

1.1 Brief introduction to cosmology

General relativity:

- Geometry of universe depends on matter and energy content
- Geometry is characterised by metric, which defines distances between points in spacetime

Homogeneous + isotropic flat universe:

$$ds^2 = -dt^2 + a(t)^2 [dx^2 + dy^2 + dz^2]$$

↑
scale factor: describes how space expands

Hubble expansion rate: $H(t) = \frac{1}{a} \frac{da}{dt}$

Friedmann equation relates H to energy density ρ :

$$H^2 = \frac{8\pi G \rho}{3} = \frac{8\pi}{3 M_p^2} \rho$$

↑ $M_p \approx 1.2 \cdot 10^{19} \text{ GeV}$
(Planck mass)

What is ρ ?

1) Universe filled with non-rel. particles (matter)

$$\rho_{nr}(t) = m \cdot n_{nr}(t) = \frac{m \cdot N}{V(t)} \sim a(t)^{-3}$$

2) Universe filled with rel. particles (radiation)

$$\rho_r(t) = E_r(t) \cdot n_r(t) \sim a(t)^{-4}$$

$$\uparrow \text{redshift: } E_r(t) \sim \lambda_r(t)^{-1} \sim a(t)^{-1}$$

$$\Rightarrow \frac{\dot{a}^2}{a^2} \sim a^{-4} \Rightarrow a(t) \sim t^{1/2}$$

$$\Rightarrow H = \frac{\dot{a}}{a} \sim t^{-1} \sim a^{-2}$$

3) Universe filled with thermal plasma with temperature T

$$f(\vec{p}) = \frac{1}{e^{E/T} \pm 1} \quad \begin{array}{l} +: \text{ fermions} \\ -: \text{ bosons} \end{array}$$

For $T \gg m$ (such that $p \approx E$)

$$S = \frac{g}{(2\pi)^3} \int E f(\vec{p}) d^3p = \begin{cases} \frac{\pi^2}{30} g T^4 & (\text{bosons}) \\ \frac{7}{8} \frac{\pi^2}{30} g T^4 & (\text{fermions}) \end{cases}$$

$$\Rightarrow H(T) = \sqrt{\frac{8\pi^3}{90}} \sqrt{g_*} \frac{T^2}{M_p} \approx 1.66 \sqrt{g_*} \frac{T^2}{M_p}$$

$\uparrow$ Total number of rel. d.o.f.

$$\Rightarrow T \sim a(t)^{-1} \sim t^{-1/2}$$

↳ Universe expands and cools down

↳ Rel. particles lose energy and become non-rel.

$$f(\vec{p}) \stackrel{T \ll m}{=} e^{-E/T} \approx e^{-(m + \frac{p^2}{2m})/T}$$

$$\Rightarrow \mathcal{S} = m \cdot g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-m/T}$$

↑
exponential
suppression

↳ "non-rel. particles annihilate away"

1.2 Boltzmann equation

↳ Differential equation for evolution of DM phase space density in the early universe:

$$\hat{L} [f(\vec{x}, \vec{p}, t)] = \hat{C} [f(\vec{x}, \vec{p}, t)]$$

↑ Liouville operator (phase space evolution) ↑ collision operator (interactions)

Simplifications: - Homogeneous + isotropic universe

$$\Rightarrow f(\vec{x}, \vec{p}, t) = f(E, t)$$

- Integrate over momenta:

$$g \int \frac{d^3 p}{(2\pi)^3} f(E, t) = n(t)$$

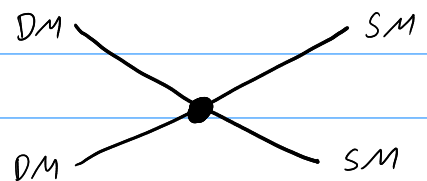
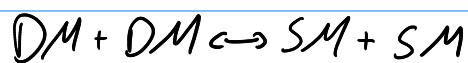
$$g \int \frac{d^3 p}{(2\pi)^3} \hat{L}[f] = \frac{dn}{dt} + 3Hn$$

For non-interacting particles: $\hat{L}[f] = 0$

$$\Rightarrow \frac{dn}{dt} + 3Hn = 0 \Rightarrow \frac{dn}{n} = -3 \frac{da}{a}$$

$$\Rightarrow n \sim a^{-3}$$

Consider general interaction



with cross section $\sigma_{ann}(\sqrt{s})$
 $\uparrow$ centre-of-mass energy

Annihilation rate $\Gamma_{ann} = \sigma_{ann} \cdot j = \sigma_{ann} \cdot v_{rel} \cdot n_{DM}$
 $\uparrow$ particle flux $\uparrow$ rel. velocity

$\sqrt{s}$ and v_{rel} vary from collision to collision

$\Rightarrow$ need to perform thermal average

$$\Gamma_{ann} = \langle \sigma_{ann} v_{rel} \rangle n_{DM}$$

$$g \int \frac{d^3p}{(2\pi)^3} \hat{C}[p_{DM}] = -n_{DM} \Gamma_{ann} + n_{SM} \Gamma_{prod}$$

$$\Rightarrow \frac{dn_{DM}}{dt} + 3H n_{DM} = \underbrace{-\langle \sigma_{ann} v_{rel} \rangle n_{DM}^2}_{\text{decreases } n_{DM}} + \underbrace{\langle \sigma_{prod} v_{rel} \rangle n_{SM}^2}_{\text{increases } n_{DM}}$$

In equilibrium the rhs vanishes

$$\Rightarrow \langle \sigma_{ann} v_{rel} \rangle (n_{DM}^{eq})^2 = \langle \sigma_{prod} v_{rel} \rangle (n_{SM}^{eq})^2$$

$$\Rightarrow \langle \sigma_{prod} v_{rel} \rangle = \langle \sigma_{ann} v_{rel} \rangle \left(\frac{n_{DM}^{eq}}{n_{SM}^{eq}} \right)^2$$

$$\Rightarrow \frac{dn_{DM}}{dt} + 3H n_{DM} = -\langle \sigma_{ann} v_{rel} \rangle \left[n_{DM}^2 - (n_{DM}^{eq})^2 \underbrace{\left(\frac{n_{SM}}{n_{SM}^{eq}} \right)^2}_{=1} \right]$$

because SM particles stay in equilibrium

Useful definitions:

$$- s = \frac{S+P}{T} = \frac{2\pi^2}{45} g_* T^3 \quad (\text{entropy density})$$

$$- Y_{DM} = \frac{n_{DM}}{s} \quad \text{and} \quad Y_{DM}^{eq} = \frac{n_{DM}^{eq}}{s} \quad (\text{yield})$$

$$- x = \frac{m_{DM}}{T} \quad (\text{inverse temperature}) \Rightarrow \frac{dx}{dt} = x H$$

$$\Rightarrow \frac{dY_{DM}}{dt} = \frac{1}{s} \frac{dn_{DM}}{dt} - \frac{1}{s^2} \frac{ds}{dt} n_{DM}$$

$\uparrow s \sim a^{-3} \Rightarrow \frac{ds}{dt} = -3Hs$

$$= \frac{1}{s} \left(\frac{dn_{DM}}{dt} + 3H n_{DM} \right)$$

$$= - \frac{1}{s} \langle \sigma_{ann} v_{rel} \rangle (n_{DM}^2 - (n_{DM}^{eq})^2)$$

$$= - Y_{DM}^{eq} \underbrace{\langle \sigma_{ann} v_{rel} \rangle}_{\Gamma_{ann}^{eq}} n_{DM}^{eq} \left(\left(\frac{n_{DM}}{n_{DM}^{eq}} \right)^2 - 1 \right)$$

$$\Rightarrow \frac{dY_{DM}}{dx} = \frac{1}{x H} \frac{dY_{DM}}{dt} = - \frac{Y_{DM}^{eq}}{x H} \Gamma_{ann}^{eq} \left(\left(\frac{Y_{DM}}{Y_{DM}^{eq}} \right)^2 - 1 \right)$$

$$\Rightarrow \boxed{\frac{x}{Y_{DM}^{eq}} \frac{dY_{DM}}{dx} = - \frac{\Gamma_{ann}^{eq}}{H} \left(\left(\frac{Y_{DM}}{Y_{DM}^{eq}} \right)^2 - 1 \right)}$$

For $\Gamma_{ann}^{eq} \ll H$ one finds $\frac{dY_{DM}}{dx} \approx 0$

$\Rightarrow$ interactions negligible compared to expansion

For $\Gamma_{ann}^{eq} \gg H$, Boltzmann eq. restores equilib.

$\hookrightarrow$ For $Y_{DM} < Y_{DM}^{eq}$: $\frac{dY_{DM}}{dx} > 0 \Rightarrow Y_{DM}$ increases

$\hookrightarrow$ For $Y_{DM} > Y_{DM}^{eq}$: $\frac{dY_{DM}}{dx} < 0 \Rightarrow Y_{DM}$ decreases

$\Rightarrow Y_{DM}$ always stays close to Y_{DM}^{eq}