Quantum Technology for Fundamental Physics:

The Quantum Sensors Challenge

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The QTNM Team

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The Challenge:

Ultra-low-noise electronics is essential for the next generation of fundamental physics

BUT

most experiments and instruments fall well short of quantum-dominated sensitivity

SO

technological development is desperately needed

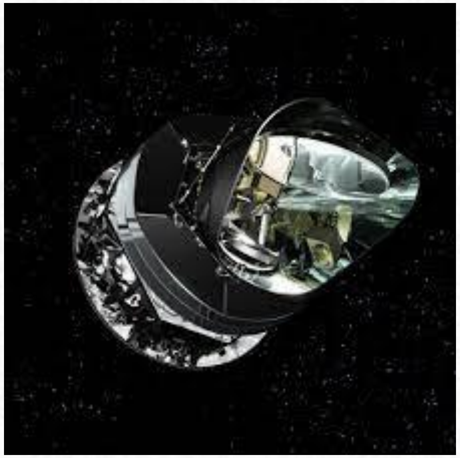
AND

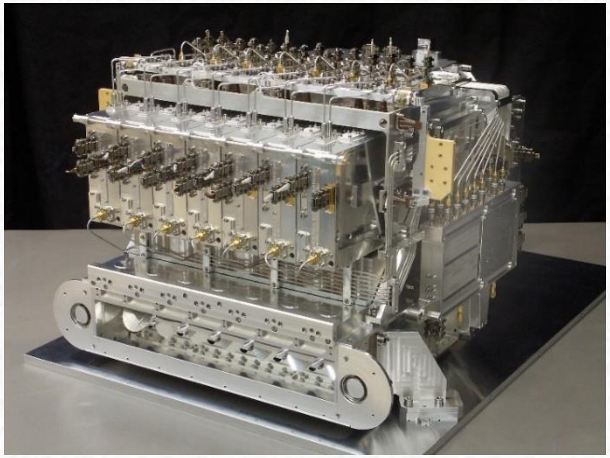
the work requires innovation and is highly intellectually rewarding in its own right

Microwave (100 GHz-1 THz) and FIR (2-30 THz) Astrophysics:

Ground based and space-based observatories needed to

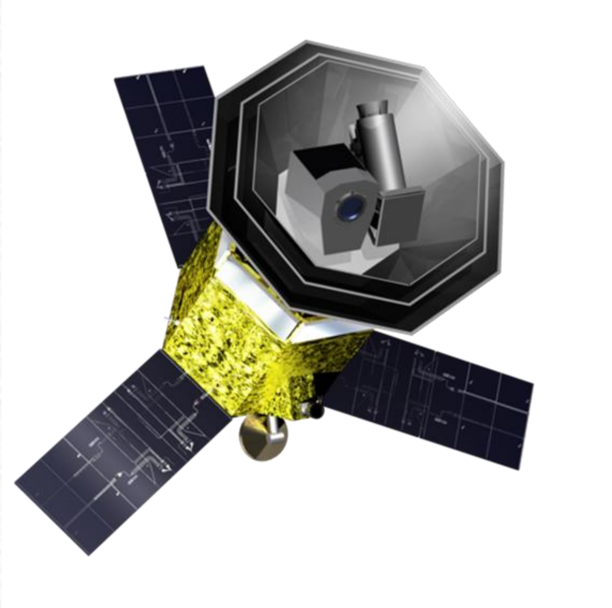
- search for the effects of gravitational waves (B-modes) in the polarization of the CMB
- study galaxy formation in the very early Universe
- study star and extra-solar planet formation in our own galaxy
- study high energy phenomena – black holes





Future superconducting imaging and spectrometer arrays (with readout) at L2:

LiteBIRD – CMB B-modes polarization mission
4,500 pixels, 40-400 GHz,

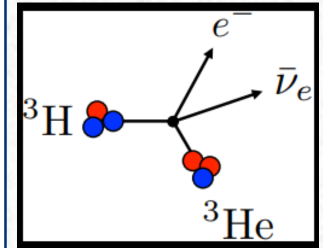


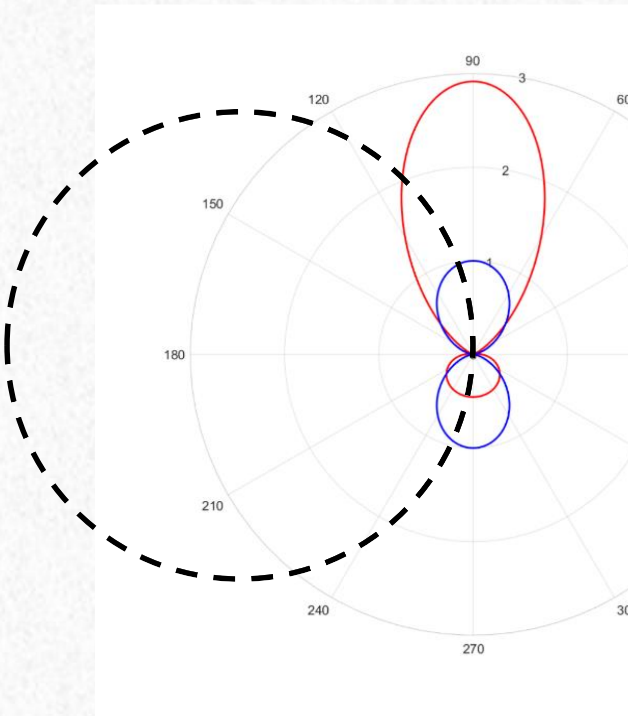
SPICA follow-on (PRIMA,FIRSST) – cooled aperture 5,000 pixels, 300-30 µm,

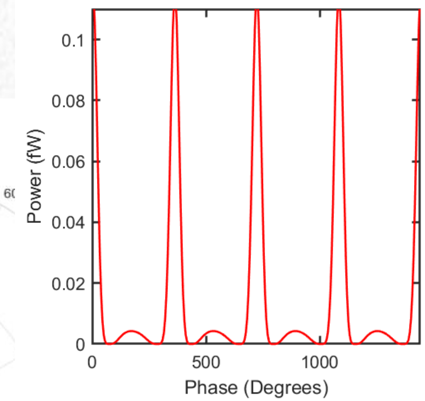
X-ray astrophysics – X-IFU Athena
4,000 pixels, 0.2 to 12 keV (~2.5-7 eV res)

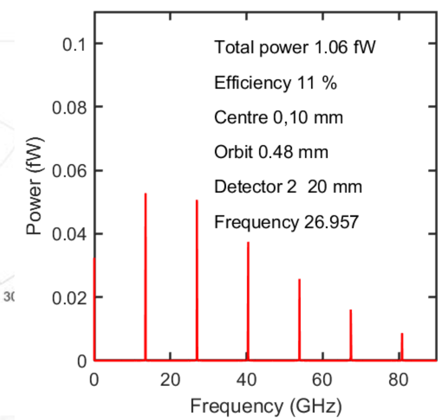
QTNM - Determine neutrino mass through cyclotron emission spectroscopy:

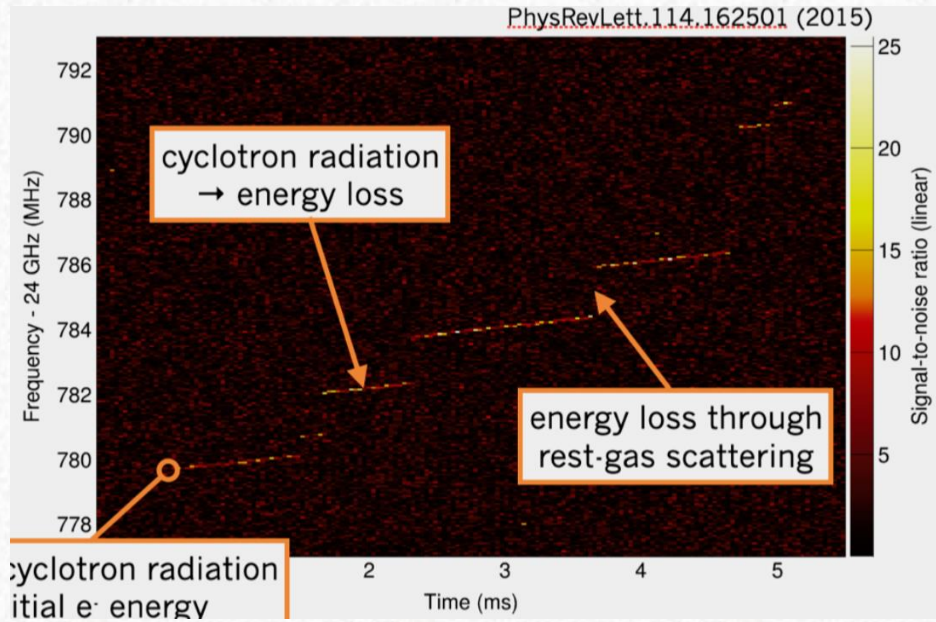
- Measure energies of individual electrons released during radioactive decay of Tritium
- Spectroscopy of synchrotron from 18.6 keV electrons in 1 T field (27 GHz, 1 ppm 1kHz, power ~1 fW for <1 mS) – only 1 Atto Joule per event
- Quantum electronics has special significance because we can't integrate!





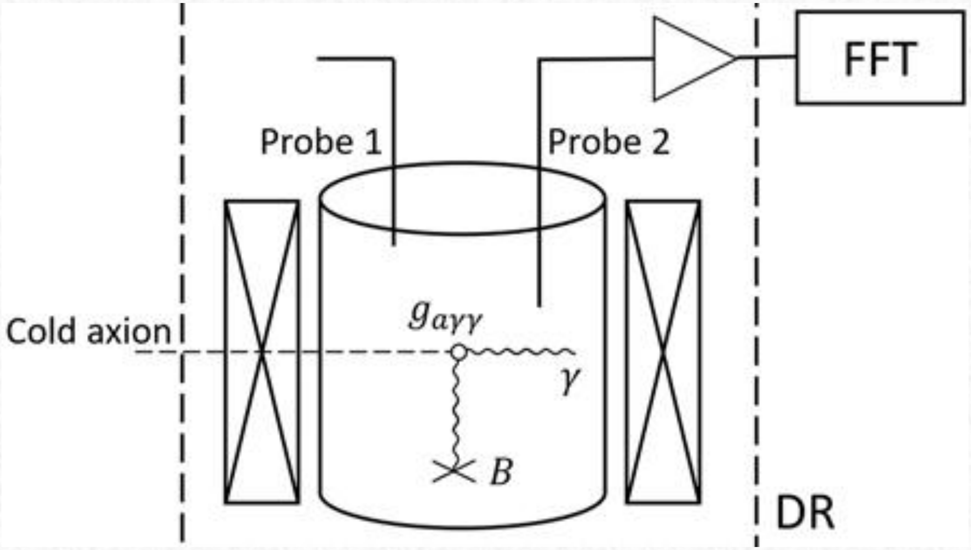


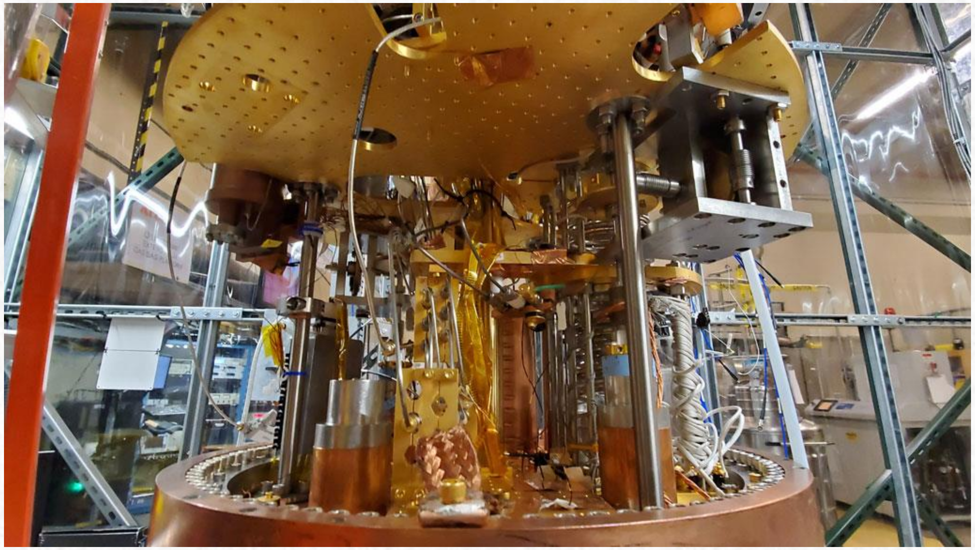


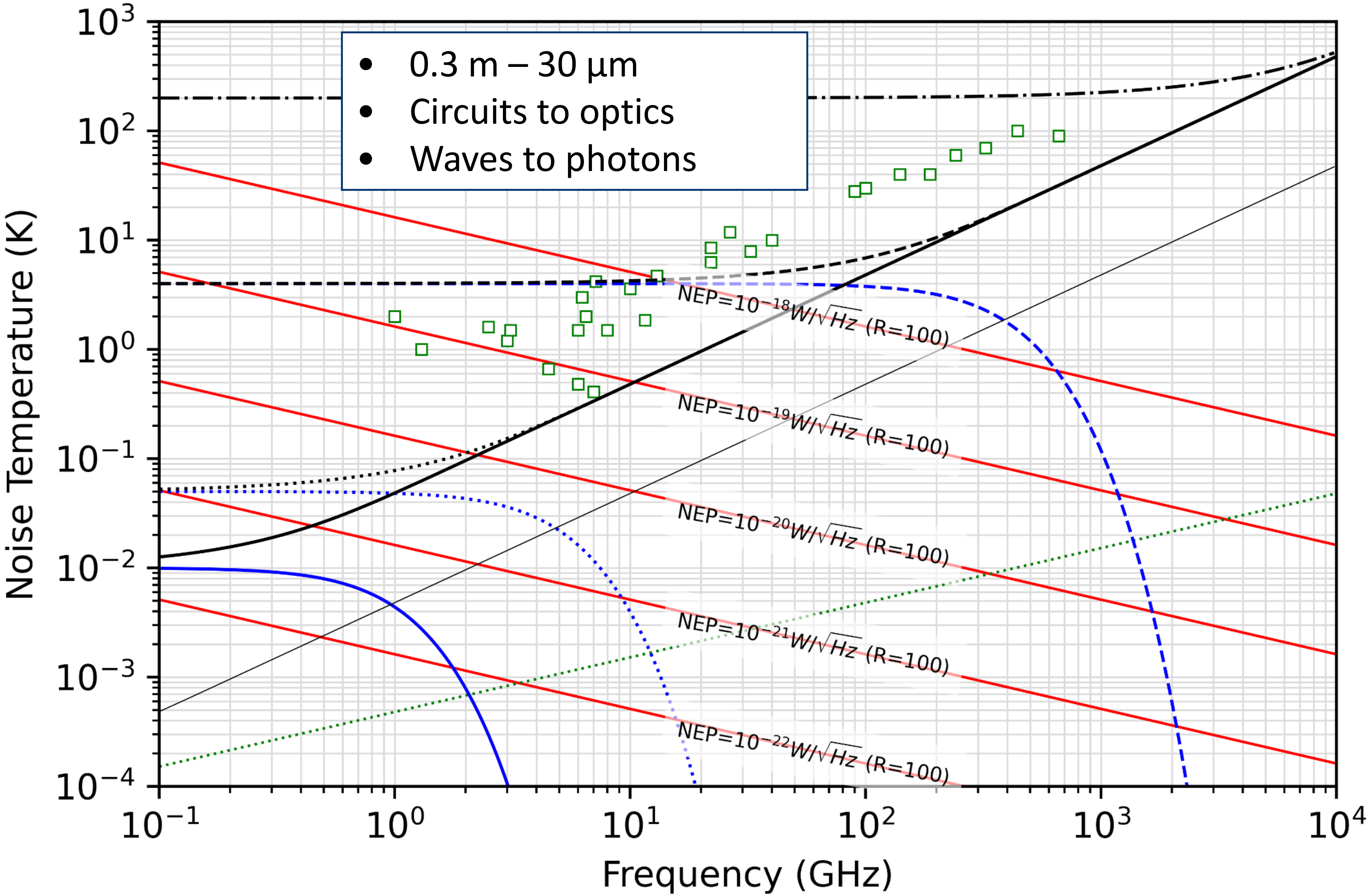


QSHS - Search for Dark Matter low-mass (μeV) weakly interacting particles (Axions):

- Collaborations with US ADMX experiment (see photo below) and ALPHA
- Probe the vacuum state of ~10 mK radiation field over 1-30 GHz to look for `unexpected' spectral features
- QSHS looking around 5 GHz, ~1 MHz features – photon rate in min^{-1}
- Quantum noise limited, and sub quantum noise limited, sensitivity
- Snowmass 2021 CF2 Wavelike Dark Matter Axion White Paper







Radio	Microwave	Submm	FIR

As one move up in frequency, how does a circuit description of behaviour turn into a photon counting description of behaviour?

Should an experiment measure field-like quantities (such as voltage) or should it measure power (such as photon rate)?

Profound difference between experiments based on measuring field-like quantities and experiments based on measuring photon fluxes and counting photons

The statistics of field fluctuations is very different from the statistics of photon rate fluctuations

Field variance (field fluctuation noise) is second order in wave amplitude, whereas power variance (photon rate variance) is fourth order

Confusingly, these are called coherent and incoherent instruments

This talk concentrates on field-like measurements of electromagnetic systems

According to quantum mechanics:

No matter how great the skill of an observer, the outcome of a single measurement on any simple physical system of any basic physical quantity is profoundly uncertain:

- Classical statistics concerns our lack of certainty or knowledge
- Quantum statistics concerns nature's lack of certainty

- Quantum mechanics applies to all dynamical variables, including electrical quantities such as voltage, current, power, electric and magnetic fields and dipoles

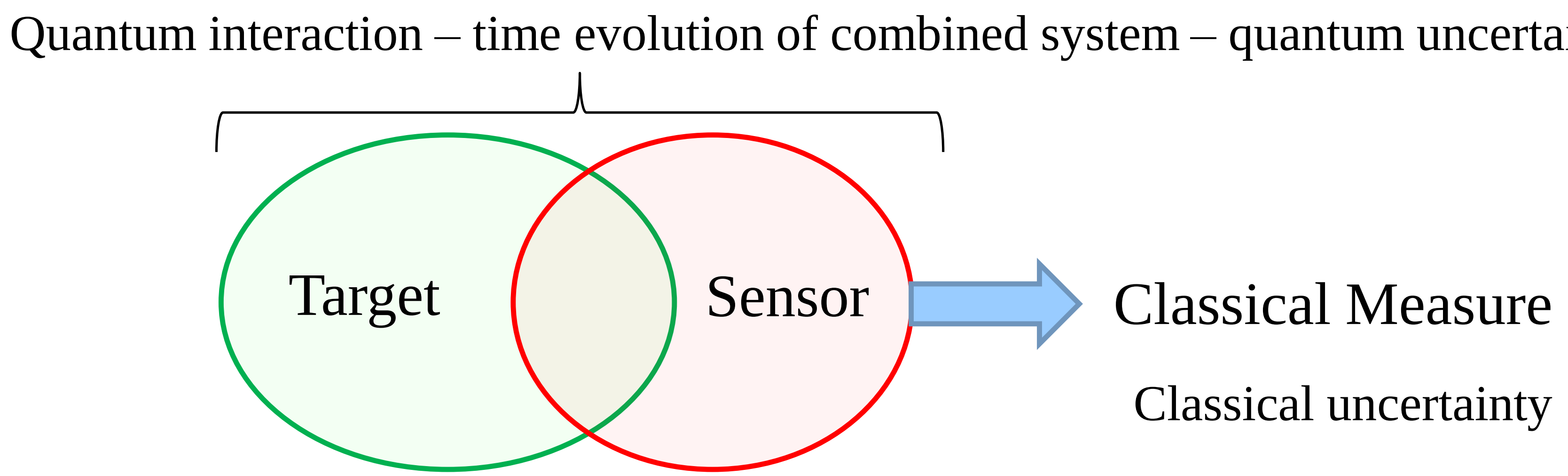
- At 10 mK to 4 K, the mysterious world of quantum mechanics is revealed, and it becomes necessary to use quantum mechanics to describe the behavior of circuits

- Quantum mechanics tracks probability distributions - the analysis of circuits becomes more complicated (inductors, capacitors, resistors, transformers, transmission lines, power detectors, transistors, mixers and amplifiers, constant voltage sources)

- Forced into asking questions about the influence of vacuum fluctuations, back action, squeezing and entanglement on the behavior of electrical circuits

What is quantum sensing (quantum information perspective) ?

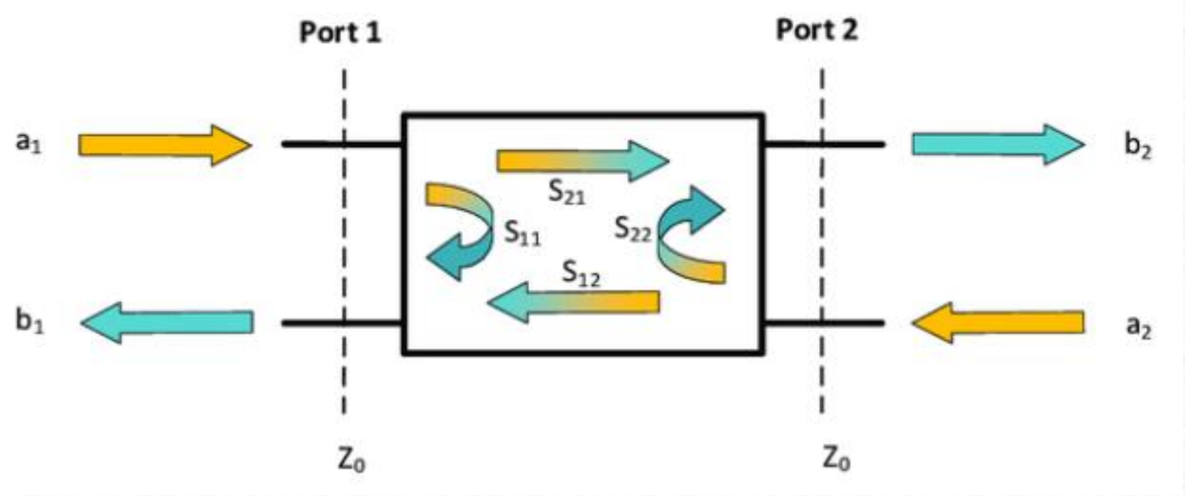
- There some target that we wish to probe, and this target must be described quantum mechanically – contains the fundamental physics.
- There is a sensor, which is often part of larger electrical circuit, which itself must be described quantum mechanically – the instrument.
- The purpose of the sensor is to create a macroscopic quantity that can be recorded, and which carries faithful information about properties of the target.
- Quantum sensing is the quantisation of the dynamical variables of the target and the interaction with the quantum behavior of the instrument carrying out the measurement



Classical analysis of circuits systems….

At short wavelengths, use scattering parameters – no need for circuits or Lagrangians!

Put the physics in a black box, and give it a name (resistor, capacitor…)



Discrete representation or connect transmission lines to ports

Voltage and current at a plane decomposed into forward and backward travelling waves

Travelling waves connected by scattering parameters

$$a(\omega) = \frac{1}{2\sqrt{Z_0}} \left[v(\omega) + i(\omega) Z_0 \right]$$

$$b(\omega) = \frac{1}{2\sqrt{Z_0}} \left[v(\omega) - i(\omega) Z_0 \right]$$

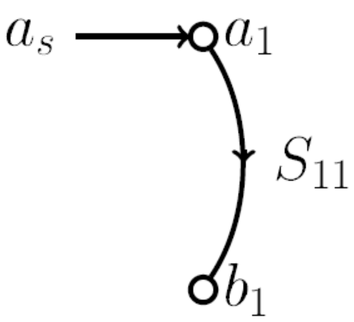
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Noise sources can also be included – stochastic quantities

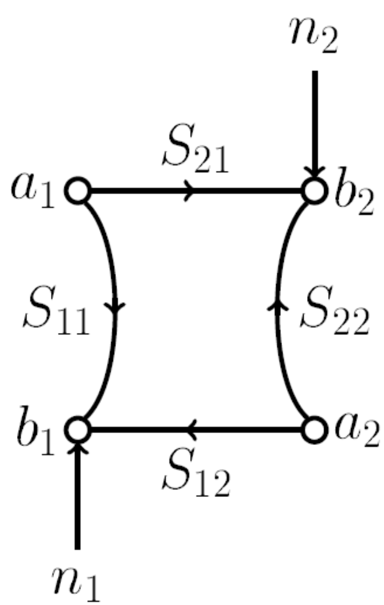
Travelling wave representation entirely equivalent to voltage or current representation

Full behavior described by directed flow graphs (used extensively in mathematics):

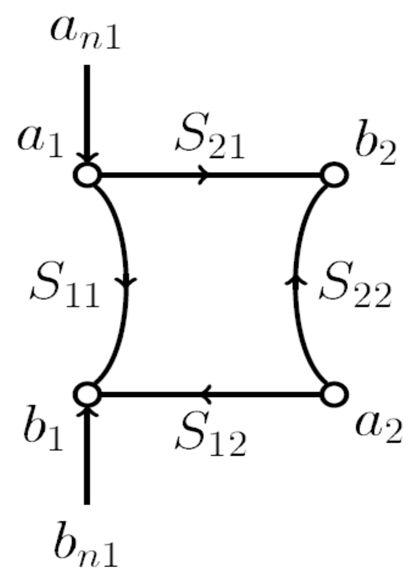
One port network, single reflection coefficient



Two port network having two internal noise sources



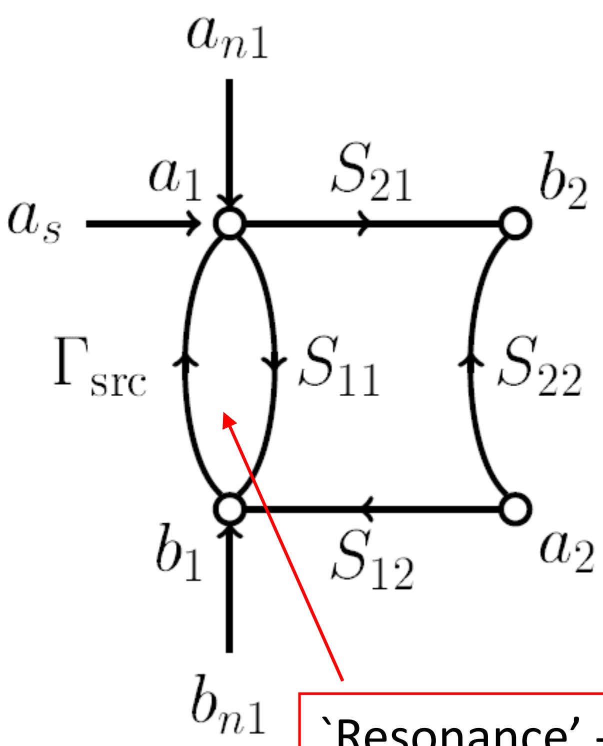
Internal noise sources referenced to the input



When referred to the input there is a noise wave effectively incident on the device, a noise wave travelling away from the input, and some complex correction coefficient

4 parameters are need to describe the noise generated by any two-port network

Consider a one port source driving a two port device (amplifier):



Generally, only know the second-order moments

$$kT_a B = \langle a_{n1} a_{n1}^* \rangle$$
$$kT_b B = \langle b_{n1} b_{n1}^* \rangle$$
$$kT_c B = \langle a_{n1} b_{n1}^* \rangle$$

These have the dimensions of power

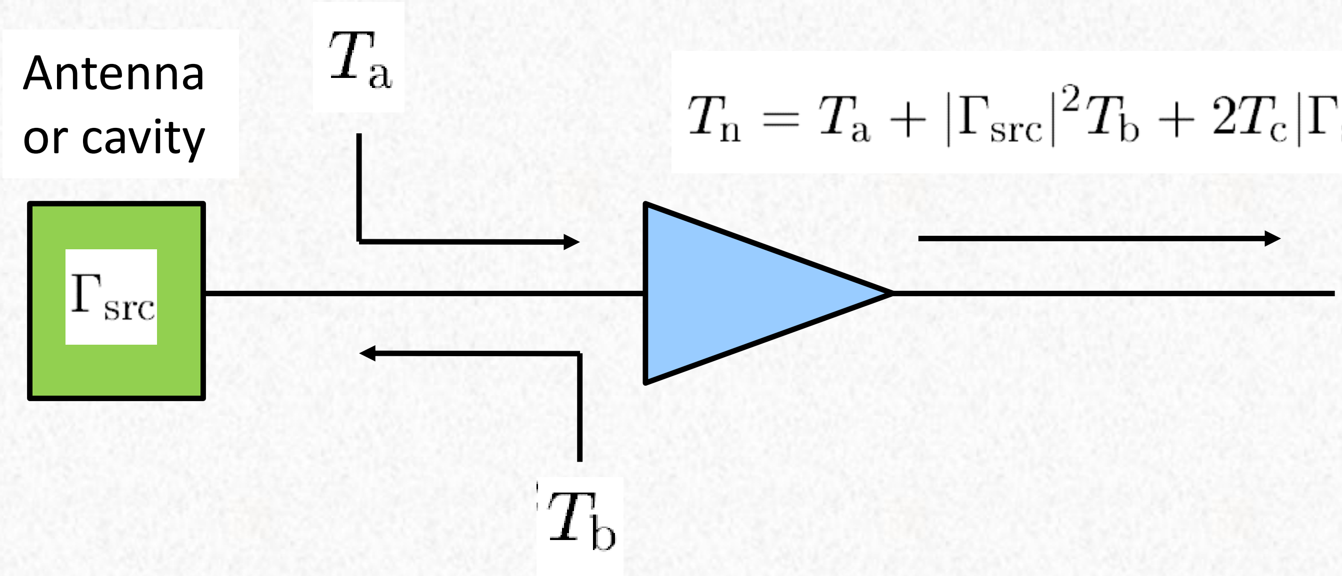
Noise temperature is a partial representation - single source

$$T_{\rm n} = T_{\rm a} + |\Gamma_{\rm src}|^2 T_{\rm b} + 2T_{\rm c}|\Gamma_{\rm src}|\cos(\phi_{\rm c} + \phi_{\rm src})$$

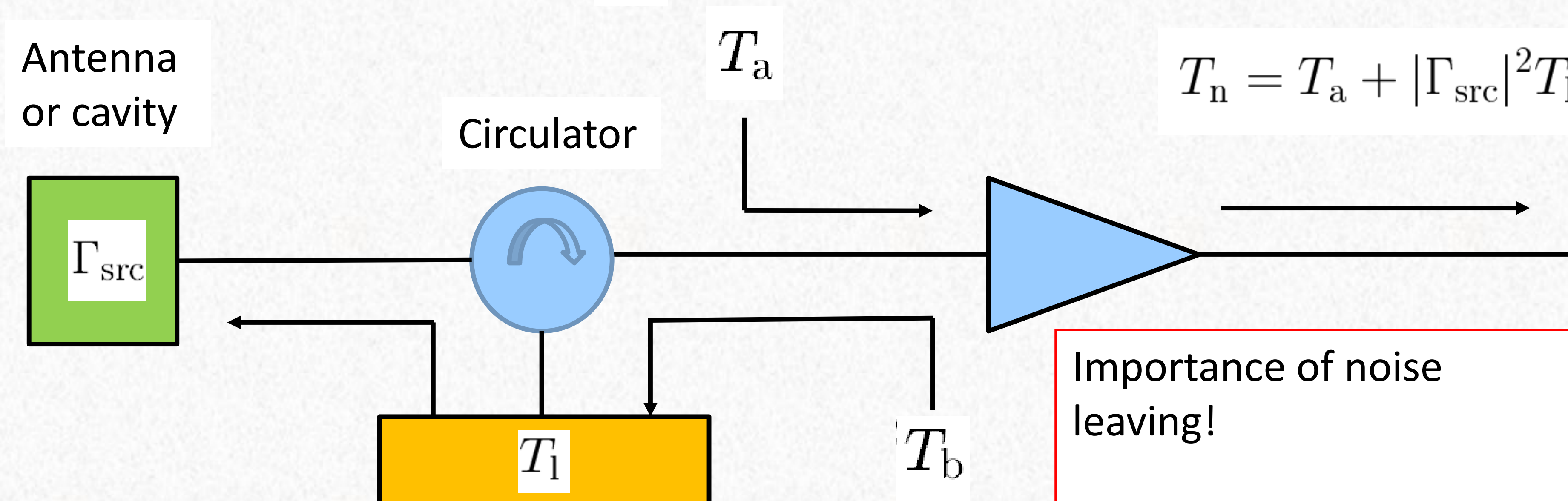
`Resonance' - standing wave on input line

- Input power matching $\Gamma_{\rm src} = S_{11}^*$
- Input noise matching maximally interferes the effects of internal sources
- Clever schemes achieve power matching and noise matching simultaneously

Flow of classical noise:



$$T_n = T_a + |\Gamma_{src}|^2 T_b + 2T_c |\Gamma_{src}| \cos(\phi_c + \phi_{src})$$

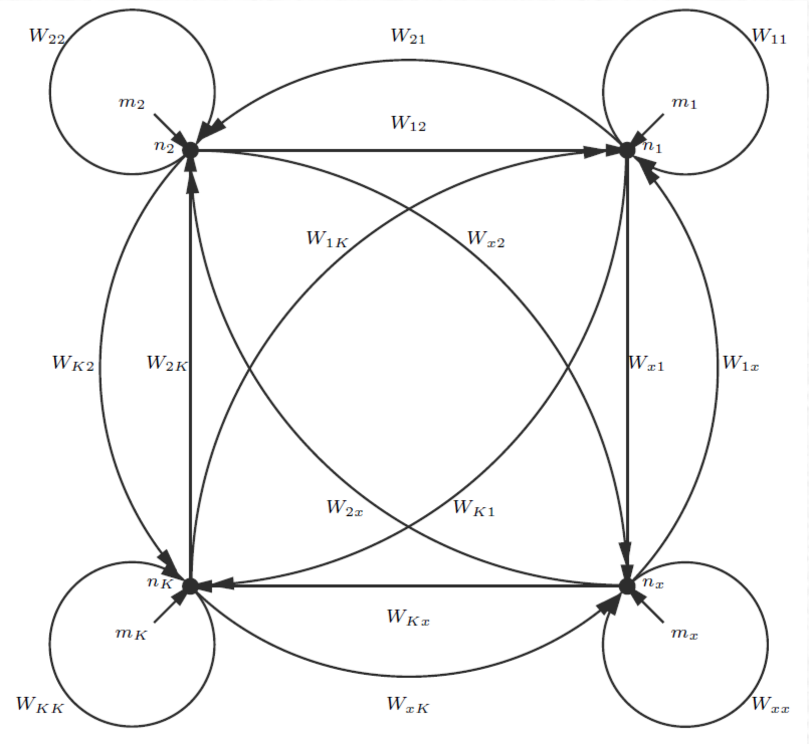


$$T_n = T_a + |\Gamma_{src}|^2 T_l$$

Importance of noise leaving!

Noise can occur due to the random nature of emission!

Complex systems are possible:



- Hours of fun working out how to reduce flow graphs, calculate dependencies, etc.
- In microwave systems, traditional approach is to use Mason's Non-Touching Loop Rule
- Generally, do not simple cascades and only have correlations – quantum correlations!
- More elegant schemes are available
- Analysis closely related to Feynman diagrams

Quantum analysis of circuits and systems….

What is the quantum version of scattering parameter analysis?

Quantise the longitudinal modes on a transmission line

- Quantum states characterise the statistics of measureable outcomes
- Quantum operators correspond to measurements, can include disturbance

$$\frac{\hat{v}^{+}(\omega)}{\sqrt{Z_0}} = \frac{1}{2\sqrt{Z_0}}\left[\hat{v}(\omega) + \hat{i}(\omega)Z_0\right] = \left[\frac{\hbar\omega}{2}\right]^{1/2} \hat{a}(\omega)$$

$$\frac{\hat{v}^{-}(\omega)}{\sqrt{Z_0}} = \frac{1}{2\sqrt{Z_0}}\left[\hat{v}(\omega) - \hat{i}(\omega)Z_0\right] = \left[\frac{\hbar\omega}{2}\right]^{1/2} \hat{b}(\omega).$$

Be careful, voltage and current commute!

Voltage and its time derivative do not commute - incompatible observables

Complex travelling wave amplitudes are not observables

– only the quadrature components are measureable, but these are subjected to Heisenberg's uncertainty relationship

Representation already brings quantum behavior …..

Annihilation operators connected through scattering parameters (still classical?):

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} + \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}$$

$$\hat{\mathsf{b}} = \mathsf{S}\hat{\mathsf{a}} + \hat{\mathsf{n}},$$

All operators act on the state space of the independent states – incoming mode amplitudes

$$\text{M-port} \quad |p_1\rangle \cdots |p_m\rangle \cdots |p_M\rangle; \quad \text{or for two-port} \quad |p_1\rangle|p_2\rangle$$

Individual incoming fields can be in coherent states, thermal states, squeezed states or some more exotic mixture

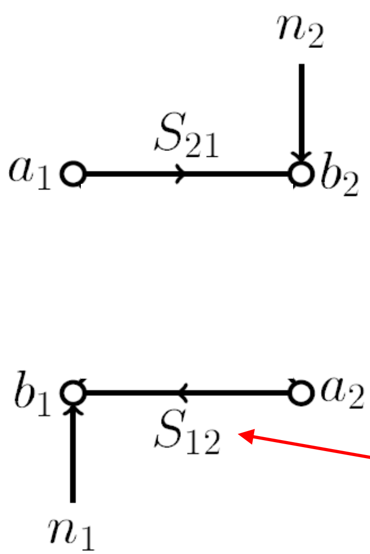
Vacuum states of seemingly unconnected ports are needed to ensure that the annihilation operators of outgoing waves satisfy bosonic commutations relationships….

Probing system by creating/annihilating photons at input, and looking for photons appearing/disappearing at output!

Scattering parameters are complex probability amplitudes – although still no QM!

Consider an ideal amplifier (designed to be match to Zo):

$$S_{11} = 0$$
$$S_{22} = 0$$



$$\hat{b}_2 = S_{21}\hat{a}_1 + \hat{n}_2$$
$$[\hat{b}_2, \hat{b}_2^\dagger] = |S_{21}|^2[\hat{a}_1, \hat{a}_1^\dagger] + [\hat{n}_2, \hat{n}_2^\dagger]$$
$$[\hat{n}_2, \hat{n}_2^\dagger] = 1 - |S_{21}|^2$$

Bad news – noise may travel backwards!

$$s^n(\omega) = \frac{\hbar\omega}{2}\langle\left\{\hat{n}_2, \hat{n}_2^\dagger\right\}\rangle \geq -\frac{\hbar\omega}{2}\langle\left[\hat{n}_2, \hat{n}_2^\dagger\right]\rangle = \frac{\hbar\omega}{2}\left(|S_{21}|^2 - 1\right)$$

$$T_{\mathrm{n}} \geq \frac{\hbar\omega}{2k}\left(1 - \frac{1}{|S_{21}|^2}\right)$$

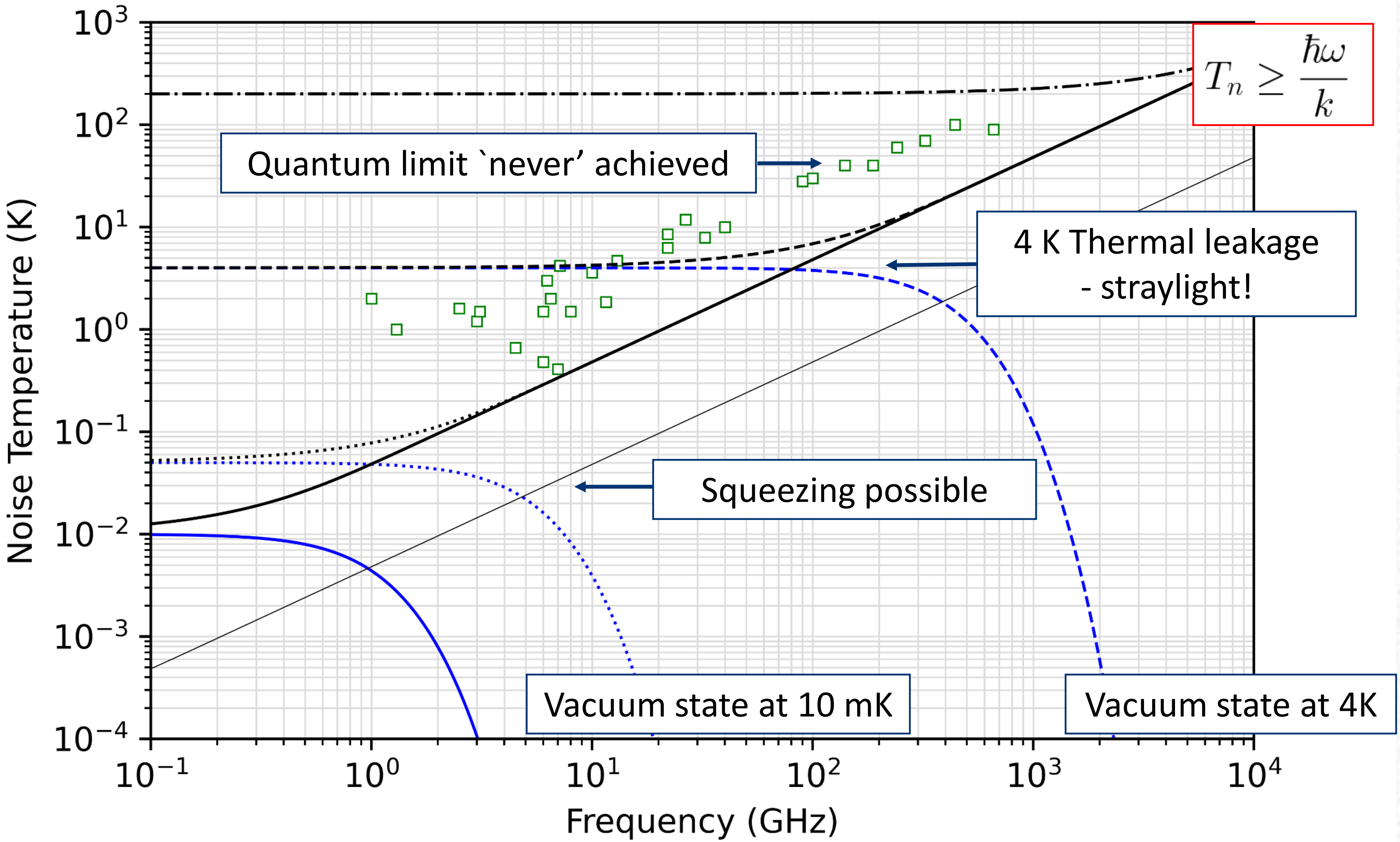
The noise temperature of any phase preserving coherent amplifier (or receiver) has a lower bound

Standard Quantum Limit – increase linearly with frequency

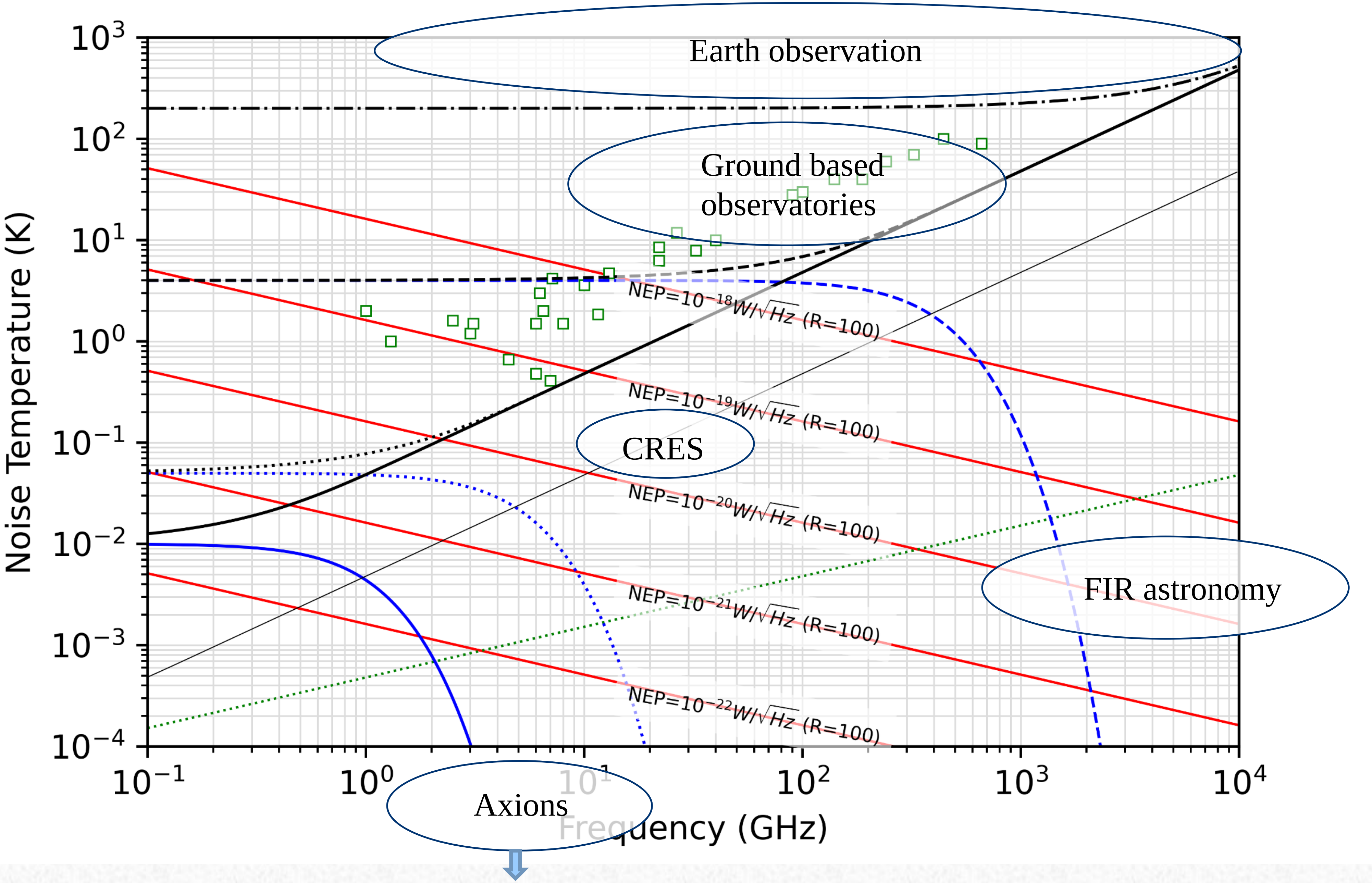
$$T_n \geq \frac{\hbar\omega}{2k}$$

Add in noise from vacuum state fluctuations of source

$$T_n \geq \frac{\hbar\omega}{k}$$



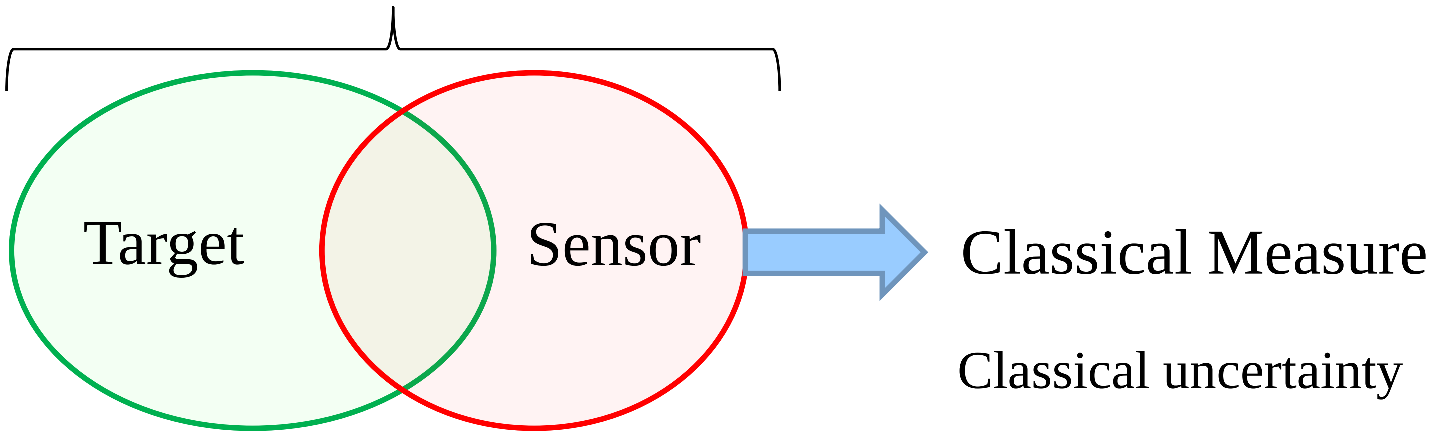
$T_n \geq \frac{\hbar\omega}{k}$
Quantum limit `never' achieved
4 K Thermal leakage
- straylight!
Squeezing possible
Vacuum state at 10 mK
Vacuum state at 4K
Noise Temperature (K)
Frequency (GHz)
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-1}
10^0
10^1
10^2
10^3
10^4



Earth observation
Ground based observatories
CRES
FIR astronomy
Axions
Noise Temperature (K)
Frequency (GHz)
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-1}
10^0
10^1
10^2
10^3
10^4
NEP=10^{-18}W/$\sqrt{Hz}$ (R=100)
NEP=10^{-19}W/$\sqrt{Hz}$ (R=100)
NEP=10^{-20}W/$\sqrt{Hz}$ (R=100)
NEP=10^{-21}W/$\sqrt{Hz}$ (R=100)
NEP=10^{-22}W/$\sqrt{Hz}$ (R=100)

Massive opportunities for pushing at the quantum limit….

Quantum interaction – time evolution of coupled system – quantum uncertainty



Forced into asking questions about the influence of vacuum fluctuations, back action, squeezing and entanglement in the context of electrical sensors and circuits

But where do the scattering parameters come from (not just scalars) ?

- Classically, they are the Fourier transforms of time-independent impulse response functions

- The output is given by convolution in the time domain or multiplication in the frequency domain

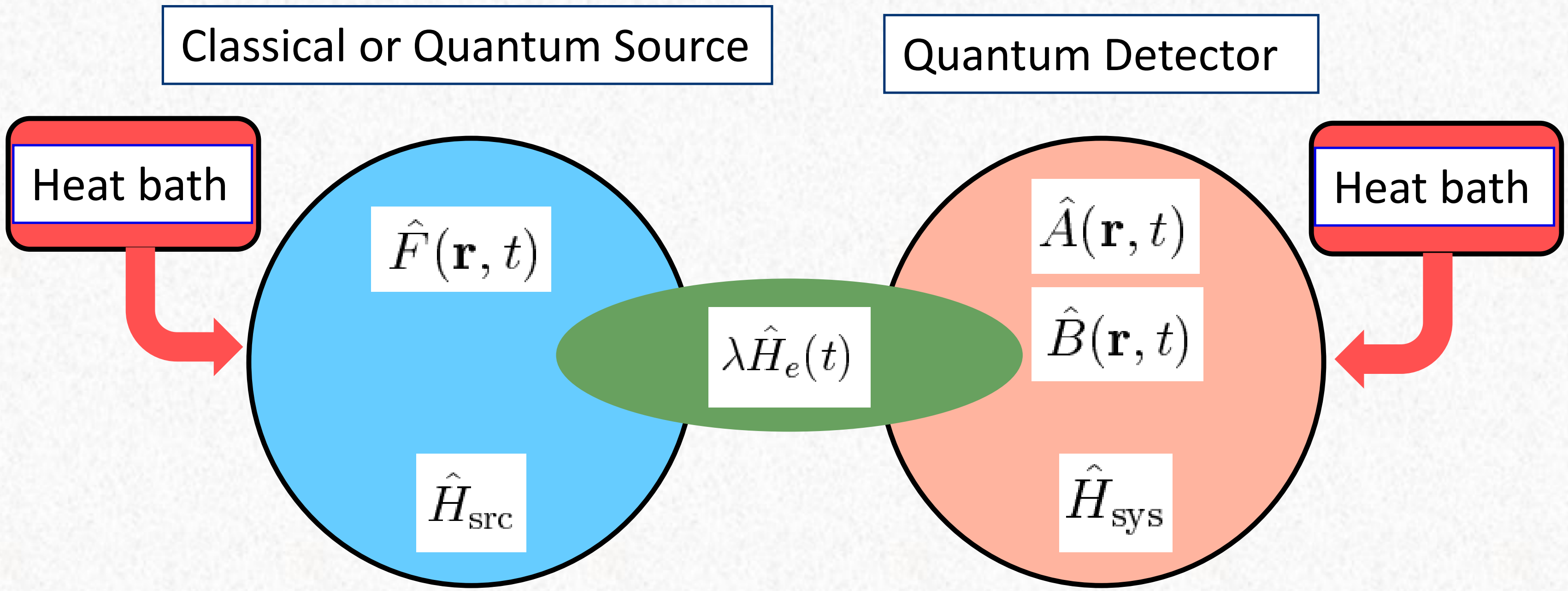
Fourier Transform

$$b(t) = \int_{-\infty}^{+\infty} s(\tau) a(t - \tau) \, \mathrm{d}\tau$$

$$b(\omega) = S(\omega) a(\omega)$$

For example:

- Electrical circuits
- complex valued surface impedance of an absorptive pixel
- optical reflection coefficient
- magnetic susceptibility



$$\begin{aligned}\hat{H}(t) &= \hat{H}_{\mathrm{sys}} + \hat{H}_{\mathrm{src}} + \lambda \hat{H}_e(t) \\ &= \hat{H}_0 + \lambda \hat{H}_e(t),\end{aligned}$$

$$|\chi\rangle = |\chi\rangle_{\mathrm{sys}} \otimes |\chi\rangle_{\mathrm{src}}$$

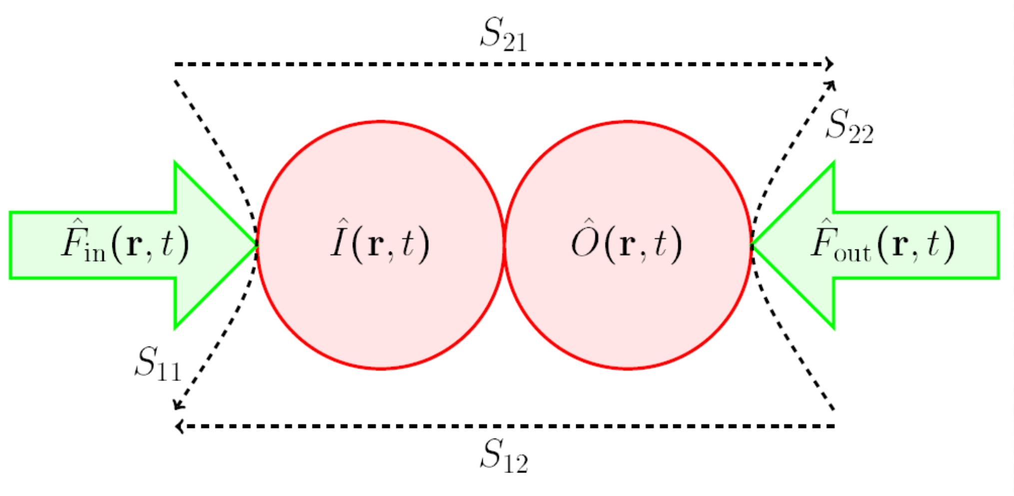
Stimulated photon absorption (power detected) ⇒

⇐ Stimulated/spontaneous photon emission

⇔ Reactive and dissipative power

Quantum response functions from first-order perturbation theory:

- Generalised force acts on some physical, measureable property of the system
- Some other measureable property is disturbed



$$\langle \Delta \hat{I}^H(\mathbf{r},t)\rangle_{t_0} = \frac{-i}{\hbar}\int_{-\infty}^{+\infty} dt'\,\theta(t-t')\int_{\mathcal{V}} \mathrm{d}^3\mathbf{r}'\,\langle\left[\hat{I}^I(\mathbf{r},t),\hat{I}^{\rm I}(\mathbf{r}',t')\right]\rangle_{t_0}\cdot\langle\hat{F}_{\rm in}(\mathbf{r}',t')\rangle_{t_0}$$

$$\langle \Delta \hat{O}^H(\mathbf{r},t)\rangle_{t_0} = \frac{-i}{\hbar}\int_{-\infty}^{+\infty} dt'\,\theta(t-t')\int_{\mathcal{V}} \mathrm{d}^3\mathbf{r}'\,\langle\left[\hat{O}^I(\mathbf{r},t),\hat{I}^{\rm I}(\mathbf{r}',t')\right]\rangle_{t_0}\cdot\langle\hat{F}_{\rm in}(\mathbf{r}',t')\rangle_{t_0}.$$

- The response function is a quantum correlation (Green's) function
- Introduction of solid-state excitations lead to excitations elsewhere – probability amplitudes – no need to track every degree of freedom

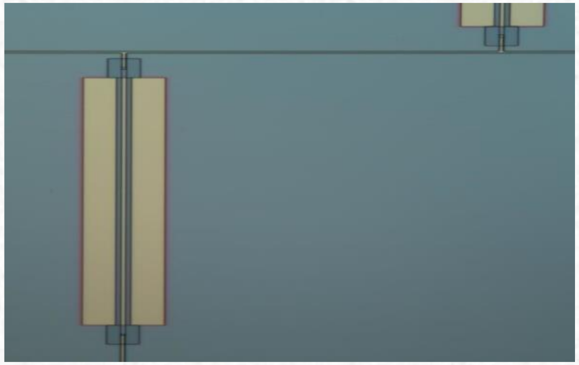
Superconducting electronics is an excellent platform:

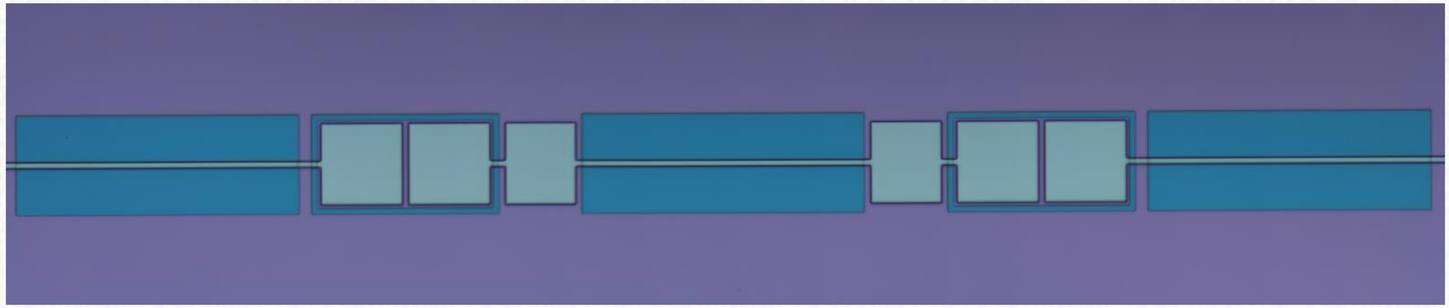
Material	Pair breaking	Wavelength range
Passive	No	Microwave to submm
SQUID	No	RF
SIS	No	Submm
TES	Yes	Submm, FIR, Optical and X-ray
KID	Yes	Submm, FIR
Paramp	No	Microwave, MMwave
SPNWD	Yes	Optical

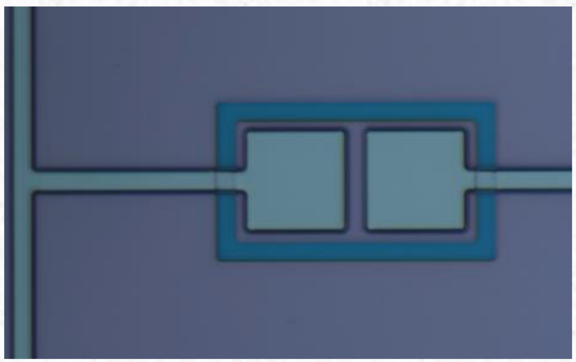
Material	T_c (K)	E_g (meV)	f_g (GHz)
NbN	16	4.8	1160
Nb	9.3	2.8	680
Ta	4.48	1.35	325
Al	1.2	0.36	90
Mo	0.9	0.27	65
Ti	0.39	0.11	26

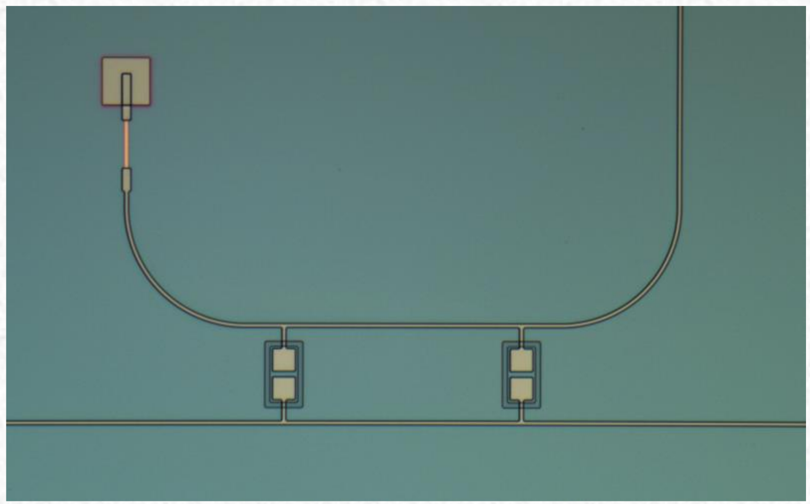
Passive 1 GHz – 1 THz components:

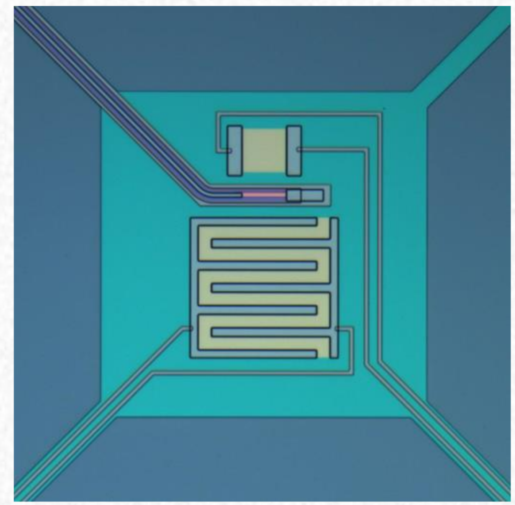
- \`Essentially’ no ohmic loss + slow wave effect due to surface inductance











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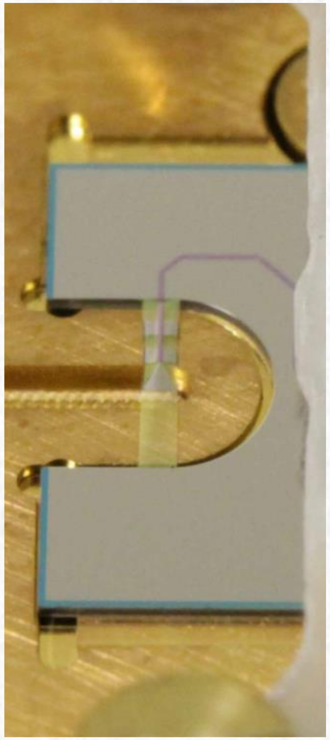
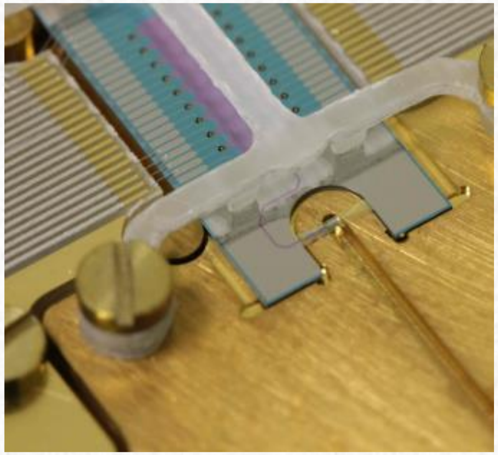
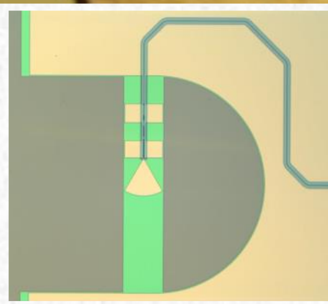
S. Zhou, D. J. Goldie, C.N. Thomas, and S. Withington, *Calculation and measurement of of critical temperature in thin superconducting multilayers*, Supercond. Sci. Tech. **31**, 105004 (2018)

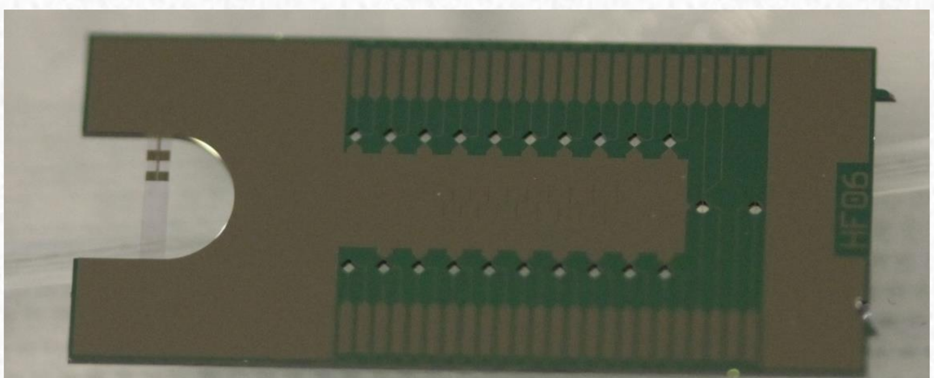
Superconducting detectors 50 GHz – 1.2 THz:

Waveguide probe on SiN – 200nm

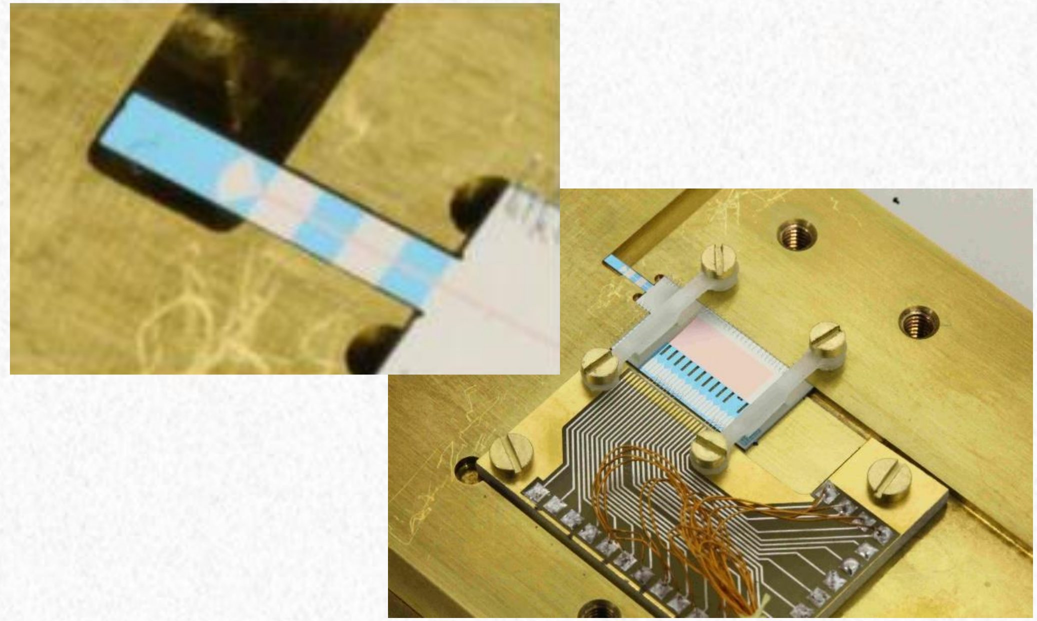




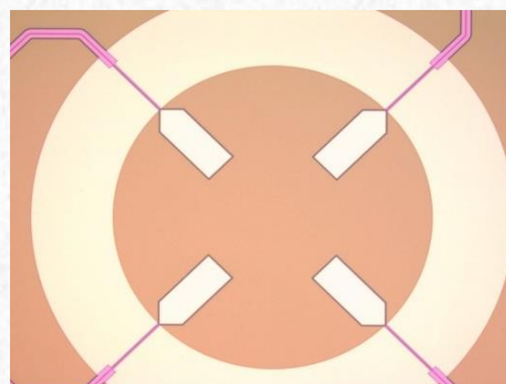




Waveguide probe in Si - 200µm

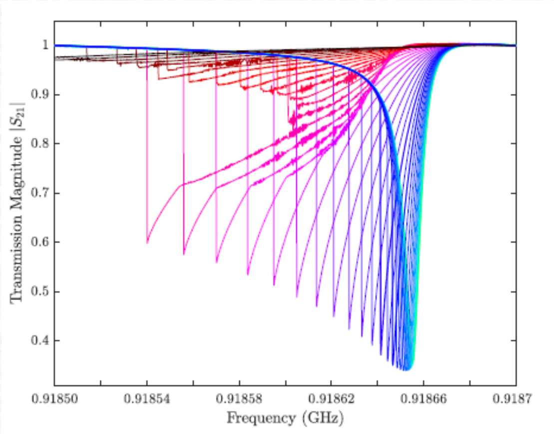


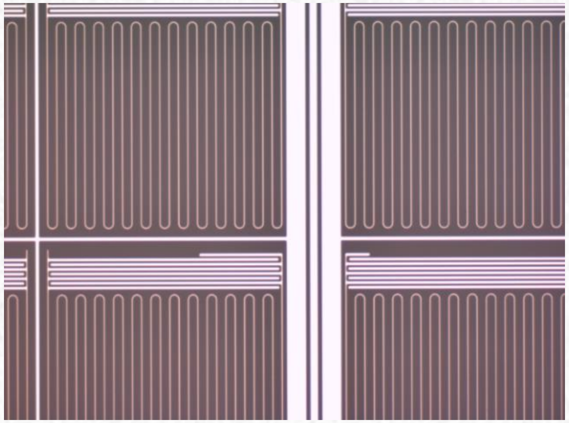
Circular waveguide probe on SiN - 200µm

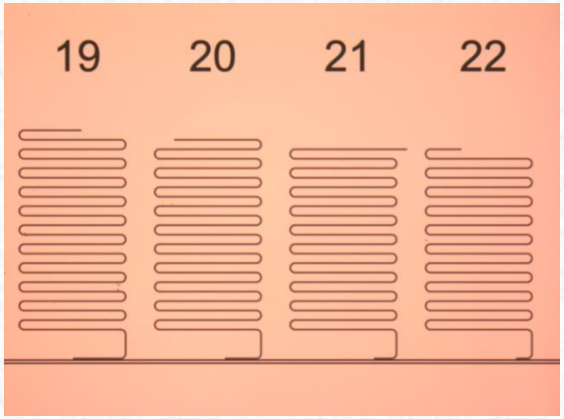


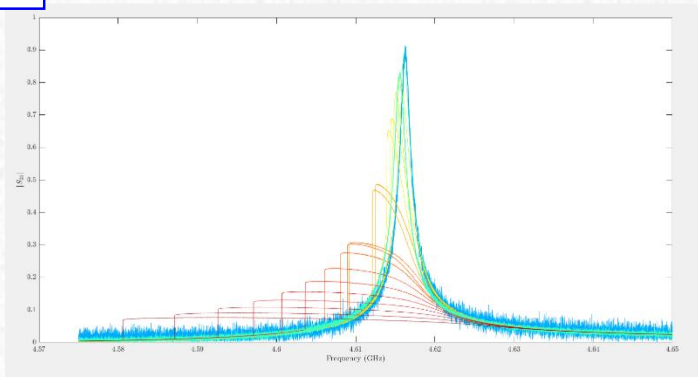
Nonlinear superconducting resonators:

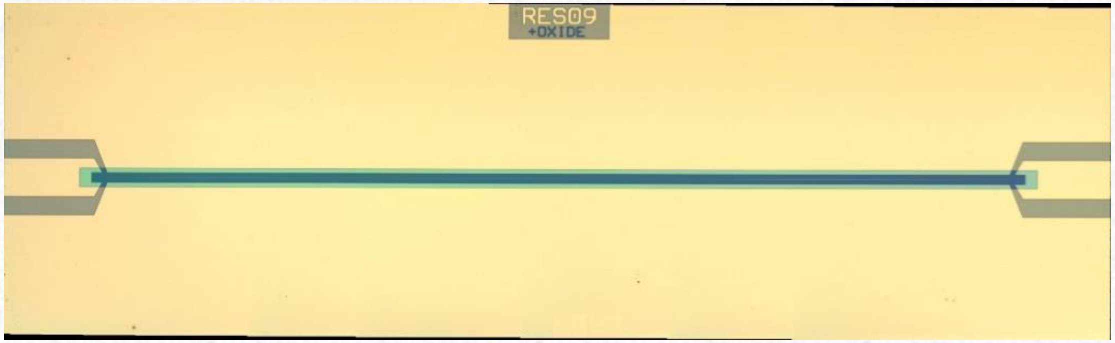
- At each frequency a certain amount of power is dissipated
- Kinetic inductance determines the resonant frequency
- Quasiparticle heating or TLS determines the Q
- As the frequency is swept, complex curves result
- Pick up samples from a resonance curve that is changing

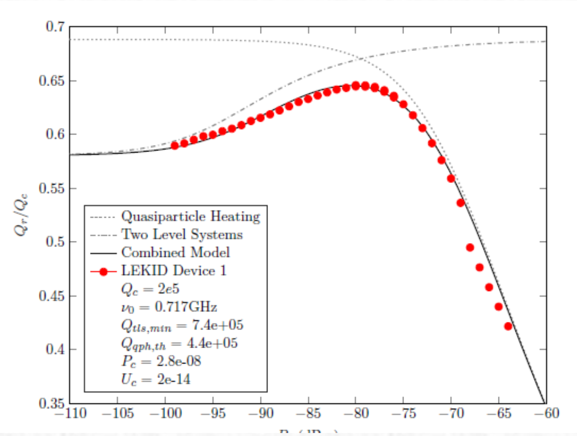












C Thomas, S Withington, Z Sun, T Skyrme, D Goldie, Nonlinear effects in superconducting thin film microwave resonators. New Journal of Physics, vol. 22 (2020).

Numerous interesting questions relating to descriptions of this kind:

Lots of fun to be had………

Professor Stafford Withington, University of Oxford:

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Questions:

Why is noise temperature only a partial representation?

What is a Coherent State?

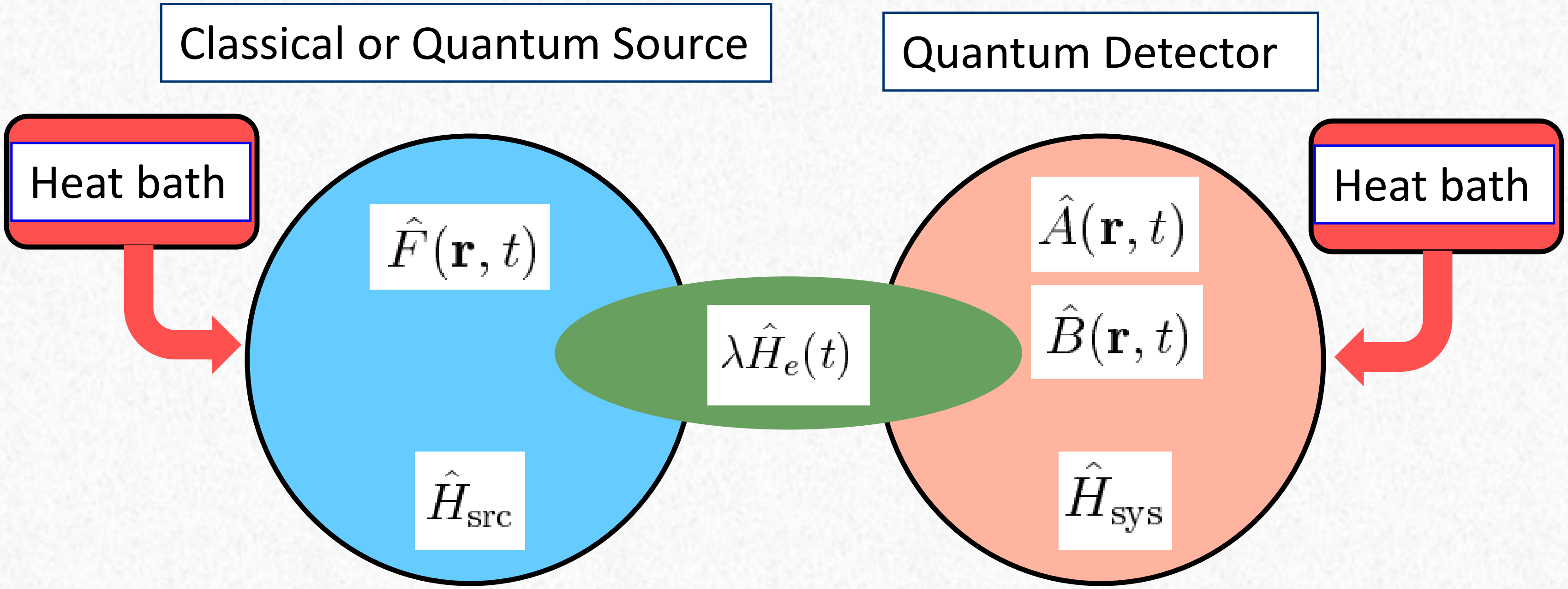
What is a Thermal State?

Why are quantum networks represented in in terms of weighted linear combinations of annihilation operators?

How are signal flow graphs solved in general terms?

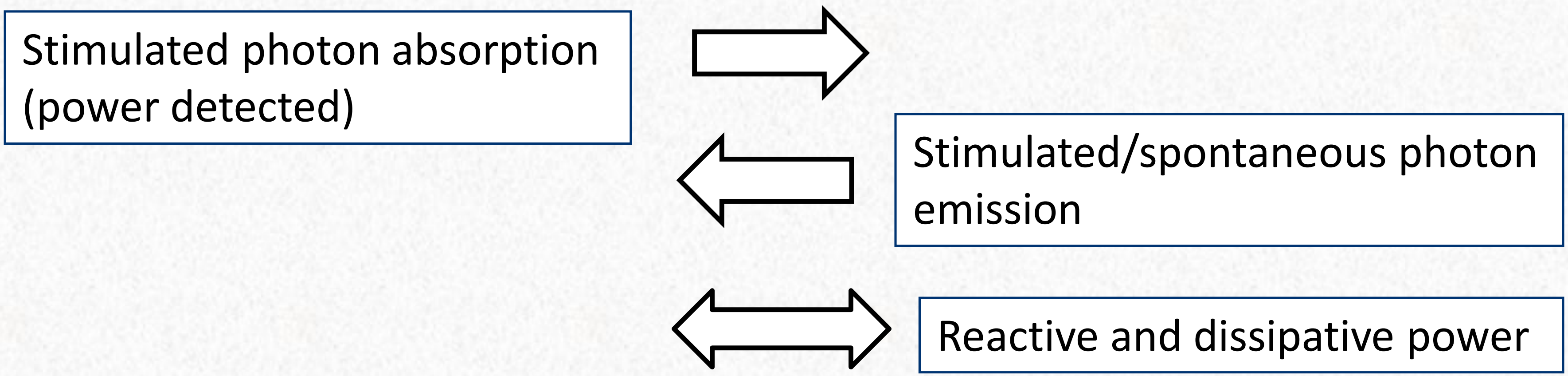
What is the nature of Quantum Response Functions

What does `state collapse’ mean in the context of an electrical circuit?



$$\hat{H}(t) = \hat{H}_{\rm sys} + \hat{H}_{\rm src} + \lambda\hat{H}_e(t) = \hat{H}_0 + \lambda\hat{H}_e(t),$$

$$|\chi\rangle = |\chi\rangle_{\rm sys} \otimes |\chi\rangle_{\rm src}$$



Coupling Hamiltonian:

$$\hat{H}_e(t) = \int \hat{F}(\mathbf{r},t)\cdot\hat{A}(\mathbf{r},t)\, d^3\mathbf{r}.$$

F(r,t) is generalized force (source) – classical or quantum

Dielectric loss Potential interacting with electron gas: $\hat{H}_e(t) = -e\int d^3\mathbf{r}\, V(\mathbf{r},t)\hat{n}(\mathbf{r})$, where $V(\mathbf{r},t)$ is the classical potential, and $\hat{n}(\mathbf{r})$ is the charge density operator. We use $F(\mathbf{r},t) \equiv -eV(\mathbf{r},t)$ and $\hat{A}(\mathbf{r}) \equiv \hat{n}(\mathbf{r})$ to give the response function which is the polarizability.

Magnetic loss Magnetic dipole moment (or spin) in magnetic field: $\hat{H}_e(t) = -\int d^3\mathbf{r}\,\hat{\mathbf{m}}(\mathbf{r})\cdot\mathbf{B}(\mathbf{r},t)$, where $\mathbf{m}(\mathbf{r})$ is the magnetic dipole moment density operator, or spin density operator as appropriate. We use $F(\mathbf{r},t) \equiv \mathbf{B}(\mathbf{r},t)$ and $\hat{A}(\mathbf{r}) \equiv \hat{\mathbf{m}}$ to give the response function, which gives the paramagnetic susceptibility.

Electromagnetic loss For a single particle having charge e, mass m, and momentum operator $\hat{\mathbf{p}}(t)$ in an electromagnetic field have vector potential $\mathbf{A}(\mathbf{r},t)$. Say $\hat{H}_e(t) = (-e/m)\hat{\mathbf{p}}(t)\cdot\mathbf{A}(\mathbf{r},t)$ and so $F(\mathbf{r},t) \equiv (-e/m)\mathbf{A}(\mathbf{r},t)$ and $\hat{\mathbf{A}}(t) \equiv \hat{\mathbf{p}}(t)$. Or, we could integrate over the momentum density operator, $\hat{\mathbf{A}}(\mathbf{r},t) \equiv \hat{\mathbf{p}}(\mathbf{r},t)$

Linear response theory according to Kubo:

Some characteristic of the system, which may respond to the applied source:

$$\hat{B}(\mathbf{r},t) = \hat{B}_0(\mathbf{r},t) + \lambda\Delta\hat{B}(\mathbf{r},t).$$

$$\begin{aligned}\hat{B}^H(\mathbf{r},t) &= \hat{U}^\dagger(t,t_0)\hat{B}^H(\mathbf{r},t_0)\hat{U}(t,t_0)\\ &= \hat{S}^\dagger(t,t_0)\hat{B}^I(\mathbf{r},t)\hat{S}(t,t_0).\end{aligned}$$

$$\hat{S}(t,t_0) = 1 - \lambda\frac{i}{\hbar}\int_{t_0}^{t} dt'\,\hat{H}_e^I(t').$$

$$\Delta\hat{B}^H(\mathbf{r},t) = \frac{-i}{\hbar}\int_{t_0}^{t} dt'\left[\hat{B}^I(\mathbf{r},t),\hat{H}_e^I(t')\right]$$

$$\hat{H}_e^I(t) = \int \hat{F}^I(\mathbf{r},t)\cdot\hat{A}^I(\mathbf{r},t)\,d^3\mathbf{r},$$

$$\langle\Delta\hat{B}^H(\mathbf{r},t)\rangle = \frac{-i}{\hbar}\int d^3\mathbf{r}'\int_{t_0}^{t} dt'\left\langle\left[\hat{B}^I(\mathbf{r},t),\hat{A}^I(\mathbf{r}',t')\right]\right\rangle_{\mathrm{sys},0}\cdot\left\langle\hat{F}^I(\mathbf{r}',t')\right\rangle_{\mathrm{src},0}.$$

If the source field is wave in a coherent state, classical response theory follows

Linear response functions:

$$\Delta\langle\hat{B}\rangle(\mathbf{r},t) = \int\int \chi(\mathbf{r},t;\mathbf{r}',t')F(\mathbf{r}',t')\,d^3\mathbf{r}'dt.$$

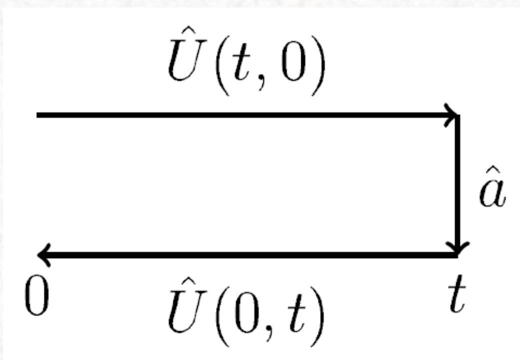
$$\chi(\mathbf{r},t;\mathbf{r}',t') = \frac{-i}{\hbar}\theta(t-t')\left\langle\left[\hat{A}(\mathbf{r},t),\hat{A}(\mathbf{r}',t')\right]^I\right\rangle,$$

$$\chi(\mathbf{r},t;\mathbf{r}',t') = \frac{-i}{\hbar}\theta(t-t')\left\langle\left[\hat{B}(\mathbf{r},t),\hat{A}(\mathbf{r}',t')\right]^I\right\rangle.$$

Uncoupled source and detector – time evolution in Heisenberg picture

$$\langle A\rangle(t) = \langle\psi(t_0)|\hat{A}^H(t)|\psi(t_0)\rangle$$

Heisenberg
$\lvert\psi^H(t)\rangle = \lvert\psi^H(t_0)\rangle$
$\hat{A}^H(t) = \hat{U}^\dagger(t,t_0)\hat{A}^s(t)\hat{U}(t,t_0)$ (time-dependent observable)
$\hat{A}^H(t) = \hat{U}^\dagger(t,t_0)\hat{A}^H(t_0)\hat{U}(t,t_0)$ (time-independent observable)
$\hat{\rho}^H(t) = \hat{\rho}^H(t_0) = \hat{\rho}^S(t_0)$
$\langle A\rangle(t) = \mathrm{Tr}\left[\hat{\rho}^H\hat{A}^H(t)\right]$



$$\hat{a}^H(t) = e^{-i\omega t}\hat{a}$$

Coupled source and detector – time evolution in Interaction Picture

$$= \hat{H}_0 + \lambda \hat{H}_e(t),$$

$$|\psi^I(t)\rangle = e^{+i\hat{H}_0(t-t_0)/\hbar}|\psi^S(t)\rangle$$
$$= e^{+i\hat{H}_0(t-t_0)/\hbar}\hat{U}(t,t_0)|\psi^S(t_0)\rangle$$

Interaction
$\lvert\psi^I(t)\rangle = \hat{S}(t,t_0)\lvert\psi^I(t_0)\rangle$
$\hat{A}^I(t) = e^{+i\hat{H}_0(t-t_0)/\hbar}\hat{A}^S(t)e^{-i\hat{H}_0(t-t_0)/\hbar}$ (time-dependent observable)
$\hat{A}^I(t) = e^{+i\hat{H}_0(t-t_0)/\hbar}\hat{A}^I(t_0)e^{-i\hat{H}_0(t-t_0)/\hbar}$ (time-independent observable)
$\hat{\rho}^I(t) = \hat{S}(t,t_0)\hat{\rho}^I(t_0)\hat{S}^\dagger(t,t_0)$
$\langle A\rangle(t) = \mathrm{Tr}\left[\hat{\rho}^I(t)\hat{A}^I(t)\right]$

$$\hat{S}(t,t_0) = \overleftarrow{\mathcal{T}}\left[\exp\left\{\left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} dt'\hat{V}^I(t')\right\}\right] \qquad t \geq t_0,$$

Thermal Density Operator to calculate expectation values:

$$\hat{\rho} = \sum_{i=1}^{I} P_i |\psi_i\rangle\langle\psi_i|$$

$$\hat{\rho} = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-E_n/kT} |\phi_n\rangle\langle\phi_n|$$

$$\hat{\rho} = \frac{1}{Z} e^{-\hat{H}/kT},$$

$$\langle A \rangle = \mathrm{Tr}\left[\hat{\rho}\hat{A}\right]$$

$$n_i = \langle \hat{n}_i \rangle = \frac{1}{e^{\hbar\omega_i/k_B T} - 1}.$$

$$(\Delta n_i)^2 = \langle \Delta \hat{n}_i^2 \rangle = \frac{e^{\hbar\omega_i/k_B T}}{(e^{\hbar\omega_i/k_B T} - 1)^2} = \frac{1}{4\sinh^2(\hbar\omega_i/2k_B T)},$$

State space is tensor product of state spaces of source and detector:

$$|\chi\rangle = |\chi\rangle_{\mathrm{sys}} \otimes |\chi\rangle_{\mathrm{src}}$$

Multi-particle Fock space of system

Multi-mode Fock space of source $|\psi\rangle = \prod_{\alpha,\mathbf{k}} |n_\alpha(\mathbf{k})\rangle,$

$$\hat{\rho} = \hat{\rho}_{\mathrm{sys}} \otimes \hat{\rho}_{\mathrm{src}}$$

Only assume that source and system are not entangled prior to interaction being applied.