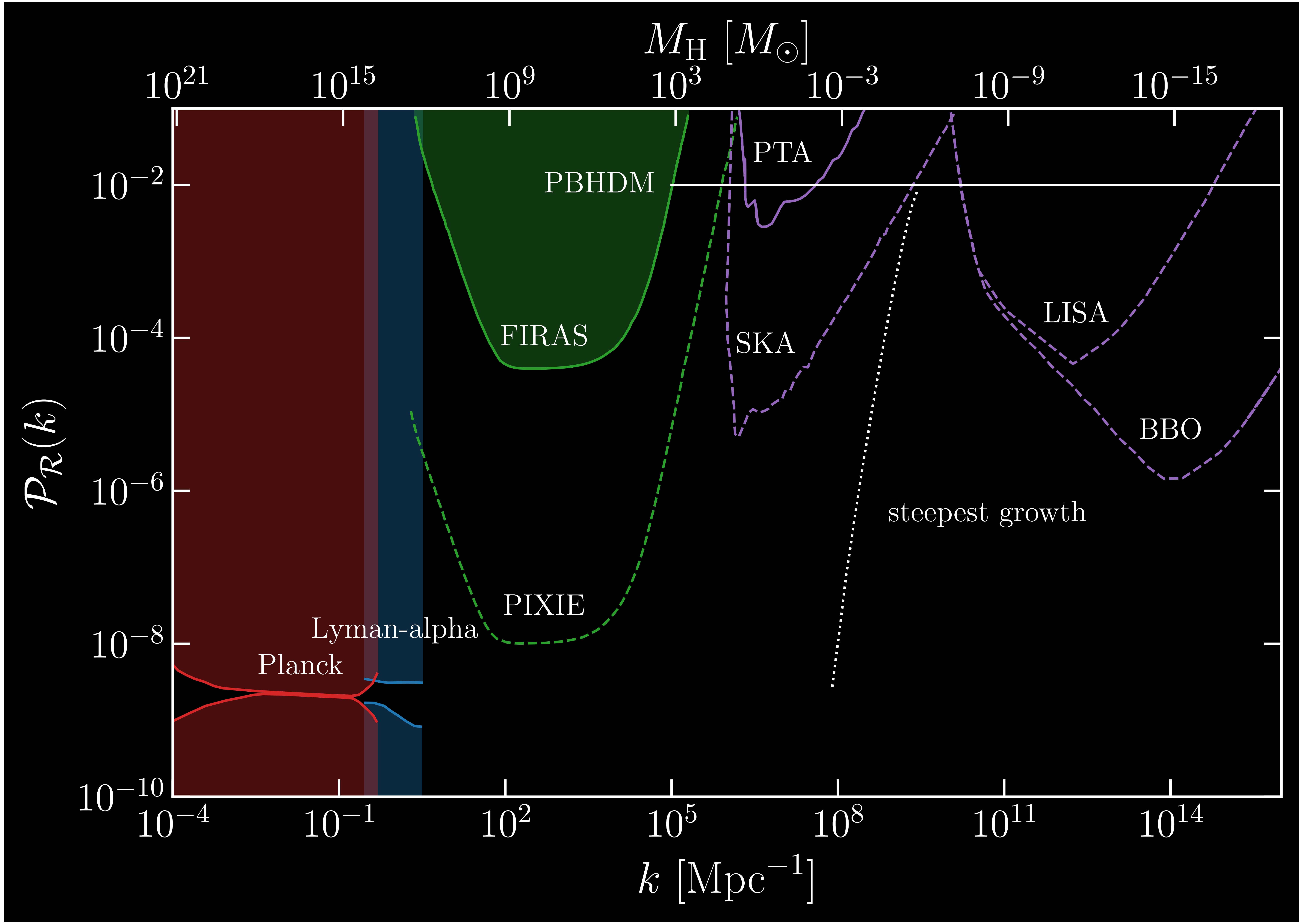


Credit: Swagat Mishra

The main way we constrain models of inflation from observation



Observational Constraints on Power Spectrum - very little on small scales



Carr et al 2020; Green and Kavanagh 2020

Since LIGO's amazing direct detection of coalescing BH binaries, PBHs have had a resurgence of interest.

For reviews and future directions see Green & Kavanagh [arXiv: 2007.10722], Carr & Kuhnel [arXiv:2006.028380, Bird et al [arXiv:2203.08967]

Form from over densities in early Universe - before nucleosynthesis - non-baryonic [Zel'dovich & Novikov; Hawking]

They evaporate (Hawking radiation), lifetime longer than age of Universe for $M>10^{15}$g — can make them a DM candidate [Hawking, Chapline]

Maybe some of the BHs in the binaries detected by LIGO-VIRGO are primordial [Bird et al, Clesse & Garcia-Bellido, Sasaki et al]

Formation

Favoured - collapse of large density perturbations (shortly after horizon entry) during radiation domination

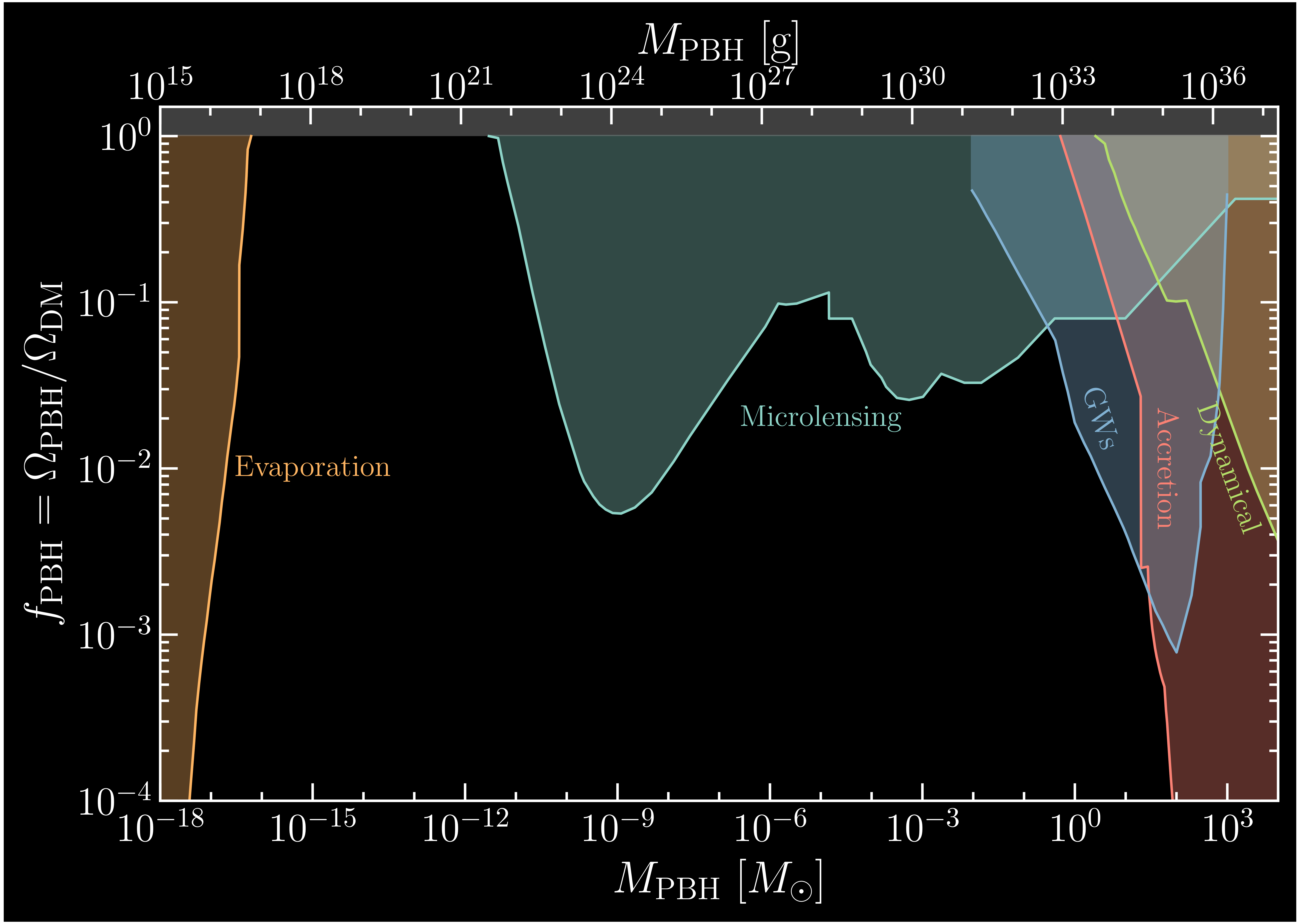
Also collapse of cosmic string loops [Hawking, Polnarev & Zemboricz], bubble collisions [Hawking, Moss & Stewart], fragmenting inflation condensates [Cotner & Kusenko]

Threshold for PBH formation [Carr] : $\delta \gtrsim \delta_c \sim w = p/\rho = 1/3$. — density contrast at horizon crossing, depends on shape of perturbation which depends on primordial power spectrum

PBH mass roughly equal to horizon mass

$$M_{\rm PBH} \sim 10^{15} g \left(\frac{t}{10^{-23}}\right) \sim M_{\rm sun} \left(\frac{t}{10^{-6} s}\right)$$

Present day bounds on PBHs as DM



Green and Kavanagh 2020

Primordial Black Holes are really really cool !

[Hawking 1971, Carr, Hawking 1974, Hawking 1974, Page 1975]

- Formed very early - typically within the first few seconds of the Hot Big Bang phase !
- We can use them to probe very small early Universe physics.
- Hawking told us, they have a temperature, and they evaporate as well as accrete.

- Hawking radiation - hard to detect.

$$T_H = \frac{\hbar c^3}{8\pi G K_B M_{\rm BH}} = 6.19 \times 10^{-8} \left(\frac{M_\odot}{M_{\rm BH}} \right) \; K$$

- Evaporation rate: $\frac{dm_{\rm BH}}{dt} = -\frac{g_\star}{3} \frac{m_{\rm Pl}^4}{m_{\rm BH}^2}$ ---> mass (t): $m_{\rm BH}^3 = m_0^2 - g_\star m_{\rm Pl}^4 t$ ---> lifetime: $\tau = \frac{m_0^3}{g_\star m_{\rm Pl}^4}$

- Initial mass of PBH evaporating today — about that of a mountain $M_c \simeq \left(\frac{t_0}{13.8 \ {\rm Gyr}} \right)^{\frac{1}{3}} \ 10^{15} \ {\rm gm}$

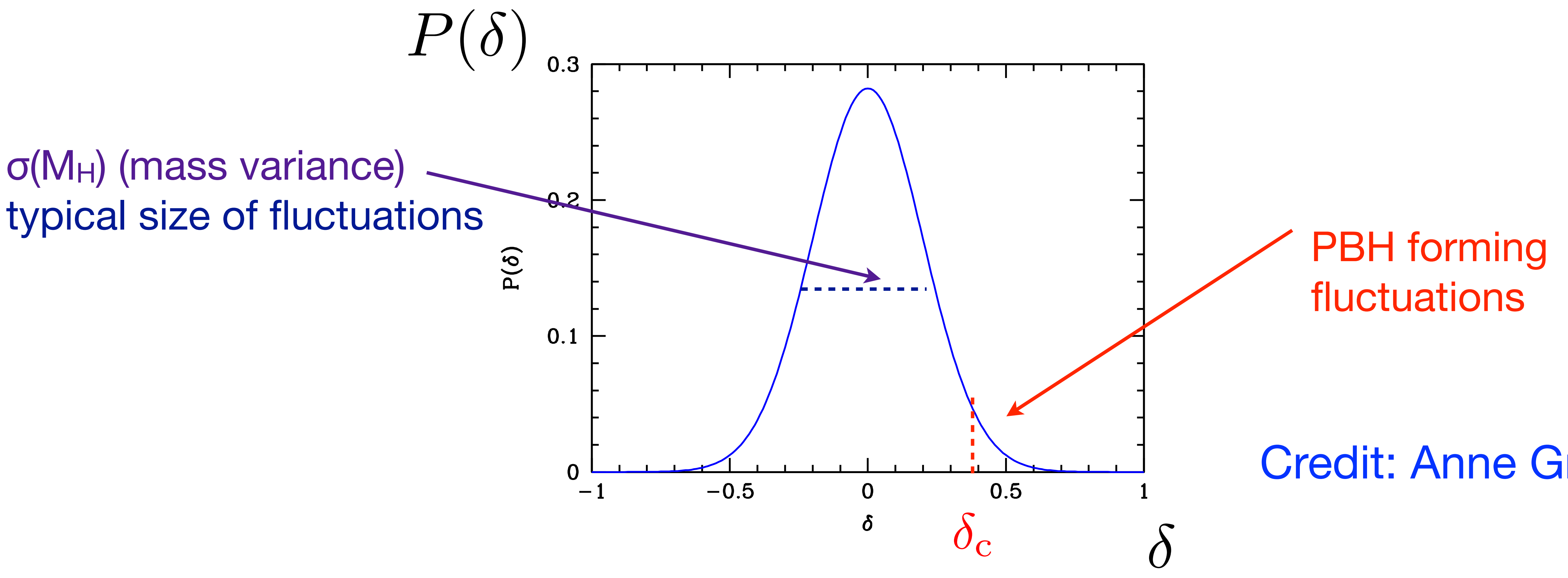
- Mass at formation $M_{\rm PBH} \simeq M_{\rm H} = 6 \times 10^4 \left(\frac{t}{1{\rm sec}} \right) M_{\rm sun}$ PBHs evaporating today formed around 10^{-23} sec into HBB phase

Initial PBH mass fraction (fraction of universe in regions dense enough to form PBHs)

$$\beta(M) \sim \int_{\delta_c}^{\infty} P(\delta(M_H))d\delta(M_H)$$

For Gaussian probability distribution :

$$\beta(M) = \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma(M_H)}\right)$$



Credit: Anne Green

but in fact β must be small, hence $\sigma \ll \delta_c$ and $\beta(M) \sim \sigma(M_H)\exp\left(-\frac{\delta_c^2}{2\sigma^2(M_H)}\right)$

But PBH are matter, so in radiation their contribution to the energy density budget grows

$$\frac{\rho_{\rm PBH}}{\rho_{\rm rad}} \propto \frac{a^{-3}}{a^{-4}} \propto a$$

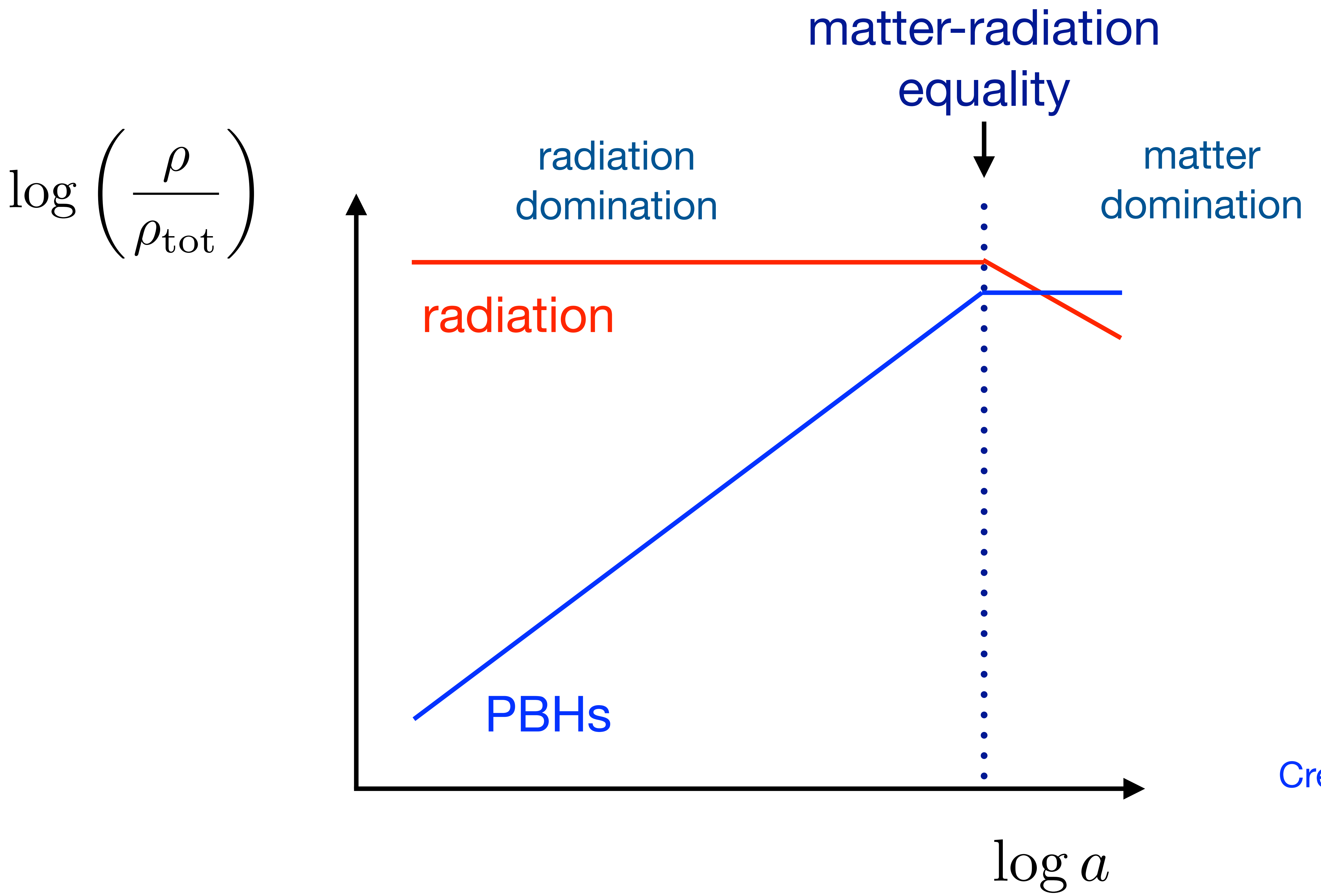
Relation between PBH initial mass function β and fraction of DM in form of PBHs, f:

$$\beta(M) \sim 10^{-9} f \left(\frac{M}{M_{\rm sun}}\right)^{1/2}$$

So β must be small but non-negligible

But PBH are matter, so in radiation their contribution to the energy density budget grows

$$\frac{\rho_{\rm PBH}}{\rho_{\rm rad}} \propto \frac{a^{-3}}{a^{-4}} \propto a$$

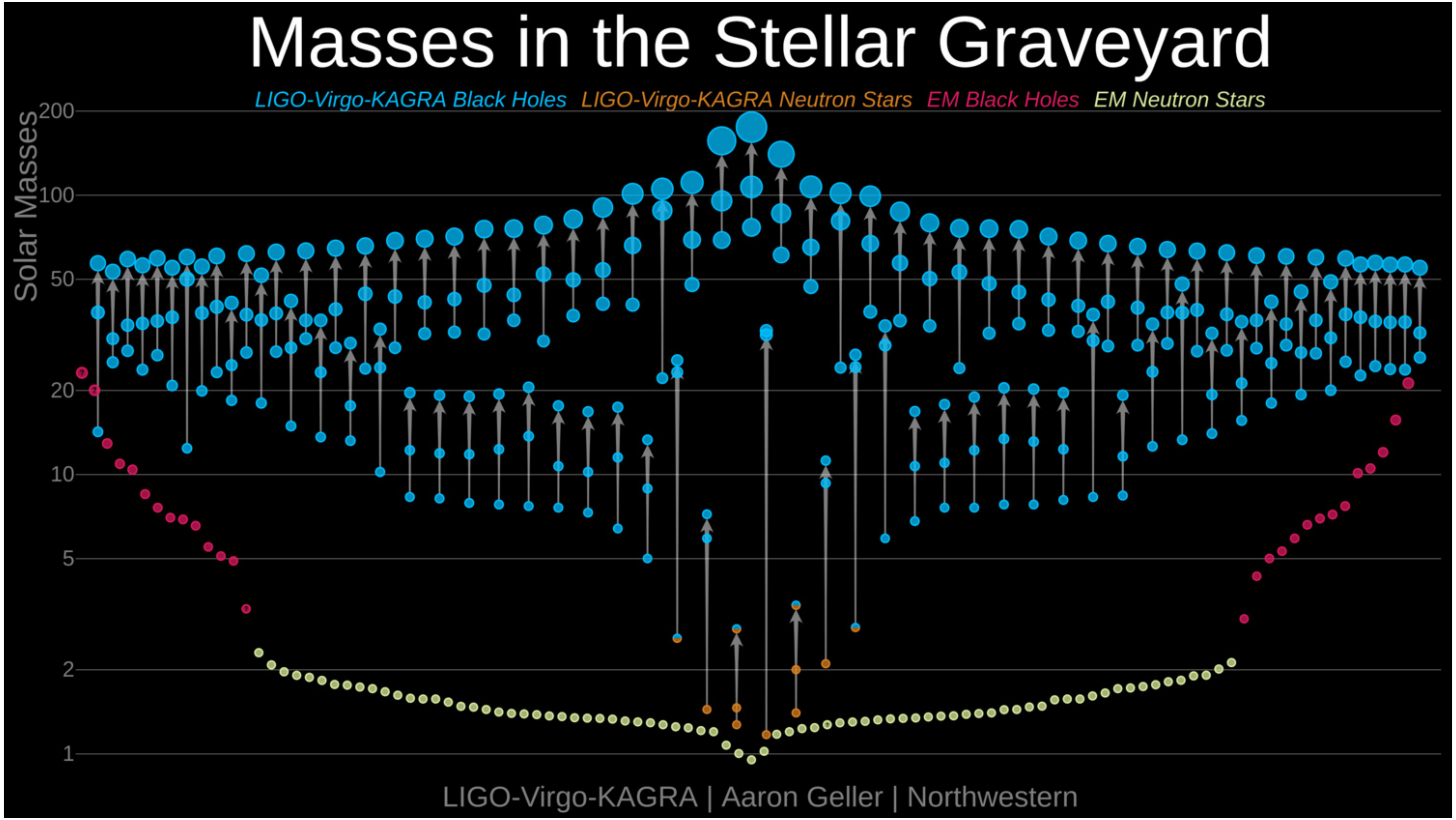


Credit: Anne Green

Black Hole Binaries discovered by LIGO-VIRGO-KAGRA

$100 M_\odot$

$10 M_\odot$



Any of them PBHs ? [Bird et al, Clesse & Garcia-Bellido, Sasaki et al]

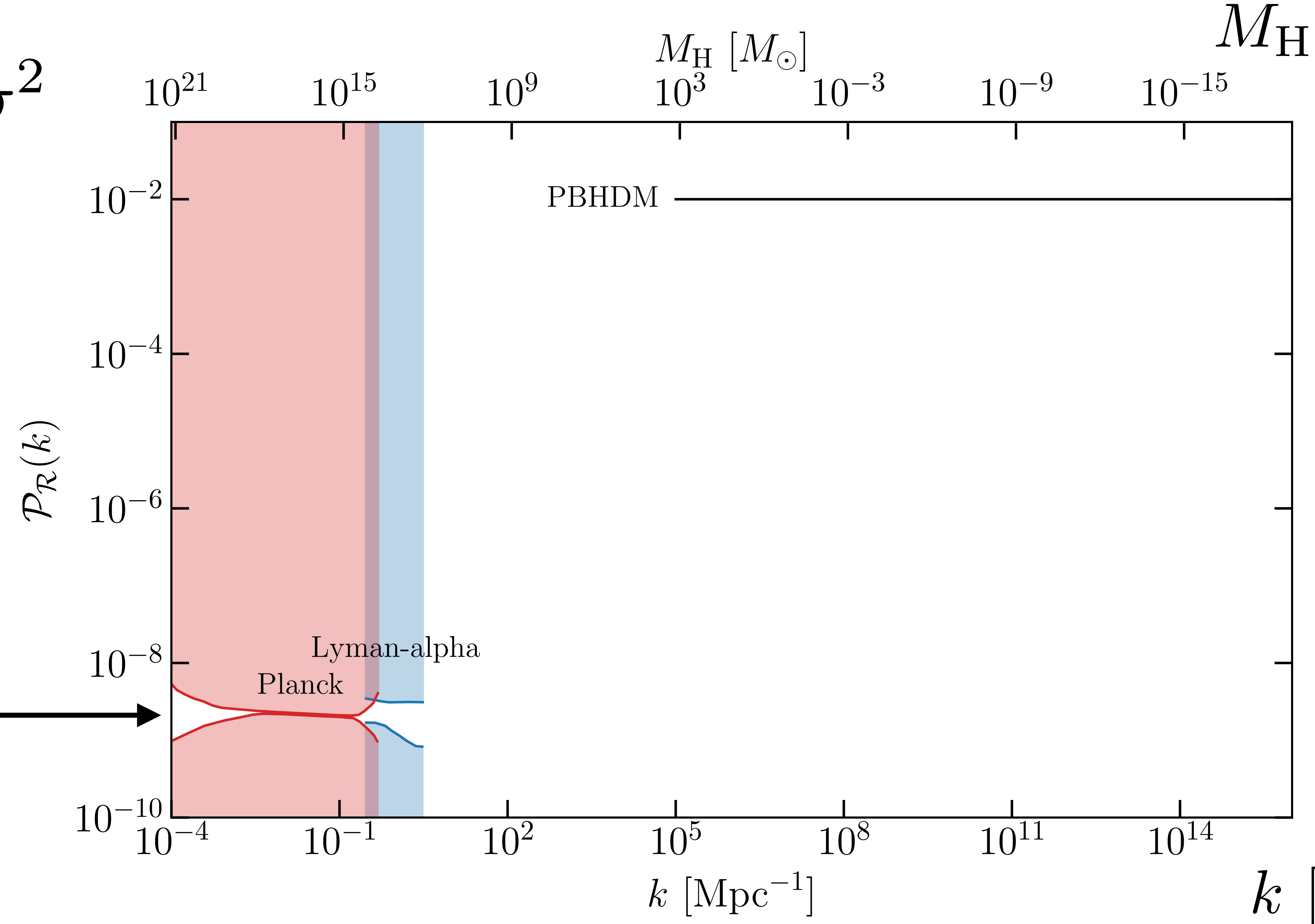
But on CMB we know primordial perturbations have amplitude $\sigma(M_H) \sim 10^{-5} \Rightarrow \beta(M) \sim \mathrm{erfc}(10^5) \sim \exp(-10^{10})$

Totally negligible if initial perturbations were close to scale invariant.

To form an interesting number of PBHs the primordial perturbations must be significantly larger ($\sigma^2(M_H)$~0.01) on small scales than on cosmological scales.

$\mathcal{P}_{\mathcal{R}}(k) \sim \sigma^2$

$M_{\rm H}\ [M_\odot]$ ~ PBH mass



amplitude required for $f_{\rm PBH} \approx 1$

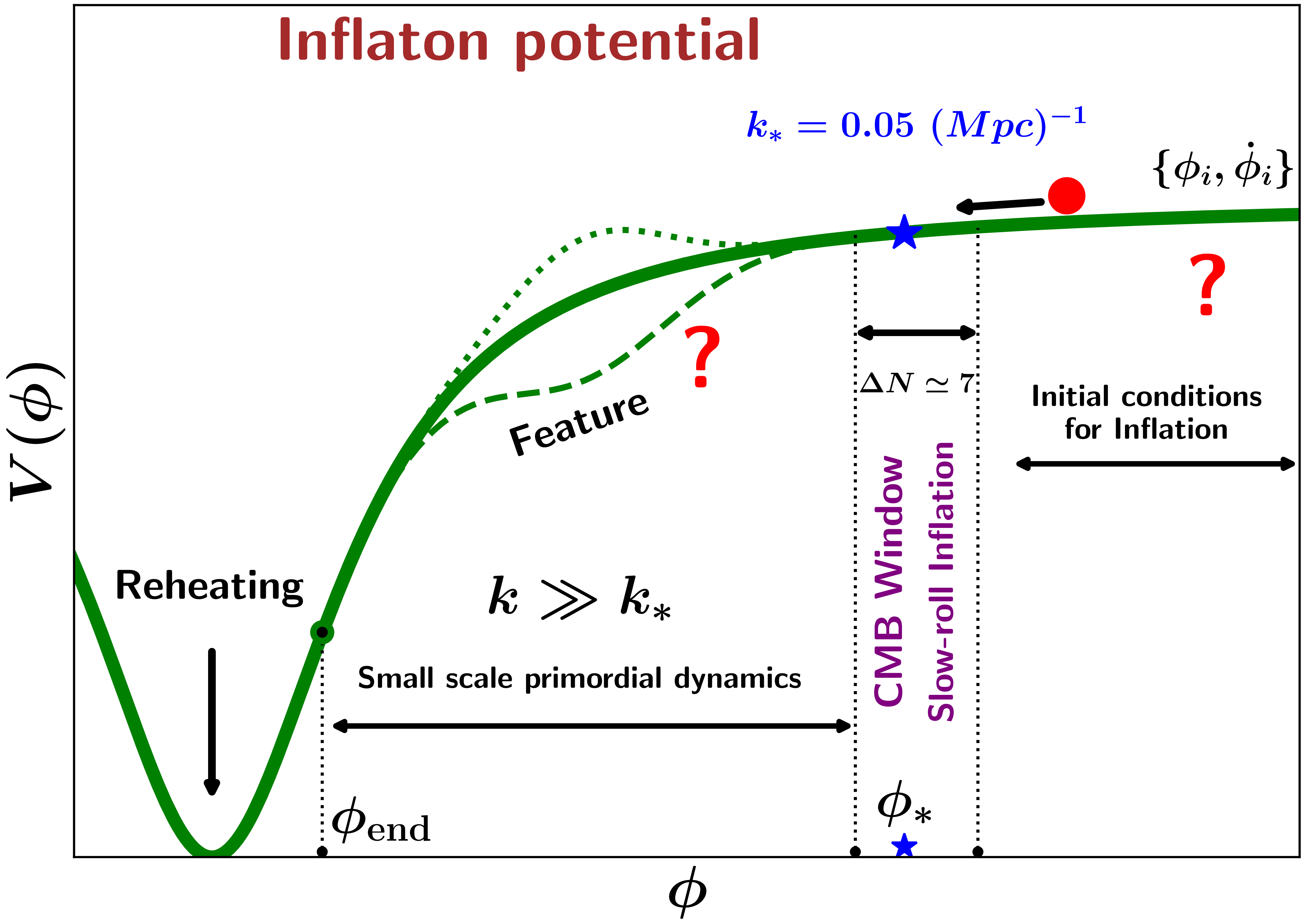
measured amplitude on cosmological scales

$k\ [\mathrm{Mpc}^{-1}]$

One approach — introduce non-gaussianity. PBHs form from rare large density fluctuations arising during inflation, change the shape of the tail of the probability distribution —> can significantly affect the PBH distribution

Credit: Anne Green

Inflation - second brief recap



$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^4x\,\sqrt{-g}\,\left[\frac{m_p^2}{2}\,R - \frac{1}{2}\,\partial_\mu\phi\,\partial_\nu\phi\,g^{\mu\nu} - V(\phi)\right]$$

$$H^2 \equiv \frac{1}{3m_p^2}\,\rho_\phi = \frac{1}{3m_p^2}\left[\frac{1}{2}\dot\phi^2 + V(\phi)\right]$$

$$\dot H \equiv \frac{\ddot a}{a} - H^2 = -\frac{1}{2m_p^2}\,\dot\phi^2\,,$$

$$\ddot\phi + 3\,H\dot\phi + V_{,\phi}(\phi) = 0\,.$$

$$\epsilon_H = -\frac{\dot H}{H^2} = \frac{1}{2m_p^2}\,\frac{\dot\phi^2}{H^2}\,,$$

$$\eta_H = -\frac{\ddot\phi}{H\dot\phi} = \epsilon_H + \frac{1}{2\epsilon_H}\,\frac{\mathrm{d}\epsilon_H}{\mathrm{d}N}\,,$$

Slow roll parameters

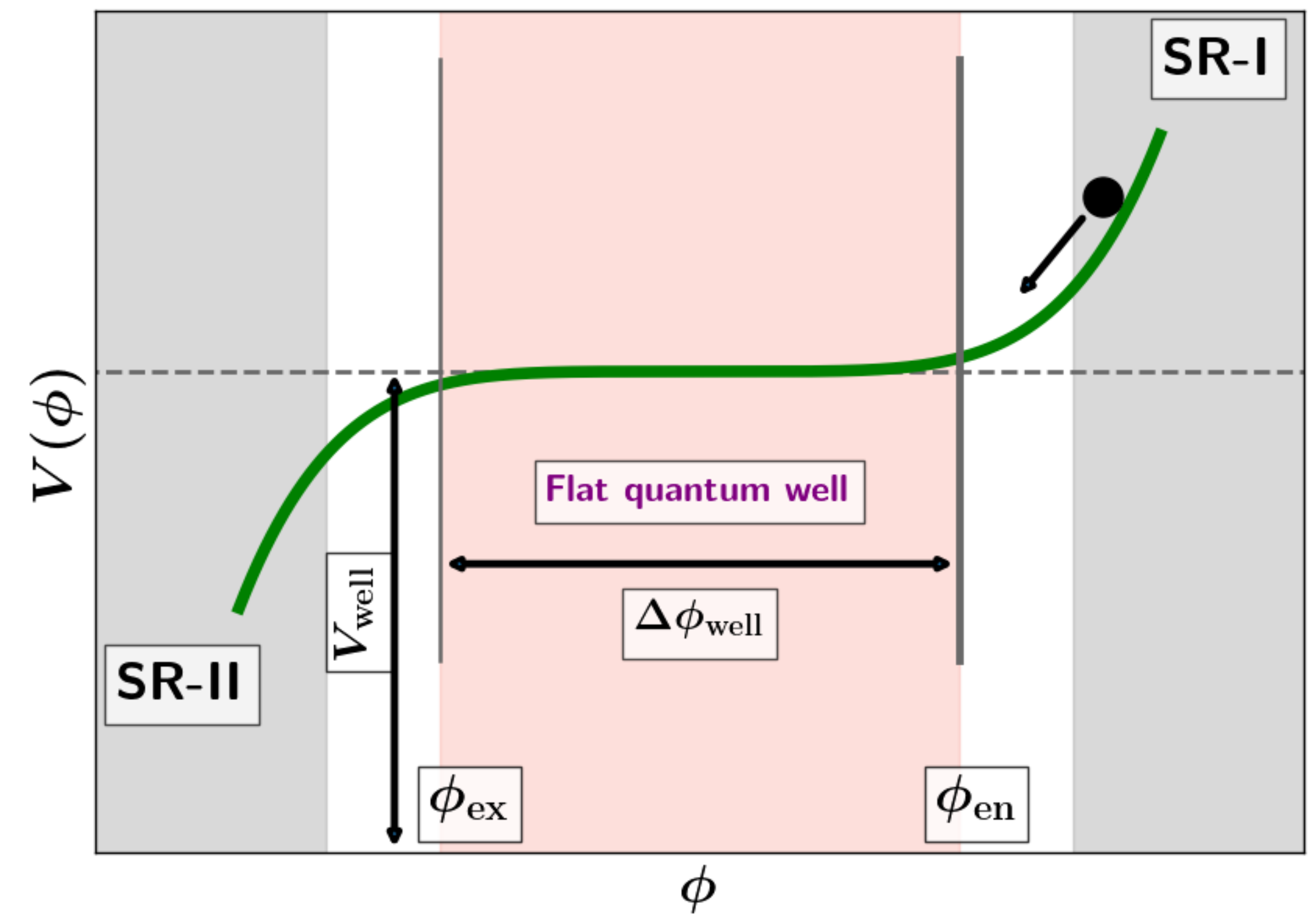
- **Quasi-de Sitter** inflation corresponds to the condition $\epsilon_H \ll 1$.
- **Slow-roll** inflation corresponds to both $\epsilon_H,\ \eta_H \ll 1$.

Introducing features into the inflaton potential - to generate the PBH abundance

Inflaton potential featuring **an approximate inflection point** or **a local bump/dip** at **low scales** slows down the inflaton leading to appreciable enhancement of scalar power-spectrum

$$P_\zeta = \frac{1}{8\pi^2}\left(\frac{H}{m_p}\right)^2 \frac{1}{\epsilon_H} \qquad \epsilon_H = \frac{1}{2m_p^2}\frac{\dot\phi^2}{H^2}$$

.



PBH formation requires enhancement of the inflationary power spectrum by a factor of 10^7 within less than 40 e-folds of expansion, the quantity $\Delta \ln \varepsilon / \Delta N$, hence $|\eta_H|$ must grow to be of order unity, so violate the second slow roll condition. A flat plateau like region in the potential can allow this.

Ultra Slow roll inflation [Kinney (2005), Inoue and Yokoyama (2002)]

At intermediate field values, inflaton enters a transient period of USR. Since $V'(\phi)\sim 0$,

$$\ddot\phi + 3H\dot\phi = 0 \Rightarrow -\ddot\phi/H\dot\phi = +3, \quad \text{hence} \quad \eta_H = +3 \quad \text{(during USR)}$$

Inflaton speed drops exponentially with number of e-folds : $\dot\phi = \dot\phi_{\rm en}\, e^{-3\,H\,(t-t_{\rm en})} \propto e^{-3\,N}$

Critical entry velocity to just get across the plateau

$$\dot\phi_{\rm cr} = -3\,H\,\Delta\phi_{\rm well}, \qquad \pi_{\rm cr} = -3\,\Delta\phi_{\rm well}. \qquad \pi = \frac{{\rm d}\phi}{{\rm d}N} = \frac{\dot\phi}{H}$$

Quantum dynamics — stochastic inflation formalism - non - perturbative approach to calc the full primordial PDF [Starobinsky 1982]

Effective long wavelength IR treatment of inflation, inflaton field is coarse grained over super Hubble scales k ≲ σaH, with const $\sigma \ll 1$.

Hubble exiting smaller scale UV modes are constantly converted into IR modes due to accelerated expansion.

Coarse grained inflaton field follows a Langevin-type-stochastic differential equation with stochastic noise terms sourced by the smaller scale UV modes, on top of classical drift terms sourced by V'(ϕ).

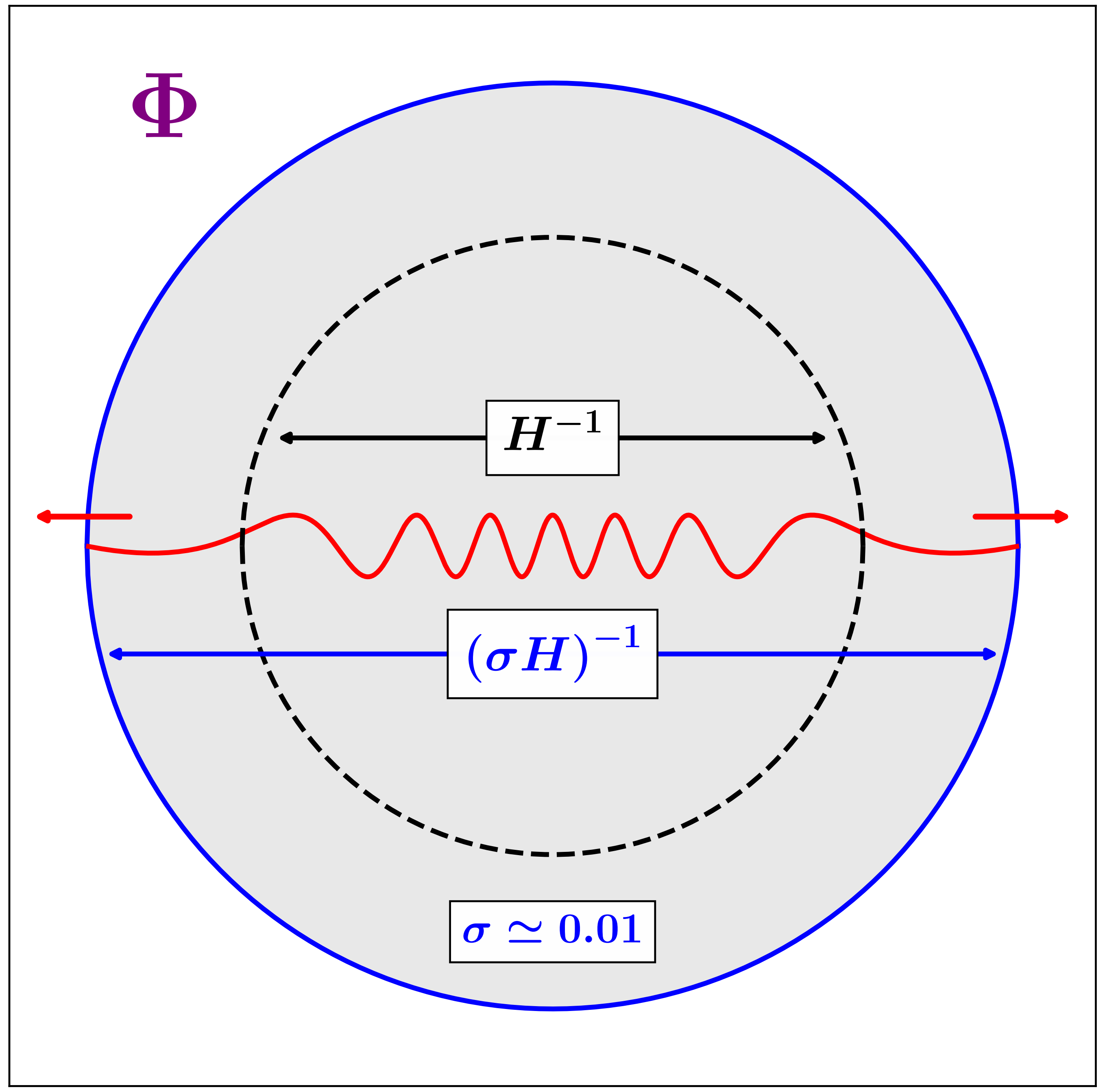
Split the Heisenberg operators of the inflaton $\hat{\phi}(N,\vec{x})$ and its conjugate momentum $\hat{\pi}_\phi = \mathrm{d}\hat{\phi}/\mathrm{d}N$ into the corresponding IR $\{\hat{\Phi},\hat{\Pi}\}$ and UV $\{\hat{\varphi},\hat{\pi}\}$ parts:

$$\hat{\phi} = \hat{\Phi} + \hat{\varphi} \quad , \quad \hat{\pi}_\phi = \hat{\Pi} + \hat{\pi}$$

where the UV fields are defined as

$$\hat{\varphi}(N,\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{\frac{3}{2}}} W\left(\frac{k}{\sigma a H}\right) \left[\phi_k(N)\,\hat{a}_{\vec{k}}\,e^{-i\vec{k}.\vec{x}} + \phi_k^*(N)\,\hat{a}_{\vec{k}}^\dagger\,e^{i\vec{k}.\vec{x}}\right]$$

$$\hat{\pi}(N,\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{\frac{3}{2}}} W\left(\frac{k}{\sigma a H}\right) \left[\pi_k(N)\,\hat{a}_{\vec{k}}\,e^{-i\vec{k}.\vec{x}} + \pi_k^*(N)\,\hat{a}_{\vec{k}}^\dagger\,e^{i\vec{k}.\vec{x}}\right]$$



Credit: Swagat Mishra

$W\,(k/\sigma a H)$ is the 'window function' Selects out modes with momentum k>σaH

With Swagat Mishra and Anne Green - e-Print:2303.17375 - JCAP 2023

Hamiltonian equations for coarse grained (IR) fields are Langevin equation

$$\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}N} = \hat{\Pi} + \hat{\xi}_\phi(N) \ ,$$
$$\frac{\mathrm{d}\hat{\Pi}}{\mathrm{d}N} = -(3-\epsilon_H)\,\hat{\Pi} - \frac{V_{,\phi}(\hat{\Phi})}{H^2} + \hat{\xi}_\pi(N) \,,$$

where the field and momentum noise operators $\hat{\xi}_\phi(N)$ and $\hat{\xi}_\pi(N)$ are given by

$$\hat{\xi}_\phi(N) = -\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{\frac{3}{2}}} \frac{d}{\mathrm{d}N} W\left(\frac{k}{\sigma a H}\right) \left[\phi_k(N)\,\hat{a}_{\vec{k}}\, e^{-i\vec{k}.\vec{x}} + \phi_k^*(N)\,\hat{a}_{\vec{k}}^\dagger\, e^{i\vec{k}.\vec{x}}\right]$$
$$\hat{\xi}_\pi(N) = -\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{\frac{3}{2}}} \frac{d}{\mathrm{d}N} W\left(\frac{k}{\sigma a H}\right) \left[\pi_k(N)\,\hat{a}_{\vec{k}}\, e^{-i\vec{k}.\vec{x}} + \pi_k^*(N)\,\hat{a}_{\vec{k}}^\dagger\, e^{i\vec{k}.\vec{x}}\right]$$

Assume Window function with sharp IR/UV cut-off
$$W\left(\frac{k}{\sigma a H}\right) = \Theta\left(\frac{k}{\sigma a H} - 1\right) .$$

- Physically, the noise terms $\hat{\xi}_\phi$ and $\hat{\xi}_\pi$ in the Langevin equations are sourced by the constant outflow of UV modes into the IR modes
- As UV mode exits the cut-off scale $k = \sigma a H$ to become part of the IR field on super-Hubble scales, IR field receives a '*quantum kick*' with typical amplitude $\sim \sqrt{\langle 0|\hat{\xi}(N)\hat{\xi}(N')|0\rangle}$, where $|0\rangle$ is usually taken to be the Bunch-Davies vacuum.
- Given that $\sigma \ll 1$, this happens on ultra super-Hubble scales, where the UV modes must have already become classical fluctuations..

With $\xi_i = \{\xi_\phi, \xi_\pi\}$, equal-space noise correlators (auto-correlators) are

$$\langle \xi_i(N)\, \xi_j(N') \rangle = \Sigma_{ij}(N)\, \delta_D(N - N') \,,$$

where the noise correlation matrix Σ_{ij} is

$$\Sigma_{ij}(N) = (1 - \epsilon_H)\, \frac{k^3}{2\pi^2}\, \phi_{i_k}(N) \phi^*_{j_k}(N) \Big|_{k=\sigma a H} .$$

The noise correlation matrix is important !

Equivalent Fokker-Planck equation - time evolution of the PDF of $\{\Phi, \Pi\}$, subject to appropriate bcds.

$$\frac{\partial}{\partial \mathcal{N}} P_{\Phi_i}(\mathcal{N}) = \left[D_i \frac{\partial}{\partial \Phi_i} + \frac{1}{2} \Sigma_{ij} \frac{\partial^2}{\partial \Phi_i \partial \Phi_j} \right] P_{\Phi_i}(\mathcal{N})$$

where

$$D_i = \left\{ \Pi, -(3 - \epsilon_H)\, \Pi - \frac{V_{,\phi}(\Phi)}{H^2} \right\}$$

1. Absorbing boundary at $\phi^{(\mathrm{A})}$

$$P_{\Phi=\phi^{(\mathrm{A})},\Pi}(\mathcal{N}) = \delta_D(\mathcal{N}) \,,$$

Closer to ϕ at end of inflation

2. Reflecting boundary at $\phi^{(\mathrm{R})}$

$$\frac{\partial}{\partial \Phi} P_{\Phi=\phi^{(\mathrm{R})},\Pi}(\mathcal{N}) = 0 \,.$$

Closer to ϕ at cmb scale

Characteristic function: $\chi_{\mathcal{N}}(q;\Phi_i)$, given by Fourier transform of the PDF $P_{\Phi_i}(\mathcal{N})$

$$\chi_{\mathcal{N}}(q;\Phi_i) \equiv \langle e^{i\,q\,\mathcal{N}}\rangle = \int_{-\infty}^{\infty} e^{i\,q\,\mathcal{N}} P_{\Phi_i}(\mathcal{N})\,\mathrm{d}\mathcal{N}\,,$$

CF then satisfies

$$\left[D_i \frac{\partial}{\partial \Phi_i} + \frac{1}{2}\Sigma_{ij}\frac{\partial^2}{\partial\Phi_i\partial\Phi_j} + iq\right]\chi_{\mathcal{N}}(q;\Phi_i) = 0\,,$$

with bcs

$$\chi_{\mathcal{N}}(q;\phi^{(\mathrm{A})},\Pi) = 1\,, \quad \frac{\partial}{\partial\Phi}\chi_{\mathcal{N}}(q;\phi^{(\mathrm{R})},\Pi) = 0\ .$$

Usual approach: assume noise matrix elements Σ_{ij} are of the de Sitter-type:

$$\Sigma_{\phi\phi} = (H/2\pi)^2\,, \qquad \Sigma_{\phi\pi},\,\Sigma_{\pi\pi} \simeq 0\,.$$

Quantum diffusion across a flat segment of the inflaton potential [Pattsion et al 2021]. Intro

$$f = \frac{\Phi - \phi_{\mathrm{ex}}}{\Delta\phi_{\mathrm{well}}}\,, \quad y = \frac{\Pi}{\pi_{\mathrm{cr}}}\,, \quad \mu^2 \simeq \frac{\Delta\phi^2_{\mathrm{well}}}{m_p^2}\frac{1}{v_{\mathrm{well}}}\,, \quad v_{\mathrm{well}} = V_{\mathrm{well}}/m_p^4\,,$$

f is the fraction of the flat well which remains to be traversed; y is the momentum relative to the critical momentum, V_{well} is the height of the flat quantum well.

Free stochastic diffusion : $\pi_{\rm en} \ll \pi_{\rm cr} \Rightarrow y_{\rm en} \ll 1 \longrightarrow$ the classical drift term can be ignored [Ezquiaga et al [2020], Pattison et al [2021]]

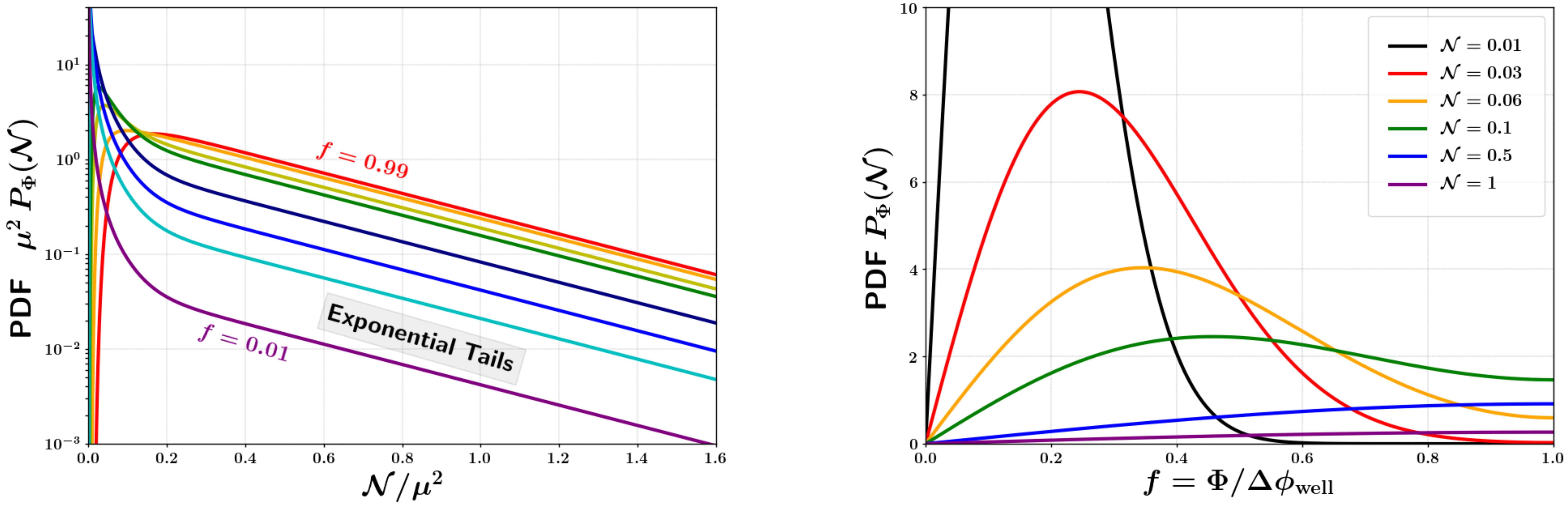
$$P_f(\mathcal{N}) = \sum_{n=0}^{\infty} A_n \sin\left[(2n+1)\,\frac{\pi}{2}\,f\right] \mathrm{e}^{-\Lambda_n \mathcal{N}} ,$$

where

$$A_n = (2n+1)\frac{\pi}{\mu^2} , \qquad \Lambda_n = (2n+1)^2 \frac{\pi^2}{4}\frac{1}{\mu^2} ,$$

For $\mathcal{N} \gg 1$, PDF has an exponential tail

$$P_\Phi(\mathcal{N}) \simeq A_0\, e^{-\Lambda_0 \mathcal{N}} .$$



(a) (b)

• But when power spectrum sufficiently amplified for an interesting abundance of PBHs, $\pi_{\rm en} \simeq \pi_{\rm cr} \Rightarrow y_{\rm en} \simeq 1$.

• Then, both classical drift and stochastic diffusion become important (at least initially during the entry into the USR segment).

• Furthermore, the de Sitter approximations for the noise matrix elements might breakdown during the transition into the USR phase [Ahmadi et al 2022] .

• Consequently, it becomes important to estimate the noise matrix elements more accurately.

Case 1: Noise matrix elements in stochastic inflation with featureless potential – slow roll case

Evolution of modes $\{\phi_k, \pi_k\}$ given via Mukhanov-Sasaki equation which in terms of conformal time τ is

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0 \ ,$$

where

$$\begin{aligned} z &= a m_p \sqrt{2\epsilon_H} \ , \\ \frac{z''}{z} &= (aH)^2 \left[2 + 2\epsilon_H - 3\eta_H + 2\epsilon_H^2 + \eta_H^2 - 3\epsilon_H\eta_H - \frac{1}{aH}\eta_H'\right] \end{aligned}$$

and in the spatially flat gauge: $$\phi_k = \frac{v_k}{a} \ , \quad \pi_k = \frac{\mathrm{d}}{\mathrm{d}N}\left(\frac{v_k}{a}\right)$$

Early times, all mode sub horizon -> impose Bunch Davies i.c

$$\lim_{k\tau \longrightarrow -\infty} v_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}$$

Intro new time variable:

$$T = -k\tau = \frac{k}{aH}$$

MS-eqn becomes :

$$\frac{\mathrm{d}^2 v_k}{\mathrm{d}T^2} + \left(1 - \frac{\nu^2 - \frac{1}{4}}{T^2}\right) v_k = 0,$$

$$\nu^2 = \frac{1}{(aH)^2} \frac{z''}{z} + \frac{1}{4}.$$

For slow-roll inflation, $\nu^2 \geq 9/4$ at early times and increases monotonically towards the end of inflation.

Case of Pure dS limit, both $\epsilon_H, \eta_H = 0$, leading to $z''/z = 2a^2H^2$ and $\nu^2 = 9/4$.

Obtain mode solution:

$$v_k(T) = \frac{1}{\sqrt{2k}} \left(1 + \frac{i}{T}\right) e^{iT}$$

And exact noise matrix elements, (recall evaluated at $k = \sigma a H$, hence when $T = \sigma$)

$$\Sigma_{\phi\phi} = (1+\sigma^2)\left(\frac{H}{2\pi}\right)^2$$

$$\mathrm{Re}(\Sigma_{\phi\pi}) = -\sigma^2 \left(\frac{H}{2\pi}\right)^2$$

$$\Sigma_{\pi\pi} = \sigma^4 \left(\frac{H}{2\pi}\right)^2$$

For $\sigma = 0.01$ say have $\Sigma_{\phi\phi} : \Sigma_{\phi\pi} : \Sigma_{\pi\pi} = 1 : 10^{-4} : 10^{-8}$ - which is why $\Sigma_{\phi\pi}$ and $\Sigma_{\pi\pi}$ usually ignored.

Case of slow roll inflation where $\epsilon_H, \eta_H \ll 1$, the slow-roll parameters **but** do not exactly vanish.

For realistic SR potentials, ν is roughly equal to 3/2 and evolves slowly and monotonically. We obtain

$$v_k(T) = e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2k}} \sqrt{T}\, H_\nu^{(1)}(T)\,,$$

And on superhorizon scales:

$$\Sigma_{\phi\phi} = 2^{2(\nu-\frac{3}{2})} \left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2 \left(\frac{H}{2\pi}\right)^2 T^{2(-\nu+\frac{3}{2})}\,,$$

$$\mathrm{Re}\,(\Sigma_{\phi\pi}) = -2^{2(\nu-\frac{3}{2})} \left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2 \left(\frac{H}{2\pi}\right)^2 \left(-\nu+\frac{3}{2}\right) T^{2(-\nu+\frac{3}{2})}\,,$$

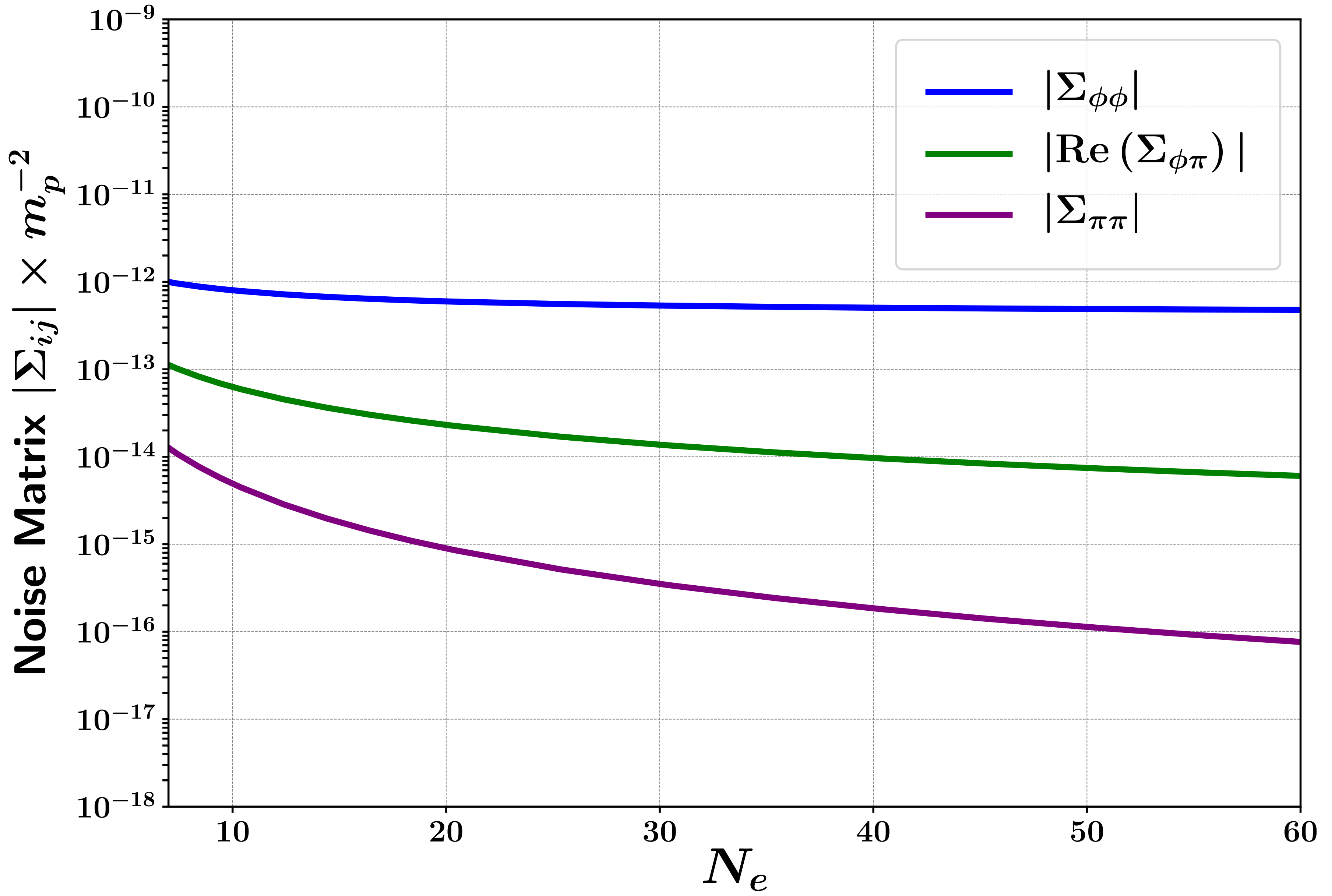
$$\Sigma_{\pi\pi} = 2^{2(\nu-\frac{3}{2})} \left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2 \left(\frac{H}{2\pi}\right)^2 \left(-\nu+\frac{3}{2}\right)^2 T^{2(-\nu+\frac{3}{2})}\,.$$

Note, the hierarchy of noise terms no longer necessarily present

For D-brane KKLT type potential

$$V(\phi) = V_0 \frac{\phi^2}{M^2+\phi^2}$$

we find for large N_e, $\Sigma_{\phi\phi} : |\mathrm{Re}(\Sigma_{\phi\pi})| : \Sigma_{\pi\pi} = 1 : 10^{-2} : 10^{-4}$ unlike de Sitter case.



Case of potentials with a slow-roll violating feature, like USR with $\epsilon_H \ll 1$, while $\eta_H \geq 1$

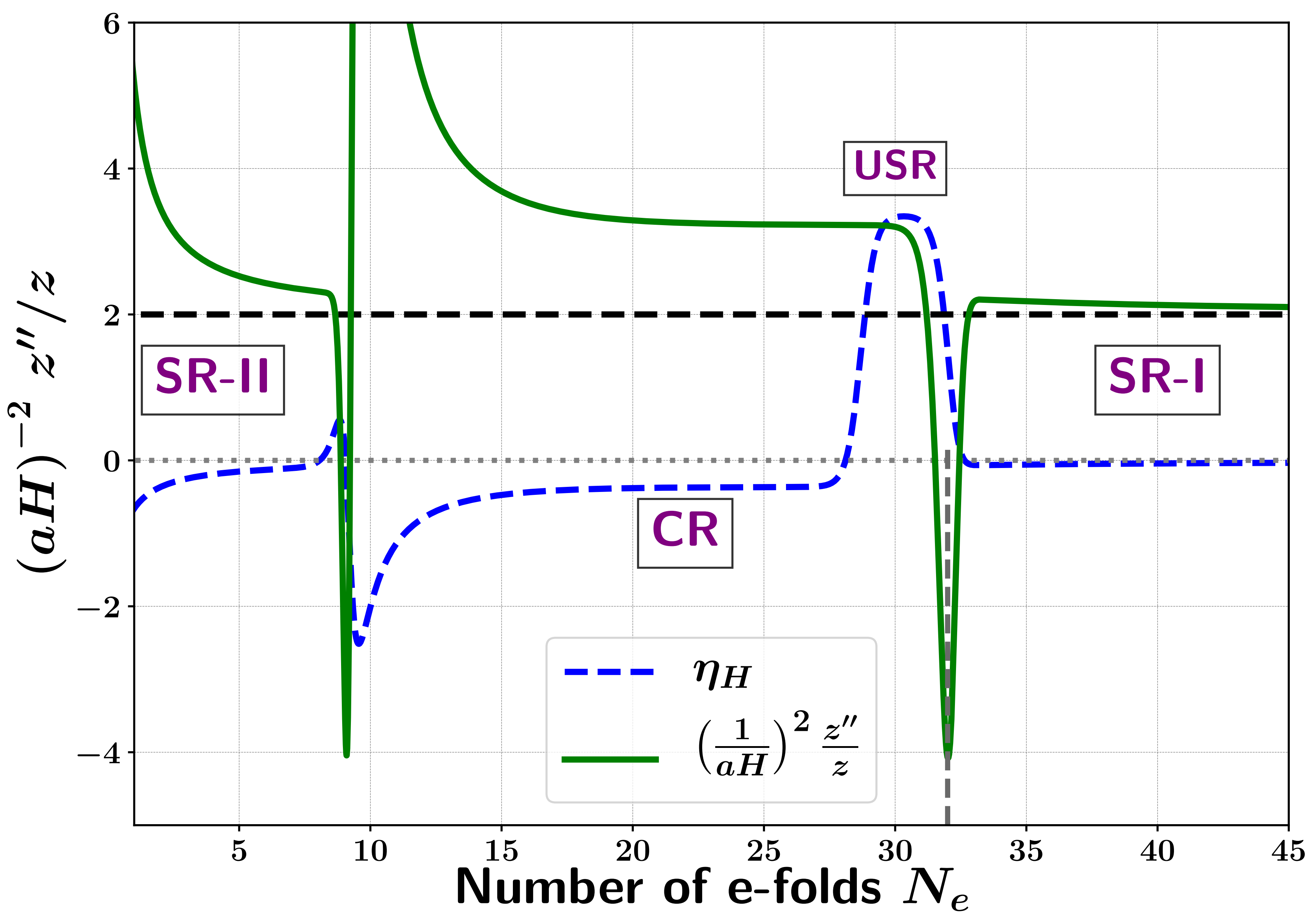
Dynamics undergoes number of phases driven by η_H. We now have :

$$\frac{1}{(aH)^2}\,\frac{z''}{z} \simeq 2 - 3\,\eta_H + \eta_H^2 + \tau\,\frac{\mathrm{d}\eta_H}{\mathrm{d}\tau}$$

Specific example, a modified KKLT potential with an additional tiny Gaussian bump-like feature [Mishra et al 2019]:

$$V_{\rm b}(\phi) = V_0\,\frac{\phi^2}{M^2+\phi^2}\,\left[1 + A\,\exp\left(-\,\frac{1}{2}\,\frac{(\phi-\phi_0)^2}{\tilde{\sigma}^2}\right)\right],$$

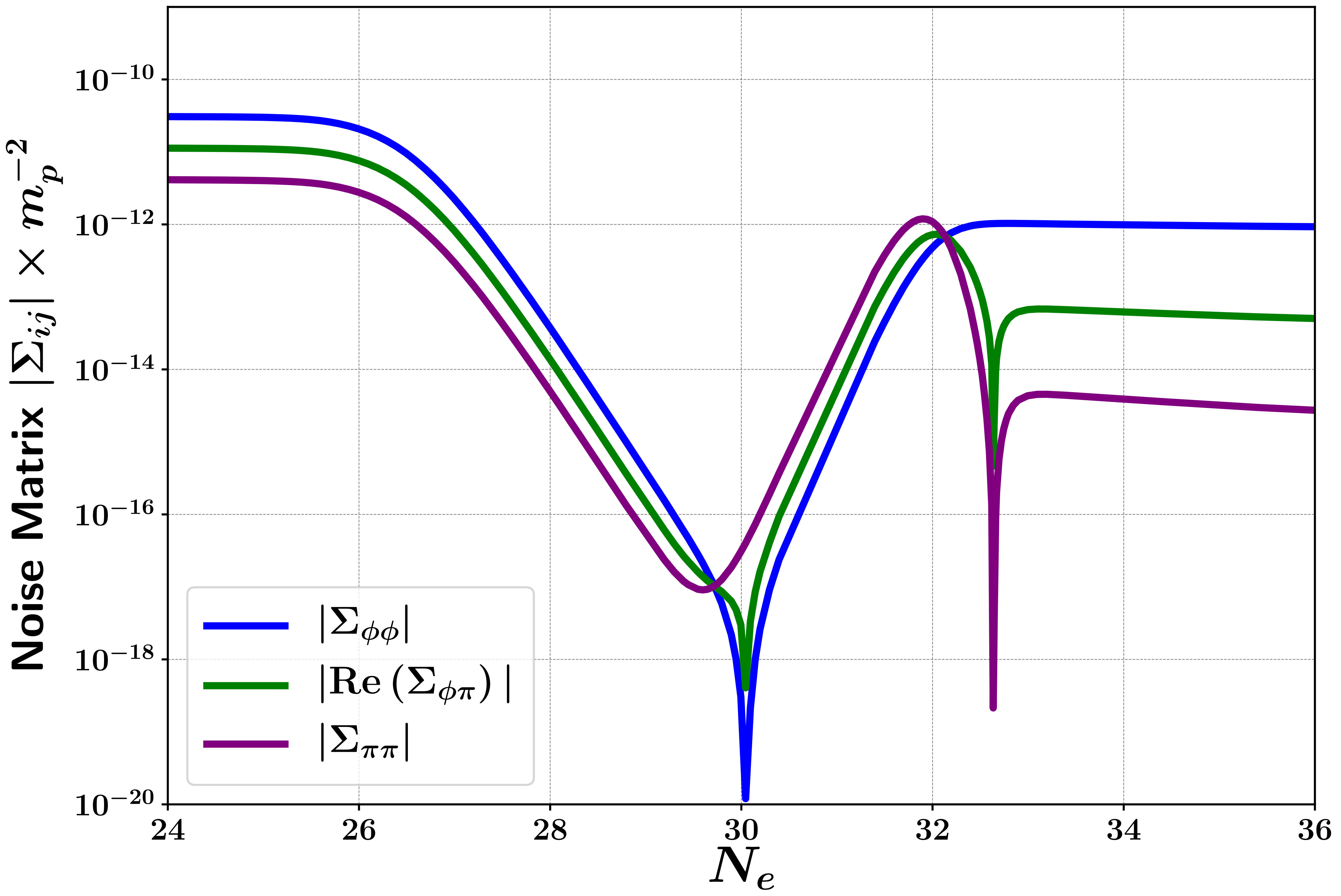
where A, $\tilde{\sigma}$ and ϕ_0 represent the height, width and position of the bump respectively.

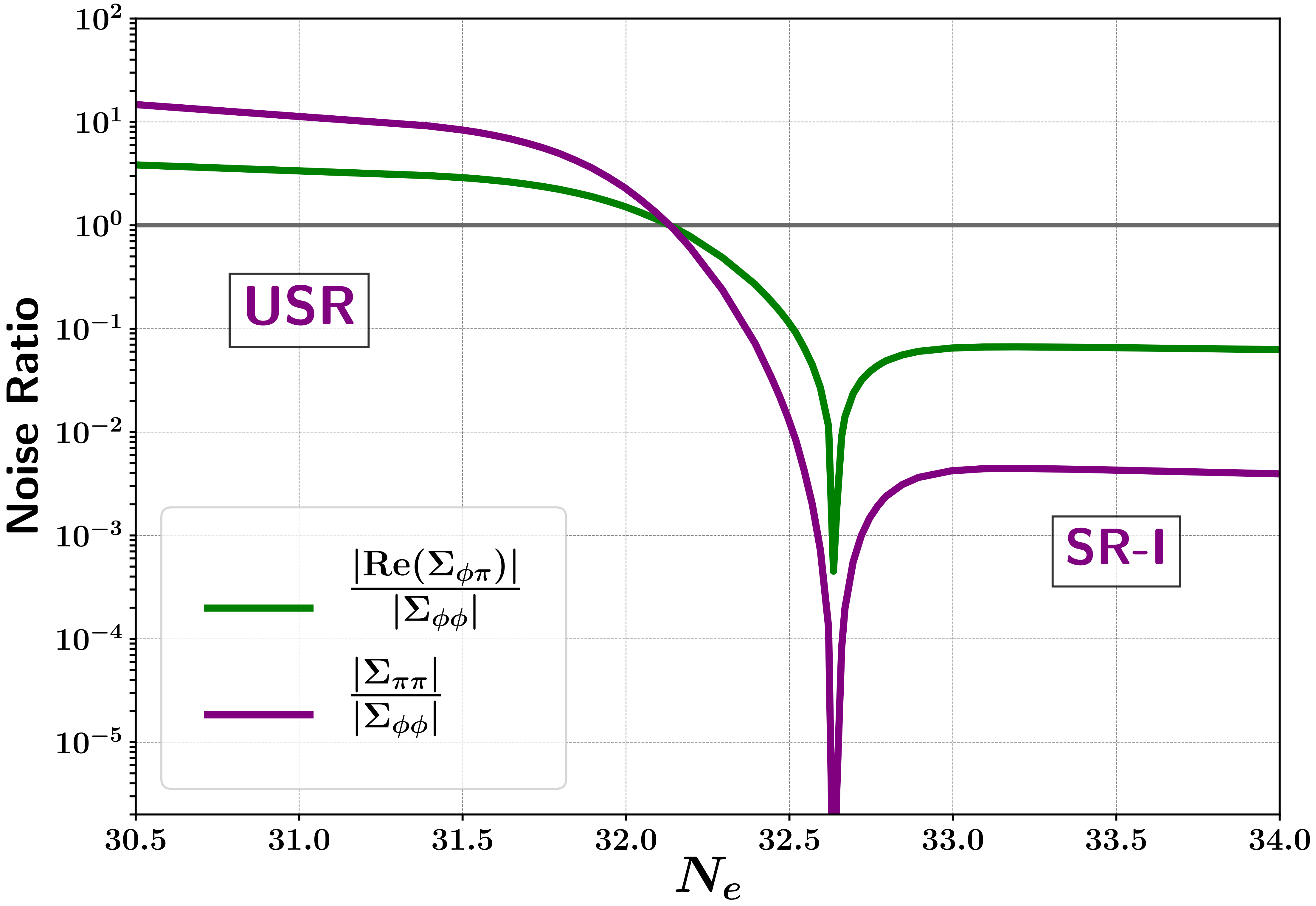


Fixed $M = 0.5\,m_p$, and bump parameters to be $A = 1.87\times10^{-3}$, $\tilde{\sigma} = 1.993\times10^{-2}$ and $\phi_0 = 2.005\,m_p$.

Gives amplification of the scalar power-spectrum, $\mathcal{P}_\zeta$, by a factor of 10^7 relative to its value on CMB scales.

Numerical noise matrix elements, Σ_{ij} - note the switching of dominant terms during USR





Noise ratios in the SR-1 to USR region

With Swagat Mishra and Anne Green - e-Print:2303.17375

Outstanding steps to calculate the PBH mass fraction

Have calculated the stochastic noise matrix elements Σ_{ij}, for a sharp transition from SR to USR

Aim is to determine the PDF of the number of e-folds, $P_{\Phi,\Pi}(\mathcal{N})$, by solving the adjoint Fokker-Planck eqn

$$\frac{\partial}{\partial \mathcal{N}} P_{\Phi_i}(\mathcal{N}) = \left[D_i \frac{\partial}{\partial \Phi_i} + \frac{1}{2} \Sigma_{ij} \frac{\partial^2}{\partial \Phi_i \partial \Phi_j} \right] P_{\Phi_i}(\mathcal{N}) .$$

$$P_{\Phi,\Pi}(\mathcal{N}) = \sum_{n=0}^{\infty} B_n(\Phi,\Pi)\, e^{-\Lambda_n \mathcal{N}}$$

$B_n(\Phi,\Pi)$ to be determined from b.c. and expressions for Σ_{ij}

Then calculate the mass fraction of PBHs $\beta_{\rm PBH}$.

$$\beta(\Phi,\Pi) \equiv \int_{\zeta_c}^{\infty} P(\zeta_{\rm cg})\, \mathrm{d}\zeta_{\rm cg} = \int_{\zeta_c + \langle \mathcal{N}(\Phi,\Pi) \rangle}^{\infty} P_{\Phi,\Pi}(\mathcal{N})\, \mathrm{d}\mathcal{N}$$

$$\beta(\Phi,\Pi) = \sum_{n=0}^{\infty} \frac{B_n(\Phi,\Pi)}{\Lambda_n} \exp\left[-\Lambda_n \left[\zeta_c + \sum_{m=0}^{\infty} \frac{B_m(\Phi,\Pi)}{\Lambda_m^2} \right] \right]$$

Of course this might not be possible !

$$\beta^{\rm G}(\Phi,\Pi) \simeq \frac{\sigma_{\rm cg}}{\sqrt{2\pi}\zeta_c} \exp\left[-\frac{\zeta_c^2}{2\sigma_{\rm cg}^2} \right]$$

Compare with Gaussian PDF for typical fluctuations in the perturbative approach, $\beta^{\rm G}(\Phi,\Pi)$

$$\sigma_{\rm cg}^2(\Phi,\Pi) = \int_{k(\Phi,\Pi)}^{k_e} \frac{\mathrm{d}k}{k} \mathcal{P}_\zeta(k) \ .$$

Conclusions

PBHs are black holes that could have formed In the early universe

They could in principle have any mass and therefore could be dark matter candidates without the need of new particles.

PBHs can form due to the gravitational collapse of large fluctuations - require modification from standard slow roll inflation

An accurate calculation of the full PDF of the perturbations is required to calculate their abundance.

Stochastic inflation is a powerful framework for computing the cosmological correlators non-perturbatively.

However to correctly account for the back-reaction effect of small scale (UV) fluctuations, on the long wavelength coarse-grained (IR) field, it is essential to compute the noise matrix elements accurately.

Since most single field inflationary potentials with a PBH-forming feature violate the slow-roll conditions, a precise calculation of the stochastic noise matrix elements beyond slow roll is required.

Have seen some rich structure in the noise terms and shown how poor the de Sitter type solution can be in determining the noise across the SRi-USR regimes.

We saw a sharp decline of the noise terms after the transition and expect this will decrease the amount of quantum diffusion of the IR fields across the PBH-forming feature.

Therefore we expect the tail of the PDF to decline less rapidly than what is usually found using the pure dS approximation without any transitions [see also Ahmadi et al 2022].

Conclusions cont…

Have not discussed many elements of PBH physics:

Role in Information paradox [Hawking 1971,1974]
Role as a catalysis of Ewk phase transition [Gregory et al 2014]
Possible role of PBH Planck mass relics in dark matter constraints [Zeldovich 1984, MacGibbon 1987]
Alternative formation mechanisms such as collapsing cosmic string loops
or from bubble collisions. [Hawking Moss & Stewart 1982]
Baryogenesis scenarios from PBH evaporations [Zeldovich & Starobinski 1976]
PBHs decay by evaporation - interesting attractor solution where PBHs in equilibrium with radiation in both radiation dominated and matter dominated universe - might lead to interesting new features. [Barrow et al 1992]

For objects that as far as we know have never been detetced, PBHs offer staggering constraints on cosmological models.

When discussing inflation model building and particle cosmology - Subir often had a pretty impressive colleague and friend to challenge assumptions

I learnt a great deal from both of them

Thank you Subir for your collegiality and friendship over quite a few years now.

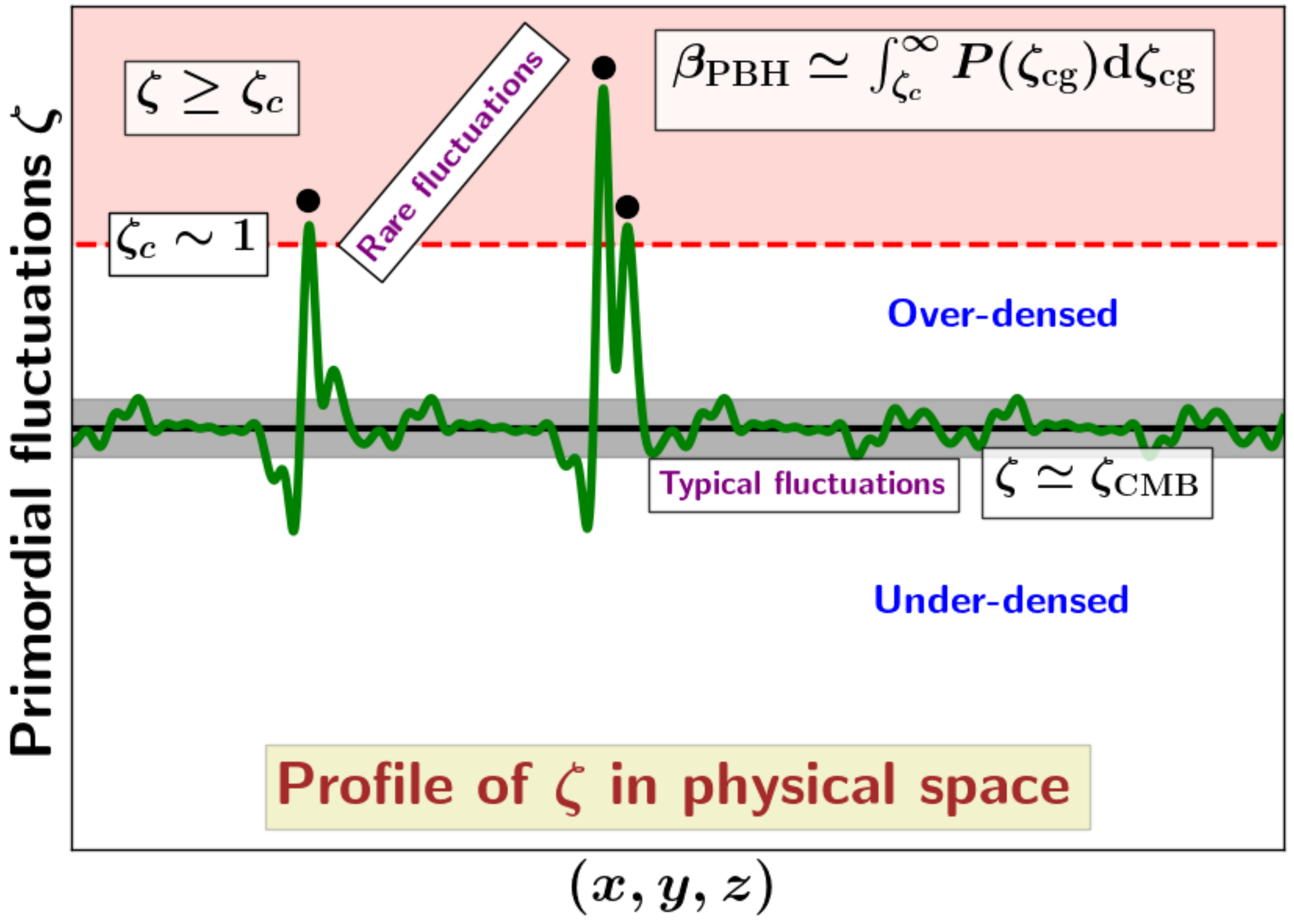
No one has done more to try and keep us honest as we get over excited when interpreting the latest cosmological observations.

Me included !

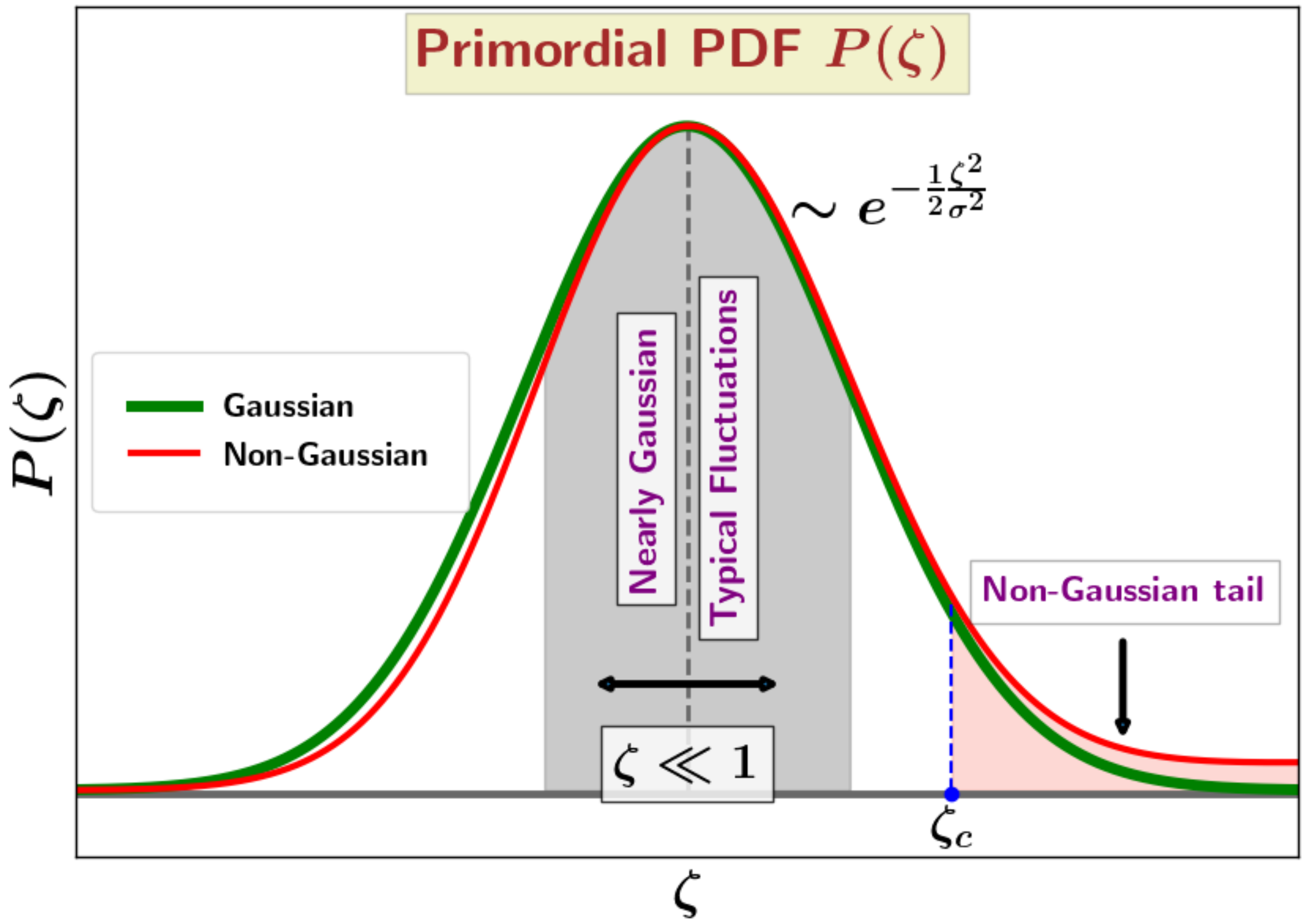
Extra slides

Why might a non-gaussian PDF of Primordial Fluctuations help with creating PBHs ?

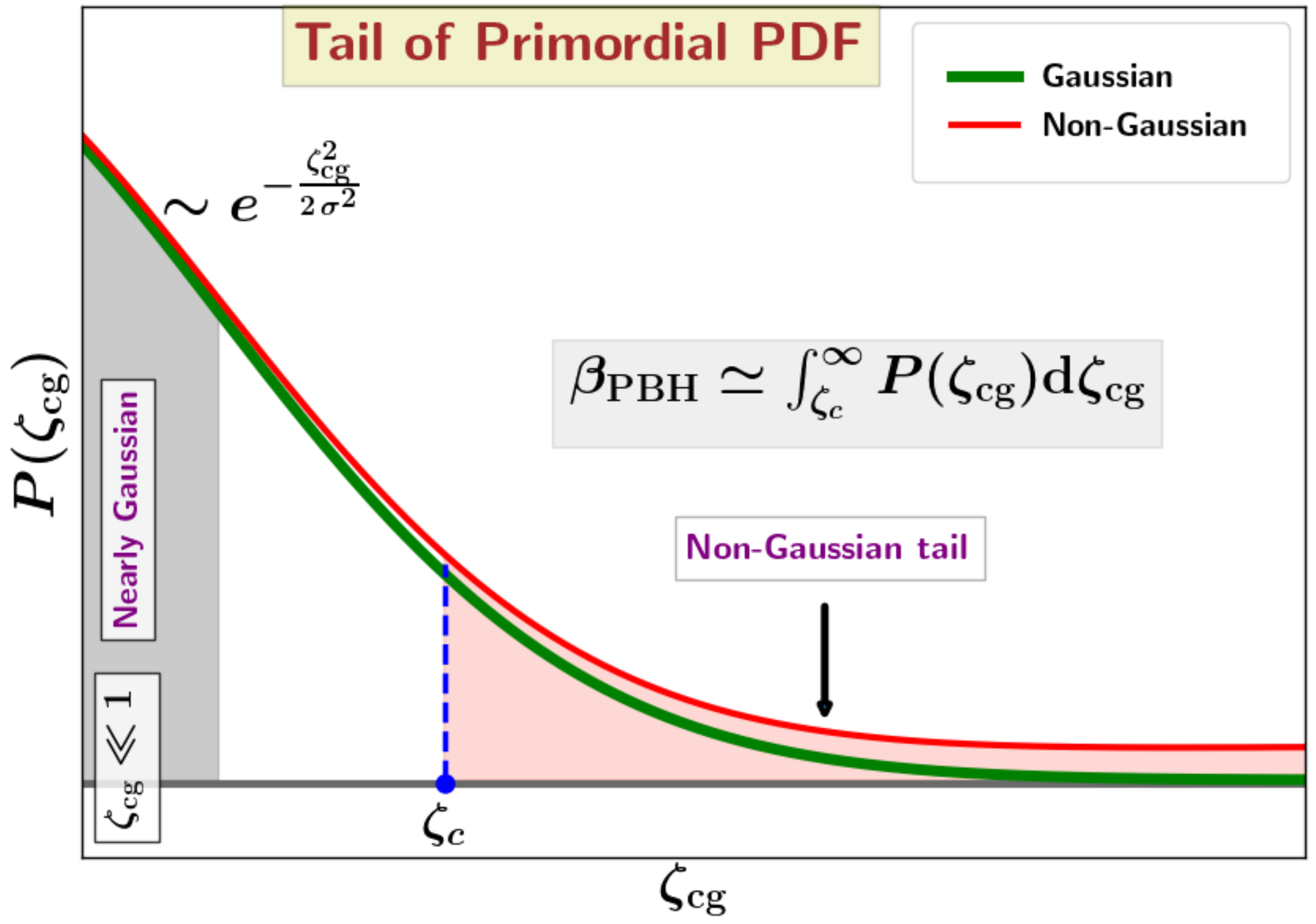
We expect PBHs to form from rare peaks in the fluctuations in the density contrast



For small fluctuations we expect the PDF to be Gaussian

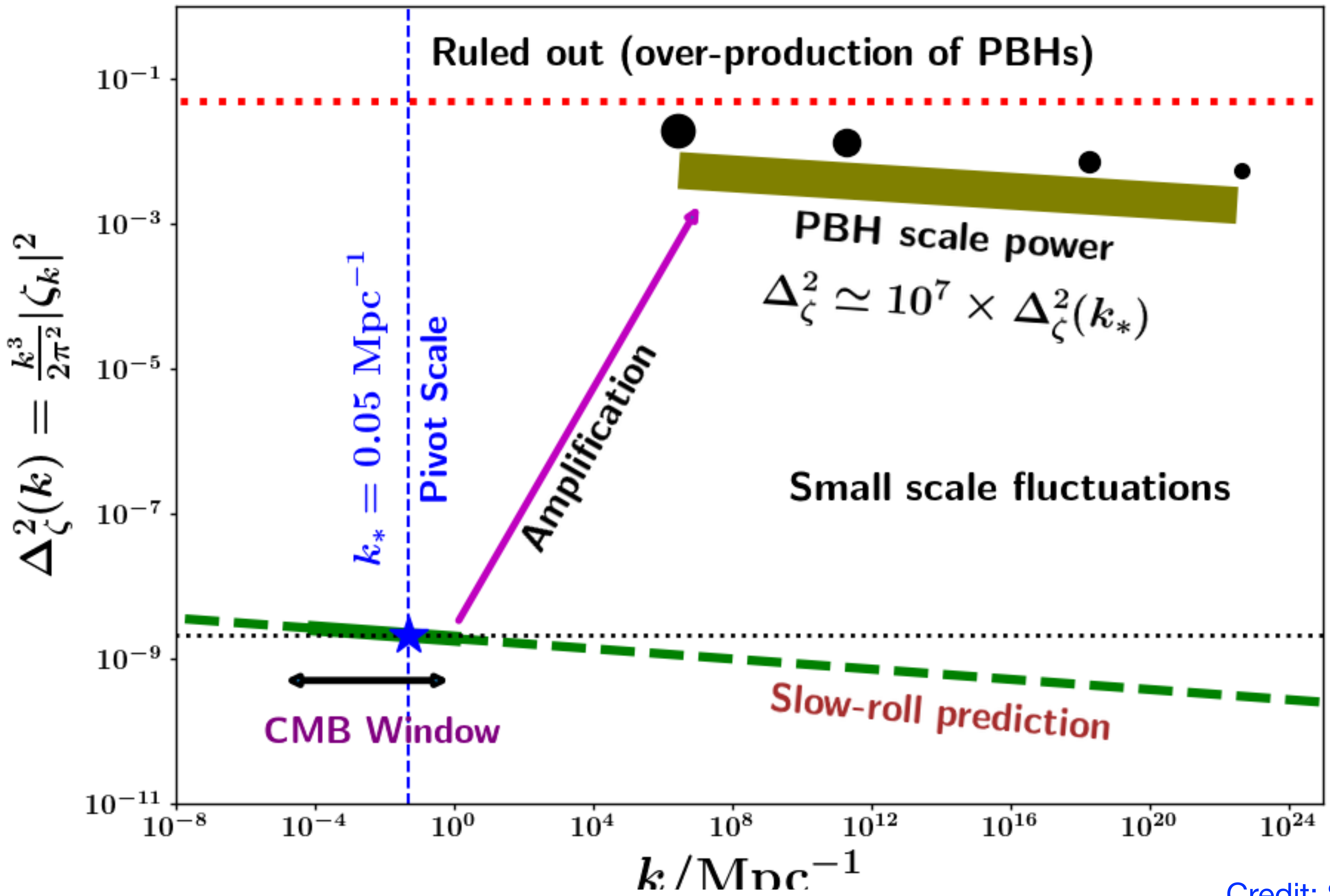


But deviations from Gaussian for large fluctuations could increase the PDF enhancing the likelihood of forming PBHs



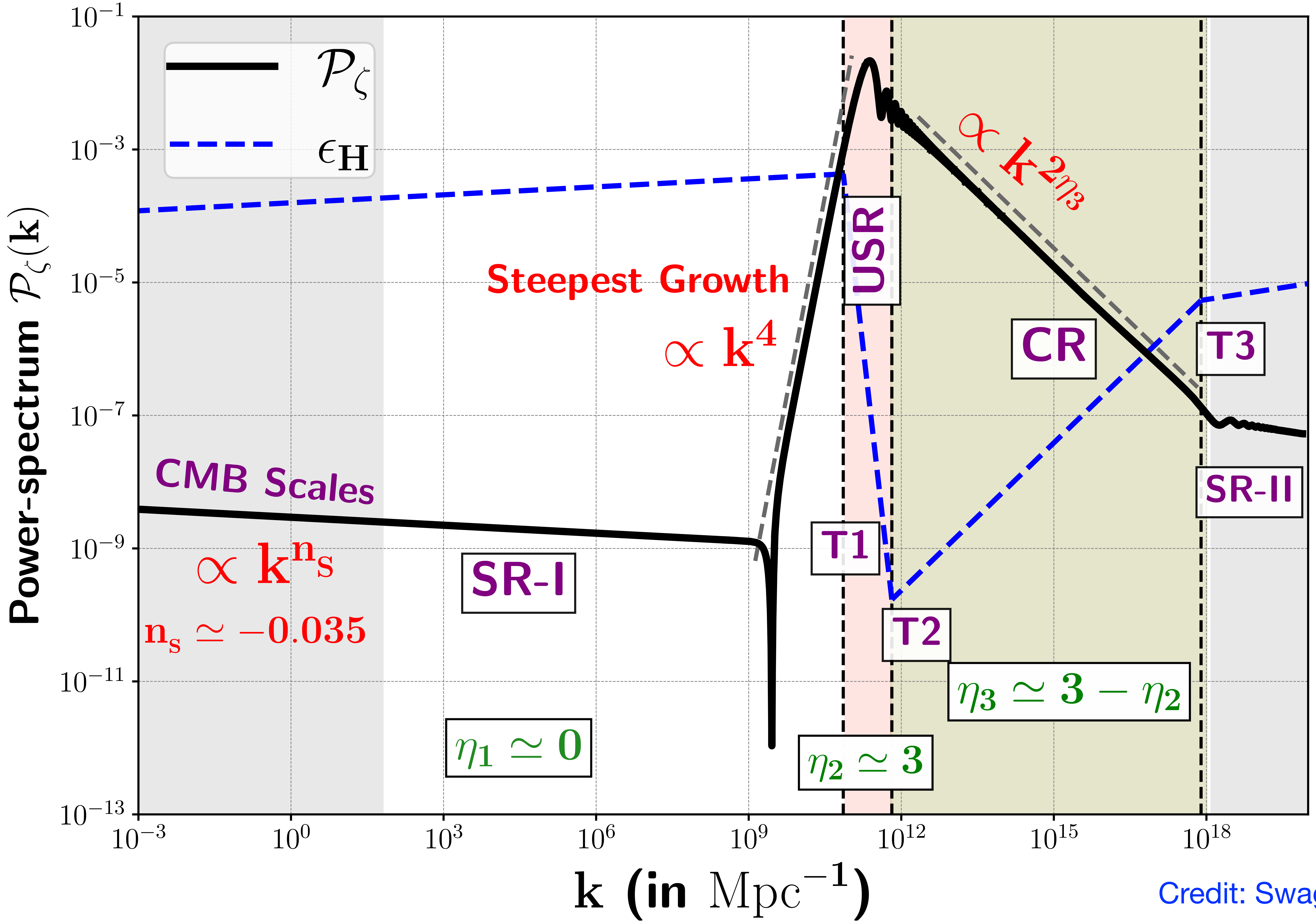
Credit: Swagat Mishra

Required amplification for interesting PBH scenarios



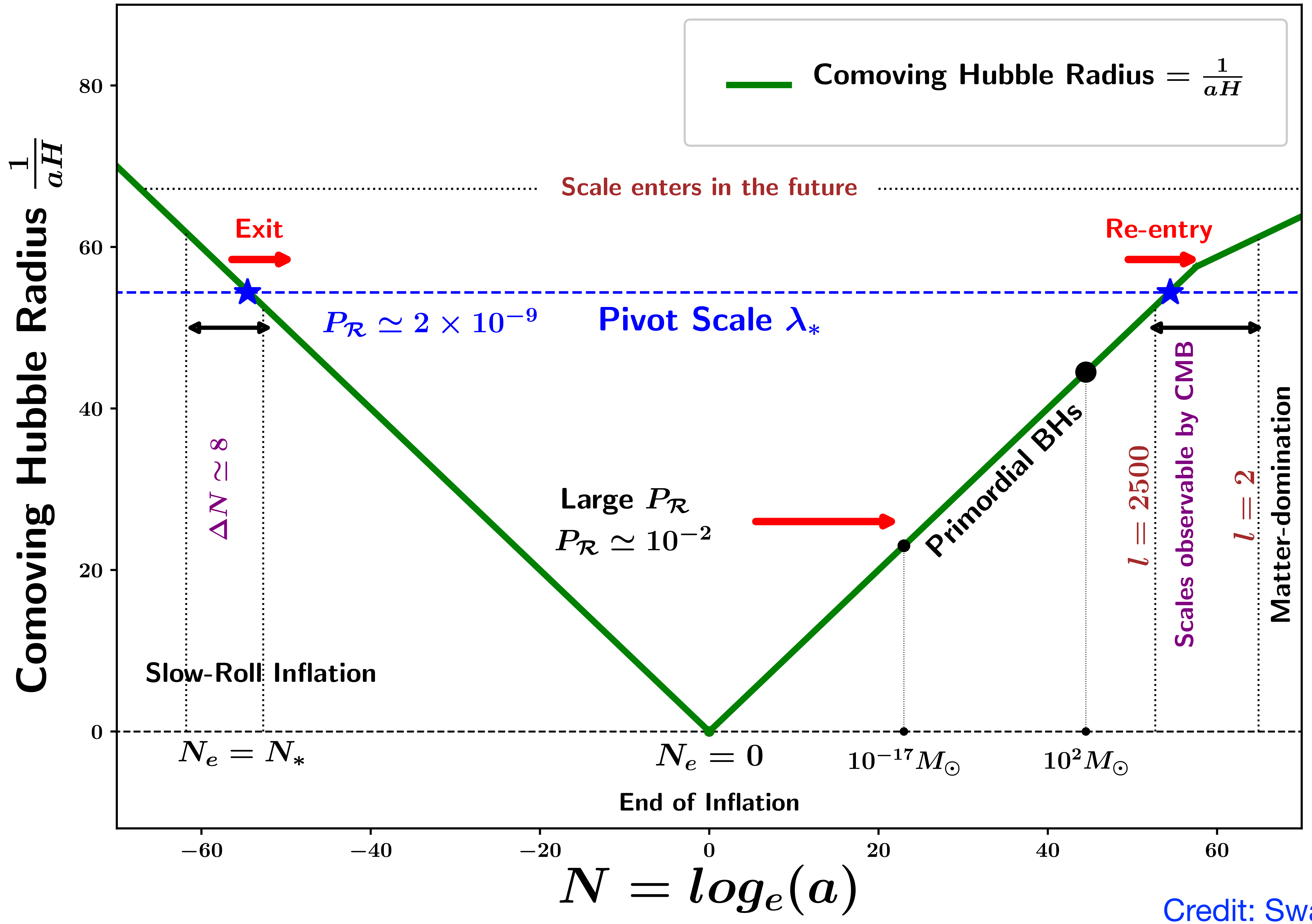
Credit: Swagat Mishra

In terms of a power spectrum generated from inflation we require



Credit: Swagat Mishra

PBH size fluctuations re-enter on different scales



Credit: Swagat Mishra

Analytic treatment of instantaneous transition - works really nicely

Ansatz - motivated by numerical results

$$\eta_H(\tau) = \eta_1 + (\eta_2 - \eta_1)\ \Theta(\tau - \tau_1)$$

Assume piecewise constant η_H - makes instantaneous (yet finite) transition η_1->η_2 at time $\tau = \tau_1$

Obtain

$$\nu^2 - \frac{1}{4} \equiv \frac{z''}{z}\,\tau^2 = \mathcal{A}\,\tau\,\delta_D(\tau - \tau_1) + \nu_1^2 - \frac{1}{4} + \left(\nu_2^2 - \nu_1^2\right)\ \Theta(\tau - \tau_1)\,,$$

where

$$\mathcal{A} = \eta_2 - \eta_1\,, \quad \nu_{1,2}^2 - \frac{1}{4} = 2 - 3\,\eta_{1,2} + \eta_{1,2}^2\,.$$

Note the delta function - gives the rapid dip

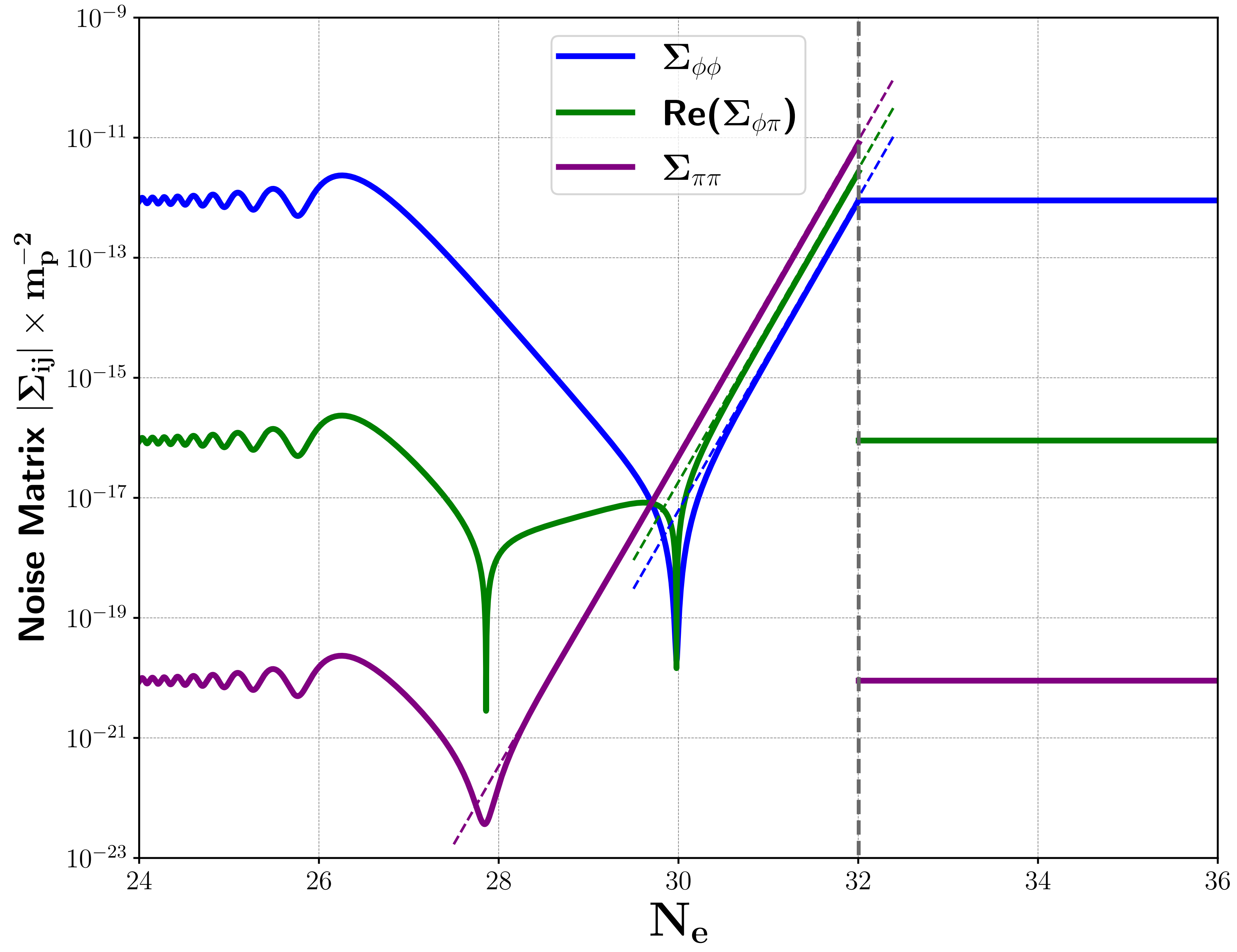
Nosie matrix elements

$$\Sigma_{\phi\phi} = \left(\frac{H}{2\pi}\right)^2 T^2 \left|\sqrt{2k}\,v_k(T)\right|^2\Big|_{T=\sigma}\,,$$

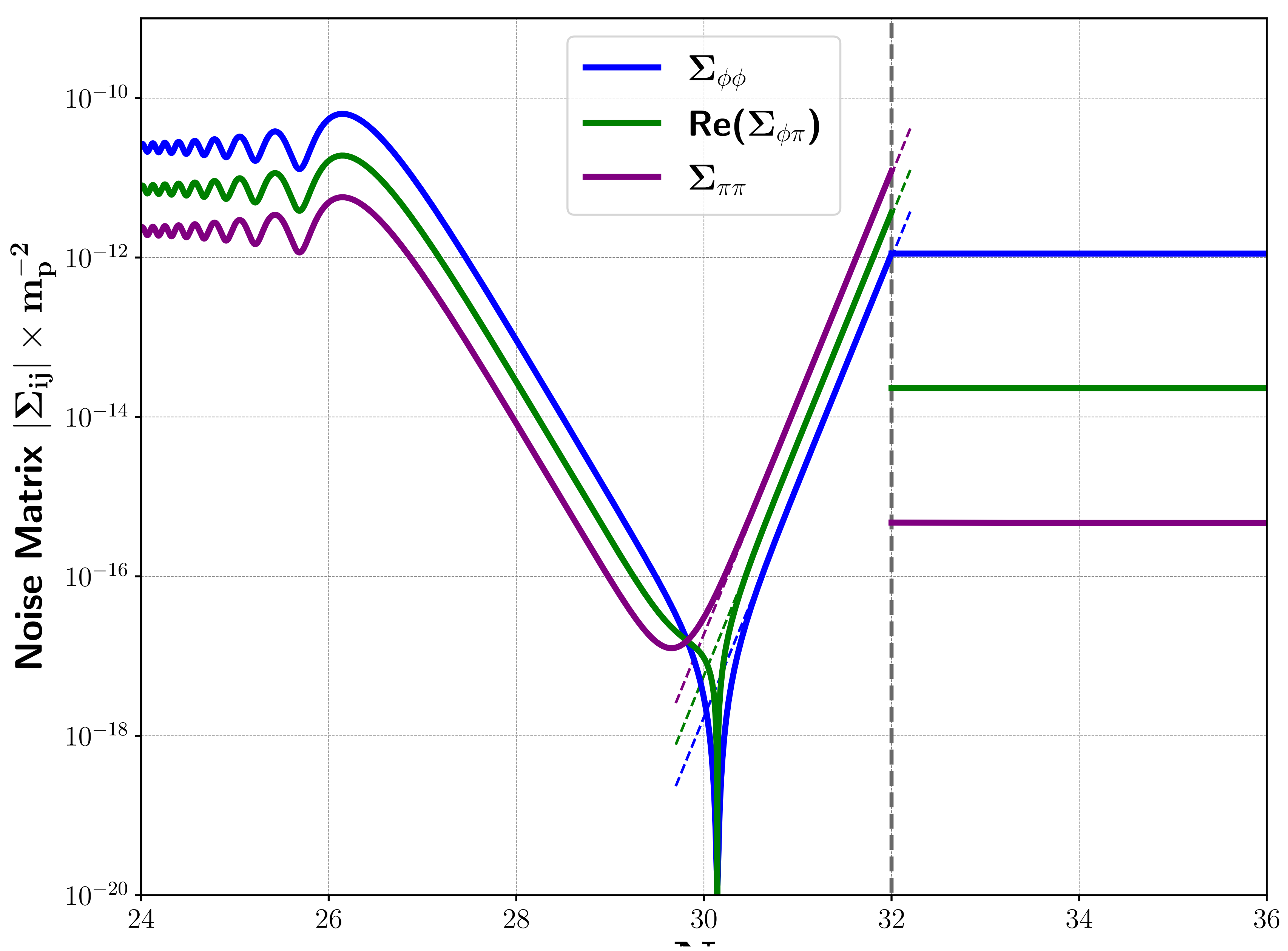
$$\mathrm{Re}\left(\Sigma_{\pi\phi}\right) = -\left(\frac{H}{2\pi}\right)^2 T^2 \mathrm{Re}\left(\sqrt{2k}v_k^*(T)\left[T\frac{\mathrm{d}}{\mathrm{d}T}\left(\sqrt{2k}v_k(T)\right) + \sqrt{2k}v_k(T)\right]\right)\Big|_{T=\sigma}$$

$$\Sigma_{\pi\pi} = \left(\frac{H}{2\pi}\right)^2 T^2 \left|T\,\frac{\mathrm{d}}{\mathrm{d}T}\left(\sqrt{2k}\,v_k(T)\right) + \sqrt{2k}\,v_k(T)\right|^2\Big|_{T=\sigma}\,,$$

de Sitter case: $\nu_1 = \nu_2 = 3/2$



Instantaneous transition - from SRI: $\nu_1 = 1.52$ to USR with $\nu_2 = 1.8$



Features of analytic solution

Pre transition epoch $T \geq T_1$ with $\nu = \nu_1$

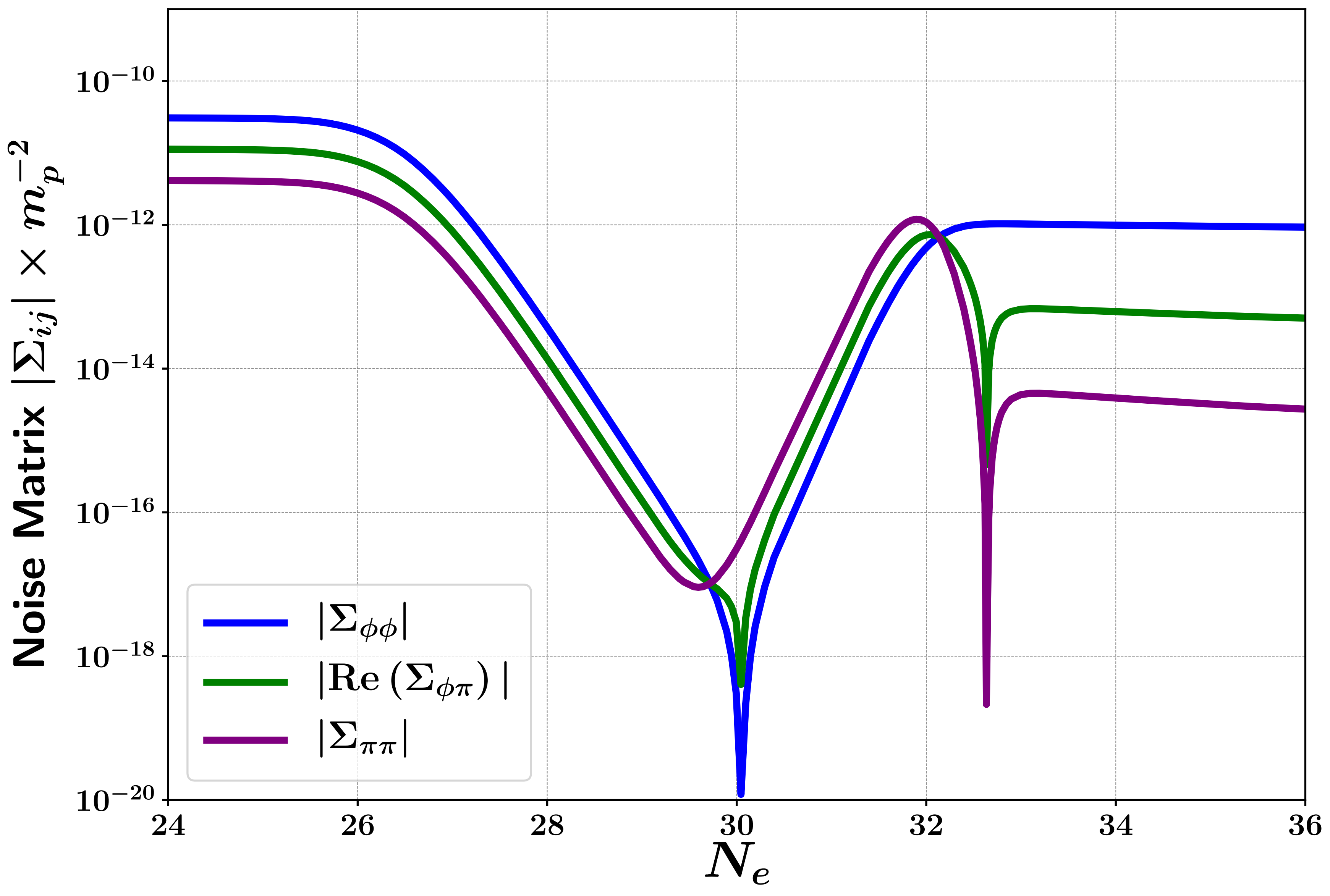
$$\Sigma_{\phi\phi} : |\mathrm{Re}(\Sigma_{\phi\pi})| : \Sigma_{\pi\pi} \to 1 : \left(\nu_1 - \frac{3}{2}\right) : \left(\nu_1 - \frac{3}{2}\right)^2$$

Immediately after transition epoch $\Sigma_{ij} \propto e^{2\mathcal{A}N_e}$ ($\mathcal{A} \equiv \eta_2 - \eta_1 = 3.32$) and

$$\Sigma_{\phi\phi} : |\mathrm{Re}(\Sigma_{\phi\pi})| : \Sigma_{\pi\pi} \to 1 : \mathcal{A} : \mathcal{A}^2$$

Sufficiently late times, $T \ll T_1$, same as above but with $\nu = \nu_2$,

$$\Sigma_{\phi\phi} : |\mathrm{Re}(\Sigma_{\phi\pi})| : \Sigma_{\pi\pi} \to 1 : \left(\nu_2 - \frac{3}{2}\right) : \left(\nu_2 - \frac{3}{2}\right)^2$$



Full numerical solution