Perturbative QCD at NNLO and N^3LO: recent progress in antenna subtraction

Turin Phenomenology Seminar

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Durham University

IP3

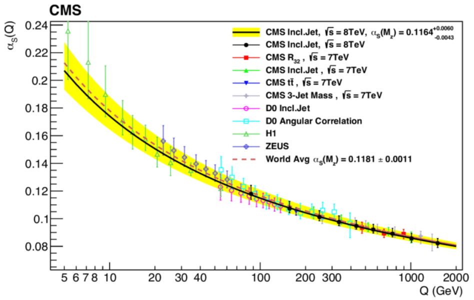
THE ROYAL SOCIETY Newton International Fellowship

Outline

- Introduction
- Antenna Subtraction
    - Generalised Antenna Functions
- Applications
    - e+e- → 4 jets @ NNLO
    - e+e- → 2 jets @ N^3LO

INTRODUCTION

QCD in high-energy collisions



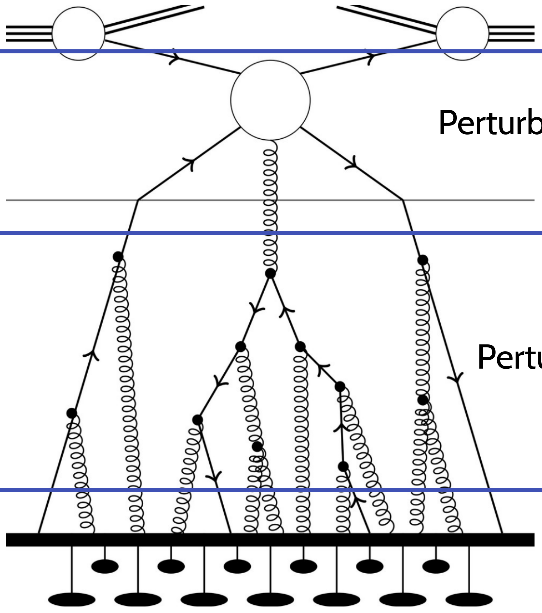
- Asymptotic freedom as Q→∞
- Confinement as Q→0

$$\alpha_s(100\ \mathrm{GeV}) \approx 10\,\alpha_{EW} \approx 0.1$$

$$Q \approx \Lambda_{\mathrm{QCD}}$$

Proton structure
Parton Distribution Functions

QCD factorization
Fits to data



$$Q \approx \sqrt{s}$$

Hard scattering

Perturbative QCD (fixed-order)
Infrared subtraction
Multi-loop calculations

$$Q \ll \sqrt{s}$$

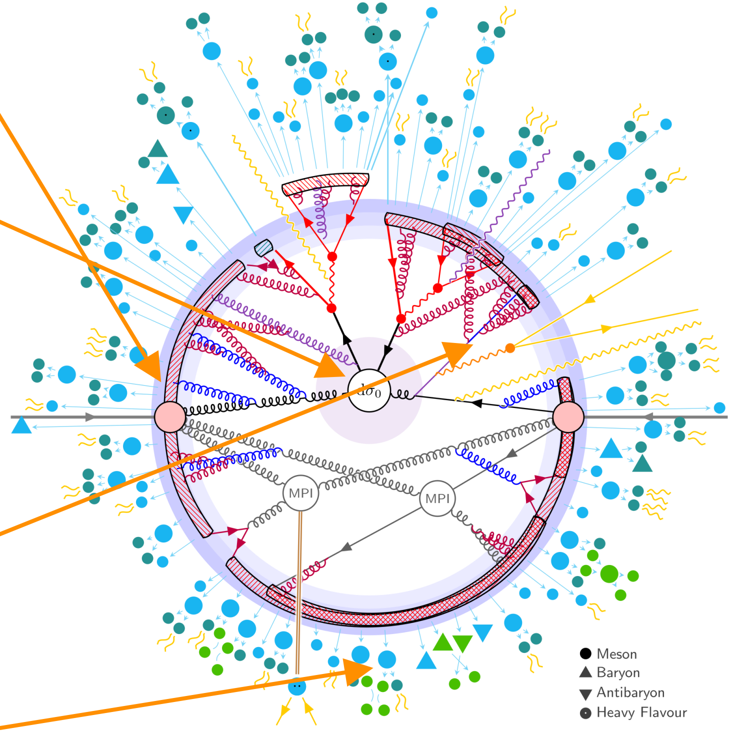
Parton showers

Perturbative QCD (all-order)
Shower algorithms
Matching

$$Q \approx \Lambda_{\mathrm{QCD}}$$

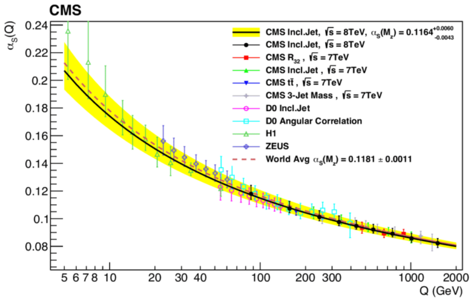
Hadronization

Non-perturbative QCD
Analytic models
Phenomenological models, tuning



[Pythia manual]

QCD in high-energy collisions



- Asymptotic freedom as $Q \to \infty$
- Confinement as $Q \to 0$

$$\alpha_s(100\ \mathrm{GeV}) \approx 10\, \alpha_{EW} \approx 0.1$$

$Q \approx \Lambda_{\mathrm{QCD}}$

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$Q \approx \sqrt{s}$

Hard scattering

Perturbative QCD (fixed-order)
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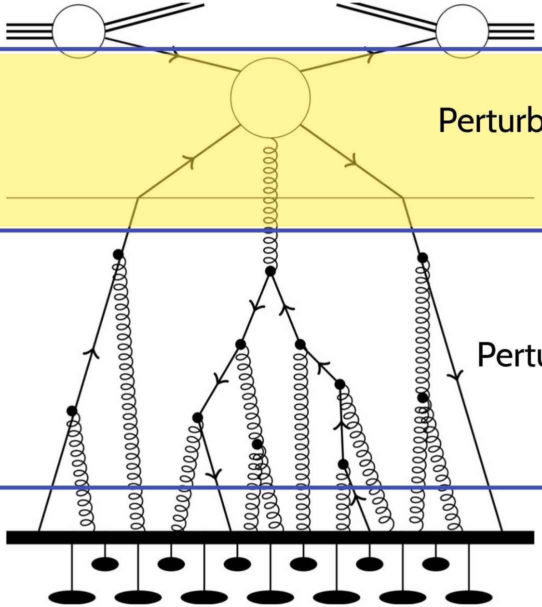
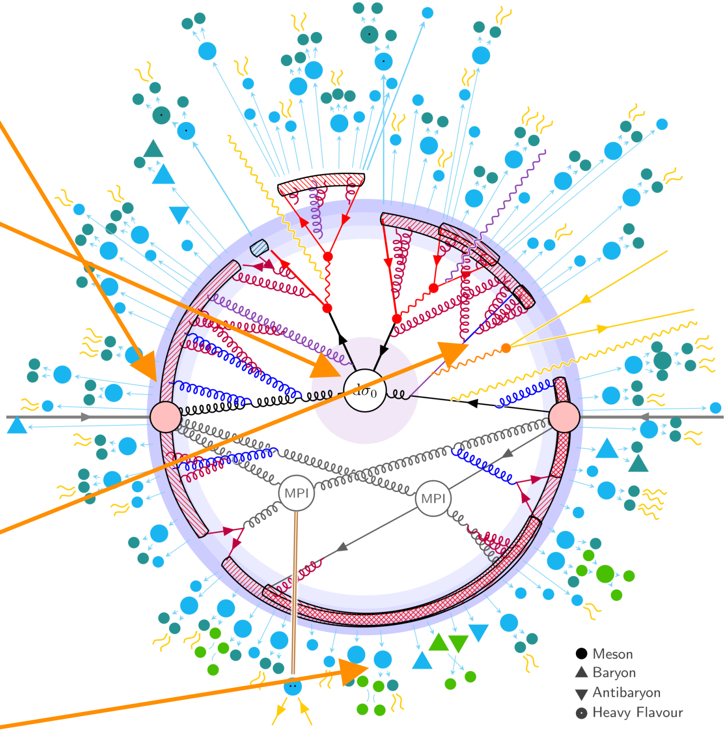
Hadronization



Non-perturbative QCD
Analytic models
Phenomenological models, tuning



[Pythia manual]

Fixed-order calculations in perturbative QCD

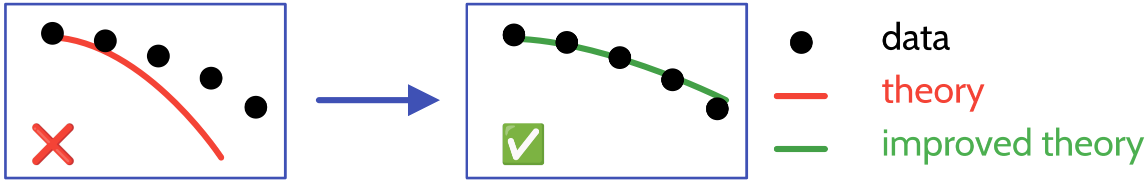
$$\alpha_s \approx 0.1$$

$$\sigma = \sigma_0 + \left(\frac{\alpha_s}{2\pi}\right)\sigma_1 + \left(\frac{\alpha_s}{2\pi}\right)^2\sigma_2 + \left(\frac{\alpha_s}{2\pi}\right)^3\sigma_3 + \dots$$

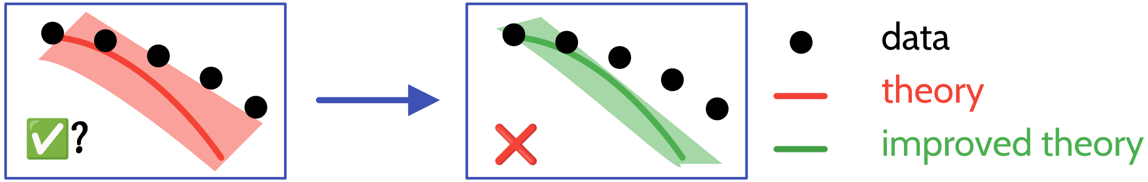
Leading Order (LO)	Next-to-Leading Order (NLO)	Next-to-Next-to Leading Order (NNLO)	Next-to-Next-to-Next-to Leading Order (N^3LO)
	O(10%) - O(100%)	O(1%) - O(10%)	≤ O(1%)

Motivations:

- Improve accuracy

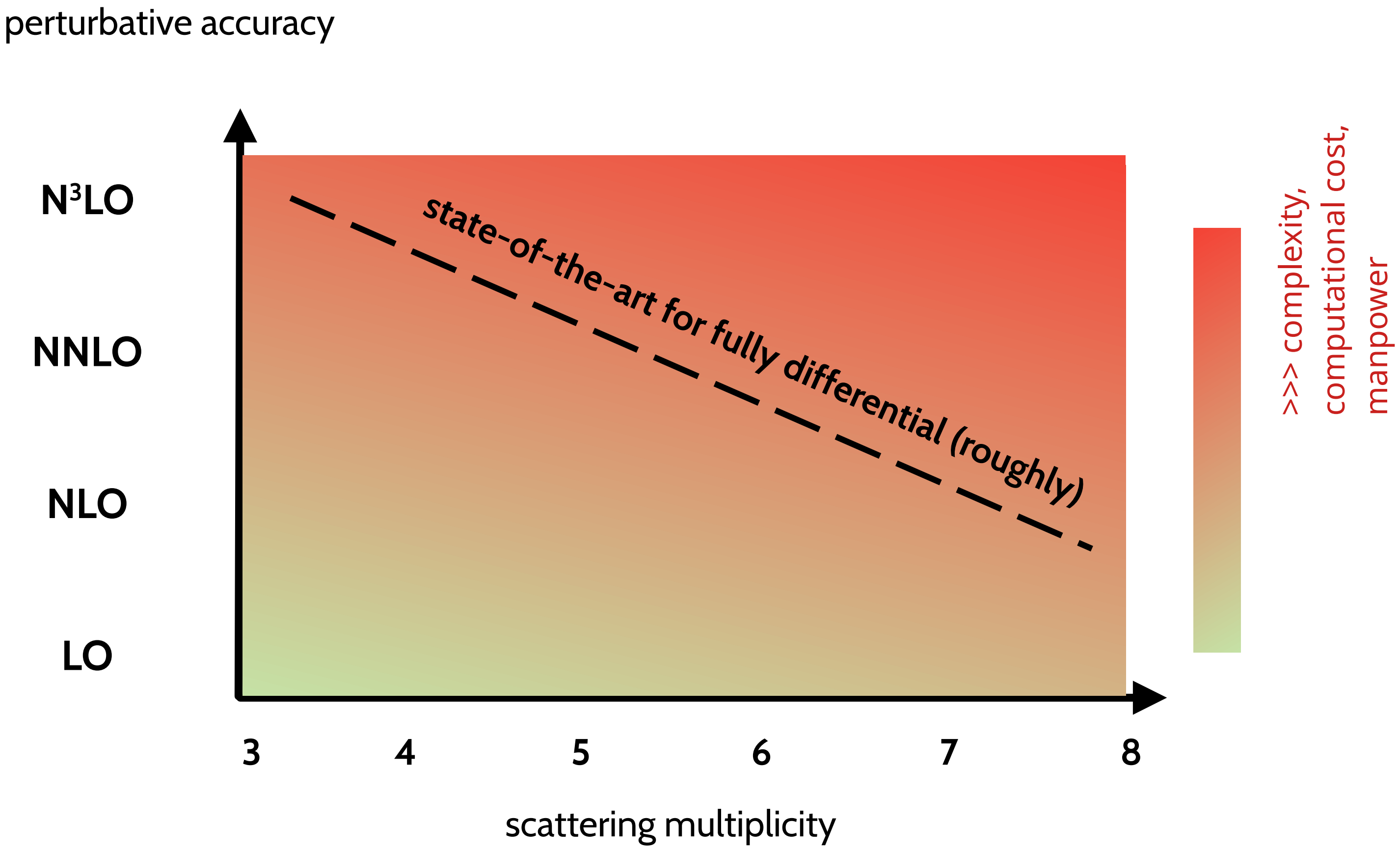


- Reduce theory uncertainties (missing higher orders)



- Push the understanding/assumptions of perturbation theory

Fixed-order calculations in perturbative QCD



Other possible "complexity" axes:

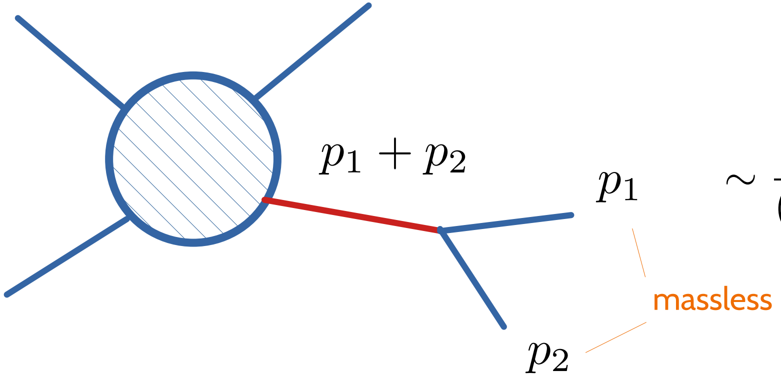
- parton/jet multiplicity
- scales/masses
- less/more differential

Reasons for this complexity:

- multi-loop scattering amplitudes
- numerics, computational cost
- **infrared singularities**

Infrared singularities

Infrared singularities arise in any QFT with massless states when **massless propagators go on-shell:**



$$\sim \frac{1}{(p_1+p_2)^2} = \frac{1}{2E_1E_2(1-\cos\theta_{12})}$$ diverges when

energies angle between p_1 and p_2

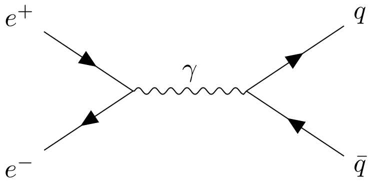
soft limit

$E_{1,2} \to 0$

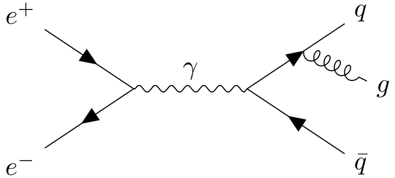
collinear limit

$\theta_{12} \to 0$

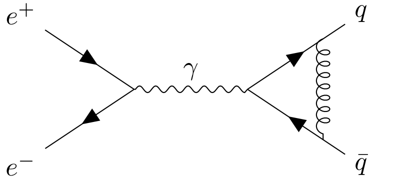
This happens when, on top of a LO process, one considers:



- **real corrections**: additional particles are emitted, which can approach the soft/collinear limits. These singularities are **implicit** in the matrix elements and become manifest after phase-space integration;



- **virtual corrections**: particles circulating in the quantum loops can approach the soft/collinear limits. These singularities are **explicit** in the matrix elements;



Infrared safety

The cancellation of infrared singularities is guaranteed for **sufficiently inclusive** observables.

[Bloch,Nordsieck 1937] abelian (QED)

[Kinoshita 1962]
[Lee,Nauenberg 1964] non-abelian (QCD)

Such observables, labeled **infrared safe,** should not be sensitive to soft/collinear emissions (inclusive over soft/collinear radiation).

So, for IR-safe observables, thy can't we directly compute the **sum of virtuals and reals**?

- we want to be fully differential over different phase-spaces;
- no analytical control apart from special observables (fully-inclusive XS, ...)
- numerical integration in 4 dimensions

there are promising approaches to do so for differential predictions (loop-tree duality), but their effectiveness still needs to be demonstrated

Infrared divergences need to be properly **regularized and subtracted from each component**.

This is done within subtraction schemes.

Factorisation of infrared singularities

Subtraction schemes rely on the **universal behaviour of QCD** in infrared limits

- factorisation of infrared singularities of loop amplitudes:

$$|A^1\rangle = I^1\,|A^0\rangle + |A^1_{\rm fin}\rangle\ , \qquad |A^2\rangle = I^1\,|A^1\rangle + I^2\,|A^0\rangle + |A^2_{\rm fin}\rangle$$

[Catani ‘98] [Bern,De Freitas,Dixon ‘03]
[Gardi,Magnea ‘09] [Becher,Neubert ‘09]

scattering amplitudes in **colour space**

Infrared operators in **colour space**

- factorization of real-emission scattering amplitudes in soft and collinear limits:

colour charge

eikonal factor

$$|A(q,p_1,\dots,p_n)\rangle \sim \sum_{i=1}^{n} \boldsymbol{T}_i\,\frac{p_i^\mu}{p_i\cdot q}\,|A(p_1,\dots,p_n)\rangle$$

[Altarelli,Parisi ‘77] [Ellis,Marchesini,Webber ‘87]
[Berends,Giele ‘89] [Campbell,Glover ‘98] [Catani,Grazzini ‘00]

$$|A(\dots,p_i,p_j,\dots)|^2 \sim \frac{2}{s_{ij}}P_{I\leftarrow ij}(z)|A(\dots,p_I,\dots)|^2$$

splitting function

Subtraction at NLO

$$\int_n \equiv$$ Integration over an n-particle phase space

NLO correction to the partonic cross section:

$$\mathrm{d}\sigma_{NLO} = \int_n \mathrm{d}\sigma^V + \int_{n+1} \mathrm{d}\sigma^R$$

infrared divergent infrared divergent

Subtraction at NLO:

$$\mathrm{d}\sigma_{NLO} = \int_n [\mathrm{d}\sigma^V - \mathrm{d}\sigma^V_{\mathrm{sub.}}] + \int_{n+1} [\mathrm{d}\sigma^R - \mathrm{d}\sigma^R_{\mathrm{sub.}}]$$

constructed by exploiting the factorisation of QCD amplitudes in infrared limits

virtual subtraction term

real subtraction term

with $\mathrm{d}\sigma^V_{\mathrm{sub.}} = -\int_1 \mathrm{d}\sigma^R_{\mathrm{sub.}}$ to recover the original result.

Subtraction at NNLO

$$\int_n \equiv$$ Integration over an n-particle phase space

NNLO correction to the partonic cross section:

$$\mathrm{d}\sigma_{NNLO} = \int_n \mathrm{d}\sigma^{VV} + \int_{n+1} \mathrm{d}\sigma^{RV} + \int_{n+2} \mathrm{d}\sigma^{RR}$$

infrared divergent — infrared divergent — infrared divergent

Subtraction at NNLO:

$$\mathrm{d}\sigma_{NNLO} = \int_n [\mathrm{d}\sigma^{VV} - \mathrm{d}\sigma^{VV}_{\mathrm{sub.}}] + \int_{n+1} [\mathrm{d}\sigma^{RV} - \mathrm{d}\sigma^{RV}_{\mathrm{sub.}}] + \int_{n+2} [\mathrm{d}\sigma^{RR} - \mathrm{d}\sigma^{RR}_{\mathrm{sub.}}]$$

double-virtual subtraction term

real-virtual subtraction term

double-real subtraction term

with:

$$\mathrm{d}\sigma^{RR}_{\mathrm{sub.}} = \mathrm{d}\sigma^{RR,1}_{\mathrm{sub.}} + \mathrm{d}\sigma^{RR,2}_{\mathrm{sub.}} \qquad \mathrm{d}\sigma^{RV}_{\mathrm{sub.}} = \mathrm{d}\sigma^{RV,1}_{\mathrm{sub.}} - \int_1 \mathrm{d}\sigma^{RR,1}_{\mathrm{sub.}} \qquad \mathrm{d}\sigma^{VV}_{\mathrm{sub.}} = -\int_1 \mathrm{d}\sigma^{RV,1}_{\mathrm{sub.}} - \int_2 \mathrm{d}\sigma^{RR,2}_{\mathrm{sub.}}$$

single-unresolved — double-unresolved

Subtraction at N^{k}LO

$$\int_n \equiv$$ Integration over an n-particle phase space

N^{k}LO correction to the partonic cross section:

each term in the sum is infrared divergent

$$\mathrm{d}\sigma_{N^k LO} = \sum_{i=0}^{k} \int_{n+i} \mathrm{d}\sigma^{R^i V^{k-i}}$$

Subtraction at N^{k}LO:

$$\mathrm{d}\sigma_{N^k LO} = \sum_{i=0}^{k} \int_{n} [\mathrm{d}\sigma^{R^i V^{k-i}} - \mathrm{d}\sigma^{R^i V^{k-i}}_{\mathrm{sub.}}]$$

with:

$$\mathrm{d}\sigma^{R^i V^{k-i}}_{\mathrm{sub.}} = \sum_{j=1}^{i} \mathrm{d}\sigma^{V^{k-i} R^i, j}_{\mathrm{sub.}} - \sum_{j=1}^{k-i} \int_j \mathrm{d}\sigma^{V^{k-i-j} R^i, j}_{\mathrm{sub.}}$$

Subtraction at N^{k}LO

$$\int_n \equiv$$ Integration over an n-particle phase space

N^{k}LO correction to the partonic cross section:

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$$\mathrm{d}\sigma_{N^kLO} = \sum_{i=0}^{k} \int_{n+i} \mathrm{d}\sigma^{R^iV^{k-i}}$$

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$$\mathrm{d}\sigma_{N^kLO} = \sum_{i=0}^{k} \int_{n} [\mathrm{d}\sigma^{R^iV^{k-i}} - \mathrm{d}\sigma^{R^iV^{k-i}}_{\mathrm{sub.}}]$$

How to build these?

How to perform this integration?

with:

$$\mathrm{d}\sigma^{R^iV^{k-i}}_{\mathrm{sub.}} = \sum_{j=1}^{i} \mathrm{d}\sigma^{V^{k-i}R^i,j}_{\mathrm{sub.}} - \sum_{j=1}^{k-i} \int_j \mathrm{d}\sigma^{V^{k-i-j}R^i,j}_{\mathrm{sub.}}$$

Is infrared subtraction "solved" at N^kLO?

	NLO	NNLO	N^3LO
Given matrix elements and resources, N^kLO corrections can **in principle** be computed			
New N^kLO calculations are **actually** streamlined			
N^kLO calculations can be easily reproduced and performed by anyone			
N^kLO event generation is possible			

Is infrared subtraction "solved" at N^{k}LO?

	NLO	NNLO	N^{3}LO
Given matrix elements and resources, N^{k}LO corrections can **in principle** be computed	✓		
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N^{k}LO event generation is possible	✓		

NLO

Fully general IR subtraction techniques:

- Dipole subtraction [Catani,Seymour '96]
- FKS subtraction [Frixione,Kunszt,Signer '96]

+

Automation of one-loop amplitudes:

- OneLoop, QCDLoop, LoopTools, ...
- Recola, OpenLoops, NJET, ...

General-purpose Monte Carlo event generators

Public tools:

- MadGraph5;
- Sherpa;
- Herwig;
- POWHEG BOX;
- MoCaNLO;
- ...

+ matching to parton showers:

- MC@NLO
- POWHEG

Is infrared subtraction “solved” at N^{k}LO?

	NLO	NNLO	N^{3}LO
Given matrix elements and resources, N^{k}LO corrections can **in principle** be computed	✓	✓ *	
New N^{k}LO calculations are **actually** streamlined	✓	✓ **	
N^{k}LO calculations can be easily reproduced and performed by anyone	✓	●	
N^{k}LO event generation is possible	✓	✗	

NNLO

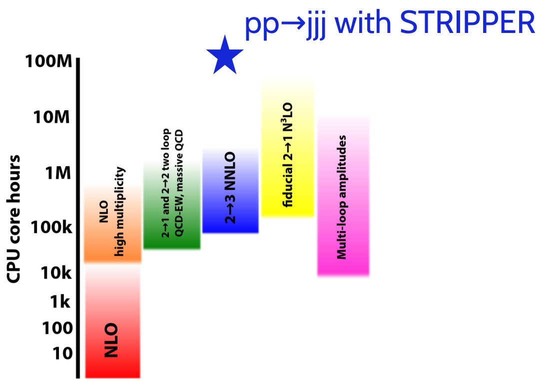
* sector-improved residue subtraction, antenna subtraction, nested soft-collinear subtraction, local analytic sector subtraction

** STRIPPER

Public tools: MCFM, MATRIX, NNLOJET, History, ...
Non-public tools: STRIPPER, ...

Several proposed/implemented approaches:

- Antenna subtraction; [Gehrmann,Gehrmann-De Ridder,Glover '05], [Currie,Glover,Wells '13]
- qT-slicing; [Catani,Grazzini '07]
- Sector-improved residue subtraction; [Czakon '10] [Czakon,Heymes '14]
- N-jettiness slicing; [Gaunt,Stahlhofen,Tackmann,Walsh '14]
- CoLoRFul subtraction; [Del Duca,Duhr,Kardos,Somogyi,Szor,Trocsanyi,Tulipant '16]
- Local analytic sector subtraction; [Magnea,Maina,Pelliccioli,Signorile-Signorile,Torrielli,Uccirati '17]
- Nested soft-collinear subtraction; [Caola,Melnikov,Rontsch '17]
- Projection-to-Born; [Cacciari,Dreyer,Karlberg,Salam,Zanderighi '18]



[Febres Cordero,von Manteuffel,Neumann '22]

State of the art of NNLO calculations: 2→3 processes

Sector improved residue subtraction, STRIPPER

- Higher-order QCD corrections to top-quark pair production in association with a jet — pp→ttj, LC VV
- Isolated photon production in association with a jet pair through next-to-next-to-leading order in QCD — pp→ɣjj
- NNLO QCD corrections to event shapes at the LHC — pp→jjj, LC VV
- Next-to-next-to-leading order QCD corrections to $Wb\bar{b}$ production at the LHC (PHYSICAL REVIEW D 106, 074016 (2022)) — pp→Wbb, LC VV, mb=0
- Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC (PHYSICAL REVIEW LETTERS 127, 152001 (2021)) — pp→jjj, LC VV
- NNLO QCD corrections to diphoton production with an additional jet at the LHC — pp→ɣɣj, LC VV
- NNLO QCD corrections to three-photon production at the LHC — pp→ɣɣɣ, LC VV

qT slicing, MATRIX

- NNLO QCD predictions for $W\gamma\gamma$ production at the LHC — pp→Wɣɣ, LC VV
- Precise predictions for $t\bar{t}H$ production at the LHC: inclusive cross section and differential distributions — pp→Htt, soft approx. massification (LC VV)
- Precise Predictions for the Associated Production of a W Boson with a Top-Antitop Quark Pair at the LHC (PHYSICAL REVIEW LETTERS 131, 231901 (2023)) — pp→Wtt, soft approx. massification (LC VV)
- Associated production of a W boson and massive bottom quarks at next-to-next-to-leading order in QCD (PHYSICAL REVIEW D 107, 074032 (2023)) — pp→Wbb, massification (LC VV)
- Higgs Boson Production in Association with a Top-Antitop Quark Pair in Next-to-Next-to-Leading Order QCD (PHYSICAL REVIEW LETTERS 130, 111902 (2023)) — pp→Htt, soft approx. massification (LC VV)
- Triphoton production at hadron colliders in NNLO QCD — pp→ɣɣɣ, LC VV

Antenna subtraction, NNLOJET

- Precise Predictions for Event Shapes in Diphoton Production at the LHC (PHYSICAL REVIEW LETTERS 134, 171901 (2025)) — pp→ɣɣj
- Automation of antenna subtraction in colour space: gluonic processes — pp→jjj, LC, gluons

qT slicing + PS, MiNNLOPS

- Higgs boson production in association with massive bottom quarks at NNLO+PS — pp→Hbb, massification
- Next-to-next-to-leading order event generation for Z-boson production in association with a bottom-quark pair — pp→Zbb, massification (LC VV)

Matteo Marcoli

A lot of recent progress on NNLO formalisms and tools

Antenna subtraction

NNLOJET: a parton-level event generator for jet cross sections at NNLO QCD accuracy

NNLOJET Collaboration

Generalised antenna functions for higher-order calculations

Elliot Fox, Nigel Glover and Matteo Marcoli

The colourful antenna subtraction method

T. Gehrmann, E.W.N. Glover and M. Marcoli

A general algorithm to build real-radiation antenna functions for higher-order calculations

Oscar Braun-White, Nigel Glover and Christian T Preuss

Nested soft-collinear subtraction

Integrated subtraction terms and finite remainders for arbitrary processes with massless partons at colliders in the nested soft-collinear subtraction scheme

Federica Devoto, Kirill Melnikov, Raoul Röntsch, Chiara Signorile-Signorile, Davide Maria Tagliabue, Matteo Tresoldi

Towards a general subtraction formula for NNLO QCD corrections to processes at hadron colliders: final states with quarks and gluons

Federica Devoto, Kirill Melnikov, Raoul Röntsch, Chiara Signorile-Signorile, Davide Maria Tagliabue, Matteo Tresoldi

A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation

Federica Devoto, Kirill Melnikov, Raoul Röntsch, Chiara Signorile-Signorile and Davide Maria Tagliabue

history: A tool for fully-differential cross sections at next-to-next-to-leading order

Sven Yannick Klein and Lukas Simon

Local analytic sector subtraction

Advances in Local Analytic Sector Subtraction: massive NLO and elements of NNLO automation

Gloria Bertolotti, Giovanni Limatola, Paolo Torrielli, and Sandro Uccirati

NNLO subtraction for any massless final state: a complete analytic expression

Gloria Bertolotti, Lorenzo Magnea, Giovanni Pelliccioli, Alessandro Ratti, Chiara Signorile-Signorile, Paolo Torrielli and Sandro Uccirati

Loop-Tree Duality

Local finiteness for real-virtual corrections to electroweak production in partonic collisions

Charalampos Anastasiou, Julia Karlen, Yao Ma, George Sterman

NNLO QCD corrections to $\gamma\gamma \to Q\bar{Q}$ from Local Unitarity combined with Coulomb resummation and NLO EW effects

Zeno Capatti, Mathijs Fraaije, Valentin Hirschi, Lucien Huber, Ben Ruijl, Hua-Sheng Shao

Numerical integration of loop integrals through local cancellation of threshold singularities

D. Kermanschah

Vacuum amplitudes and time-like causal unitary in the loop-tree duality

The LTD Collaboration, Selomit Ramírez-Uribe, Andrés E. Rentería-Olivo, David F. Rentería-Estrada, Jorge J. Martinez de Lejarza, Prasanna K. Dhani, Leandro Cieri, Roger J. Hernández-Pinto, German F. R. Sborlini, William J. Torres Bobadilla, and Germán Rodrigo

Towards an automated generator based on OpenLoops + LTD

Stefano Pozzorini

based on ongoing work with

Gloria Bertolotti, Nicolò Giraudo, Florian Herren and Jonas Lindert

see also talk by Gloria Bertolotti today at 11 AM

HP2, Torino, September 10–13 2024

CoLoRFul subtraction

NNLOCAL: completely local subtractions for color-singlet production in hadron collisions

V. Del Duca, C. Duhr, L. Fekésházy, F. Guadagni, P. Mukherjee, G. Somogyi, F. Tramontano and S. Van Thurenhout

CoLoRFul for hadron collisions: Integrating the counterterms

S. Van Thurenhout, V. Del Duca, C. Duhr, L. Fekésházy, F. Guadagni, P. Mukherjee, G. Somogyi and F. Tramontano

CoLoRFulNNLO for hadron collisions: regularizing initial-state double real emissions

V. Del Duca, G. Somogyi, F. Tramontano

NNLO grids with STRIPPER

HighTEA: high energy theory event analyser

Michał Czakon, Zahari Kassabov, Alexander Mitov, Rene Poncelet and Andrei Popescu

qT slicing, NNLO grids with MATRIX

Soft contributions to heavy quark production in arbitrary kinematics

Simone Devoto, Javier Mazzitelli

MATRIX HAWAII: PineAPPL interpolation grids with MATRIX

S. Devoto, T. Ježo, S. Kallweit, C. Schwand

Slicing with jets

q_T-slicing with multiple jets

Rong-Jun Fu, Rudi Rahn, Ding Yu Shao, Wouter J. Waalewijn, and Bin Wu

NNLO predictions with nonlocal subtractions and fiducial power corrections in GENEVA

Simone Alioli, Georgios Billis, Alessandro Broggio and Giovanni Stagnitto

Jet Production at NNLO: Exploring a New Scheme

Luca Buonocore, Massimiliano Grazzini, Flavio Guadagni, Jürg Haag, Stefan Kallweit and Luca Rottoli

Is infrared subtraction “solved” at N^{k}LO?

	NLO	NNLO	N^{3}LO
• Given matrix elements and resources, N^{k}LO corrections can **in principle** be computed	✓	✓ *	✗
• New N^{k}LO calculations are **actually** streamlined	✓	✓	✗
• N^{k}LO calculations can be easily reproduced and performed by anyone	✓	●	✗
• N^{k}LO event generation is possible	✓	✗	✗

N^{3}LO

Very few techniques available for differential calculations, applicable to specific (simple) processes:

- Projection to Born;
- Slicing (qT, 0-jettiness);
- Antenna subtraction;

State of the art of N^3LO calculations

(Semi-)Inclusive:

- gg→H [Anastasiou,Durh,Dulat,Herzog,Mistleberger '15] [Mistleberger '18]
- gg→HH [Chen,Li,Shao,Wang '19,'20]
- VBF H [Dreyer,Karlberg '16]
- VBF HH [Dreyer,Karlberg '18]
- pp→ɣ/Z/W
- bb→H [Durh,Dulat,Mistleberger '20] [Durh,Mistleberger '21]
- pp→Z/W H [Baglio,Durh,Mistleberger,Szafron '22]
- e^+e^-→jj (heavy quarks) [Chen,Guan,He,Liu,Ma '22] (SIA) [He,Xing,Yang,Zhu '25]

Fully differential:

- gg→H [Cieri,Chen,Gehrmann,Glover,Huss '18] * [Billis,Dehnadi,Ebert,Michel,Tackmann '21] * [Chen,Gehrmann,Glover,Huss,Mistleberger,Pelloni '21] †
- pp→ɣ/Z/W [Chen,Gehrmann,Glover,Huss,Yang,Zhu '21,'22] * [Chen,Gehrmann,Glover,Huss,Monni,Re,Rottoli,Torrielli '22] * [Campbell,Neumann '22,'23] *
- DIS [Currie,Gehrmann,Glover,Huss,Niehues,Vogt '18] †
- H→bb [Mondini,Schiavi,Williams '19] †
- e^+e^-→jj [Chen,Jakubcik,MM,Stagnitto '25] ‡
- pp→ɣɣ [Czakon,Eschment,Generet,Poncelet '26] *

Several phenomenologically relevant results despite the extreme complexity.

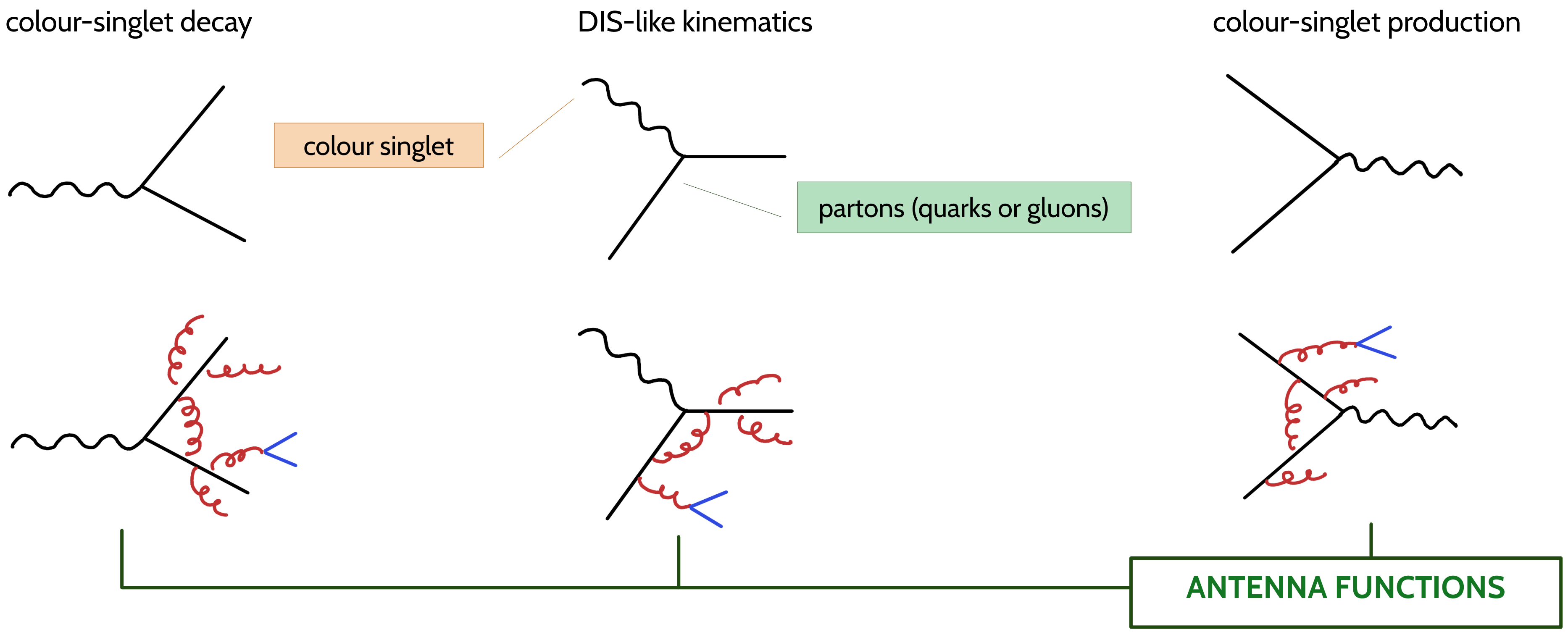
Available techniques are applicable to limited cases.
New approaches must be developed for more complicated processes.

* qT slicing
† Projection-to-Born
‡ Antenna Subtraction

ANTENNA SUBTRACTION

Antenna subtraction

Underlying idea: the behaviour of QCD matrix elements in soft/collinear limit is **universal**.
Exploit simple scattering processes as **prototypes** to describe the singular behaviour in more complicated reactions.

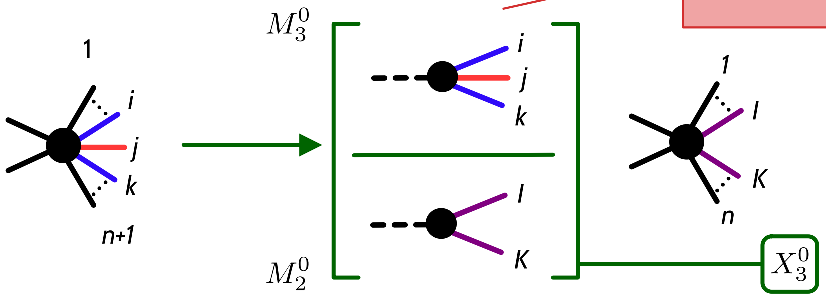


These (colour-ordered) matrix elements can be used to construct subtraction terms!
The two original partons constitute the **colour dipole** (antenna) emitting radiation.

[Gehrmann-De Ridder,Gehrmann,Glover ‘03,’04,’05]

Antenna factorisation

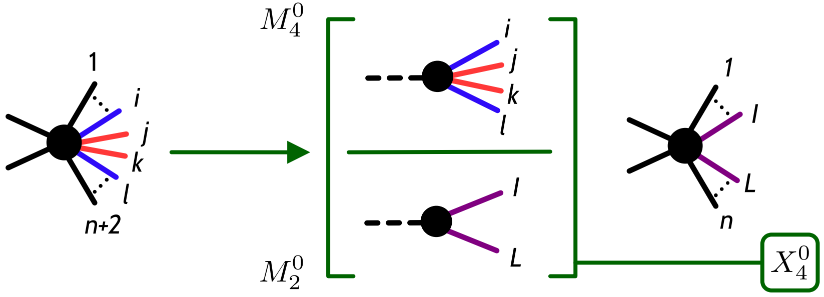
Squared colour-ordered amplitudes for colour-singlet decay



3-parton tree antenna (NLO, real)

$$X_3^0 = \frac{M_3^0}{M_2^0}$$

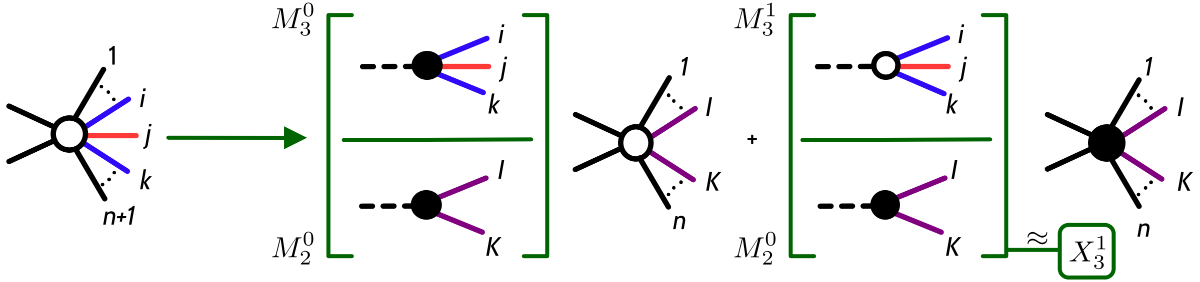
$$\mathcal{X}_3^0 \propto \int \mathrm{d}\Phi_3 \, X_3^0$$



4-parton tree antenna (NNLO, double-real)

$$X_4^0 = \frac{M_4^0}{M_2^0}$$

$$\mathcal{X}_4^0 \propto \int \mathrm{d}\Phi_4 \, X_4^0$$



Unresolved **tree-level** factor

Unresolved **one-loop** factor

3-parton 1-loop antenna (NNLO, real-virtual)

$$X_3^1 = \frac{M_3^1}{M_2^0} - X_3^0 \frac{M_1^2}{M_2^0}, \qquad \mathcal{X}_3^1 \propto \int \mathrm{d}\Phi_3 \, X_3^1$$

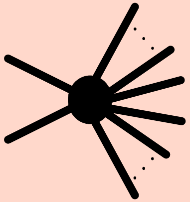
Antenna subtraction at NNLO

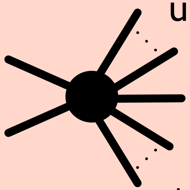
[Gehrmann-De Ridder,Gehrmann,Glover '05] [Currie,Glover,Wells '13]

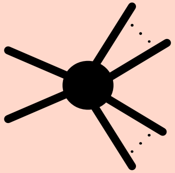
NLO sub-sector

iterated limits

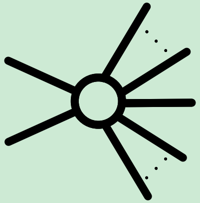
RR:

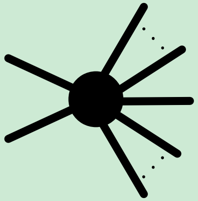

$n+2$

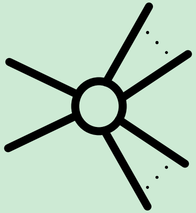
→ $X_3^0 \cdot$  $n+1$ (single unresolved)

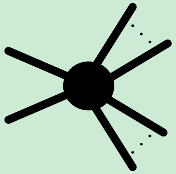
$+$ $X_4^0 \cdot$ / $X_3^0 X_3^0 \cdot$  n (double unresolved)

RV:

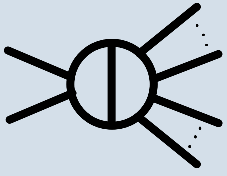

$n+1$

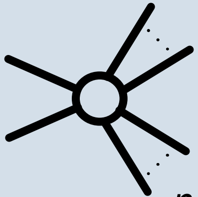
→ $\mathcal{X}_3^0 \cdot$  $n+1$ (removes ε-poles)

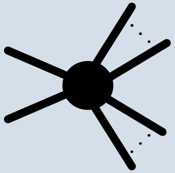
$+$ $X_3^0 \cdot$  n

$+$ $X_3^1 \cdot$ / $\mathcal{X}_3^0 X_3^0 \cdot$  n

VV:


n

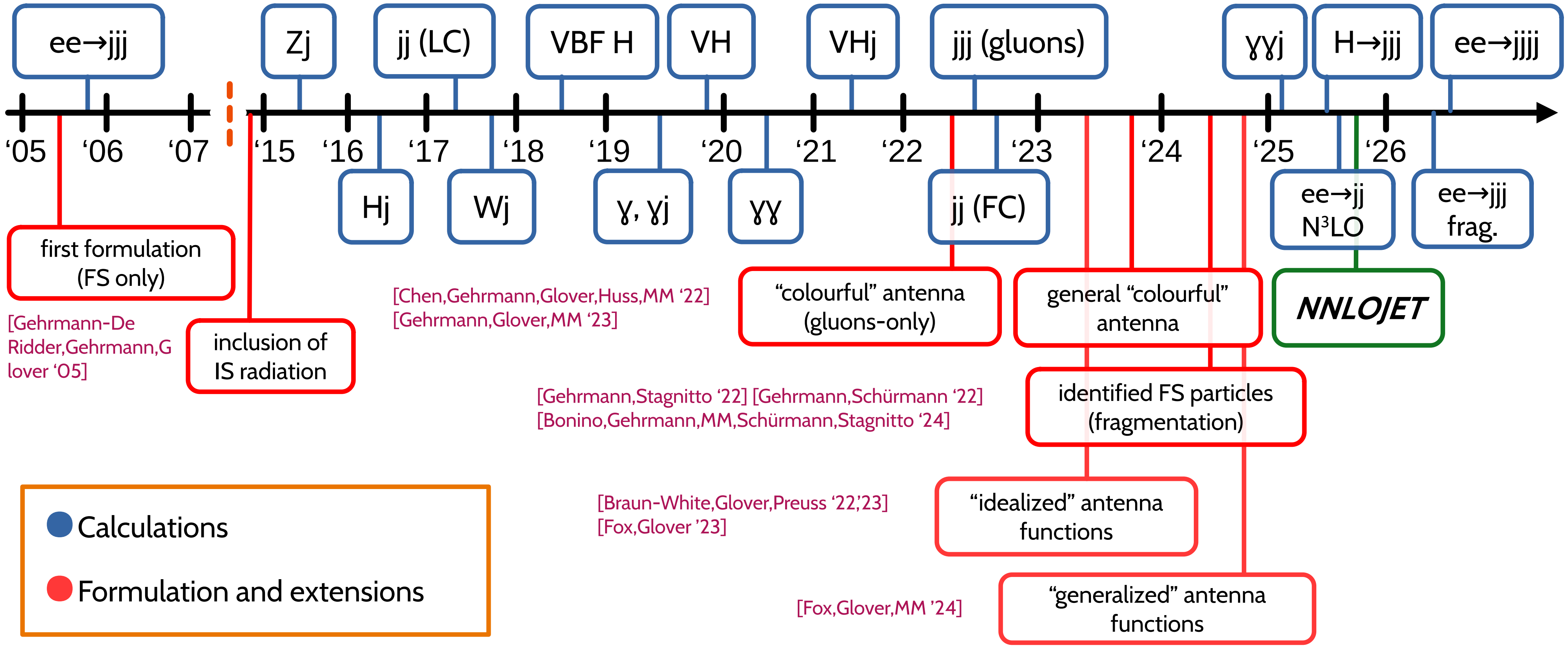
→ $\mathcal{X}_3^0 \cdot$  n

$+$ $\mathcal{X}_4^0 \cdot$ / $\mathcal{X}_3^1 \cdot$ / $\mathcal{X}_3^0 \otimes \mathcal{X}_3^0 \cdot$  n

Antenna subtraction

Now public! [NNLOJET collaboration: Huss *et al.* ‘25]

Successfully applied at NNLO to a variety of processes within the ***NNLOJET*** Monte Carlo framework.



Antenna subtraction

Now public! [NNLOJET collaboration: Huss *et al.* '25]

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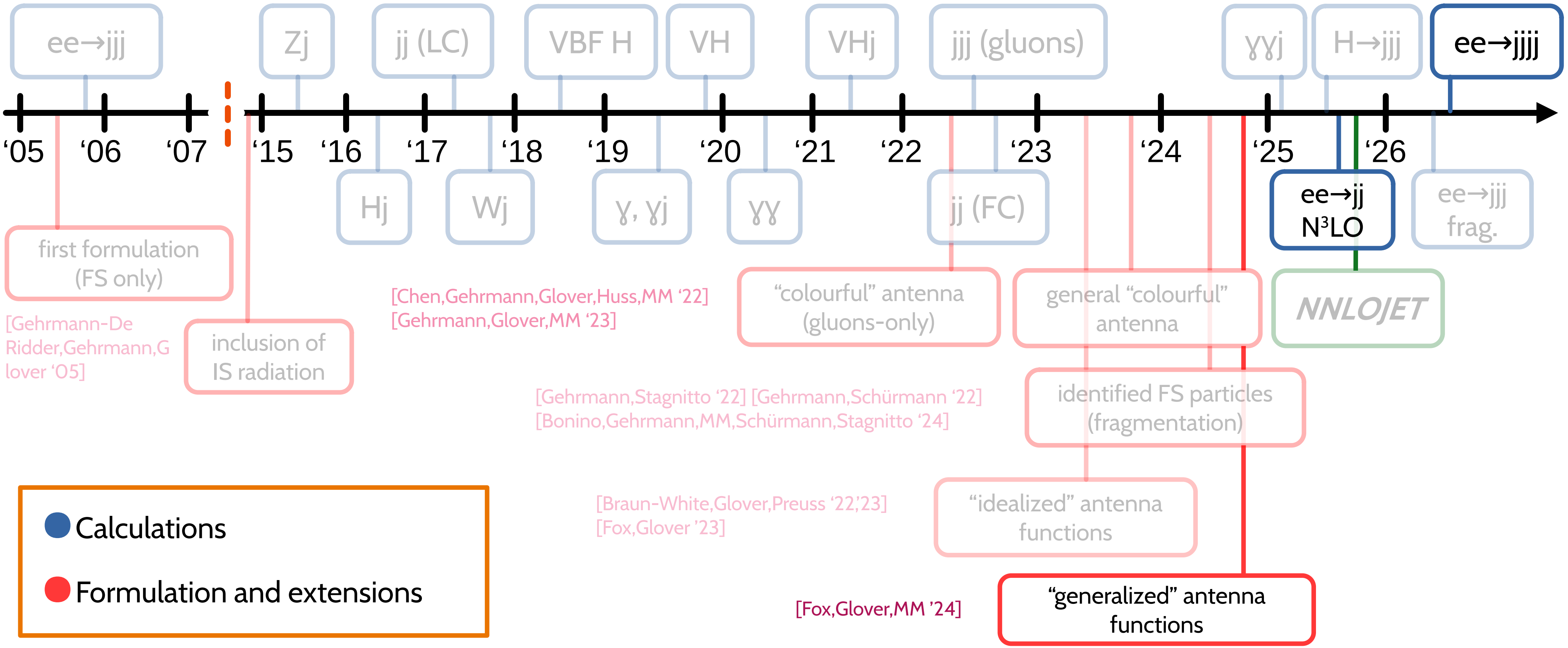


ANTENNA SUBTRACTION
Generalised Antenna Functions

Singularities cancellation in colour space

Recently, different NNLO subtraction schemes reached a **more general formulation** and explicitly demonstrated the **cancellation of infrared singularities** for **individual structures in colour space**.

[Chen,Gehrmann,Glover,Huss,MM '22,'23]
[Bertolotti,Magnea,Pelliccioli,Ratti,Signorile-Signorile,Torrielli,Uccirati '23]
[Devoto,Melnikov,Rontsch,Signorile-Signorile,Tagliabue,Tresoldi '24,'25]

Infrared insertion	Structures in colour space	
single dipole	$\langle A^0_{n+2} \vert \boldsymbol{T}_i \cdot \boldsymbol{T}_j \vert A^0_{n+2} \rangle$	$i \neq j$
iterated dipole two common radiators	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j \right)^2 \vert A^0_{n+2} \rangle$	$i \neq j$
iterated dipole one common radiator	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k \right) \left(\boldsymbol{T}_k \cdot \boldsymbol{T}_j \right) \vert A^0_{n+2} \rangle$	$i \neq j, k,\ j \neq k$
iterated dipole no common radiator	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j \right) \left(\boldsymbol{T}_k \cdot \boldsymbol{T}_l \right) \vert A^0_{n+2} \rangle$	$i \neq j, k, l,\ j \neq k, l,\ k \neq l$

The Colourful Antenna Subtraction Method

Idea:

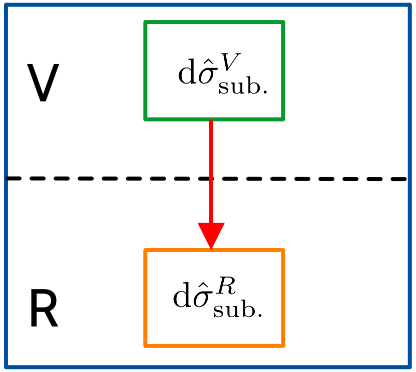
- rewrite the singularity structure of loop matrix elements to extract information about real-emission counterterms
- propagate the colour-correlations through the layers of an NNLO subtraction scheme

[Chen, Gehrmann, Glover, Huss, MM '22]

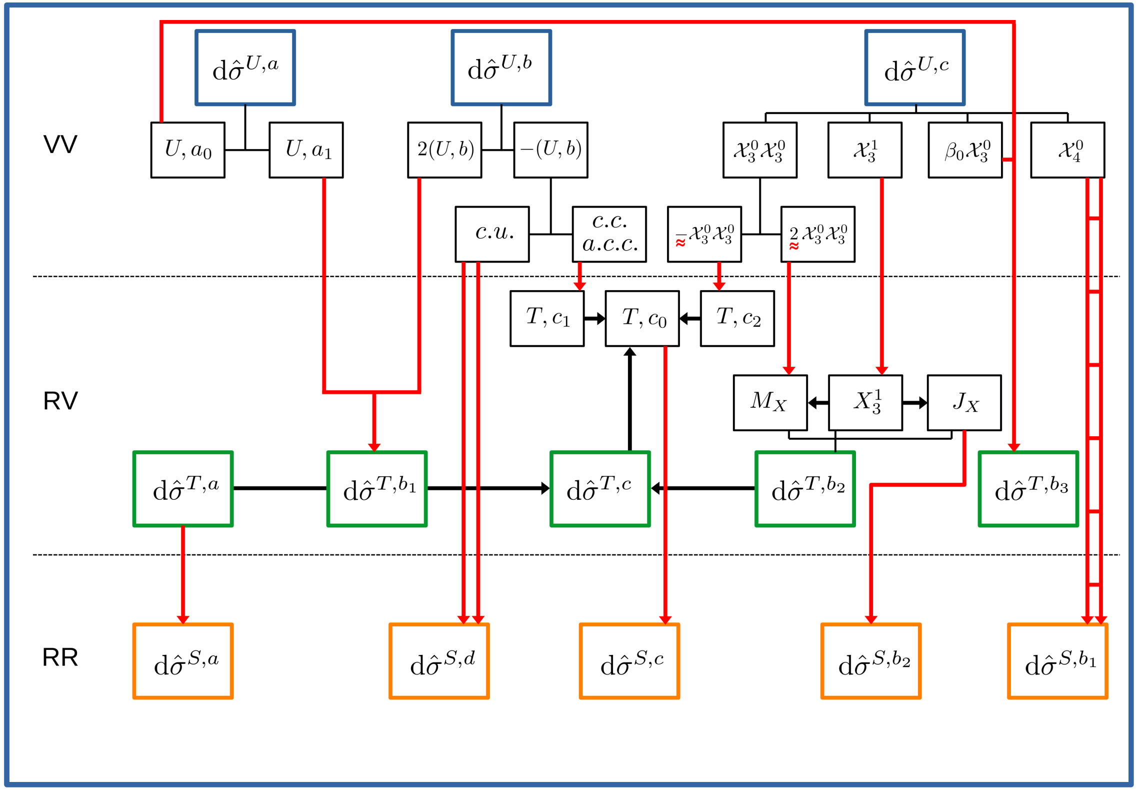
[Gehrmann, Glover, MM '23]

see also [Magnea,Maina,Pelliccioli,Signorile-Signorile,Torrielli,Uccirati '18]

NLO

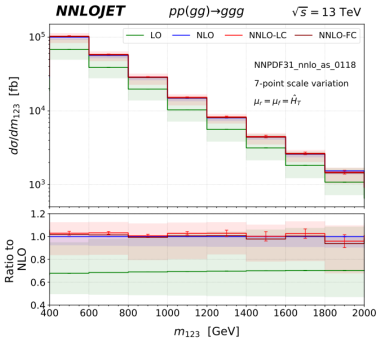


NNLO

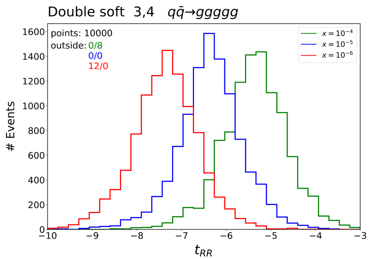


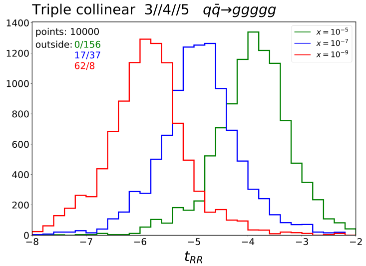
Still not there yet …

[Chen, Gehrmann, Glover, Huss, MM '22]



[Gehrmann, Glover, MM '23]





It is possible to generate automatically subtraction terms:

- ✅ not limited by multiplicity or colour factor
- ✅ validated against existing subtraction terms
- ✅ candidate subtraction terms for pp→jjj @NNLO (most challenging process)

but:

- ❌ very large expressions with many iterated terms (thousands of terms for pp→jjj @NNLO)
- ❌ very hard to recognize patterns, painful and slow debugging cycle
- ❌ computational time for the evaluation of subtraction terms catches up with the matrix elements

We can perform calculations by brute force, but **we must be missing something …**

Singularities cancellation in colour space

The problem must be in our building blocks, so we went back to see if we could improve them...

[Fox,Glover,MM '24]

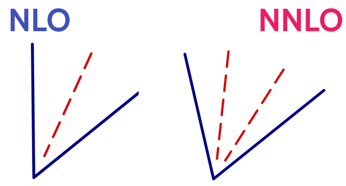
Infrared insertion	Structures in colour space		Topology	Antenna
single dipole	$\langle A^0_{n+2} \vert \boldsymbol{T}_i \cdot \boldsymbol{T}_j \vert A^0_{n+2} \rangle$	$i \neq j$		
iterated dipole two common radiators	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j \right)^2 \vert A^0_{n+2} \rangle$	$i \neq j$		
iterated dipole one common radiator	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k \right) \left(\boldsymbol{T}_k \cdot \boldsymbol{T}_j \right) \vert A^0_{n+2} \rangle$	$i \neq j, k,\ j \neq k$		
iterated dipole no common radiator	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j \right) \left(\boldsymbol{T}_k \cdot \boldsymbol{T}_l \right) \vert A^0_{n+2} \rangle$	$i \neq j, k, l,\ j \neq k, l,\ k \neq l$		

Singularities cancellation in colour space

hard radiator

unresolved emission

[Fox,Glover,MM '24]

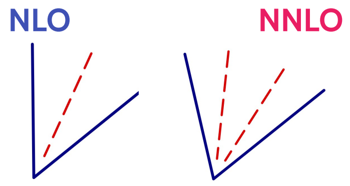
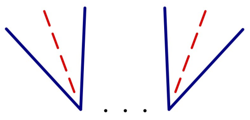
Infrared insertion	Structures in colour space		Topology	Antenna
single dipole	$\langle A^0_{n+2} \vert \boldsymbol{T}_i \cdot \boldsymbol{T}_j \vert A^0_{n+2} \rangle$	$i \neq j$		$X^0_3,\ X^0_4$
iterated dipole two common radiators	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j \right)^2 \vert A^0_{n+2} \rangle$	$i \neq j$		
iterated dipole one common radiator	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k \right) \left(\boldsymbol{T}_k \cdot \boldsymbol{T}_j \right) \vert A^0_{n+2} \rangle$	$i \neq j, k,\ j \neq k$		
iterated dipole no common radiator	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j \right) \left(\boldsymbol{T}_k \cdot \boldsymbol{T}_l \right) \vert A^0_{n+2} \rangle$	$i \neq j, k, l,\ j \neq k, l,\ k \neq l$		

Singularities cancellation in colour space

hard radiator

unresolved emission

[Fox,Glover,MM ‘24]

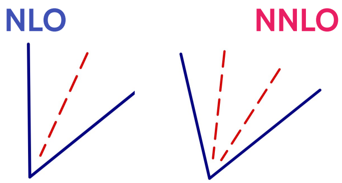
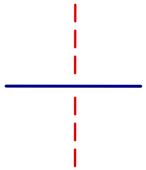
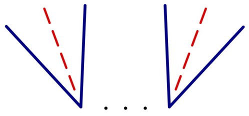
Infrared insertion	Structures in colour space	Topology	Antenna
single dipole	$\langle A^0_{n+2} \vert \boldsymbol{T}_i \cdot \boldsymbol{T}_j \vert A^0_{n+2}\rangle \quad i \neq j$		$X^0_3,\ X^0_4$
iterated dipole two common radiators	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j\right)^2 \vert A^0_{n+2}\rangle \quad i \neq j$		
iterated dipole one common radiator	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k\right)\left(\boldsymbol{T}_k \cdot \boldsymbol{T}_j\right) \vert A^0_{n+2}\rangle \quad i \neq j, k,\ j \neq k$		
iterated dipole no common radiator	$\langle A^0_{n+2} \vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j\right)\left(\boldsymbol{T}_k \cdot \boldsymbol{T}_l\right) \vert A^0_{n+2}\rangle \quad i \neq j, k, l,\ j \neq k, l,\ k \neq l$		$X^0_3 X^0_3$

Singularities cancellation in colour space

hard radiator

unresolved emission

[Fox,Glover,MM '24]

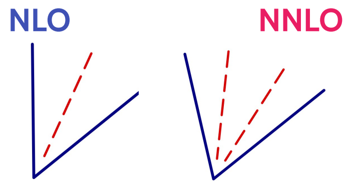
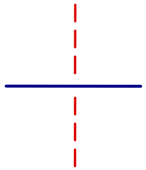
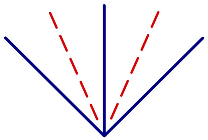
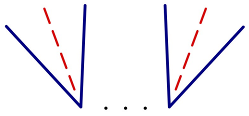
Infrared insertion	Structures in colour space		Topology	Antenna
single dipole	$\langle A^0_{n+2}\vert \boldsymbol{T}_i \cdot \boldsymbol{T}_j \vert A^0_{n+2}\rangle$	$i \neq j$		$X^0_3, \ X^0_4$
iterated dipole two common radiators	$\langle A^0_{n+2}\vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j\right)^2 \vert A^0_{n+2}\rangle$	$i \neq j$		$\widetilde{X}^0_4$
iterated dipole one common radiator	$\langle A^0_{n+2}\vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k\right)\left(\boldsymbol{T}_k \cdot \boldsymbol{T}_j\right) \vert A^0_{n+2}\rangle$	$i \neq j, k, \ j \neq k$		
iterated dipole no common radiator	$\langle A^0_{n+2}\vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j\right)\left(\boldsymbol{T}_k \cdot \boldsymbol{T}_l\right) \vert A^0_{n+2}\rangle$	$i \neq j, k, l, \ j \neq k, l, \ k \neq l$		$X^0_3 X^0_3$

Singularities cancellation in colour space

hard radiator

unresolved emission

[Fox,Glover,MM '24]

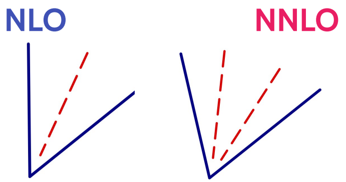
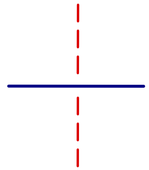
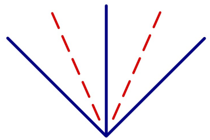
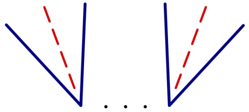
Infrared insertion	Structures in colour space		Topology	Antenna
single dipole	$\langle A^0_{n+2}\vert \boldsymbol{T}_i \cdot \boldsymbol{T}_j \vert A^0_{n+2}\rangle$	$i \neq j$		$X^0_3,\ X^0_4$
iterated dipole two common radiators	$\langle A^0_{n+2}\vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j\right)^2 \vert A^0_{n+2}\rangle$	$i \neq j$		$\widetilde{X}^0_4$
iterated dipole one common radiator	$\langle A^0_{n+2}\vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k\right)\left(\boldsymbol{T}_k \cdot \boldsymbol{T}_j\right) \vert A^0_{n+2}\rangle$	$i \neq j, k,\ j \neq k$		
iterated dipole no common radiator	$\langle A^0_{n+2}\vert \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j\right)\left(\boldsymbol{T}_k \cdot \boldsymbol{T}_l\right) \vert A^0_{n+2}\rangle$	$i \neq j, k, l,\ j \neq k, l,\ k \neq l$		$X^0_3 X^0_3$

Singularities cancellation in colour space

hard radiator

unresolved emission

[Fox,Glover,MM '24]

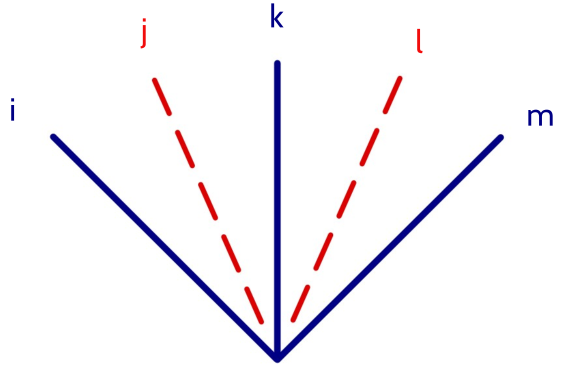
Infrared insertion	Structures in colour space		Topology	Antenna
single dipole	$\langle A^0_{n+2} \| \boldsymbol{T}_i \cdot \boldsymbol{T}_j \| A^0_{n+2} \rangle$	$i \neq j$		$X^0_3, \; X^0_4$
iterated dipole two common radiators	$\langle A^0_{n+2} \| \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j\right)^2 \| A^0_{n+2} \rangle$	$i \neq j$		$\widetilde{X}^0_4$
iterated dipole one common radiator	$\langle A^0_{n+2} \| \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k\right)\left(\boldsymbol{T}_k \cdot \boldsymbol{T}_j\right) \| A^0_{n+2} \rangle$	$i \neq j, k, \; j \neq k$		**?**
iterated dipole no common radiator	$\langle A^0_{n+2} \| \left(\boldsymbol{T}_i \cdot \boldsymbol{T}_j\right)\left(\boldsymbol{T}_k \cdot \boldsymbol{T}_l\right) \| A^0_{n+2} \rangle$	$i \neq j, k, l, \; j \neq k, l, \; k \neq l$		$X^0_3 X^0_3$

Generalised antenna functions

[Fox,Glover,MM '24]

most complicated and inefficient sector of antenna subtraction

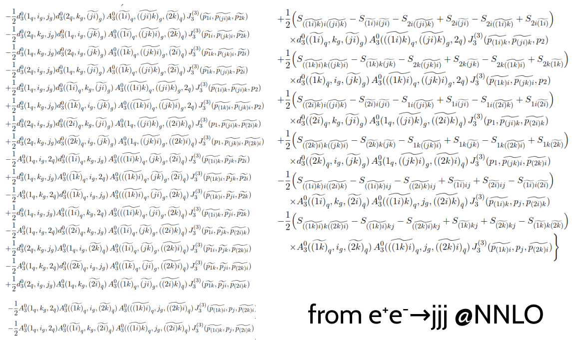
Almost colour-connected emissions: only one shared hard radiator



NOT fully iterated: the two emissions "feel" each other through the recoil on the shared radiator

traditional antenna functions can be used,
but a **very complicated** sequence of iterated structures is needed, plus **Large-Angle-Soft-Terms**

[Gehrmann De-Ridder,Gehrmann,Glover,Heinrich '07]
[Weinzierl '08] [Currie,Glover,Wells '13]



from $e^+e^-\to jjj$ @NNLO

Ideally we want:

$$M^0_{n+2}(\ldots,i,j,k,l,m\ldots)$$

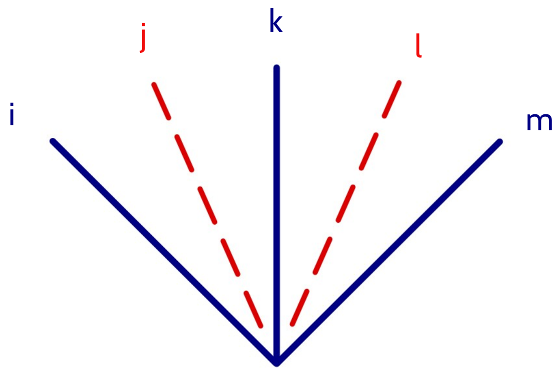
5→3 momentum mapping

generalized three-hard-radiator antenna function

$$X^0_{5,3}(i^h,j,k^h,l,m^h)M^0_n(\ldots,\widetilde{(ijk)},\widetilde{(ijklm)},\widetilde{(klm)},\ldots)$$

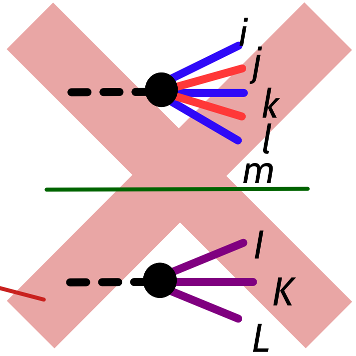
Generalised antenna functions

[Fox,Glover,MM '24]



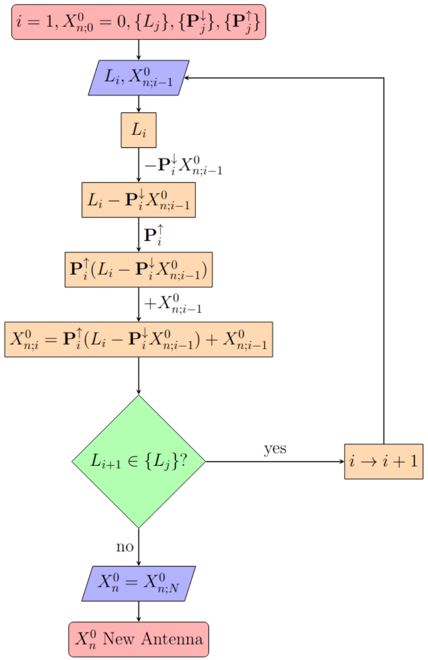
Not possible with matrix element-based antenna functions

non-trivial function of the three-particle phase space



With the **designer antenna algorithm**, it is possible to construct antenna functions with **more than two hard radiators: generalized antenna functions.**

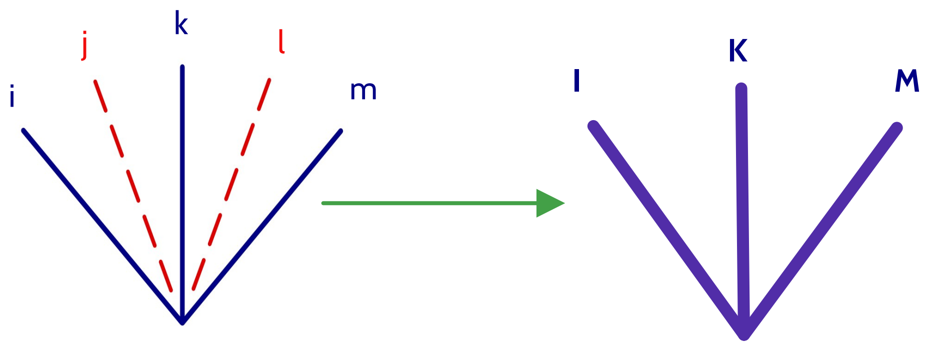
- set of desired limits
- add an unresolved factor to the antenna
- remove overlap between limits
- ensure physical propagators (integrability)
- update antenna
- repeat till no more limits



[Braun-White,Glover,Preuss '22,'23]

Mapping and analytic integration

$$\mathrm{map}_{5\to3}: \quad \begin{aligned} p_I &= p_i + p_j - \frac{s_{ij}}{s_{ik}+s_{jk}} p_k \\ p_K &= \left(1 + \frac{s_{ij}}{s_{ik}+s_{jk}} + \frac{s_{lm}}{s_{lk}+s_{mk}}\right) p_k \\ p_M &= p_l + p_m - \frac{s_{lm}}{s_{lk}+s_{mk}} p_k \end{aligned}$$



iterated dipole mapping

Only one new (class of) integral(s):

$$\begin{aligned} &I(b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9) \\ &= \int_0^1 z_1^{b_1}(1-z_1)^{b_2} dz_1 \int_0^1 z_2^{b_3}(1-z_2)^{b_4} dz_2 \\ &\times \int_0^1 \int_0^1 y_1^{b_5}(1-y_1)^{b_6} y_2^{b_7}(1-y_2)^{b_8}(1-y_1(1-y_2))^{b_9} dy_1\, dy_2 \\ &= \frac{\Gamma(1+b_1)\Gamma(1+b_2)}{\Gamma(2+b_1+b_2)} \frac{\Gamma(1+b_3)\Gamma(1+b_4)}{\Gamma(2+b_3+b_4)} \\ &\times \frac{\Gamma(1+b_5)\Gamma(1+b_6)}{\Gamma(2+b_5+b_6)} \frac{\Gamma(1+b_7)\Gamma(1+b_8)}{\Gamma(2+b_7+b_8)} \\ &\times {}_3F_2([1+b_5, 1+b_8, -b_9], [2+b_5+b_6, 2+b_7+b_8], 1) \end{aligned}$$

$$\begin{aligned} s_{ij} &= y_1(1-y_2)s_{IK}, \\ s_{jk} &= (1-y_1)(1-y_2)(1-z_1)s_{IK}, \\ s_{ik} &= z_1(1-y_1)(1-y_2)s_{IK}, \\ s_{kl} &= (1-y_1)(1-y_2)(1-z_2)s_{KM}, \\ s_{km} &= z_2(1-y_1)(1-y_2)s_{KM}, \\ s_{lm} &= y_2 s_{KM}, \\ s_{ijk} &= (1-y_2)s_{IK}, \\ s_{klm} &= (1-y_1(1-y_2))s_{KM} \end{aligned}$$

[Fox,Glover,MM ‘24]

Generalised antenna functions: validation

[Fox,Glover,MM ‘24]

NNLO correction to event shapes in e⁺e⁻ annihilation:

- ✅ perfect agreement with original method
- ✅ much simpler subtraction terms
- ✅ up to 5-10x faster

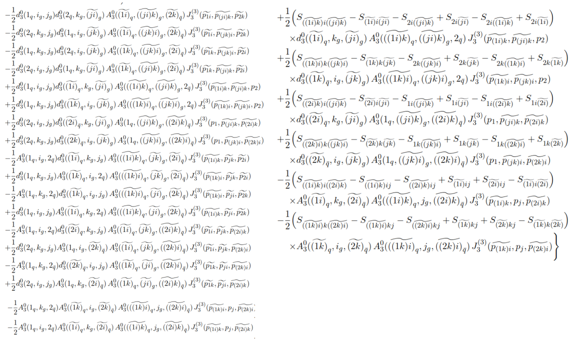
This:

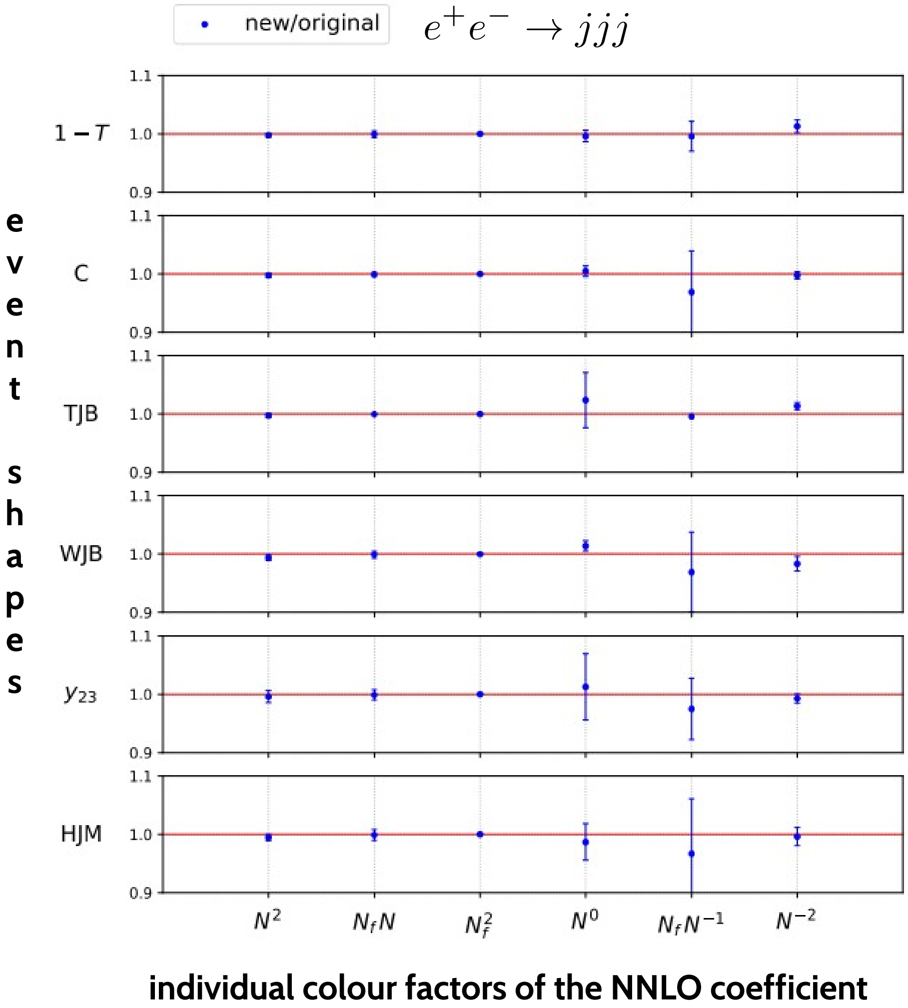
$$+\ A^0_{5,3}(1,i,j,k;2)\, M^0_3((\{1ij\}),(\{1ijk2\}),(\{jk2\})))\, J^{(3)}_3(\{p\}_3)$$

$$-\ D^0_3(1,i,j)\, D^0_3(2,k,(\widetilde{ij}))\, M^0_3((\widetilde{1i}),[\underline{(\widetilde{ji})}k],[k2])\, J^{(3)}_3(\{p\}_3)$$

$$-\ D^0_3(2,k,j)\, D^0_3(1,i,(\widetilde{kj}))\, M^0_3([1i],[\underline{(\widetilde{kj})}i],(\widetilde{2k}))\, J^{(3)}_3(\{p\}_3)$$

Instead of this:

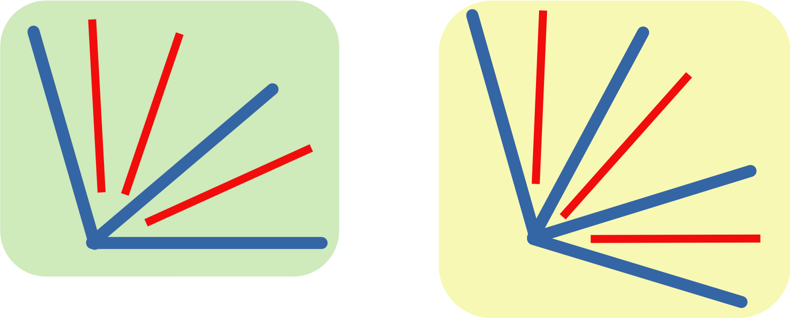




Generalised antenna functions at N^3LO

[work in progress with Elliot Fox and Nigel Glover]

RRR:

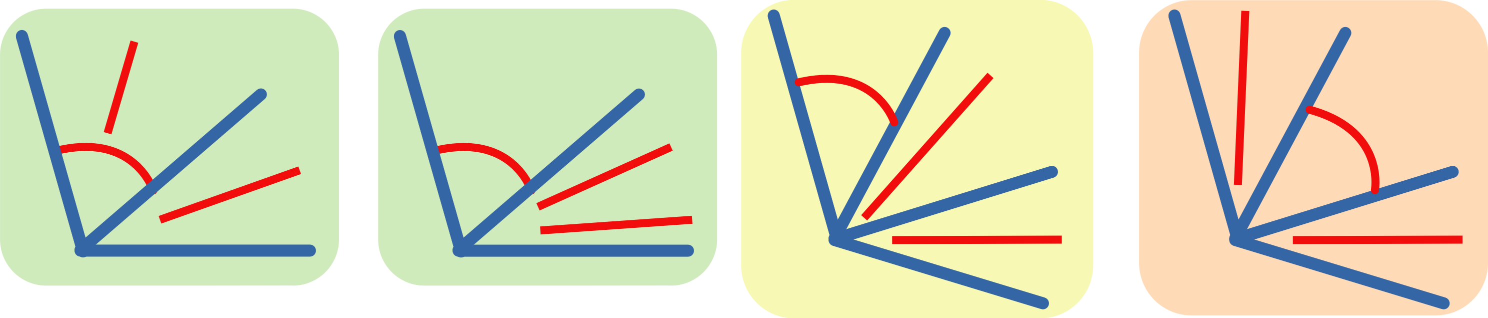


Starting from:
$e^+e^- \to 3j$
$pp \to V+j$

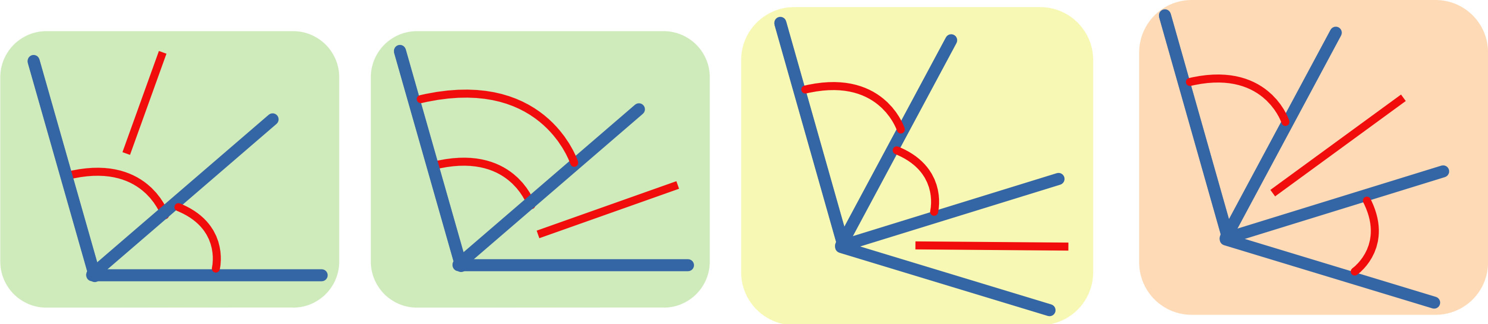
Starting from:
$e^+e^- \to 4j$
$pp \to 2j$

Special case for:
$e^+e^- \to 3j$
$pp \to V+j$
General case for:
$e^+e^- \to 4j$
$pp \to 2j$

RRV:



RVV:



APPLICATIONS

Jet production in electron-positron annihilation

[Chen, Jakubčík, MM, Stagnitto 2505.10618]

[Chen, Chicherin, Fox, Glover, MM, Sotnikov, Sun, Zhang, Zoia 2602.18185]

Inclusive XS		2 jets	3 jets	4 jets	5 jets	6 jets	7 jets	…
2008	N^{4}LO							
1991	N^{3}LO	2025						
	NNLO	2003	2007	2026				
	NLO		1981	1997	2010	2012	2012	
	LO							…

References in the backup slides

APPLICATIONS

$e^+e^- \to$ 4 jets @ NNLO

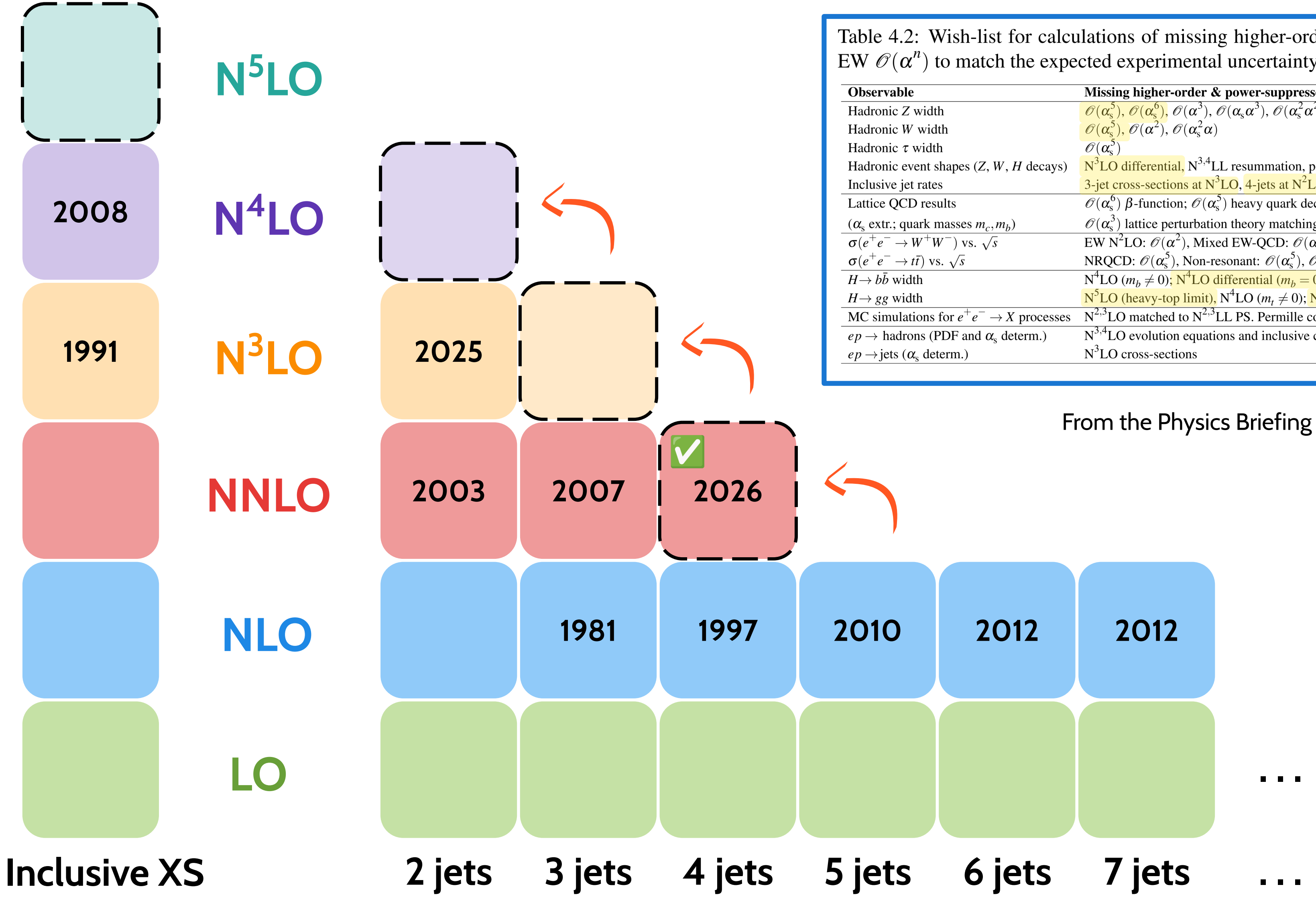


Table 4.2: Wish-list for calculations of missing higher-order perturbative QCD $\mathcal{O}(\alpha_s^n)$ and/or EW $\mathcal{O}(\alpha^n)$ to match the expected experimental uncertainty at future e^+e^- and ep colliders.

Observable	Missing higher-order & power-suppressed corrections
Hadronic Z width	$\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha_s^6)$, $\mathcal{O}(\alpha^3)$, $\mathcal{O}(\alpha_s\alpha^3)$, $\mathcal{O}(\alpha_s^2\alpha^2)$
Hadronic W width	$\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha_s^2\alpha)$
Hadronic τ width	$\mathcal{O}(\alpha_s^5)$
Hadronic event shapes (Z, W, H decays)	N^3LO differential, $N^{3,4}$LL resummation, power corrections
Inclusive jet rates	3-jet cross-sections at N^3LO, 4-jets at N^2LO, 5-jets at NLO
Lattice QCD results	$\mathcal{O}(\alpha_s^6)$ β-function; $\mathcal{O}(\alpha_s^5)$ heavy quark decoupling; $\mathcal{O}(\alpha_s^4)$ static potential
(α_s extr.; quark masses m_c, m_b)	$\mathcal{O}(\alpha_s^3)$ lattice perturbation theory matching (lattice coupling to $\alpha_s^{\overline{MS}}$ etc.)
$\sigma(e^+e^- \to W^+W^-)$ vs. $\sqrt{s}$	EW N^2LO: $\mathcal{O}(\alpha^2)$, Mixed EW-QCD: $\mathcal{O}(\alpha_s\alpha^2)$, $\mathcal{O}(\alpha_s^2\alpha)$
$\sigma(e^+e^- \to t\bar{t})$ vs. $\sqrt{s}$	NRQCD: $\mathcal{O}(\alpha_s^5)$, Non-resonant: $\mathcal{O}(\alpha_s^5)$, $\mathcal{O}(\alpha_s^3)$ differential; QED: $\mathcal{O}(\alpha^3)$ at NNLL
$H \to b\bar{b}$ width	N^4LO ($m_b \neq 0$); N^4LO differential ($m_b = 0$)
$H \to gg$ width	N^5LO (heavy-top limit), N^4LO ($m_t \neq 0$); N^4LO differential, N^3LO differential ($m_t \neq 0$)
MC simulations for $e^+e^- \to X$ processes	$N^{2,3}$LO matched to $N^{2,3}$LL PS. Permille control of non-pQCD effects (hadronization, CR, ...)
$ep \to$ hadrons (PDF and α_s determ.)	$N^{3,4}$LO evolution equations and inclusive cross-sections
$ep \to$ jets (α_s determ.)	N^3LO cross-sections

From the Physics Briefing Book [de Blas et al. 2511.03883]

References in the backup slides

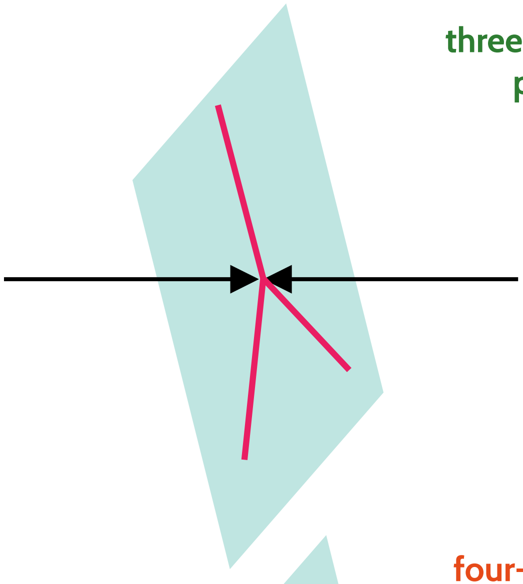
What’s interesting about four-jet production?

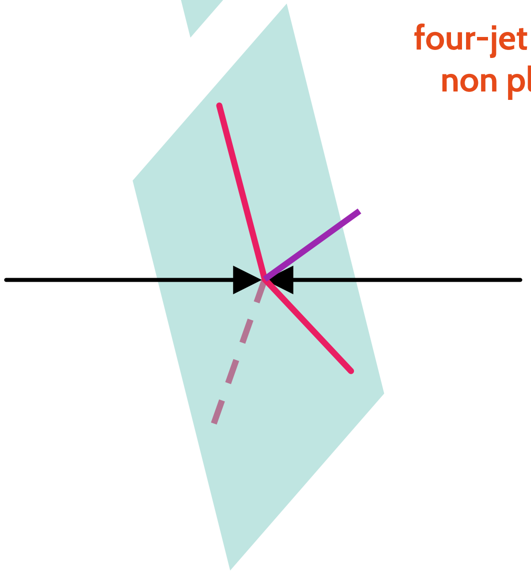
- Four-jet events deviate from the **planar configuration.**
  Several **event shape variables** have been considered: thrust-minor, aplanarity, D-parameter, light jet mass, narrow jet broadening, …;

- Four-jet event shapes are significantly harder to investigate than three-jet ones: more challenging fixed-order, resummation, hadronization modelling;

- At NNLO, the complexity is comparable to cutting-edge calculations for hadronic 2→3 processes. **Highest jet-multiplicity** considered so far at NNLO;

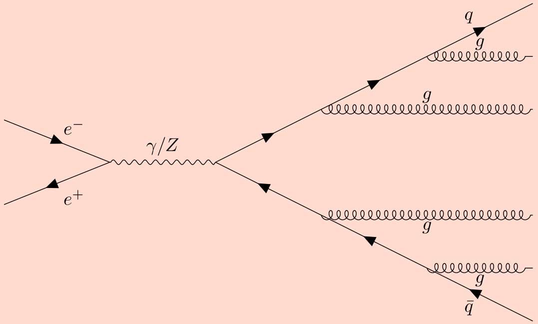
- As a first application, we compute the **four-jet rate at NNLO**;
  Results are available at https://doi.org/10.5281/zenodo.18630975.





Setup of the calculation: matrix elements

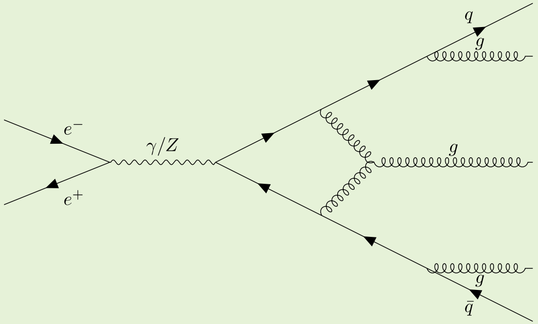
REAL-REAL



[Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller 1907.13071]

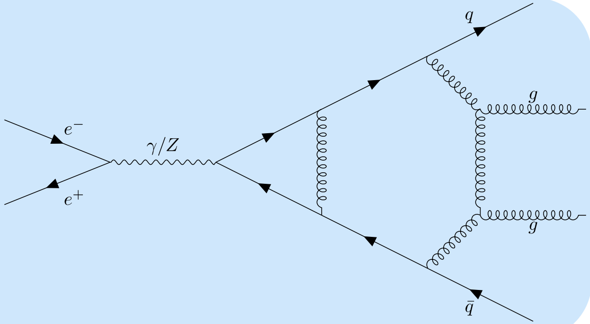
- from **OpenLoops2**
- fully decomposed into colour factors
- double-precision is enough

REAL-VIRTUAL



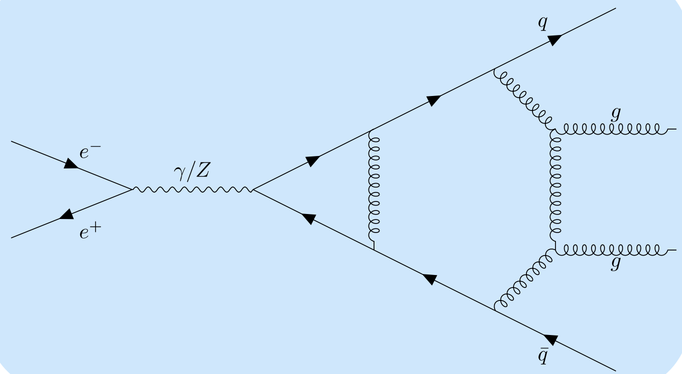
- from **OpenLoops2**
- **no colour decomposition**
- switch to built-in higher precision mode in the IR limits

VIRTUAL-VIRTUAL



- **NEW!**

Setup of the calculation: two-loop contributions



- cutting edge: **five-point** two-loop with **one external massive leg**

$$R^{\vec{a}}(\vec{p}) = \sum_{i,j} r_i(\vec{p})\ M^{\vec{a}}_{ij}\ f_j(\vec{p})$$

one-mass pentagon functions

rational coefficients

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2009.13917]
[Chicherin, Sotnikov, Zoia 2110.10111]

- crossing of the amplitudes for pp → Vjj [Abreu, Febres Corder, Ita, Klinkert, Page, Sotnikov 2110.07541] [De Laurentis, Ita, Page, Sotnikov 2503.10595]

$$\{f_j^{\text{prod}}\} \to \{f_j^{\text{decay}}\}$$

Complete set of decay pentagon functions available in the **PentagonFunctions++** public library.
Symbolic expressions are available on Zenodo at **https://doi.org/10.5281/zenodo.18630975** .

[Catani 9802439] [Sterman, Tejeda-Yeomans 0210130] [Becher, Neubert 0901.0722]

- we consider the **leading-colour approximation** for the two-loop finite remainder, defined in the **Catani scheme**

$$N_c \to \infty \qquad \text{with} \quad N_f/N_c \quad \text{kept constant}$$

- The computational cost for the integration of the two-loop contributions is negligible compared to the rest of the calculation.

Setup of the calculation: IR subtraction and numerical implementation

[Gehrmann-De Ridder, Gehrmann, Glover 0505111]

- Infrared subtraction performed with the **antenna subtraction method**:
  - colourful antenna subtraction + generalized antenna functions:
    → **simplicity and computational efficiency**

[Chen, Gehrmann, Glover, Huss, MM 2203.13531]
[Gehrmann, Glover, MM 2310.19757]
[Braun-White, Glover, Preuss 2302.12787, 2307.14999]
[Fox, Glover, MM 2410.12904]

- We integrate IR-subtracted remainders within the parton-level Monte Carlo framework ***NNLOJET***

[NNLOJET Collaboration, A. Huss *et al.* 2503.22804]

  - efficient numerical implementation, **even for challenging regions** of the phase space (three-jet limit);

- We consider:
  - **mZ** = 91.2 GeV, **α_s(mZ)** = 0.118, **N_f** = 5 massless quark flavours;
  - no top-loop contributions: expected to be negligible
  - only partial singlet contributions (no axial-singlet, …): also expected to be negligible

[Knihel, Kuhn Phys.Lett.B 224 (1989) 229-232]
[Hagiwara, Kuruma, Yamada Nucl.Phys.B 358 (1991) 80-96]
[Dixon, Signer 9609460, 9706285]
[Frederix, Frixione, Melnikov, Zanderighi 1008.5313]

- All our results are available in digital format on Zenodo at **https://doi.org/10.5281/zenodo.18630975**

Results: jet rates in electron-positron annihilation

The **n**-jet rate is given by the fraction of events with **n** resolved jets:

$$R_n(y_{\rm cut}) = \frac{\sigma_n(y_{\rm cut})}{\sigma_{\rm inc.}}$$

jet resolution parameter

exclusive n-jet cross section

inclusive hadronic cross section

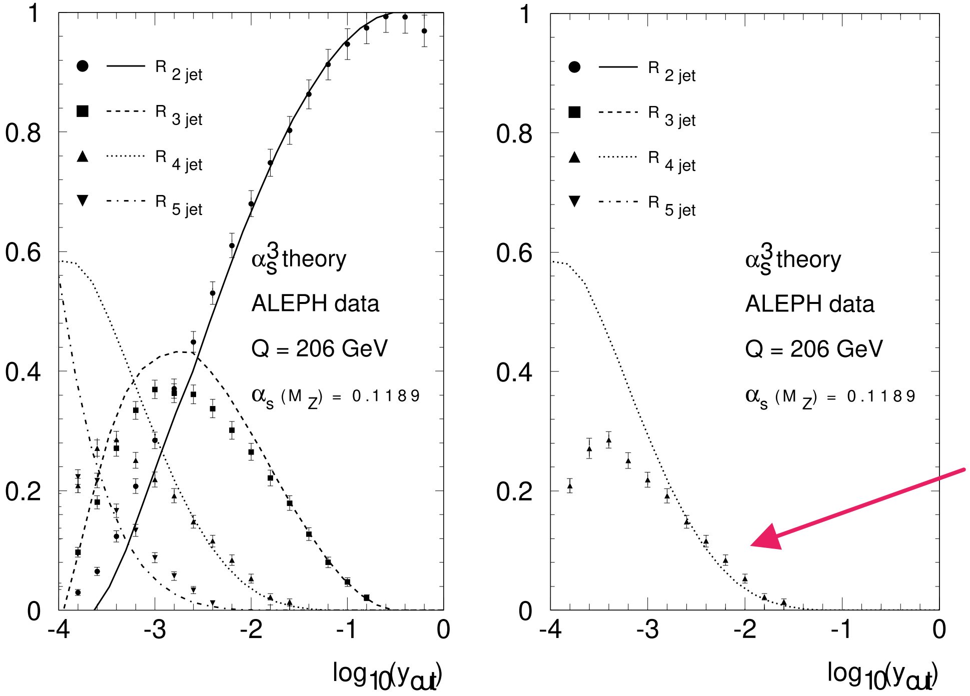
[Brown, Stirling *Phys. Lett.* B 252 (1990)]
[Catani, Dokshitzer, Olsson, Turnock, Webber *Phys. Lett.* B 269 (1991)]

We consider the Durham (k_t) jet clustering algorithm:

$$y_{ij}^D = \frac{2\min(E_i^2, E_j^2)}{Q^2}(1-\cos\theta_{ij})\,, \quad y_{ij}^D > y_{\rm cut}\ \forall\, ij$$

Jet rates at order α_s^3:

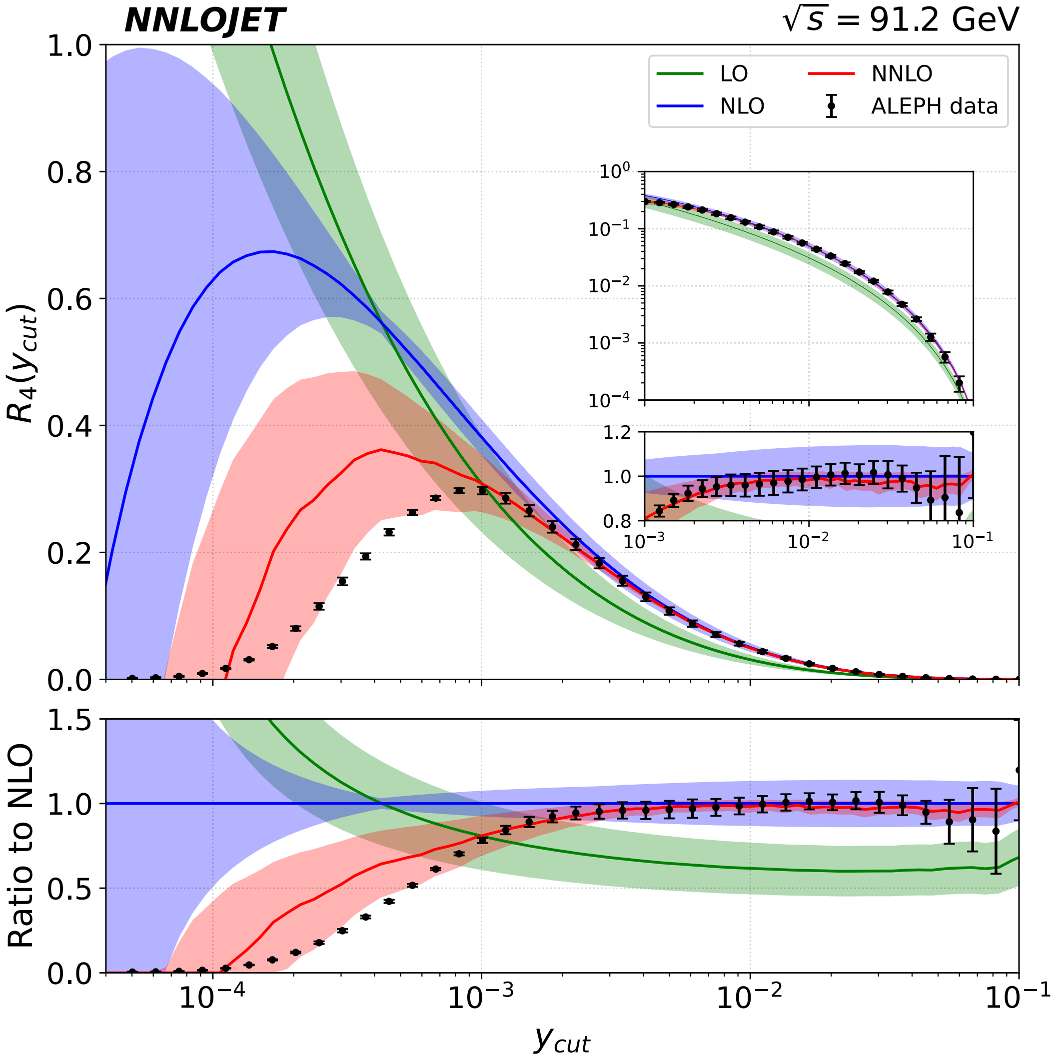
[Gehrmann-De Ridder, Gehrmann, Glover, Heinrich 0802.0813]



R_4 is only NLO-accurate

Results: the four-jet rate at NNLO

- Data from ALEPH [Heister et al. *Eur.Phys.J.C* 35 (2004) 457-486]
- We identify a **perturbative region**: **0.001 ≤ ycut ≤ 0.1**:
  - Remarkable convergence: scale uncertainties reduce from 15% at NLO to 3%-5% at NNLO (symmetrized);
  - NNLO correction amounts to a negative 2%-5%;
  - Improved agreement with data;
- For **ycut ≤ 0.001**, we observe perturbative breakdown:
  - larger scale uncertainties, non-verlapping scale bands, predictions turning negative
  - All-order resummation is needed;
  - Still NNLO reproduces logarithmic terms and the shape of the peak much better;



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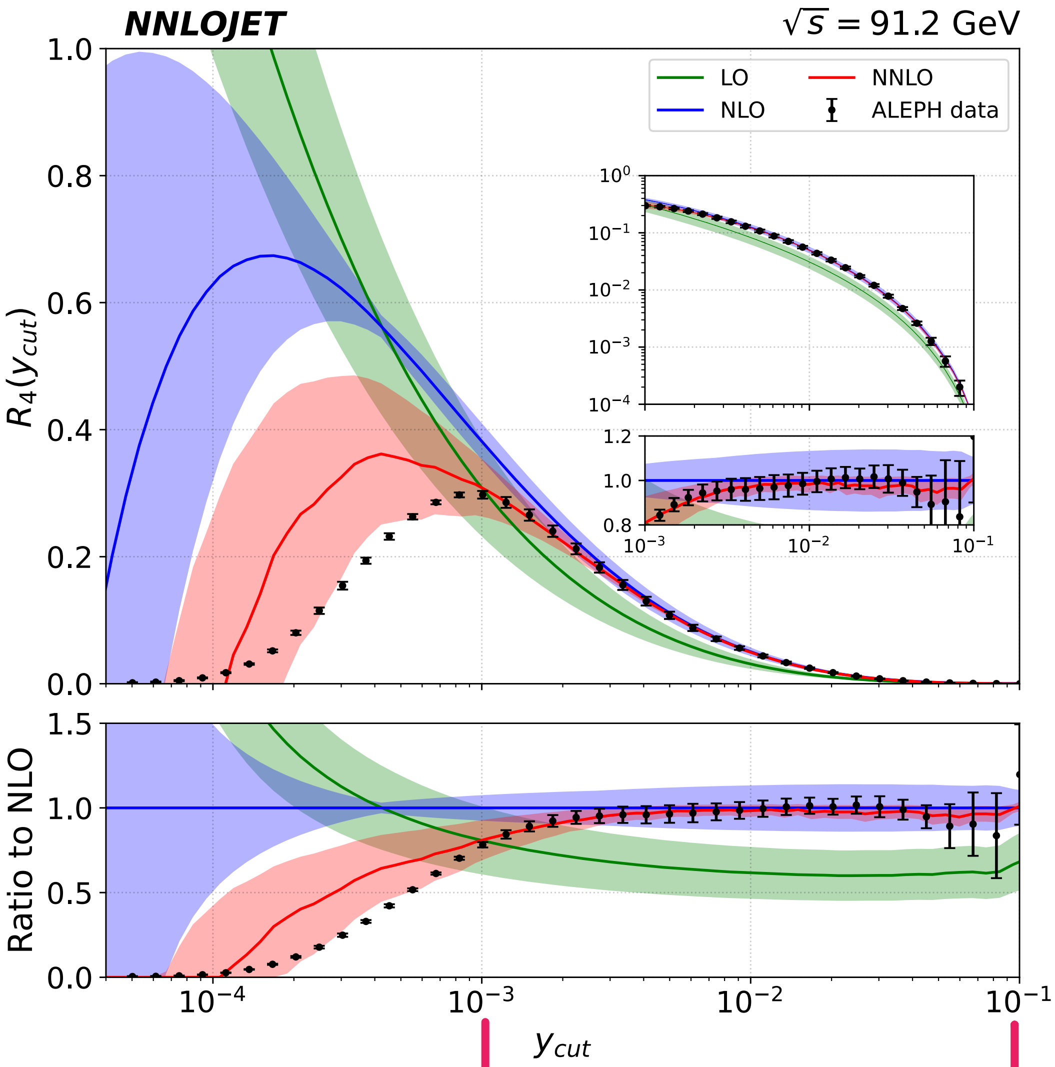

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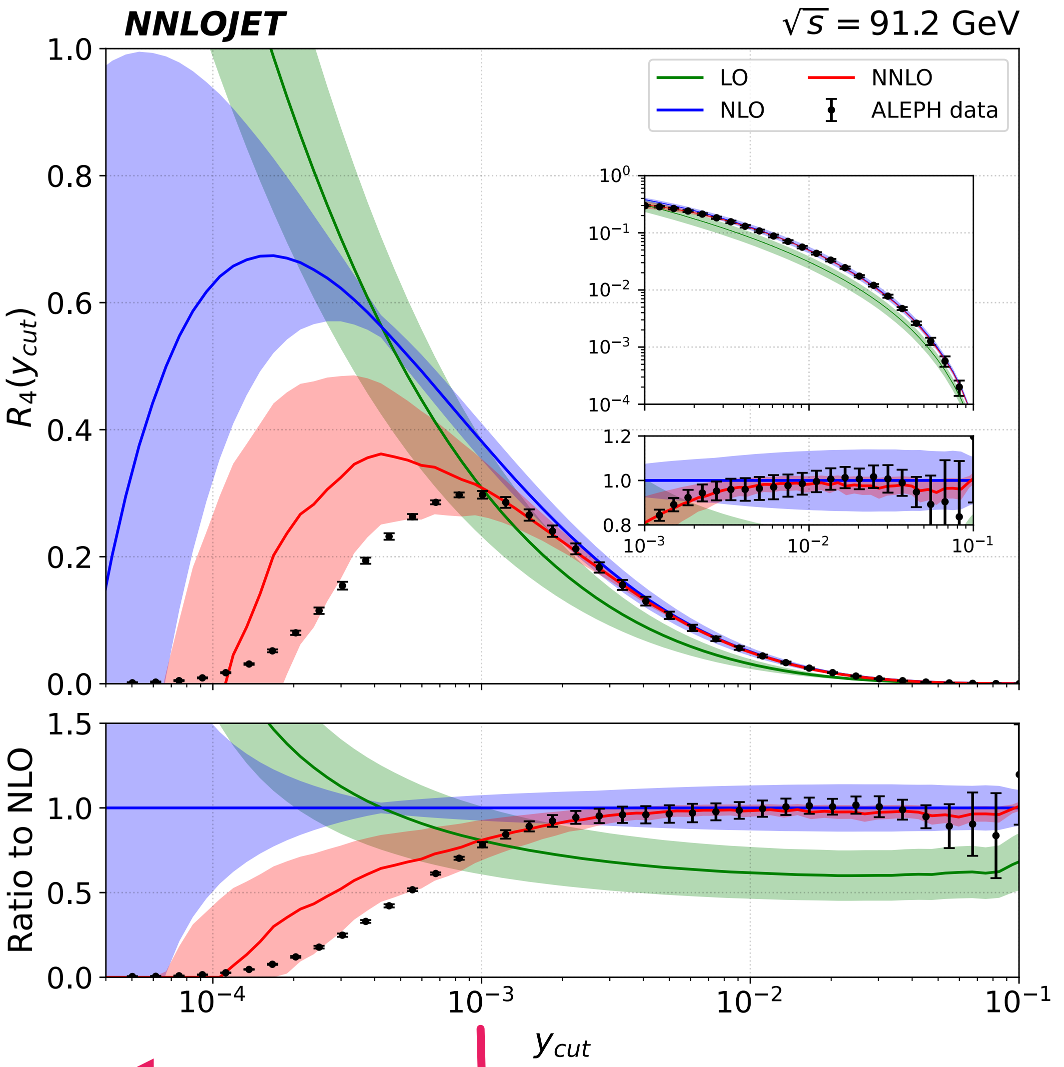
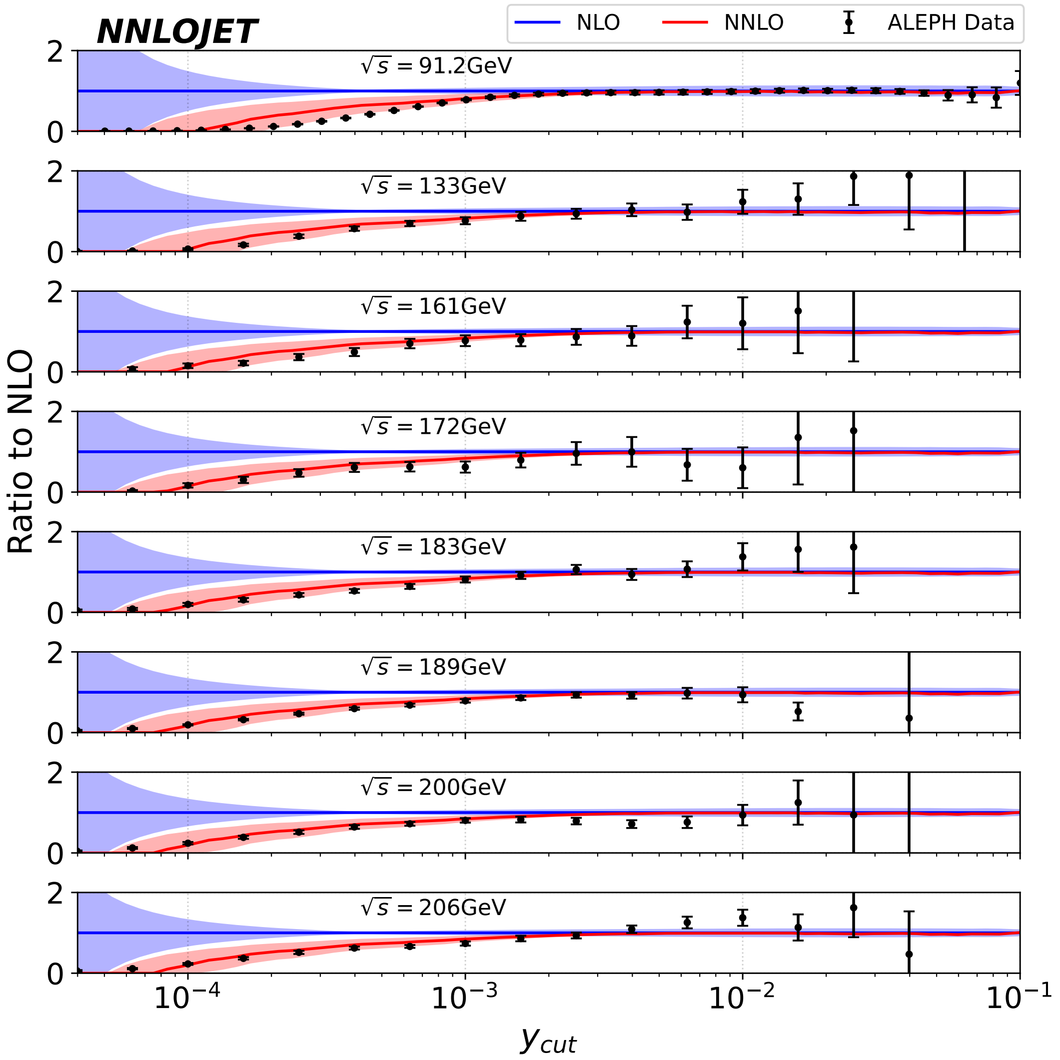


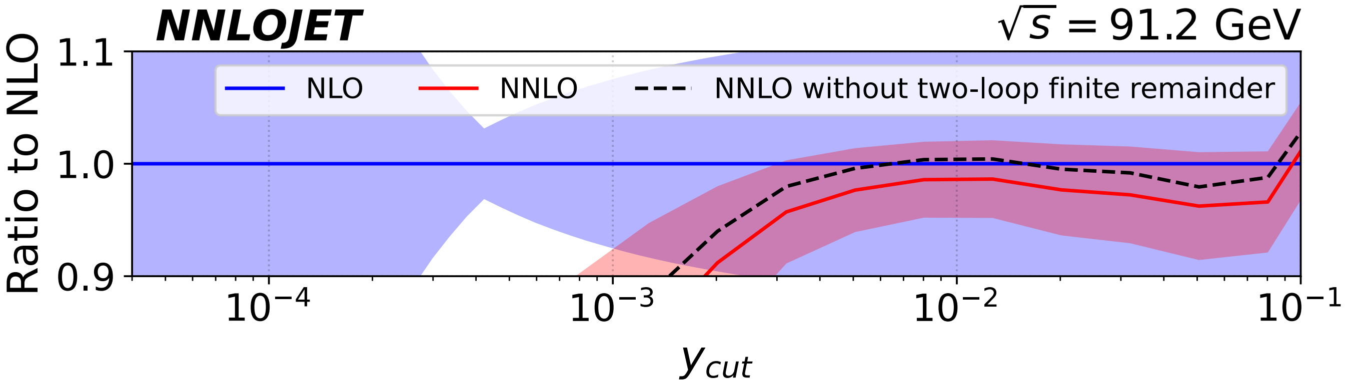
Results: energy scan

- Similar observations as at 91.2 GeV;

- You can find all plots and data at different c.o.m. energies on Zenodo at https://doi.org/10.5281/zenodo.18630975



Results: impact of the two-loop finite remainder



- In the perturbative region, the two-loop finite remainder amounts to a **negative 2%-5% of the full NNLO**;
- Here, the NNLO correction is dominated by two-loop finite-remainder.
  Though peculiar, the overall size of the finite remainder is similar to what observed in hadronic processes;
- One can estimate that the uncertainty due to **missing subleading-colour contributions** is around **1% of the full NNLO**;
- The relative size of the two-loop finite remainder is largely insensitive to the c.o.m. energy;

APPLICATIONS

$e^+e^- \rightarrow$ 2 jets @ N^3LO

Jet production at lepton colliders: $e^+e^- \to jj$ @N^{3}LO

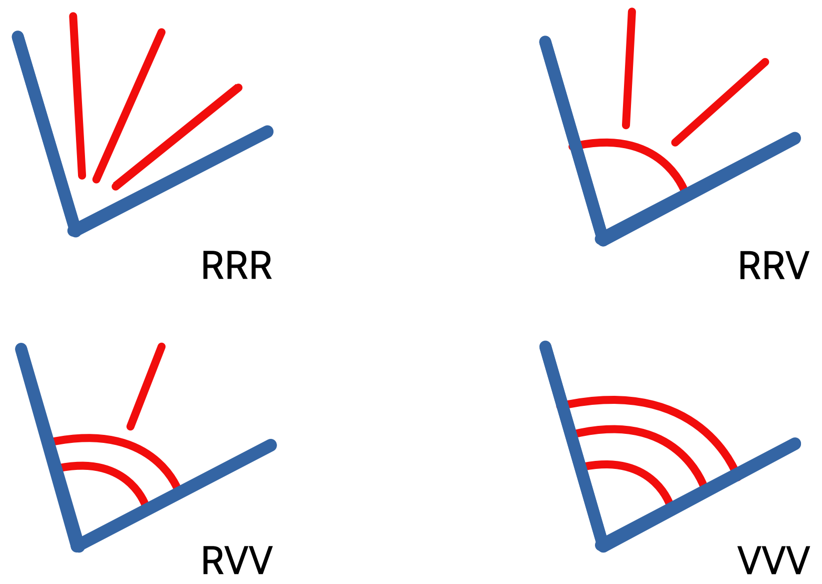
[Chen,Jakubcik,MM,Stagnitto '25]

Simple process:

- only **q-$\bar{q}$** N^{3}LO antenna functions;
- only **dipole-like correlations** at N^{3}LO;

Goals:

- definition of **N^{3}LO antenna functions**;
- exploration of **numerical challenges** (infrared stability of loop amplitudes);
- preparation of **computational framework** for more complicated processes;



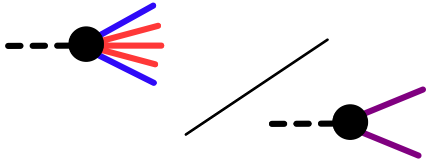
The calculation builds on top of $e^+e^- \to jjj$ @NNLO in ***NNLOJET***

[Gehrmann,Glover,Huss, Nieuhes,Zhang '17]

Final-state N³LO antenna functions (traditional ME-based ones)

	Antenna	Integrated antenna

RRR:

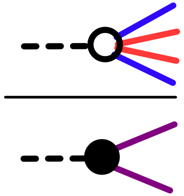


5-parton tree antenna (N3LO, triple-real)

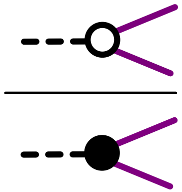
$$X_5^0 = \frac{M_5^0}{M_2^0}$$

$$\mathcal{X}_5^0 \propto \int \mathrm{d}\Phi_5 \, X_5^0$$

RRV:



$-X_4^0 \cdot$

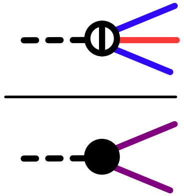


4-parton 1-loop antenna (N3LO, double-real virtual)

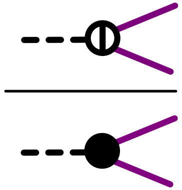
$$X_4^1 = \frac{M_4^1}{M_2^0} - X_4^0 \frac{M_2^1}{M_2^0}$$

$$\mathcal{X}_4^1 \propto \int \mathrm{d}\Phi_4 \, X_4^1$$

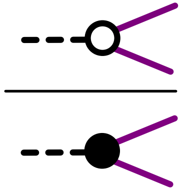
RVV:



$-X_3^0 \cdot$



$-X_3^1 \cdot$



3-parton 2-loop antenna (N3LO, real double-virtual)

$$X_3^2 = \frac{M_3^2}{M_2^0} - X_3^0 \frac{M_2^2}{M_2^0} - X_3^1 \frac{M_2^1}{M_2^0}$$

$$\mathcal{X}_3^2 = \int \mathrm{d}\Phi_3 X_3^2$$

Analytic integration of final-state N^{3}LO antenna functions

Integration of **renormalized matrix elements** for colour-singlet decay over the **fully inclusive phase space**:

$$\int \mathrm{d}\Phi_5 M_5^0, \quad \int \mathrm{d}\Phi_4 M_4^1, \quad \int \mathrm{d}\Phi_3 M_3^2, \quad \int \mathrm{d}\Phi_2 M_2^3$$

[Jakubcik,MM,Stagnitto '22]
[Chen,Jakubcik,MM,Stagnitto '23]

Two-parton three-loop, for validation:

$$\int \mathrm{d}\Phi_5 M_5^0 + \int \mathrm{d}\Phi_4 M_4^1 + \int \mathrm{d}\Phi_3 M_3^2 + \int \mathrm{d}\Phi_2 M_2^3 =$$ finite N^{3}LO inclusive XS

Master integrals from

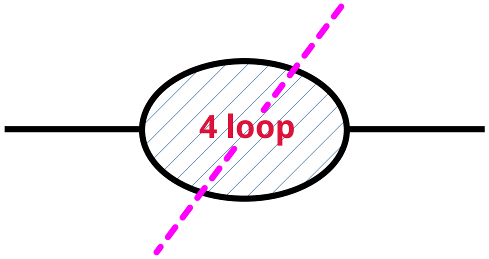
[Gituliar,Magerya,Pikelner '18]
[Magerya,Pikelner '19]

Reverse unitarity:

$$2\pi i \delta^+(p^2) \rightarrow \frac{1}{p^2 - i0} - \frac{1}{p^2 + i0}$$

[Cutkosky '60]
[Anastasiou, Melnikov '02,'03]

- Phase space and (genuine) loop integrals addressed simultaneously;
- **Systematic treatment of all four layers** within a common framework;



Local subtraction at N³LO

Subtraction at N³LO:

$$d\sigma_{N^3LO} = \int_n [d\sigma^{VVV} - d\sigma^{VVV}_{\text{sub.}}] + \int_n [d\sigma^{RVV} - d\sigma^{RRV}_{\text{sub.}}] + \int_{n+1} [d\sigma^{RRV} - d\sigma^{RRV}_{\text{sub.}}] + \int_{n+2} [d\sigma^{RRR} - d\sigma^{RRR}_{\text{sub.}}]$$

triple-virtual subtraction term

double-virtual real subtraction term

double-real-virtual subtraction term

triple-real subtraction term

with:

$$d\sigma^{RRR}_{\text{sub.}} = d\sigma^{RRR,1}_{\text{sub.}} + d\sigma^{RRR,2}_{\text{sub.}} + d\sigma^{RRR,3}_{\text{sub.}}$$

$$d\sigma^{RRV}_{\text{sub.}} = d\sigma^{RRV,1}_{\text{sub.}} + d\sigma^{RRV,2}_{\text{sub.}} - \int_1 d\sigma^{RRR,1}_{\text{sub.}}$$

$$d\sigma^{RVV}_{\text{sub.}} = d\sigma^{RVV,1}_{\text{sub.}} - \int_1 d\sigma^{RRV,1}_{\text{sub.}} - \int_2 d\sigma^{RRR,2}_{\text{sub.}}$$

$$d\sigma^{VVV}_{\text{sub.}} = -\int_1 d\sigma^{RVV,1}_{\text{sub.}} - \int_2 d\sigma^{RRV,2}_{\text{sub.}} - \int_3 d\sigma^{RRR,3}_{\text{sub.}}$$

Challenge: numerical stability of

one-loop double-unresolved quantities

Rescue-system to trigger **quadruple precision**

two-loop single-unresolved quantities

Rescue-system to trigger **Taylor expansions** of special functions

[Moch,Uwer,Weinzierl '02]

PolyLogTools [Duhr,Dulat '19]

Results

Basic checks: inclusive cross section

N^{3}LO coefficient

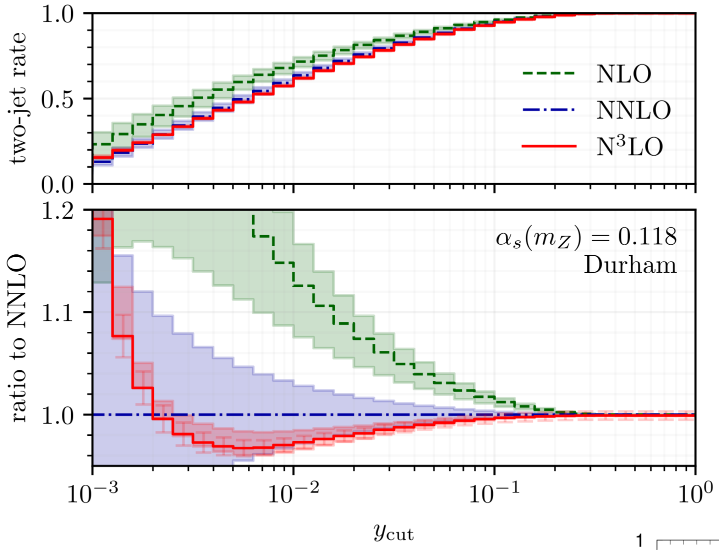
$$\sigma^{(3)} = \sigma_0 \left(\frac{\alpha_s}{2\pi}\right)^3 (-105 \pm 11)$$

Monte Carlo error.
Not so small for inclusive quantities due to large cancellations: not the *most clever* way to compute inclusive XS.

$$\sigma^{(3)}_{\text{exact}} = \sigma_0 \left(\frac{\alpha_s}{2\pi}\right)^3 (-102.14\ldots)$$

[Chetyrkin,Kühn,Kwiatkowski '95]

Two-jet rate at order O(α_s^3) (direct calculation)



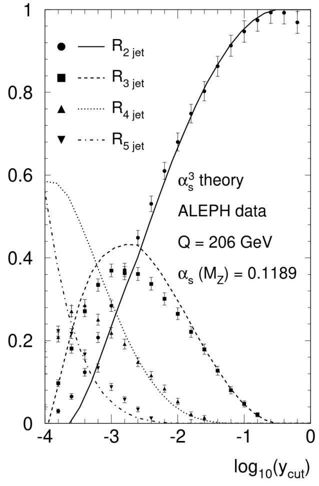
No need of previous knowledge of N^{3}LO quantities. Fiducial cuts can be implemented too

Full agreement with indirect calculation of

[Gehrmann-De Ridder, Gehrmann,Glover,Heinrich '08]

$$R_n^{(3)}(y_{\text{cut}}) = \frac{\Gamma^{(3)}_{\gamma^* \to \text{n jets}}(y_{\text{cut}})}{\Gamma^{(3)}_{\gamma^* \to \text{hadrons}}}$$

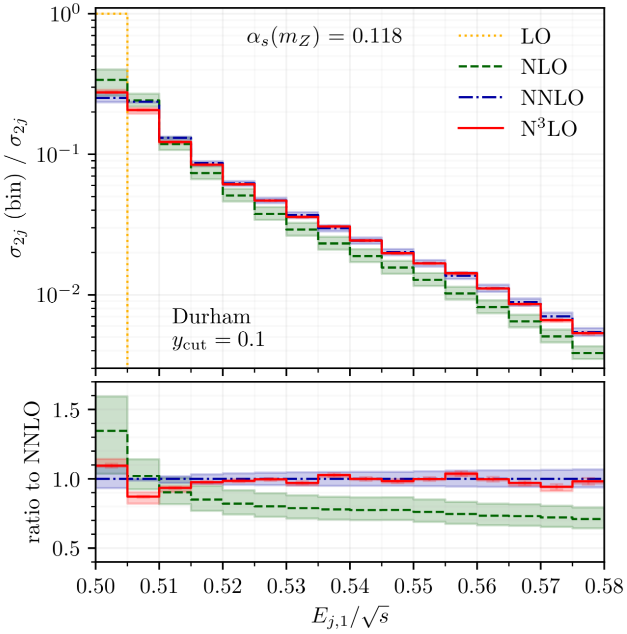
$$R_2^{(3)}(y_{\text{cut}}) = 1 - \sum_{n=3}^{5} R_n^{(3)}(y_{\text{cut}})$$

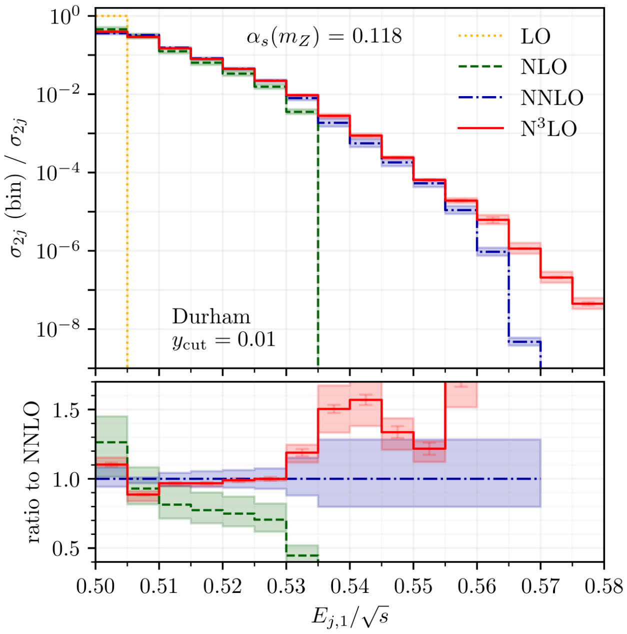


Results

Leading-jet energy:

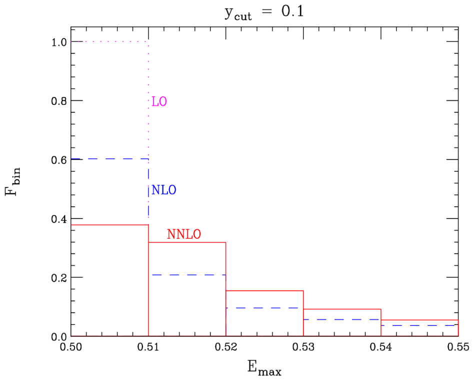
- defined on 2-jet events, bin-integrated;
- lower orders vanish quickly at large E_{j1} for smaller y_{cut} because energetic jets recoil against multiple emissions which are more likely to be clustered as three or more jets;
- again, the whole distribution can be obtained by combining $e^+e^-\to jjj$ @NNLO with the N^3LO inclusive XS. It's a proof-of-principle application.





Full agreement up to NNLO with

[Anastasiou,Melnikov,Petriello '04]



Outlook

$e^+e^- \to jj$ @N3LO: [Work in progress]

- Compute Forward-Backward asymmetry at N^3LO w.r.t. flavoured jet axis
- Study impact of different flavour jet algorithms with up to three emissions

$e^+e^- \to jjj$ @N3LO: [Work in progress]

- Goal: first local N^3LO calculation for a process with non-trivial Born kinematics
- A lot to do:
  - understand/polish N^3LO unresolved factors
  - build designer antenna functions (with 2 and 3 hard radiators)
  - integrate them
  - ensure stability of MEs (potentially very hard)

SUMMARY

NNLO may be solved, but still not so accessible. Several ongoing efforts towards automation and generalisation. N^3LO is very challenging, but many steps have been made in this direction already.

The **colourful antenna subtraction** method provides a systematic way to assemble subtraction terms, but the complexity is ultimately due to the fundamental building blocks.

Generalized antenna functions yield a simpler and more efficient formulation of final-state infrared subtraction, resulting in simpler structures and computational speed-up.

$e^+e^-\to jjjj$ @NNLO: state-of-the-art result obtained combining recent developments in antenna subtraction. Significant improvement of the predictions for the four-jet rate.

$e^+e^-\to jj$ @N3LO: first application of antenna subtraction to a fully-differential N^3LO calculation. Gradual extension to more complicated processes rely on the generalised antenna framework.

Thank you very much for your attention!

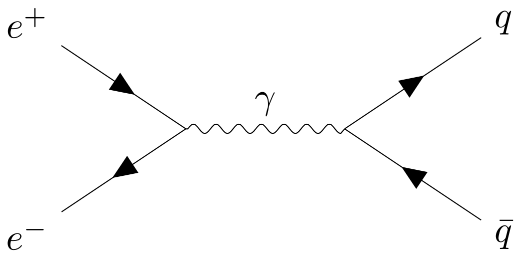
BACKUP SLIDES

A simple example: $e^+e^-\to$hadrons

$\Phi_n \equiv$ n-particle phase space

$M_n^\ell \equiv$ n-parton ℓ-loop matrix element

LO:



$$\frac{1}{2s}\int \Phi_2\, M_2^0 = \sigma_0$$

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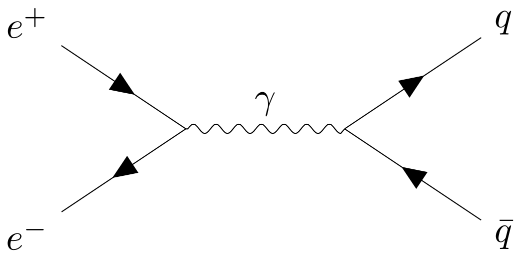
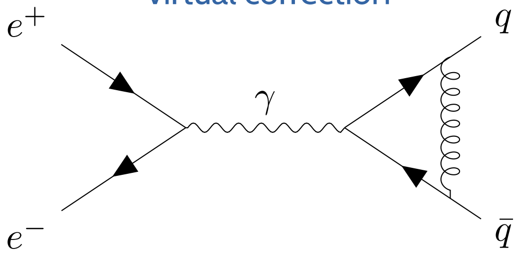


$$\frac{1}{2s}\int \Phi_2\, M_2^0 = \sigma_0$$

NLO:

(renormalized) virtual correction



dim. reg. $d = 4 - 2\epsilon$

$$\frac{1}{2s}\int \Phi_2\, M_2^1 = \frac{\alpha_s C_F}{\pi}\sigma_0\left(-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} + \text{finite terms}\right)$$

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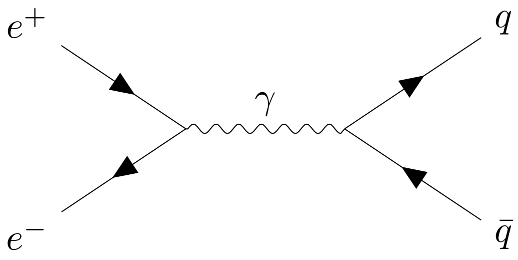

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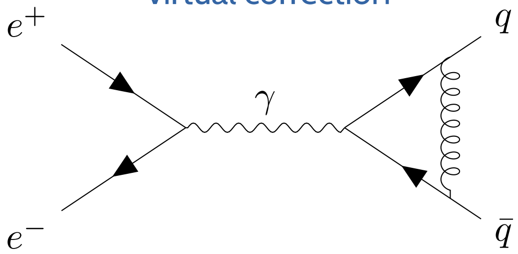

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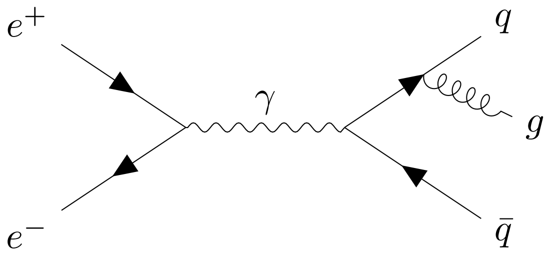
$$\frac{1}{2s}\int \Phi_2\, M_2^1 = \frac{\alpha_s C_F}{\pi}\sigma_0\left(-\frac{1}{\epsilon^2}-\frac{3}{2\epsilon}+\text{finite terms}\right)$$

real correction



$$\frac{1}{2s}\int \Phi_3\, M_3^0 = \frac{\alpha_s C_F}{\pi}\sigma_0\int_0^1 \frac{\mathrm{d}y}{y^{1+\epsilon}}\,\frac{\mathrm{d}z}{z^{1+\epsilon}}(1-y)^{-2\epsilon}(1-z)^{-\epsilon}\left[(1-y)(1-z+z^2)-\tfrac{1}{2}(z^2+y^2+y^2z^2)\right]$$

$$= \frac{\alpha_s C_F}{\pi}\sigma_0\left(+\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+\text{finite terms}\right)$$

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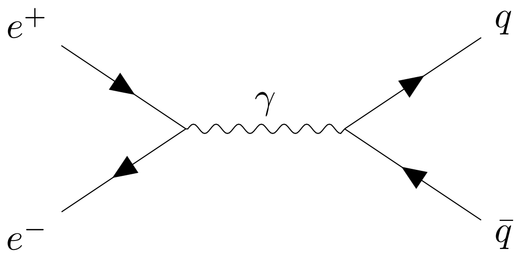
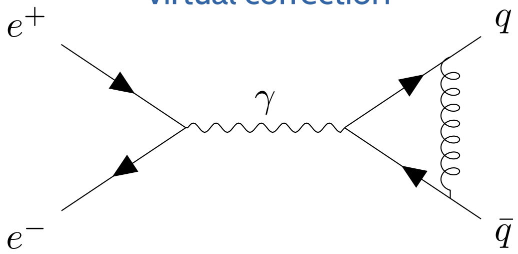


$$\frac{1}{2s}\int \Phi_2\, M_2^0 = \sigma_0$$

infrared singularities

NLO:

(renormalized) virtual correction



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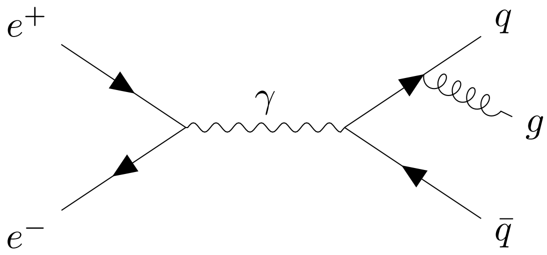
$$\frac{1}{2s}\int \Phi_2\, M_2^1 = \frac{\alpha_s C_F}{\pi}\sigma_0 \left(-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} + \text{finite terms} \right)$$

the sum is finite!

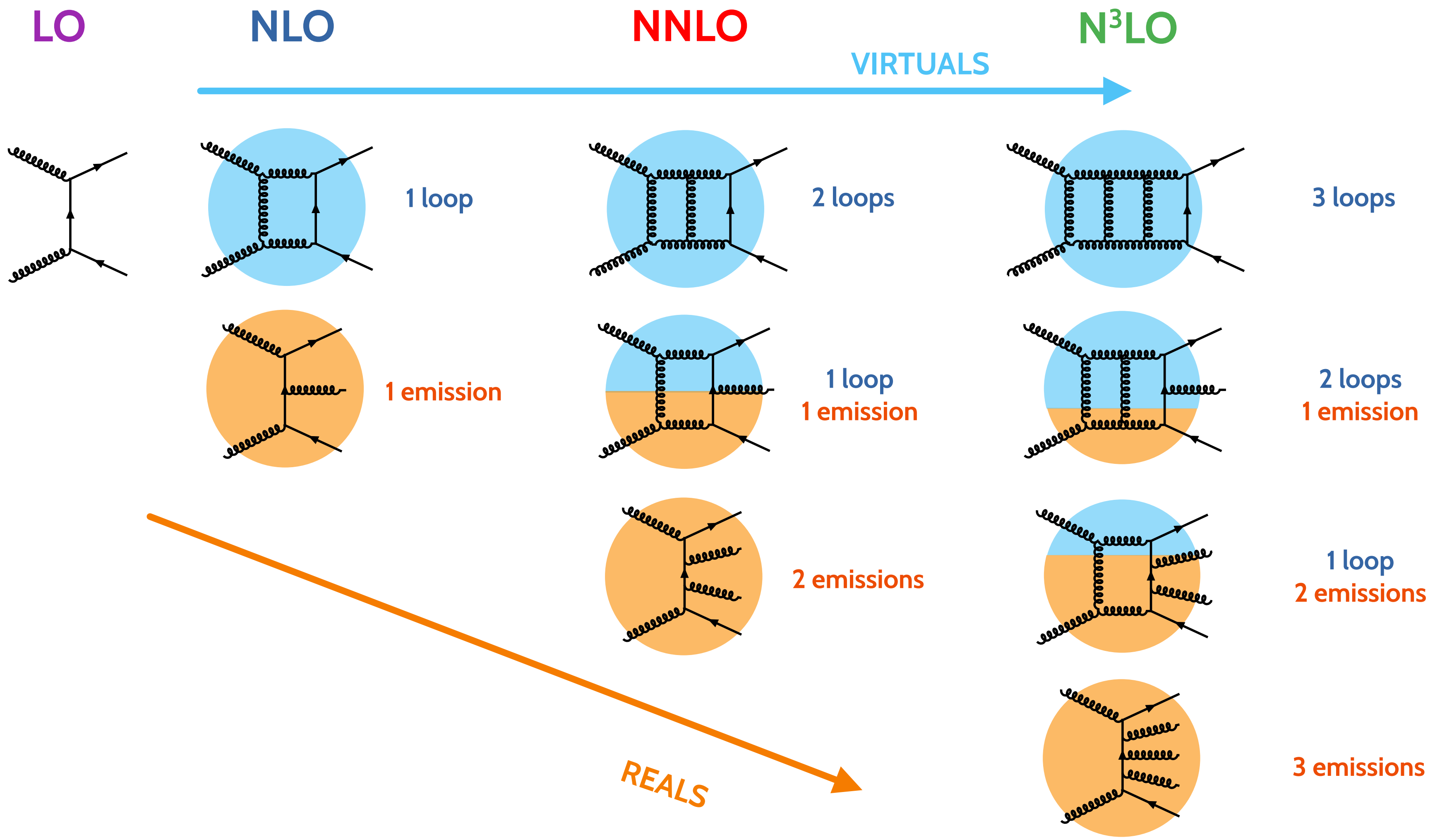
real correction



$$\frac{1}{2s}\int \Phi_3\, M_3^0 = \frac{\alpha_s C_F}{\pi}\sigma_0 \int_0^1 \frac{\mathrm{d}y}{y^{1+\epsilon}} \frac{\mathrm{d}z}{z^{1+\epsilon}} (1-y)^{-2\epsilon}(1-z)^{-\epsilon} \left[(1-y)(1-z+z^2) - \tfrac{1}{2}(z^2+y^2+y^2z^2)\right]$$

$$= \frac{\alpha_s C_F}{\pi}\sigma_0 \left(+\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \text{finite terms} \right)$$

Are all higher-order corrections infrared-finite?



Calculations with local subtraction: a recipe

unresolved factors
(soft currents, splitting functions)

Interpolate between limits:
CS dipoles, antenna functions, ...

Phase-space **sectors**:
FKS, STRIPPER,
nested soft-collinear, LASS, ...

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unresolved emissions

Analytical integration:
conceptually harder,
more efficient/stable (?)

Numerical integration (STRIPPER):
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- Combine unresolved factors and lower-order matrix elements
- phase-space **mappings**
- prevent **double-counting** of IR limits

Can be highly non-trivial for complicated processes!

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matrix elements

Careful about:

- **stability** (analytical control?)
- evaluation **time** per PS point

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Can be highly non-trivial for complicated processes!

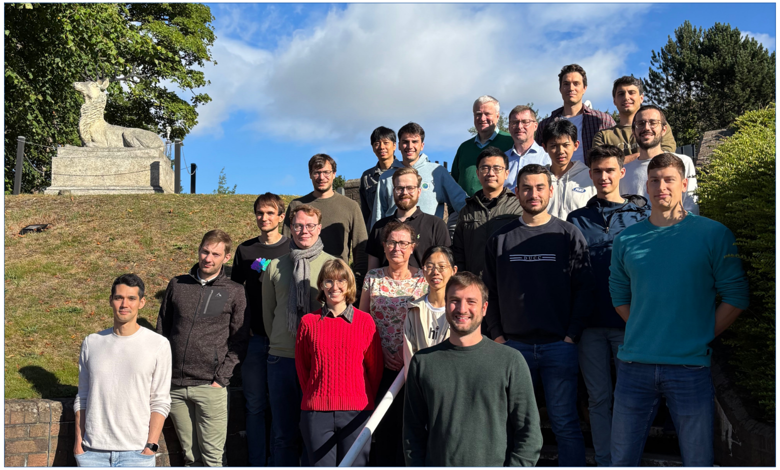
numerical integrator

Careful about:

- PS **sampling**
- MC **convergence/efficiency**
- process and observables implementation
- ...

NNLOJET

A public **parton-level event generator** implementing the **antenna subtraction** method at **NNLO**



[NNLOJET meeting 2025, Durham]

```
$ NNLOJET --listprocs
Available processes up to NNLO accuracy for hadron-hadron colliders:
*     Z      : production of a Z-boson;
*     ZJ     : production of a Z-boson plus a jet;
*     WP     : production of a positively charged W-boson;
*     WPJ    : production of a positively charged W-boson plus a jet;
*     WM     : production of a negatively charged W-boson;
*     WMJ    : production of a negatively charged W-boson plus a jet;
*     H      : production of a H-boson;
*     HJ     : production of a H-boson plus a jet;
*     H2     : production of a H-boson with decay to two colour singlets, in particular:
*       - HTO2P      : decay to two photons;
*     H2J    : production of a H-boson plus a jet with decay to two colour singlets, in particular:
*       - HTO2PJ     : decay to two photons;
*     H3     : production of a H-boson with decay to three colour singlets, in particular:
*       - HTO2L1P    : Dalitz decay to (l+l-) and photon;
*     H3J    : production of a H-boson plus a jet with decay to three colour singlets, in particular:
*       - HTO2L1PJ   : Dalitz decay to (l+l-) and photon;
*     H4     : production of a H-boson with decay to four colour singlets, in particular:
*       - HTO4E      : decay to (l+l-) (l+l-);
*       - HTO2E2MU   : decay to (l1+l1) (l2+l2-);
*       - HTO2L2N    : decay to (l1-nu1b) (l2+nu2);
*     H4J     : production of a H-boson plus a jet with decay to four colour singlets, in particular:
*       - HTO4EJ     : decay to (l+l-) (l+l-);
*       - HTO2E2MUJ  : decay to (l1+l1) (l2+l2-);
*       - HTO2L2NJ   : decay to (l1-nu1b) (l2+nu2);
*     GJ     : production of a photon plus a jet;
*     GG     : production of two photons;
*     JJ     : production of two jets;
Available processes up to NNLO accuracy for electron-positron colliders:
*     eeJJ   : production of two jets;
*     eeJJJ  : production of three jets;
Available processes up to NNLO accuracy for lepton-hadron colliders:
*     epLJ   : production of a jet through neutral-current exchange;
*     epLJJ  : production of two jets through neutral-current exchange;
Available processes up to NNLO accuracy for electron-hadron colliders:
*     epNJ   : production of a jet through charged-current exchange;
*     epNJJ  : production of two jets through charged-current exchange;
Available processes up to NNLO accuracy for positron-hadron colliders:
*     epNbJ  : production of a jet through charged-current exchange;
*     epNbJJ : production of two jets through charged-current exchange;
```

More processes and features to be added in the future.

NNLOJET: a parton-level event generator for jet cross sections at NNLO QCD accuracy

NNLOJET Collaboration

A. Huss[1,*], L. Bonino[2], O. Braun-White[3], S. Caletti[4], X. Chen[5], J. Cruz–Martinez[1], J. Currie[3], W. Feng[2], G. Fontana[2], E. Fox[3], R. Gauld[6], A. Gehrmann–De Ridder[2,4], T. Gehrmann[2], E.W.N. Glover[3], M. Höfer[7], P. Jakubčík[2], M. Jaquier[8], M. Löchner[2], F. Lorkowski[2], I. Majer[4], M. Marcoli[3], P. Meinzinger[2], J. Mo[2], T. Morgan[3], J. Niehues[3,9], J. Pires[10,11], C. T. Preuss[12,13], A. Rodriguez Garcia[4], K. Schönwald[2], R. Schürmann[2], V. Sotnikov[2], G. Stagnitto[14], D. Walker[3], S. Wells[3], J. Whitehead[15], T.Z. Yang[2] and H. Zhang[16]

1 Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland
2 Physik-Institut, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland
3 Institute for Particle Physics Phenomenology, Department of Physics, University of Durham, Durham, DH1 3LE, UK

28 Mar 2025

[NNLOJET collaboration: Huss *et al.* '25]

Infrared singularities of virtual amplitudes in colour space

[Catani '98] [Bern,De Freitas,Dixon '03]
[Gardi,Magnea '09] [Becher,Neubert '09]

IR singularity structure at **one-loop**:

$$|A^1_{n+2}\rangle = \boldsymbol{I}^{(1)}|A^0_{n+2}\rangle + \text{finite terms}$$

one-loop — tree-level

$$\boldsymbol{I}^{(1)} = \sum_{(i,j)} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\, \mathcal{I}^{(1)}_{ij}(\epsilon)$$

Colour-charge dipoles

Colour-stripped singular functions

IR singularity structure at **two-loop**:

one-loop — tree-level

$$|A^2_{n+2}\rangle = \boldsymbol{I}^{(1)}|A^1_{n+2}\rangle + \boldsymbol{I}^{(2)}|A^0_{n+2}\rangle + \text{finite terms}$$

two-loop

$$\boldsymbol{I}^{(2)}(\epsilon,\mu_r^2) = -\frac{1}{2}\sum_{(i,j)}\sum_{(k,l)} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)(\boldsymbol{T}_k \cdot \boldsymbol{T}_l)\, \mathcal{I}^{(1)}_{ij}(\epsilon)\mathcal{I}^{(1)}_{kl}(\epsilon) - \frac{\beta_0}{\epsilon}\sum_{(i,j)} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\, \mathcal{I}^{(1)}_{ij}(\epsilon) + \sum_{(i,j)} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\, \mathcal{I}^{(2)}_{ij}(\epsilon)$$

$\boldsymbol{T}_i$ SU(3) generator in the representation of parton *i*

$$\mathcal{I}^{(2)}_{ij}(\epsilon,\mu_r^2) = e^{-\epsilon\gamma_E}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon} + K\right)\mathcal{I}^{(1)}_{ij}(2\epsilon) - \mathcal{H}^{(2)}_{ij}(\epsilon)$$

Colourful antenna subtraction

Complexity

Idea: rewrite the singularity structure of loop matrix elements to extract information about real-emission counterterms

see also [Magnea,Maina,Pelliccioli,Signorile-Signorile,Torrielli,Uccirati '18]

Traditional approach

- VV
- RV
- RR

"closure" of the subtraction

analytical integration

Proliferation of IR limits at high-multiplicities

New approach

Predictable in full generality

- VV
- RV
- RR

"unintegration" or insertion of unresolved partons

"closure" of the subtraction

Integrated dipoles with antenna functions

$$\boldsymbol{\mathcal{J}}^{(1)} = \sum_{(i,j)} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\, \mathcal{J}_2^{(1)}(i,j) \qquad \boldsymbol{\mathcal{J}}^{(2)} = \sum_{(i,j)} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\, \mathcal{J}_2^{(2)}(i,j)$$

one and two-loop dipole operators

$$J_2^{(1)} = c_{\mathcal{X}_3^0} \mathcal{X}_3^0 + c_{\Gamma^{(1)}} \Gamma^{(1)} \qquad Poles\left[\mathcal{J}_2^{(1)}(i,j)\right] = Poles\left[\mathrm{Re}\left(\mathcal{I}_{ij}^{(1)}(\epsilon)\right)\right]$$

$$J_2^{(2)} = c_{\mathcal{X}_4^0} \mathcal{X}_4^0 + c_{\mathcal{X}_3^1} \mathcal{X}_3^1 + c_{\mathcal{X}_3^0 \mathcal{X}_3^0} \mathcal{X}_3^0 \mathcal{X}_3^0 + c_{\beta_0} \frac{\beta_0}{\epsilon} \mathcal{X}_3^0 \left(\frac{|s|}{\mu_r^2}\right)^{-\epsilon} + c_{\Gamma^{(2)}} \overline{\Gamma}^{(2)}$$

$$Poles\left[N_c \mathcal{J}_2^{(2)}(i,j) - \frac{\beta_0}{\epsilon} \mathcal{J}_2^{(1)}(i,j)\right] = Poles\left[\mathrm{Re}\left(\mathcal{I}_{ij}^{(2)}(\epsilon) - \frac{\beta_0}{\epsilon} \mathcal{I}_{ij}^{(1)}(\epsilon)\right)\right]$$

Written in terms of **integrated antenna functions** and mass-factorisation kernels (IS radiation)

$$Poles\left(\mathrm{d}\hat{\sigma}^{V}\right) = \mathcal{N}_V \int \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \mathrm{d}\Phi_n\, J_n^n(\Phi_n)\, Poles\left[2 \langle A_{n+2}^0 | \boldsymbol{\mathcal{J}}^{(1)} | A_{n+2}^0 \rangle\right]$$

$$Poles\left(\mathrm{d}\hat{\sigma}^{VV}\right) = \mathcal{N}_{VV} \int \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \mathrm{d}\Phi_n J_n^n(\Phi_n) Poles \Bigg\{ \times 2\Big[\langle A_{n+2}^0 | \boldsymbol{\mathcal{J}}^{(1)} | A_{n+2}^1 \rangle + \langle A_{n+2}^1 | \boldsymbol{\mathcal{J}}^{(1)} | A_{n+2}^0 \rangle - \frac{b_0 N_c}{\epsilon} \langle A_{n+2}^0 | \boldsymbol{\mathcal{J}}^{(1)} | A_{n+2}^0 \rangle - \langle A_{n+2}^0 | \boldsymbol{\mathcal{J}}^{(1)} \otimes \boldsymbol{\mathcal{J}}^{(1)} | A_{n+2}^0 \rangle + \langle A_{n+2}^0 | \boldsymbol{\mathcal{J}}^{(2)} | A_{n+2}^0 \rangle\Big] \Bigg\}$$

Infrared structure of one- and two-loop MEs in terms of integrated antenna functions

[Chen, Gehrmann, Glover, Huss, MM '22]
[Gehrmann, Glover, MM '23]

Subtraction at NLO

[Chen, Gehrmann, Glover, Huss, MM '22] [Gehrmann, Glover, MM '23]

- Use integrated dipoles to construct the virtual subtraction term

$$\mathrm{d}\sigma^{V}_{\mathrm{sub.}} = \mathcal{N}_V \int \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_n\, J^n_n(\Phi_n)\, \left[2\, \langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}|A^0_{n+2}\rangle\right]$$

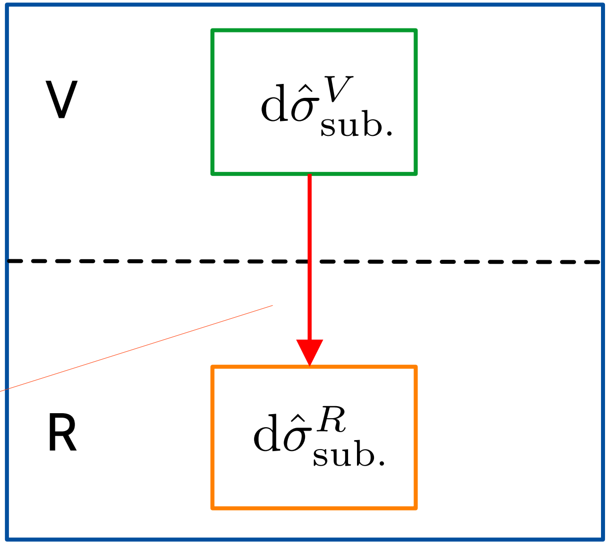
Guaranteed to remove the virtual poles

- Infer the real subtraction term through the **insertion of an unresolved parton**

$$\mathrm{d}\sigma^{R}_{\mathrm{sub.}} = -\mathcal{I}ns\left[\mathrm{d}\sigma^{V}_{\mathrm{sub.}}\right]$$

Inverse operation with respect to:

$$\mathrm{d}\sigma^{V}_{\mathrm{sub.}} = -\int_1 \mathrm{d}\sigma^{R}_{\mathrm{sub.}}$$



Convert integrated antenna functions to unintegrated ones exploiting on-to-one correspondence

virtuals: (n+2)-particle PS reals: (n+3)-particle PS

$$\mathcal{X}^0_3(s_{ij})\, A^0_{n+2}(.,i,.,j,.) \leftrightarrow X^0_3(i,u,j)A^0_{n+2}(.,\widetilde{iu},.,\widetilde{uj},.)$$

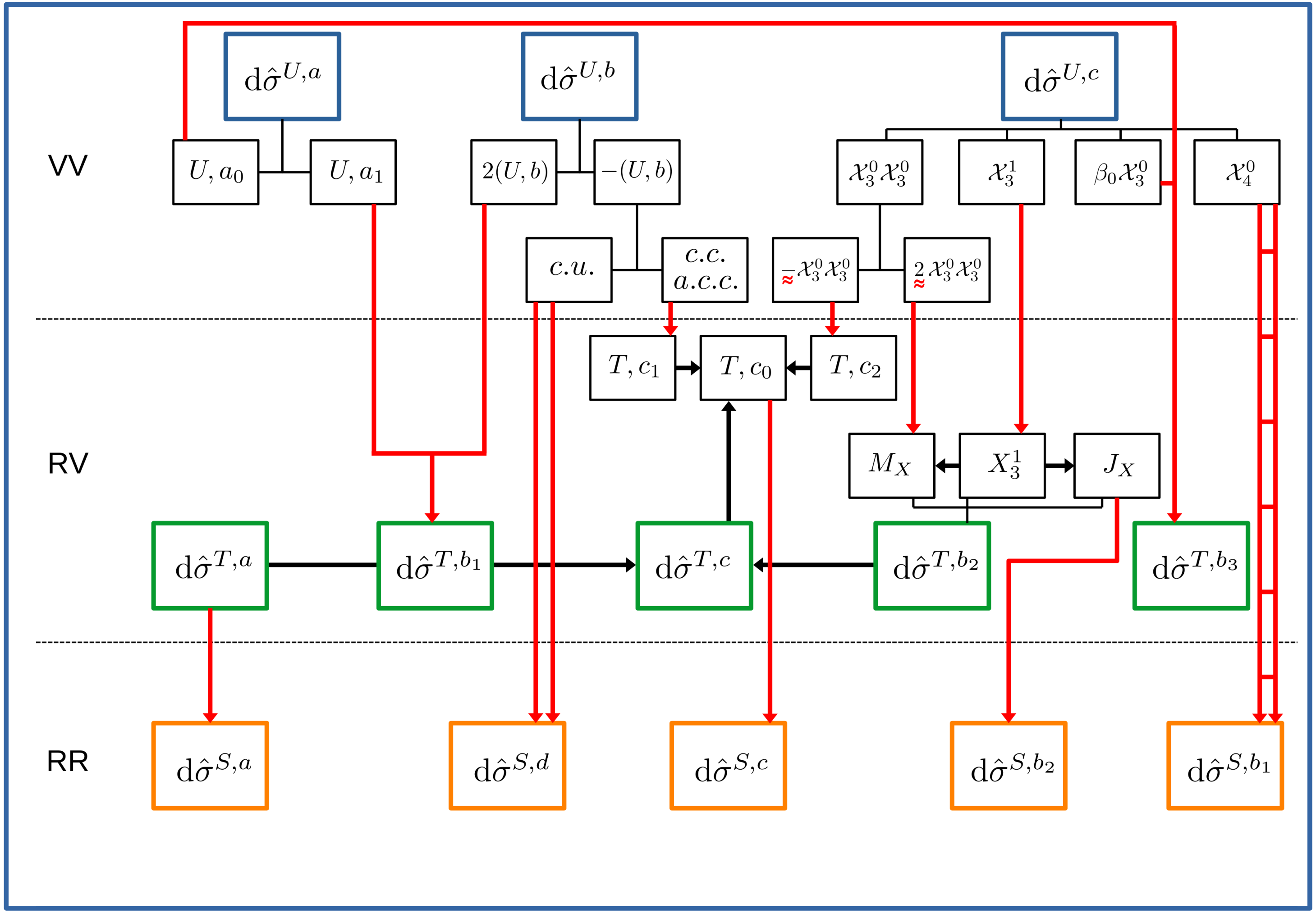
unresolved parton

momentum mapping

Subtraction at NNLO

[Chen, Gehrmann, Glover, Huss, MM '22] [Gehrmann, Glover, MM '23]

- Double virtual subtraction term from **integrated dipoles**;
- **First insertion** of an unresolved parton + generation of **new structures** for the real virtual subtraction term;
- **Second insertion** of an unresolved parton for the double-real subtraction term.



Integrated dipoles with antenna functions

$$\mathcal{J}^{(1)} = \sum_{(i,j)\geq 3} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\, \mathcal{J}_2^{(1)}(i,j) + \sum_{i\neq 1,2} (\boldsymbol{T}_1 \cdot \boldsymbol{T}_i)\, \mathcal{J}_2^{(1)}(1,i) + \sum_{i\neq 1,2} (\boldsymbol{T}_2 \cdot \boldsymbol{T}_i)\, \mathcal{J}_2^{(1)}(2,i) + (\boldsymbol{T}_1 \cdot \boldsymbol{T}_2)\, \mathcal{J}_2^{(1)}(1,2)$$

$$\mathcal{J}^{(2)} = N_c \sum_{(i,j)\geq 3} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\, \mathcal{J}_2^{(2)}(i,j) + N_c \sum_{i\neq 1,2} (\boldsymbol{T}_1 \cdot \boldsymbol{T}_i)\, \mathcal{J}_2^{(2)}(1,i) + N_c \sum_{i\neq 1,2} (\boldsymbol{T}_2 \cdot \boldsymbol{T}_i)\, \mathcal{J}_2^{(2)}(2,i) + N_c (\boldsymbol{T}_1 \cdot \boldsymbol{T}_2)\, \mathcal{J}_2^{(2)}(1,2)$$

both hard radiators in the final state

one hard radiator in the initial state

both hard radiators in the initial state

Colour decomposition:

$$\mathcal{J}_2^{(\ell)} = J_2^{(\ell)} + \frac{1}{N_c^2}\tilde{J}_2^{(\ell)} + \frac{N_f}{N_c}\hat{J}_2^{(\ell)} + \frac{N_f}{N_c^3}\hat{\tilde{J}}_2^{(\ell)} + \frac{N_f^2}{N_c^2}\hat{\hat{J}}_2^{(\ell)}$$

Colour-stripped integrated dipoles

$$J_2^{(1)} = c_{\mathcal{X}_3^0}\mathcal{X}_3^0 + c_{\Gamma^{(1)}}\Gamma^{(1)}$$

$$J_2^{(2)} = c_{\mathcal{X}_4^0}\mathcal{X}_4^0 + c_{\mathcal{X}_3^1}\mathcal{X}_3^1 + c_{\mathcal{X}_3^0\mathcal{X}_3^0}\mathcal{X}_3^0\mathcal{X}_3^0 + c_{\beta_0}\frac{\beta_0}{\epsilon}\mathcal{X}_3^0\left(\frac{|s|}{\mu_r^2}\right)^{-\epsilon} + c_{\Gamma^{(2)}}\overline{\Gamma}^{(2)}$$

Key property: written in terms of **integrated antenna functions** and MF kernels (for IS radiation)

Integrated dipoles with antenna functions

Catani's IR operators

$$Poles\left[\mathcal{J}_2^{(1)}(i,j)\right] = Poles\left[\mathrm{Re}\left(\mathcal{I}_{ij}^{(1)}(\epsilon)\right)\right]$$

$$Poles\left[N_c\mathcal{J}_2^{(2)}(i,j) - \frac{\beta_0}{\epsilon}\mathcal{J}_2^{(1)}(i,j)\right] = Poles\left[\mathrm{Re}\left(\mathcal{I}_{ij}^{(2)}(\epsilon) - \frac{\beta_0}{\epsilon}\mathcal{I}_{ij}^{(1)}(\epsilon)\right)\right]$$

$$Poles\left(\mathrm{d}\hat{\sigma}^V\right) = \mathcal{N}_V \int \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_n\, J_n^n(\Phi_n)\, Poles\left[2\left\langle A_{n+2}^0\middle|\boldsymbol{\mathcal{J}}^{(1)}\middle|A_{n+2}^0\right\rangle\right]$$

$$Poles\left(\mathrm{d}\hat{\sigma}^{VV}\right) = \mathcal{N}_{VV} \int \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_n J_n^n(\Phi_n) Poles\Bigg\{ \times 2\Big[\langle A_{n+2}^0|\boldsymbol{\mathcal{J}}^{(1)}|A_{n+2}^1\rangle + \langle A_{n+2}^1|\boldsymbol{\mathcal{J}}^{(1)}|A_{n+2}^0\rangle - \frac{b_0 N_c}{\epsilon}\langle A_{n+2}^0|\boldsymbol{\mathcal{J}}^{(1)}|A_{n+2}^0\rangle - \langle A_{n+2}^0|\boldsymbol{\mathcal{J}}^{(1)}\otimes\boldsymbol{\mathcal{J}}^{(1)}|A_{n+2}^0\rangle + \langle A_{n+2}^0|\boldsymbol{\mathcal{J}}^{(2)}|A_{n+2}^0\rangle\Big]\Bigg\}$$

Infrared structure of one- and two-loop MEs in terms of integrated antenna functions

[Chen, Gehrmann, Glover, Huss, MM '22] [Gehrmann, Glover, MM '23]

Double virtual subtraction term

Straightforward use of integrated dipoles in colour space:

$$\mathrm{d}\sigma^{U} = \mathcal{N}_{VV} \int \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2} \mathrm{d}\Phi_n J_n^n(\Phi_n)$$

$\mathrm{d}\sigma^{U,a_0}$

$$\times 2\Big\{ \langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}|A^1_{n+2}\rangle + \langle A^1_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}|A^0_{n+2}\rangle - \frac{\beta_0}{\epsilon}\langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(1)}|A^0_{n+2}\rangle$$

single one-loop insertion at one-loop

$\mathrm{d}\sigma^{U,a_1}$

$$-\langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(1)} \otimes \boldsymbol{\mathcal{J}}^{(1)}|A^0_{n+2}\rangle + \langle A^0_{n+2}|\boldsymbol{\mathcal{J}}^{(2)}|A^0_{n+2}\rangle \Big\}$$

double one-loop insertion

two-loop insertion

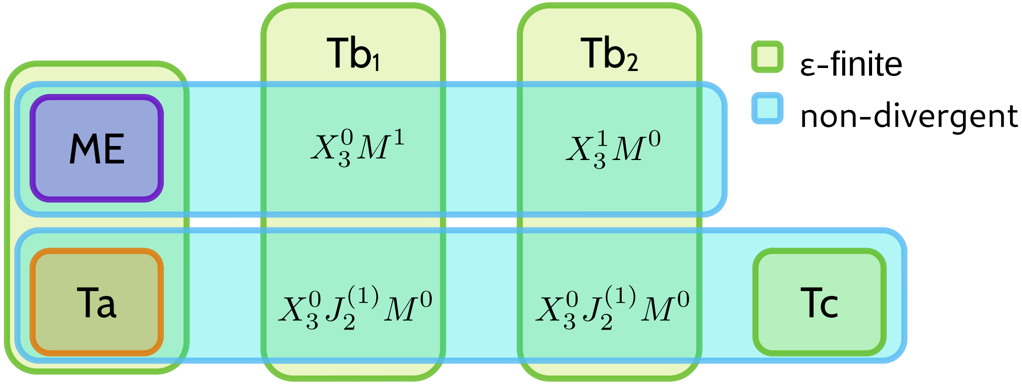
$\mathrm{d}\sigma^{U,b}$

$\mathrm{d}\sigma^{U,c}$

Real virtual subtraction term

The real-virtual subtraction term has to:

- remove explicit IR poles;
- Subtract soft and collinear behaviour;



removal of explicit poles

$$\mathrm{d}\sigma^{T,a} = \mathcal{N}_V \int \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_n\, J_n^n(\Phi_n)\, \left[2\left\langle A^0_{n+3}\right|\boldsymbol{\mathcal{J}}^{(1)}\left|A^0_{n+3}\right\rangle\right]$$

(tree-level divergence) x (one-loop ME)

$$\mathrm{d}\sigma^{T,b_1} = -\mathcal{I}ns\left[\mathrm{d}\sigma^{U,a_1}\right] - 2\mathcal{I}ns\left[\mathrm{d}\sigma^{U,b}\right] - \mathrm{d}\sigma^{MF,1,b}$$

(one-loop divergence) x (tree-level ME)

$$\mathrm{d}\sigma^{T,b_2} = -\mathcal{I}ns\left[\mathrm{d}\sigma^{U,c,\mathcal{X}_3^1}\right] + \mathrm{d}\sigma^{T,b_2,J_X} + \mathrm{d}\sigma^{T,b_2,M_X}$$

compensates for oversubtraction

$$\mathrm{d}\sigma^{T,c} = \frac{1}{2}\left[\sigma^{T,c,\mathrm{prel.}} + \sigma^{T,c,\mathcal{S}} + \sigma^{T,c,\mathrm{extra}}\right]$$

Double real subtraction term

simultaneous double insertion of two colour-connected partons:

$$\mathcal{X}_4^0(s_{ij})\, A_{n+2}^0(.,i,.,j,.)$$
$$\updownarrow$$
$$X_4^0(i,u_1,u_2,j)\, A_{n+2}^0(.,\widetilde{iu_1u_2},.,\widetilde{u_1u_2j},.)$$

single-unresolved $\mathrm{d}\sigma^{S,a} = -\mathcal{I}ns\left[\mathrm{d}\sigma^{T,a}\right]$

colour-connected double-unresolved $\mathrm{d}\sigma^{S,b_1} = -\mathcal{I}ns_2\left[\mathrm{d}\sigma^{U,c,\mathcal{X}_4^0}\right]$

removes single-uresolved from S_{b1} $\mathrm{d}\sigma^{S,b_2} = -\mathcal{I}ns\left[\mathrm{d}\sigma^{T,b_2,J_X}\right]$

almost colour-connected double-unresolved $\mathrm{d}\sigma^{S,c} = -\mathcal{I}ns\left[\mathrm{d}\sigma^{T,c_0}\right]$

colour-unconnected double-unresolved $\mathrm{d}\sigma^{S,d} = +\mathcal{I}ns\left[\mathcal{I}ns\left[\mathrm{d}\sigma^{U,b,c.u.}\right]\right]$

iterated single insertion

Mapping (in)dependence

[Fox,Glover,MM '24]

Let's consider:

- **n_p** momenta **{p}** involved in an unresolved configuration
- **n_q** spectator momenta **{q}**

mapping to absorb the recoil of unresolved radiation:

$$\{p\} \to \{\tilde{p}\}$$

generic subtraction term

$$d\sigma^S \propto \int dPS_{n_p+n_q}(\{p\},\{q\})\, X(\{p\})\, M(\{\tilde{p}\},\{q\})\, J^{(n_{\tilde{p}}+n_q)}_{n_{jets}}(\{\tilde{p}\},\{q\})$$

full phase-space — unresolved factor (antenna function) — resolved matrix element — measurement function

selects n_{jets} jets applies fiducial cuts

The mapping is chosen to induce a **factorization of the phase space**

$$dPS_{n_p+n_q}(\{p\},\{q\}) = dPS_X(\{p\}/\{\tilde{p}\})\, dPS_{n_{\tilde{p}}+n_q}(\{\tilde{p}\},\{q\})$$

unresolved phase-space

resolved phase-space: the measurement function acts on it

$$d\sigma^S \propto \int dPS_{n_{\tilde{p}}+n_q}(\{\tilde{p}\},\{q\}) \left[\int X(\{p\}) dPS_X(\{p\}/\{\tilde{p}\})\right] M(\{\tilde{p}\},\{q\})\, J^{(n_{\tilde{p}}+n_q)}_{n_{jets}}(\{\tilde{p}\},\{q\})$$

$$= \int dPS_{n_{\tilde{p}}+n_q}(\{\tilde{p}\},\{q\})\, \mathcal{X}(\{\tilde{p}\})\, M(\{\tilde{p}\},\{q\})\, J^{(n_{\tilde{p}}+n_q)}_{n_{jets}}(\{\tilde{p}\},\{q\})$$

integrated unresolved factor

two mappings are **equivalent** if they yield the same $\mathcal{X}(\{\tilde{p}\})$

Mapping (in)dependence [Fox,Glover,MM '24]

Both traditional and designer antenna functions have only **physical propagators** and are defined over the **inclusive radiation PS**, which leads to simpler integration

two hard radiators: $n_{\widetilde{p}} = 2\,, \qquad \{p_1, \ldots, p_{n_p}\} \to \{\widetilde{p}_A, \widetilde{p}_B\}$

momentum conservation

$$s_{1\ldots n_p} \equiv (p_1 + \cdots + p_{n_p})^2 = (\widetilde{p}_A + \widetilde{p}_B)^2 \equiv s_{AB}$$

only scale in the process!

$$\mathcal{X}\left(\{\widetilde{p}_A, \widetilde{p}_B\}\right) = C(\epsilon)(s_{AB})^{\alpha}$$

any momentum-conserving mapping gives same result

three hard radiators: $n_{\widetilde{p}} = 3\,, \qquad \{p_1, \ldots, p_{n_p}\} \to \{\widetilde{p}_A, \widetilde{p}_B, \widetilde{p}_C\}$

momentum conservation

$$s_{1\ldots n_p} \equiv (p_1 + \cdots + p_{n_p})^2 = (\widetilde{p}_A + \widetilde{p}_B + \widetilde{p}_C)^2 \equiv s_{ABC}$$

$$\mathcal{X}\left(\{\widetilde{p}_A, \widetilde{p}_B, \widetilde{p}_C\}\right) = \sum_i C_i(\epsilon)(s_{AB})^{\alpha_i}(s_{AC})^{\beta_i}(s_{BC})^{\gamma_i} + \ldots$$

many "unfixed" scales, different result for different mappings

Application: Hadronic Higgs Decays

Hadronic Higgs decays (H→jjj @ NNLO):

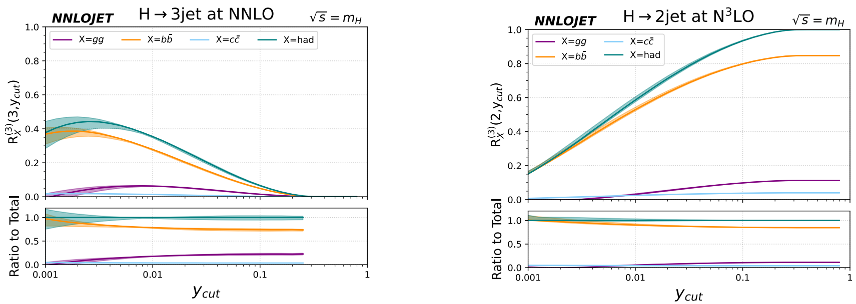
- differences in QCD radiation patterns between decay modes:
  - Yukawa decay to quarks
  - Top-loop induced decay to gluons

[Mondini,Schiavi,Williams '19]

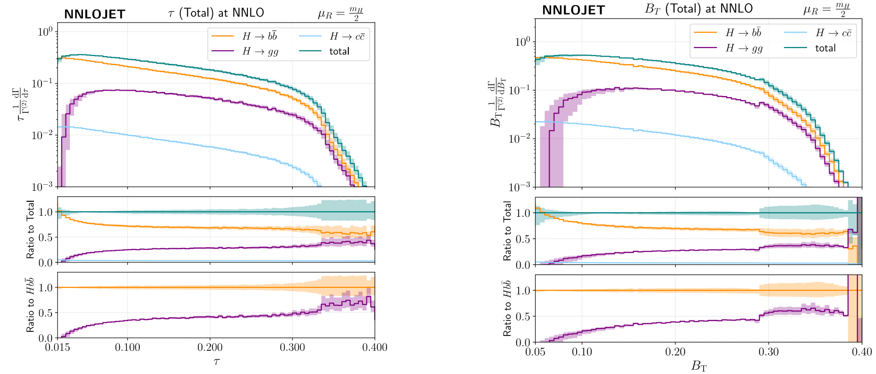
- Exploit large of H→hadrons sample from e^+e^-→ZH at future colliders (FCC-ee)
- Streamlined and efficient implementation thanks to generalised antenna functions
- Matching to resummation: good stability of fixed-order calculation in the two-particle limit
- Utility for the extraction of the strange Yukawa?

[Fox,Gehrmann-De Ridder,Gehrmann,Glover,MM,Preuss '25]

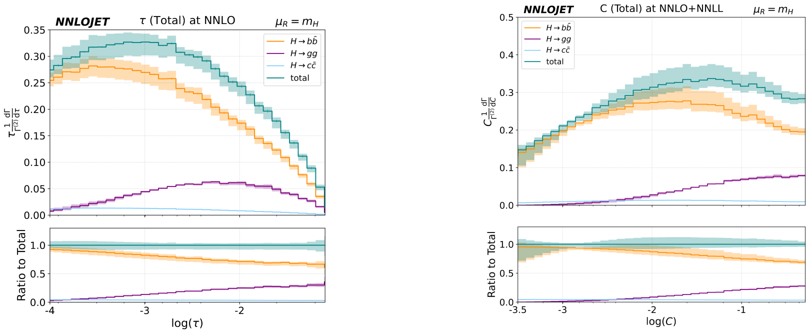
Jet rates



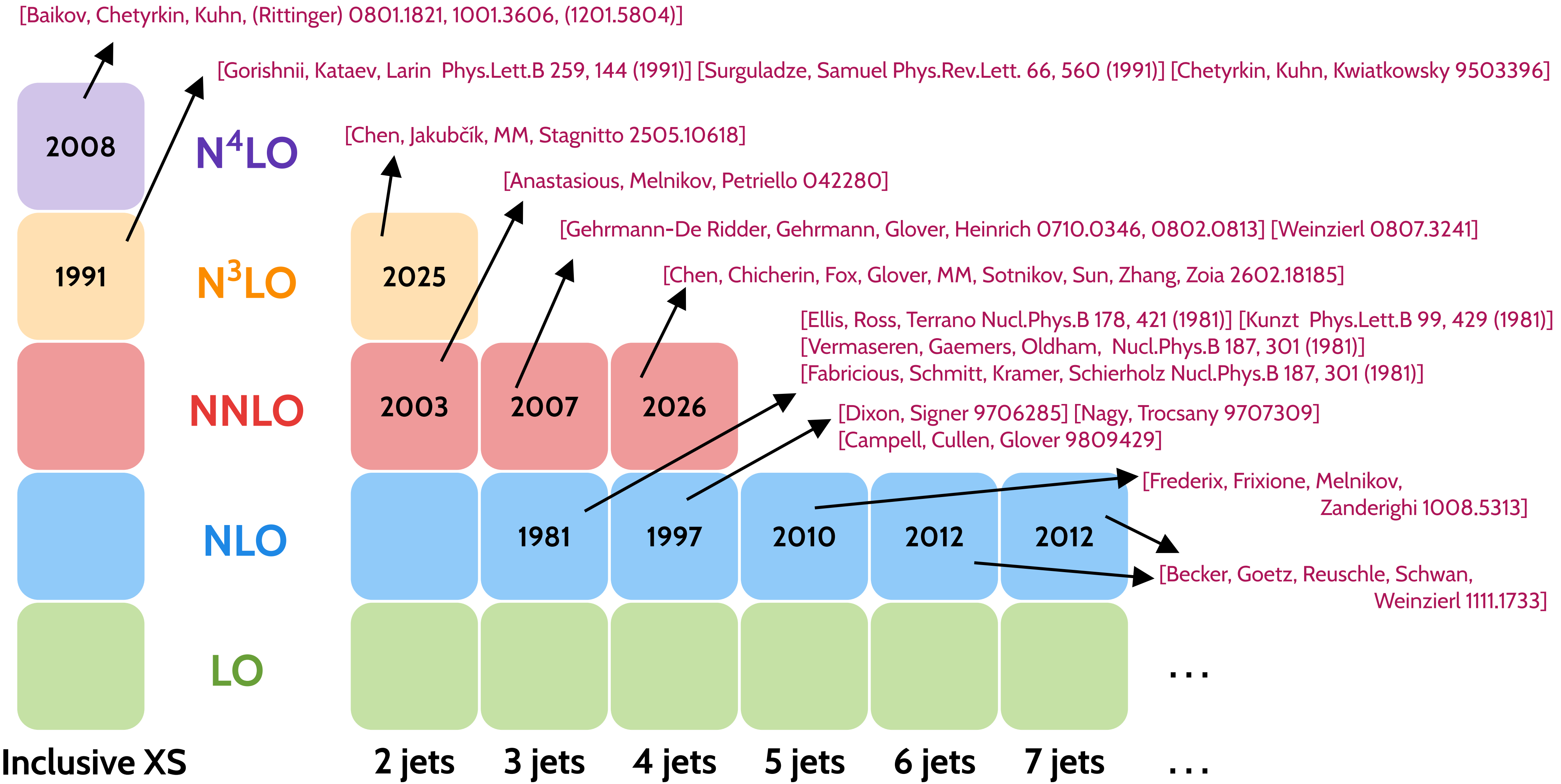
Event shapes @NNLO



Event shapes matched NNLO+NNLL



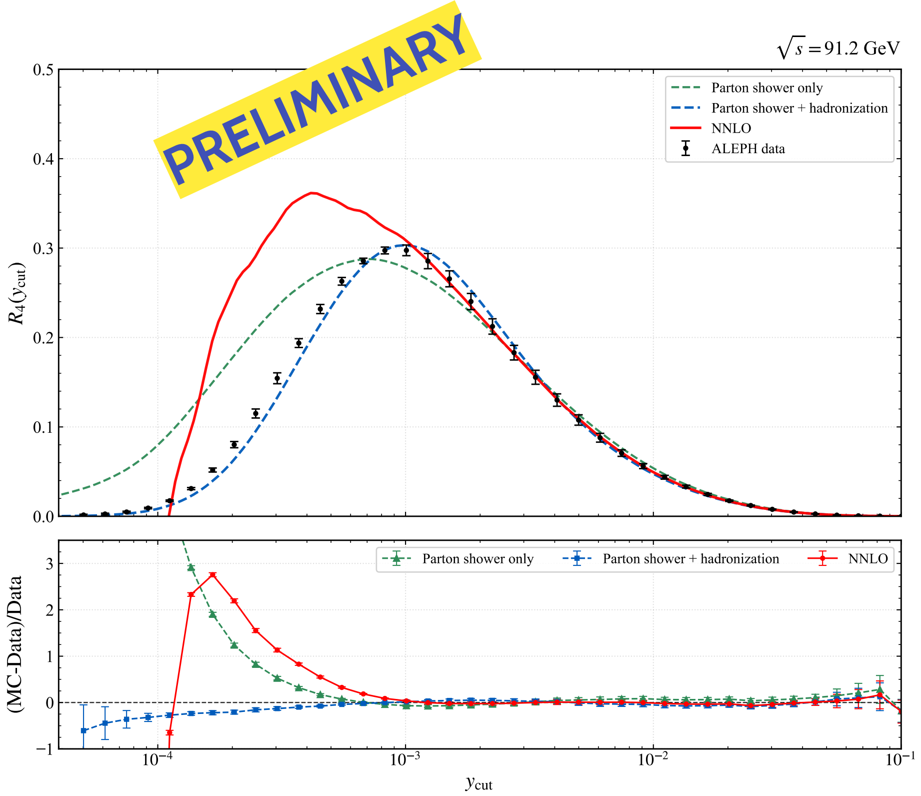
Jet production in electron-positron annihilation: references



Impact of hadronization effects

Hadronization corrections are expected to be small in the perturbative region:

- Clustering particles via the Durham algorithm reduces the sensitivity to hadronization effects;
- Fixed-order predictions seems to reproduce data very well;
- We verified explicitly by running Pythia+Vincia.



[Thanks to Huiting Sun for producing this plot]

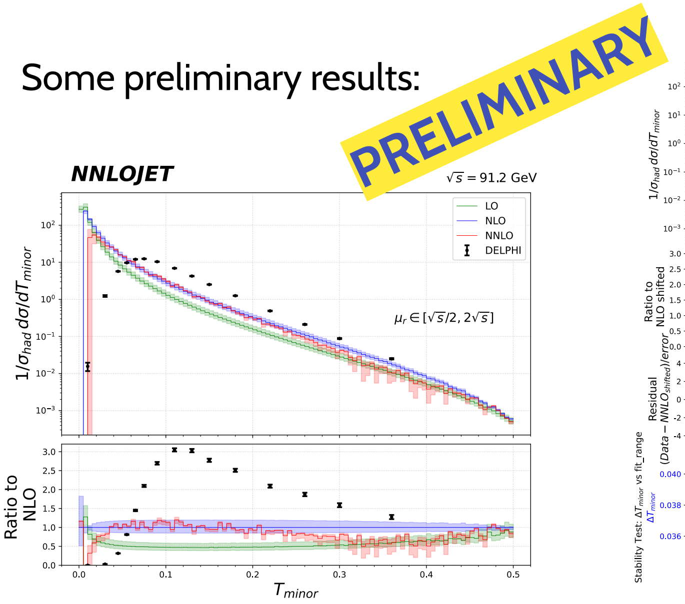
Four-jet event shapes: work in progress

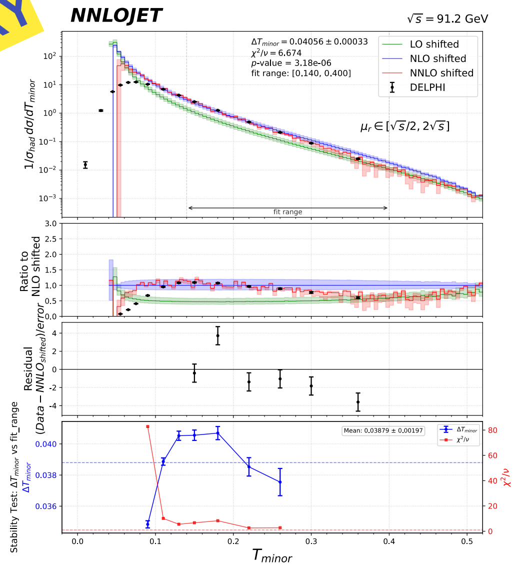
[Campbell, Cullen, Glover 9809429]

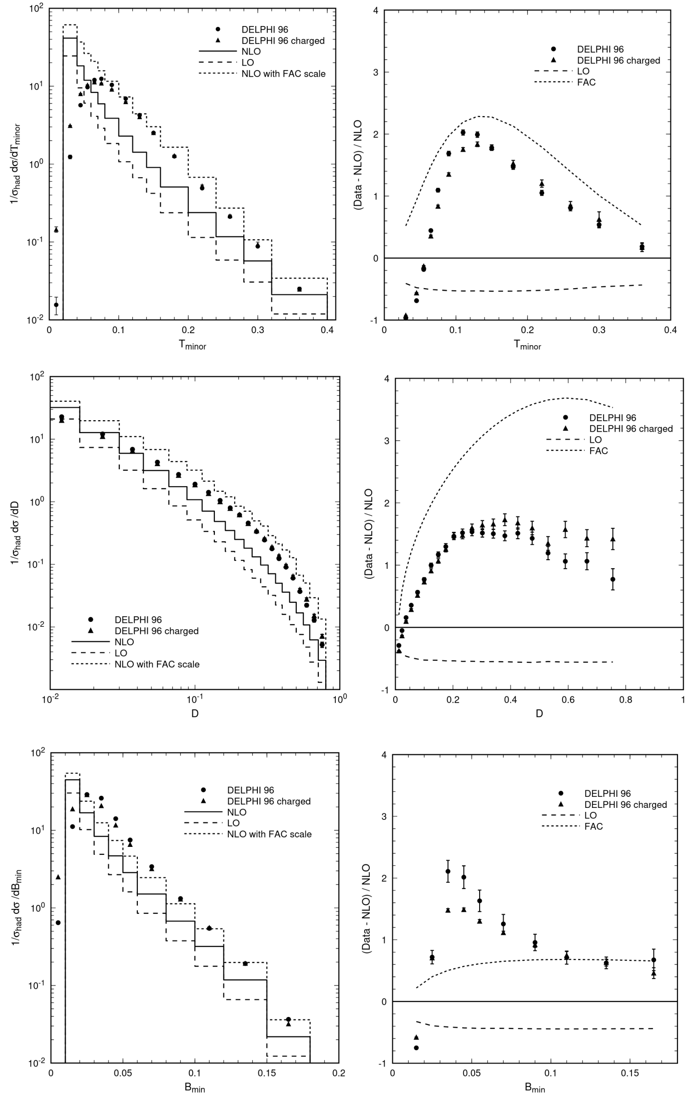
- Fixed-order predictions are quite off data, suggesting large non-perturbative corrections. This can't be fixed by NNLO or higher-orders;
- Hadronization corrections are challenging to model;

[Banfi, Dokshitzer, Marchesini, Zanderighi 0010267, 0101205, 0104162]
[Burby, Glover 0101226]

- Some preliminary results:







NLO

NLO

NLO

[Thanks to Huiting Sun for producing these plots]