Q1. $\partial_t(\phi+\underline{\delta\phi}) + \underbrace{\tfrac{1}{2}\nabla(\phi+\delta\phi)\cdot\nabla(\phi+\delta\phi)}_{\frac{1}{2}|\nabla\phi|^2 + \underline{\nabla\phi\cdot\nabla\delta\phi} + \frac{1}{2}|\nabla\delta\phi|^2} + gh + \underline{g\delta h} = 0$

linear terms

linearise: $(\partial_t + \nabla\phi\cdot\nabla)\delta\phi + g\delta h = 0$

Ⓐ $(\partial_t + \vec{v}\cdot\nabla)\delta\phi + g\delta h = 0$

$\partial_t(h+\underline{\delta h}) + \underbrace{\nabla\cdot((h+\delta h)\nabla(\phi+\delta\phi))}_{\nabla\cdot(h\nabla\phi + \underline{h\nabla\delta\phi} + \underline{\delta h\nabla\phi} + \delta h\nabla\delta\phi)} = 0$

$\nabla\phi = \vec{v}$

Ⓑ $(\partial_t + \nabla\cdot\vec{v})\delta h + \nabla\cdot(h\nabla\delta\phi) = 0$

acts on everything to the right

act on Ⓐ with $(\partial_t + \nabla\cdot\vec{v})$ then sub. in Ⓑ

$(\partial_t + \nabla\cdot\vec{v})(\partial_t + \vec{v}\cdot\nabla)\delta\phi + g(\partial_t + \nabla\cdot\vec{v})\delta h = 0$

$$\underline{(\partial_t + \nabla\cdot\vec{v})(\partial_t + \vec{v}\cdot\nabla)\delta\phi - g\nabla\cdot(h\nabla\delta\phi) = 0}$$

Q2. $h = \text{const}$ & $\vec{v} = \text{const}$

$\therefore$ plane waves $\delta\phi = Ae^{i\vec{k}\cdot\vec{x} - i\omega t}$ are solutions

const. amplitude

$\partial_t \to -i\omega$, $\nabla \to i\vec{k}$

wave equation gives

$(-i\omega + i\vec{k}\cdot\vec{v})(-i\omega + i\vec{v}\cdot\vec{k}) - ig\vec{k}\cdot(ih\vec{k}) = 0$

$\underline{(\omega - \vec{v}\cdot\vec{k})^2 = ghk^2}$, $k = \|\vec{k}\|$

comoving frame: $\vec{x}' = \vec{x} - \vec{v}t$, $t' = t$ inverse transformation $\vec{x} = \vec{x}' + \vec{v}t'$, $t = t'$

$\delta\phi \sim e^{i\vec{k}\cdot(\vec{x}'+\vec{v}t') - i\omega t'}$

$\sim e^{i\vec{k}\cdot\vec{x}' - i\underbrace{(\omega - \vec{v}\cdot\vec{k})}_{\Omega\,=\,\text{comoving frequency}}t'}$

$\therefore$ dispersion relation in this frame

$\Omega^2 = ghk^2 \longrightarrow \Omega = \pm\sqrt{gh}\,k$

phase velocity: $v_p = \frac{\Omega}{k} = \pm\sqrt{gh}$

group velocity: $v_g = \partial_k\Omega = \pm\sqrt{gh}$

$v_p = v_g$ for "relativistic" dispersion relation (i.e. $\Omega^2 \sim k^2$)

wave speed $= \underline{c = |v_g| = \sqrt{gh}}$

Q3. Write out components

Ⓒ $\partial_t(f^{tt}\partial_t\delta\phi) + \partial_t(f^{ti}\partial_i\delta\phi) + \partial_i(f^{it}\partial_t\delta\phi) + \partial_i(f^{ij}\partial_j\delta\phi) = 0$

Write wave equation in this form.
i.e. convert from vector notation to index notation.

$$\partial_t^2\delta\phi + \underbrace{\nabla\cdot(v\nabla\delta\phi)}_{\partial_i(v^i\partial_t\phi)} + \underbrace{\partial_t(v\cdot\nabla\delta\phi)}_{\partial_t(v^i\partial_i\delta\phi)} + \underbrace{\nabla\cdot(v\,v\cdot\nabla\delta\phi)}_{\partial_i(v^iv^j\partial_j\delta\phi)} - \underbrace{\nabla\cdot(c^2\nabla\delta\phi)}_{\partial_i(c^2\partial^i\delta\phi)=\partial_i(c^2\delta^{ij}\partial_j\delta\phi)} = 0$$

Ⓓ $-\partial_t^2\delta\phi - \partial_i(v^i\partial_t\delta\phi) - \partial_t(v^i\partial_i\delta\phi) + \partial_i((c^2\delta^{ij} - v^iv^j)\partial_j\delta\phi) = 0$

comparing Ⓓ with Ⓒ

$f^{tt} = -1$, $f^{it} = -v^i$, $f^{ti} = -v^i$, $f^{ij} = c^2\delta^{ij} - v^iv^j$

$$f^{\mu\nu} = \begin{pmatrix} -1 & -v^j \\ -v^i & c^2\delta^{ij} - v^iv^j \end{pmatrix}$$

Q4. want $\sqrt{-g}\,g^{\mu\nu} = f^{\mu\nu}$ & validity condition

constant $\therefore$ components of $g^{\mu\nu}$ are proportional to those of $f^{\mu\nu}$

let $g^{\mu\nu} = B\begin{pmatrix} -1 & -v^j \\ -v^i & c^2\delta^{ij} - v^iv^j \end{pmatrix}$ with $B$ constant.

compare with matrix $M$ given in question

$B = \frac{1}{c^2 C}$, $b^i = v^i$, $a = c$

$$\therefore \quad g^{\mu\nu} = \frac{1}{c^2C}\begin{pmatrix} -1 & -v^j \\ -v^i & c^2\delta^{ij} - v^iv^j \end{pmatrix}$$

$$g_{\mu\nu} = C\begin{pmatrix} c^2 - v^iv^i & -v^j \\ -v^i & \delta^{ij} \end{pmatrix}$$

$$g = -c^2C^{d+1} \longrightarrow \sqrt{-g} = cC^{\frac{d+1}{2}}$$

$$\sqrt{-g}\,g^{\mu\nu} = \frac{C^{\frac{d-1}{2}}}{c}f^{\mu\nu}$$

to make identification, require

$$C^{\frac{d-1}{2}} = c = \sqrt{gh}$$

this cannot be satisfied for $d=1$

$$(\partial_t + \nabla\cdot v)(\partial_t + v\cdot\nabla)\delta\phi - \nabla\cdot(c^2\nabla\delta\phi) = 0$$

$\Updownarrow$ provided $\boxed{d>1}$

$$\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\delta\phi) = 0, \quad g_{\mu\nu} = c^{\frac{2}{d-1}}\begin{pmatrix} c^2 - v^iv^i & -v^j \\ -v^i & \delta^{ij} \end{pmatrix}$$