

Gravitational Waves — BND School 2026

Theory | Detectors | Sources and detection

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0 Outline

- ① Lecture 1: Theory — waves, and how to compute them
- ② Lecture 2: Gravitational-wave detectors
- ③ Lecture 3: Sources and their detection

0 Three lectures

Following the KU Leuven course script. Full derivations and exercises: in the script.

- 1 **Theory** — linearised gravity and waves (§1.1–1.2); energy carried (§1.4); post-Newtonian expansion (§2.2); numerical relativity (§3.1–3.2); perturbed black holes (§4.1).
- 2 **Detectors** — effect on matter (§1.3), as the way in; detector frame (§6.1); interferometers (§6.2); noise (§6.6); time-delay interferometry (§6.3); pulsar timing (§6.4).
- 3 **Sources and detection** — generation, and two sources (Ch. 5); data, matched filtering, unmodelled searches, Bayesian inference (Ch. 7).

Level

Advanced physics students, *not* gravitational-wave specialists. No general-relativity course assumed — but no mathematics hidden either. Where a result matters: what goes in, which step does the work, what comes out. Bookkeeping steps quoted.

0 Conventions, fixed once

- ▶ Signature $(-, +, +, +)$. Greek $\mu, \nu = 0, 1, 2, 3$; Latin $i, j = 1, 2, 3$; summation implied.
- ▶ $x^0 = ct$. Linearised theory: indices raised and lowered with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.
- ▶ $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu = -c^{-2} \partial_t^2 + \nabla^2$.
- ▶ Fourier: $\tilde{a}(f) = \int dt a(t) e^{-i2\pi f t}$.
- ▶ $G = c = 1$ in Chapters 3 and 4 only; elsewhere both kept explicit.

Numbers worth memorising

$GM_\odot/c^3 = 4.93 \mu\text{s}$	solar mass, as a time
$GM_\odot/c^2 = 1.48 \text{ km}$	solar mass, as a length
$c^5/G = 3.6 \times 10^{52} \text{ W}$	natural luminosity of gravity
$1 \text{ Mpc} = 3.09 \times 10^{22} \text{ m}$	

1 Outline

- ① Lecture 1: Theory — waves, and how to compute them
- ② Lecture 2: Gravitational-wave detectors
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Lecture 1

Theory: gravitational waves, and how to compute them

Script §1.1–1.2, §1.4 | §2.2 | §3.1–3.2 | §4.1

1 Lecture 1: the plan

- 1 **Linearised general relativity** (§1.1) — weak field on flat space; what is left of general covariance; field equations in harmonic gauge.
- 2 **Gravitational waves** (§1.2) — vacuum solutions; residual gauge freedom; transverse–traceless gauge; two polarisations.
- 3 **Energy and momentum** (§1.4) — why subtle, and the answer. *Concept and result only.*
- 4 **Post-Newtonian expansion** (§2.2) — expansion parameter; ordering of the metric components. *Concept and table of orders.*
- 5 **Numerical relativity** (§3.1–3.2) — $3 + 1$ decomposition; Hamiltonian and momentum constraints.
- 6 **Perturbed black holes** (§4.1) — scalar perturbations of Schwarzschild; quasinormal modes.

The thread

Items 4–6: three different ways past the limits of item 1.
Effect on *matter* (§1.3) opens Lecture 2, where it is needed.

1 §1.1 Linearised general relativity: the starting point

Easiest way in: the regime where the field is *weak*. Metric = flat Minkowski + small correction.

Linearised metric [script Def. 1.1]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{for } |h_{\mu\nu}| \ll 1.$$

Einstein equations to first order in $h_{\mu\nu}$; all higher orders dropped.

Careful from the start

Metric components are frame dependent. The statement really being made: there exists a *specific* frame in which $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ holds, over a sufficiently large region.

1 §1.1.1 Coordinate transformations: what survives

Einstein equations are invariant under $x^\mu \rightarrow x'^\mu(x)$:

$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x).$$

Fixing the background $\eta_{\mu\nu}$ **breaks** that invariance. A much smaller family survives.

Gauge transformations

$$x^\mu \longrightarrow x'^\mu = x^\mu + \xi^\mu(x),$$

all $|\partial_\rho \xi_\mu|$ at most of order $|h_{\mu\nu}|$. To lowest order:

$$h_{\mu\nu} \longrightarrow h'_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

Global Lorentz transformations

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$

$$h'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x).$$

So $h_{\mu\nu}$ is a Lorentz tensor — caveat: boosts must not spoil $|h_{\mu\nu}| \ll 1$.

Four free functions ξ^μ — exactly $A_\mu \rightarrow A_\mu - \partial_\mu \Lambda$, one index up.

1 §1.1.2 Linearising the field equations

Riemann tensor at leading order — the $\Gamma\Gamma$ terms drop:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} (\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}).$$

Linearised Riemann tensor: gauge invariant

Substitute $h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$: every extra term cancels pairwise. Curvature is physical; $h_{\mu\nu}$ is not.

Convenient: the **trace-reversed** perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad h = \eta^{\mu\nu} h_{\mu\nu}.$$

Since $\bar{h} \equiv \eta^{\mu\nu} \bar{h}_{\mu\nu} = h - 2h = -h$, the relation inverts:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}.$$

1 §1.1.2 The harmonic gauge

In terms of $\bar{h}_{\mu\nu}$:

$$\square \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$

Three terms to kill. Gauge transformations act on $\bar{h}_{\mu\nu}$ as

$$\bar{h}_{\mu\nu} \longrightarrow \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho),$$

so impose the **harmonic gauge** — also Lorentz, Hilbert, de Donder [script Def. 1.2]:

$$\partial^\nu \bar{h}_{\mu\nu} = 0.$$

Linearised Einstein field equations [script Def. 1.3]

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Wave equation with a source, one per component. Formally $\square A_\mu = -\mu_0 j_\mu$, with more indices.

1 §1.1.3 A conservation law comes for free

Take ∂^ν of the field equation; use $\partial^\nu \bar{h}_{\mu\nu} = 0$:

$$\partial^\nu T_{\mu\nu} = 0$$

Energy–momentum conservation *in the linearised theory*. Full theory: $\nabla^\nu T_{\mu\nu} = 0$, covariant derivative.

Why it matters later

A *flat-space* law — usable under an ordinary integral sign, with integration by parts. In Lecture 3 this is the step that converts moments of the *stresses* T^{ij} into second time derivatives of moments of the *mass density* T^{00} : the quadrupole formula.

And its price

Conserved total energy and momentum $\Rightarrow$ **no monopole, no dipole radiation**. Leading emission is quadrupolar. Back to this in §5.1.

1 §1.2 Gravitational waves: the retarded solution

$T_{\mu\nu}$ non-zero only inside a finite region $\mathcal{V}$. Solution at $(t, \mathbf{x})$:

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int_{\mathcal{V}} \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'.$$

[derived in §5.1; Lecture 3]

Read the argument of $T_{\mu\nu}$

Unlike the Newtonian potential: no instantaneous knowledge of what happens inside $\mathcal{V}$. Time lags $|\mathbf{x} - \mathbf{x}'|/c$ — light-travel times from $\mathbf{x}'$ to $\mathbf{x}$.

Like electromagnetism: no instantaneous action at a distance.

1 §1.2.1 Vacuum solutions

Outside the source $T_{\mu\nu} = 0$:

$$\square \bar{h}_{\mu\nu} = 0 \quad \Longleftrightarrow \quad \left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \bar{h}_{\mu\nu} = 0.$$

A wave equation — waves at the speed of light. Solutions: superpositions of plane waves,

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \cos(\omega t - \mathbf{k} \cdot \mathbf{x}), \quad \omega = c|\mathbf{k}|,$$

$A_{\mu\nu}$ of constant components.

The field has dynamics of its own

Such solutions exist *with* $T_{\mu\nu} = 0$ *everywhere*. Unphysical by themselves — but they make the point: geometry is a field, independent of matter.

Propagation speed c : a testable prediction, and tested — GW170817 versus its gamma-ray burst.

1 §1.2.2 The residual gauge freedom

A priori: $\bar{h}_{\mu\nu}$ has 10 independent components. Most are gauge artefacts.

- ▶ Harmonic gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$ eliminates 4. Six left.
- ▶ Not spoiled by a further transformation with $\square \xi_\mu = 0$.
If $\square \xi_\mu = 0$ then also $\square \xi_{\mu\nu} = 0$, where $\xi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho$ — because $\square$ commutes with ∂_μ .
- ▶ So: 4 more functions $\xi_\mu(x)$, spoiling neither the harmonic gauge nor $\square \bar{h}_{\mu\nu} = 0$.

Step	Left	Why
symmetric $h_{\mu\nu}$	10	4×4 symmetric
harmonic gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$	6	4 conditions, from ξ^μ
residual gauge $\square \xi^\mu = 0$	2	4 more free functions, in vacuum

Massless spin- s in four dimensions: two helicity states $\pm s$. Photon two, graviton two. Not an accident of the algebra.

1 §1.2.2 Spending the residual freedom

Four functions, spent as follows.

- ▶ $\xi_0(x)$: kill the trace, $\bar{h} = 0$. Then $\bar{h}_{\mu\nu} \rightarrow h_{\mu\nu}$.
- ▶ $\xi_i(x)$, $i = 1, 2, 3$: kill $h_{0\mu}(x)$.

With $\bar{h}_{\mu\nu} = h_{\mu\nu}$, the $\mu = 0$ harmonic condition:

$$\partial^0 h_{00} + \partial^i h_{0i} = 0 \quad \xrightarrow{h_{0i}=0} \quad \partial^0 h_{00} = 0.$$

h_{00} is time independent.

What h_{00} is

The *static* part of the interaction — Newtonian potential from the total mass, no contribution from motion. The wave is the time-dependent part. Set $h_{00} = 0$.

Spatial part ($\mu = i$): $\partial^j h_{ij} = 0$. And $h = 0$ becomes $h^i_i = 0$.

All gauge freedom spent. **Two** degrees of freedom left.

1 §1.2.2 The transverse–traceless gauge

Transverse–traceless gauge [script Def. 1.4]

$$h_{0\mu} = 0, \quad h^i{}_i = 0, \quad \partial^j h_{ij} = 0.$$

Write h_{ij}^{TT} . Plane-wave solutions:

$$h_{ij}^{\text{TT}} = e_{ij}(\mathbf{k}) \cos(k_\mu x^\mu), \quad k_\mu = (\omega/c, \mathbf{k}), \quad \omega = c|\mathbf{k}|,$$

with $e_{ij}(\mathbf{k})$ the **polarisation tensor**.

Single plane wave: $\partial^j h_{ij} = 0$ becomes algebraic,

$$k^j h_{ij}^{\text{TT}} = 0 \quad \Longleftrightarrow \quad n^j h_{ij}^{\text{TT}} = 0, \quad \hat{\mathbf{n}} = \mathbf{k}/|\mathbf{k}|.$$

Non-zero components of h_{ij}^{TT} : in the plane transverse to $\hat{\mathbf{n}}$.

Caveat worth keeping: TT gauge exists only *outside* the source.

1 §1.2.2 Example: a monochromatic wave along z

z axis along $\hat{n}$. Symmetry + transversality + tracelessness leave

$$h_{ij}^{\text{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos [\omega(t - z/c)].$$

Two functions h_+ , $h_\times$: the **two polarisations**.

As a line element:

$$\begin{aligned} ds^2 = & -c^2 dt^2 + dz^2 + \left[1 + h_+ \cos [\omega(t - z/c)] \right] dx^2 \\ & + \left[1 - h_+ \cos [\omega(t - z/c)] \right] dy^2 + 2h_\times \cos [\omega(t - z/c)] dx dy. \end{aligned}$$

Read the metric

dx^2 coefficient grows while dy^2 shrinks; reversed half a period later. A *quadrupolar* stretch and squeeze, transverse to propagation. Turning this into a statement about *matter* is §1.3 — which opens Lecture 2, since it is the way into detectors.

1 §1.4 Energy and momentum: why this is delicate

Take $h_{\mu\nu}$ solving the *linearised* vacuum equations, $G_{\mu\nu}^{(1)}[h_{\rho\sigma}] = 0$. Then $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ generally does *not* solve the full equations — typically not even to second order:

$$G_{\mu\nu}[h_{\rho\sigma}] = G_{\mu\nu}^{(1)}[h_{\rho\sigma}] + G_{\mu\nu}^{(2)}[h_{\rho\sigma}] + \dots, \quad G_{\mu\nu}^{(2)} \neq 0 \text{ in general.}$$

Add a smaller correction $h_{\mu\nu}^{(2)} = \mathcal{O}(h^2)$, keep the first-order equations:

$$G_{\mu\nu}^{(1)}[h_{\rho\sigma}^{(2)}] = \frac{8\pi G}{c^4} t_{\mu\nu}, \quad t_{\mu\nu} = -\frac{c^4}{8\pi G} G_{\mu\nu}^{(2)}[h_{\rho\sigma}].$$

What this says

To second order, $h_{\mu\nu}$ curves spacetime exactly as ordinary matter of stress-energy $t_{\mu\nu}$ would. And $t_{\mu\nu}$ is symmetric and conserved, $\partial^\mu t_{\mu\nu} = 0$.

But $t_{\mu\nu}$ is *not* gauge invariant. In general relativity: no local notion of gravitational energy density.

1 §1.4 The cure: average over a region

Evaluate $t_{\mu\nu}$ not at a point but *averaged over a small spatial volume*. That average is gauge invariant:

$$T_{\mu\nu}^{\text{GW}} = -\frac{c^4}{8\pi G} \left\langle R_{\mu\nu}^{(2)} - \frac{1}{2}\eta_{\mu\nu}R^{(2)} \right\rangle.$$

Why the average simplifies so drastically

$R_{\mu\nu}^{(2)}$: a dozen terms quadratic in h . But inside $\langle \dots \rangle$ — integrate by parts, discard boundary terms. And $h_{\mu\nu}$ depends on time only through a retarded combination (wave along z : $\partial_0 h_{\mu\nu} = -\partial_z h_{\mu\nu}$), so *time and spatial derivatives are on the same footing*. With $\partial_\mu h^{\mu\nu} = 0$, $h = 0$, $\square h_{\mu\nu} = 0$, all terms but one vanish:

$$\left\langle R_{\mu\nu}^{(2)} \right\rangle = -\frac{1}{4} \left\langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \right\rangle, \quad \left\langle R^{(2)} \right\rangle = 0.$$

That is the whole content of §1.4. The rest is the algebra of the cancellation.

1 §1.4 The results

Energy–momentum tensor of gravitational waves [script Def. 1.6]

$$T_{\mu\nu}^{\text{GW}} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle$$

TT gauge, energy density: $T_{\text{GW}}^{00} = \frac{c^2}{32\pi G} \langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \rangle = \frac{c^2}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$. Far field, radially outgoing: $h_{ij}^{\text{TT}} = r^{-1} f_{ij}(t - r/c)$, hence $T_{\text{GW}}^{0r} = T_{\text{GW}}^{00}$ up to $\mathcal{O}(r^{-2})$. So:

Energy and momentum radiated [script Def. 1.7 and Def. 1.8]

$$\frac{dE_{\text{GW}}}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega \langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \rangle, \quad \frac{dP_{\text{GW}}^k}{dt} = -\frac{c^3 r^2}{32\pi G} \int_S d\Omega \langle \dot{h}_{ij}^{\text{TT}} \partial^k h_{ij}^{\text{TT}} \rangle.$$

Waves carry energy away, and can deposit it. Input to the radiation-reaction argument of Lecture 3 — and to the Hulse–Taylor test.

1 §2.1 Where linearised theory runs out

Linearised theory assumes: background flat and *fixed*; source velocity small; the two *independent*. For the strongest sources, none of that holds.

Virial theorem $\langle E_k \rangle = -\frac{1}{2} \langle E_p \rangle$ ties field strength to velocity:

$$\left(\frac{v}{c}\right)^2 \sim \frac{2GM}{c^2 R} = \frac{R_S}{R}, \quad R_S = \text{Schwarzschild radius.}$$

Two small parameters, but only one small parameter

No fast source is also weakly gravitating. A binary at $v/c = 0.3$ sits at $R \sim 10 R_S$.

Post-Newtonian theory: weakly self-gravitating, slow internally and as a whole.

$$\epsilon \sim \frac{v}{c} \sim \sqrt{\frac{R_S}{R}} \ll 1, \quad \frac{|T^{0i}|}{T^{00}} = \mathcal{O}(\epsilon), \quad \frac{|T^{ij}|}{T^{00}} = \mathcal{O}(\epsilon^2).$$

1 §2.2 Principles of the post-Newtonian formalism

Goal: extend linearised theory to weakly self-gravitating systems with slow internal and bulk motion. Tool: expansion in $\epsilon = v/c$.

The naive route, and its limitation

Compute $\mathcal{O}(\epsilon^n)$ corrections to the equations of motion; get the radiation from the multipole expansion (§5.1), itself an expansion in ϵ .

Limited: emission *subtracts energy from the motion*, and waves at one order are themselves sources at higher order. Still valid at 1PN.

Counting convention

Orders counted in steps of $\mathcal{O}(\epsilon^2)$. The n PN order carries contributions up to $\mathcal{O}(\epsilon^{2n})$.

Newtonian starting point: $g_{00} = -1 - 2\phi$, $g_{0i} = 0$, $g_{ij} = \delta_{ij}$, with the *instantaneous* potential $\phi = -G \int d^3x' T^{00}(t, x')/|x - x'|$ — no retardation.

1 §2.2.1 Which orders can appear where

Two arguments fix the structure before any calculation.

1. Time-reversal symmetry. No radiation reaction $\Rightarrow$ time-reversal symmetry. And $t \rightarrow -t$ implies $v \rightarrow -v$:

- ▶ g_{00}, g_{ij} : only *even* powers of ϵ ;
- ▶ g_{0i} : only *odd* powers.

Together — $^{(n)}g_{\mu\nu}$ being the $\mathcal{O}(\epsilon^n)$ piece:

$$g_{00} = -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + \dots$$

$$g_{0i} = {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \dots$$

$$g_{ij} = \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + {}^{(6)}g_{ij} + \dots$$

Similarly $T^{00} = {}^{(0)}T^{00} + {}^{(2)}T^{00} + \dots$, $T^{i0} = {}^{(1)}T^{i0} + \dots$, $T^{ij} = {}^{(2)}T^{ij} + \dots$

2. Slow motion. Momentum one order above energy; stresses one above momentum:

$$\frac{|T^{0i}|}{T^{00}} = \mathcal{O}(\epsilon), \quad \frac{|T^{ij}|}{T^{00}} = \mathcal{O}(\epsilon^2).$$

Through $G_{\mu\nu} \propto T_{\mu\nu}$: $g_{00} = \mathcal{O}(\epsilon^n)$ gives $g_{0i} = \mathcal{O}(\epsilon^{n-1})$, $g_{ij} = \mathcal{O}(\epsilon^{n-2})$.

1 §2.2.1 The table of post-Newtonian orders

	0PN	1PN	2PN	...	n PN
g_{00}	$-1 + {}^{(2)}g_{00}$	${}^{(4)}g_{00}$	${}^{(6)}g_{00}$	...	${}^{(2n+2)}g_{00}$
g_{0i}	0	${}^{(3)}g_{0i}$	${}^{(5)}g_{0i}$	...	${}^{(2n+1)}g_{0i}$
g_{ij}	δ_{ij}	${}^{(2)}g_{ij}$	${}^{(4)}g_{ij}$	...	${}^{(2n)}g_{ij}$

[script Tab. 2.1]

How to read it

One column per PN order. The three components enter at *different* powers of ϵ within the same order: at 1PN, g_{00} to ϵ^4 but g_{ij} only to ϵ^2 . Getting this bookkeeping right is most of the work.

Beyond 1PN

Naive scheme fails. Needed: relaxed Einstein equations, expansions in far and near zones, matching between them [script §2.5].

1 §3 Numerical relativity: the problem

Stronger fields, faster objects $\Rightarrow$ no approximation scheme represents spacetime faithfully. Solve the field equations on a computer instead.

But Einstein wrote them covariantly

No explicit coordinate choice, hence *no explicit distinction between space and time*. To evolve anything, we must undo some of Einstein's hard work.

3 + 1 formalism: split spacetime into 3D space + 1D time; rewrite the field equations in that split. Then evolve from initial data

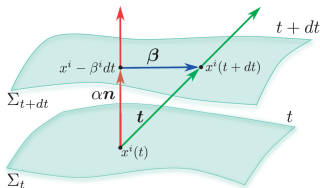
$$\{g_{\mu\nu}, \partial_t g_{\mu\nu}\} \Big|_{t=0}.$$

The mathematical setting

A **Cauchy problem** — partial differential equations plus boundary conditions. Boundary a spatial hypersurface: an *initial value problem*.

1 §3.1.1 Foliating spacetime

Cut spacetime into space-like hypersurfaces covering it, each labelled by t — a *universal time function*, not the proper time of any particular observer.



[script Fig. 3.1]

Between Σ_t and Σ_{t+dt} : three objects.

- ▶ **Three-metric** γ_{ij} — proper distance within a slice, $dl^2 = \gamma_{ij}dx^i dx^j$.
- ▶ **Lapse** α — proper time of an observer moving normal to the slices, per unit universal time: $d\tau = \alpha(t, x^i) dt$.
- ▶ **Shift** β^i — relative velocity between those normal (*Eulerian*) observers and lines of constant spatial coordinate:
 $x^i_{t+dt} = x^i_t - \beta^i(t, x^j) dt$.

Lapse and shift are gauge

The foliation is *not unique*: α , β^i freely specifiable. Choosing them well is what made numerical relativity work.

1 §3.1.1 The four-metric in 3 + 1 form

Line element in terms of lapse, shift, three-metric:

$$ds^2 = -(\alpha^2 + \beta^i \beta_i) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j,$$

indices raised and lowered with the three-metric, $\beta_i = \gamma_{ij} \beta^j$. In matrix form:

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^k \beta_k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}.$$

Count the pieces

γ_{ij} : 6. α : 1. β^i : 3. Total 10 — the components of $g_{\mu\nu}$, repackaged: 6 dynamical, 4 gauge.

$-\alpha^2$: how far apart successive slices are in proper time. β_i : how the spatial coordinates slide sideways between slices.

1 §3.1.2 Coordinate-independent form

Same structure, no coordinate system. Start from the unit normal:

$$n^\mu = \frac{1}{\alpha} (1, -\beta^i)$$

— tied directly to the Eulerian observers.

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

spatial metric as a **projector**;

$$\alpha = (-\nabla t \cdot \nabla t)^{-1/2}, \quad n^\mu = -\alpha \nabla^\mu t,$$

minus sign: n^μ future-pointing. And

$$\beta^i = -\alpha (\vec{n} \cdot \nabla x^i).$$

Time vector

$$t^\mu = \alpha n^\mu + \beta^\mu, \quad \beta^\mu = (0, \beta^i).$$

From $t^\mu n_\mu = -\alpha$:

$$t^\mu \nabla_\mu t = 1,$$

and, coordinate-independently,

$$\beta_\mu = \gamma_{\mu\nu} t^\nu.$$

αn^μ moves you off the slice; β^μ slides you along it.

1 §3.1.3 Two kinds of curvature

Curvature is split the same way space and time were.

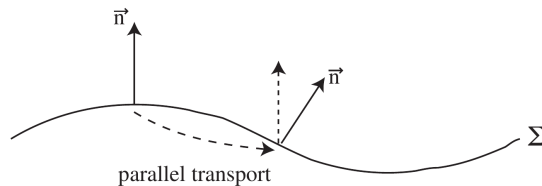
Intrinsic. Internal geometry *on* a slice, measurable from within it — the 3D Riemann tensor as a function of γ_{ij} :

$${}^{(3)}R^\alpha{}_{\beta\gamma\nu} = R^\alpha{}_{\beta\gamma\nu}[\gamma_{ij}].$$

Extrinsic. How the slice is *embedded* in the full spacetime:

$$K_{\mu\nu} = -\gamma_\mu{}^\lambda \nabla_\lambda n_\nu,$$

$\gamma_\mu{}^\lambda$ projecting onto the slice.



$K_{\mu\nu}$: difference between the unit normal at $\vec{x}$ and the parallel transport of the normal from $\vec{x} + \delta\vec{x}$ to $\vec{x}$ [script Fig. 3.2].

A cylinder: zero intrinsic curvature, non-zero extrinsic. Both needed — γ_{ij} and K_{ij} are the dynamical variables.

1 §3.2.1 A useful analogy: Maxwell's equations

Electromagnetism in the same language:

$$\begin{aligned}\nabla \cdot \vec{E} &= 4\pi\rho, & \nabla \cdot \vec{B} &= 0, \\ \partial_t \vec{E} &= \nabla \times \vec{B} - 4\pi\vec{j}, & \partial_t \vec{B} &= -\nabla \times \vec{E}.\end{aligned}$$

Constraints

Divergence equations — no time derivatives. A statement about the field throughout space, at any one instant.

Evolution

Curl equations — time derivatives present. They take the system from one instant to the next.

Now do exactly this to the Einstein field equations.

From here on $G = c = 1$.

1 §3.2.2 Decomposing the Riemann tensor

Dynamics sits in the Einstein equations. To put them in $3 + 1$ form: contract the Riemann tensor with the normal n^μ and with the projector γ^μ_ν . Different projections relate 4D $R_{\mu\nu\rho\sigma}$ to 3D ${}^{(3)}R_{\alpha\beta\delta\lambda}$ and K_{ij} .

Gauss–Codazzi — full spatial projection

$$\gamma_\alpha^\mu \gamma_\beta^\nu \gamma_\delta^\rho \gamma_\lambda^\sigma R_{\mu\nu\rho\sigma} = {}^{(3)}R_{\alpha\beta\delta\lambda} + K_{\alpha\delta}K_{\beta\lambda} - K_{\alpha\lambda}K_{\beta\delta}$$

Codazzi–Mainardi — mixed projection

$$\gamma_\alpha^\delta \gamma_\beta^\kappa \gamma_\mu^\lambda n^\nu R_{\delta\kappa\lambda\nu} = D_\beta K_{\alpha\mu} - D_\alpha K_{\beta\mu}, \quad D_\mu \equiv \gamma^\alpha_\mu \nabla_\alpha.$$

These two relations: the entire technical input for what follows.

1 §3.2.3 The Hamiltonian constraint

Partial projection of the Riemann tensor, two factors of $\gamma^{\mu\nu}$:

$$\gamma^{\alpha\mu}\gamma^{\beta\nu}R_{\alpha\beta\mu\nu} = (g^{\alpha\mu} + n^\alpha n^\mu)(g^{\beta\nu} + n^\beta n^\nu)R_{\alpha\beta\mu\nu} = R + 2n^\mu n^\nu R_{\mu\nu} = 2n^\mu n^\nu G_{\mu\nu}.$$

Same object, from Gauss–Codazzi:

$$\gamma^{\alpha\mu}\gamma^{\beta\nu}R_{\alpha\beta\mu\nu} = {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}, \quad K \equiv K^\mu{}_\mu.$$

Combine; use $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and $\rho \equiv n^\mu n^\nu T_{\mu\nu}$:

Hamiltonian constraint [script Def. 3.3]

$${}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi\rho$$

No explicit time derivatives — a constraint, in the sense of the Maxwell analogy. Gravitational counterpart of $\nabla \cdot \vec{E} = 4\pi\rho$.

1 §3.2.4 The momentum constraint

Mixed contraction of the Einstein tensor:

$$\gamma^{\alpha\mu} n^\nu G_{\mu\nu} = \gamma^{\alpha\mu} n^\nu R_{\mu\nu} = 8\pi \gamma^{\alpha\mu} n^\nu T_{\mu\nu}.$$

Same object, from Codazzi–Mainardi:

$$\gamma^{\alpha\mu} n^\nu G_{\mu\nu} = D^\alpha K - D_\mu K^{\alpha\mu}.$$

Identify the momentum density $j^\alpha \equiv -\gamma^{\alpha\mu} n^\nu T_{\mu\nu}$; combine:

Momentum constraint [script Def. 3.4]

$$D_\mu (K^{\alpha\mu} - \gamma^{\alpha\mu} K) = 8\pi j^\alpha$$

What the constraints are for

Conditions on the data *on a single slice*. γ_{ij} and K_{ij} are not free: four constraints, one Hamiltonian and three momentum. Finding data that satisfies them is the *initial data problem* [script §3.3].

1 §3.2 The full system

Carried further, the same projections give the evolution equations. With the constraints: *equivalent* to the Einstein field equations.

Constraints — no time derivatives

$$^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi\rho, \quad D_\mu (K^{\alpha\mu} - \gamma^{\alpha\mu}K) = 8\pi j^\alpha$$

Evolution — extrinsic curvature (the dynamics)

$$\begin{aligned} \partial_t K_{ij} = & \beta^k \partial_k K_{ij} + K_{ki} \partial_j \beta^k + K_{kj} \partial_i \beta^k - D_i D_j \alpha \\ & + \alpha \left[^{(3)}R_{ij} + K K_{ij} - 2K_{ik} K^k_j \right] + 4\pi\alpha [\gamma_{ij}(S - \rho) - 2S_{ij}] \end{aligned}$$

$S_{\alpha\beta} = \gamma_\alpha^\mu \gamma_\beta^\nu T_{\mu\nu}$, S its trace.

Evolution — spatial metric (purely kinematic)

$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$ — not from the field equations, but the geometric statement that K_{ij} relates the metrics of neighbouring slices.

1 §3.2.8 Does the evolution preserve the constraints?

In the continuum: yes

Bianchi identity — constraints satisfied initially remain satisfied under evolution.

Discretised: not necessarily

Hence different evolution schemes, at various levels of stability.

- ▶ **Free evolution.** Initial data solving the constraints, then all 12 evolution equations for γ_{ij} , K_{ij} . Constraints only monitored, as a diagnostic.
- ▶ **Constrained evolution.** Solve some or all constraints at every time step; evolve the rest.
- ▶ **Constrained metric evolution.** Impose an extra metric condition, e.g. conformal flatness.

Beyond this: well-posedness and hyperbolicity; BSSN and Z4 reformulations; moving-puncture gauge [script §3.4–3.5] — what made binary black-hole simulations work in 2005.

1 §4.1 Perturbed black holes: the setting

After merger: one distorted black hole. Its relaxation is perturbation theory on a fixed background. Cleanest case — a *test massless scalar field* on Schwarzschild:

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad f(r) = 1 - \frac{2M}{r}.$$

Tortoise coordinate r_* :

$$dr_* = \frac{dr}{f(r)} \iff \frac{\partial}{\partial r_*} = f(r) \frac{\partial}{\partial r}, \quad r_* = r + 2M \log \left(\frac{r}{2M} - 1 \right).$$

What r_* does

$r_* \rightarrow +\infty$ as $r \rightarrow \infty$; $r_* \rightarrow -\infty$ at the horizon $r \rightarrow 2M$ — the horizon is pushed infinitely far away. Radial null geodesics: $t \pm r_* = \text{const.}$

So the problem looks like flat-space wave propagation on an infinite line.

1 §4.1 A coordinate that is regular on the horizon

With $v = t + r_*$:

$$ds^2 = -f(r) dv^2 + 2 dv dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

which extends smoothly to the interior $r < 2M$.

Why we bother

Schwarzschild coordinates are singular at $r = 2M$ — a property of the coordinates, not the geometry. We must decide which boundary condition at the horizon is physical, and the answer is only visible in a chart that is regular there. Back to this in three slides.

v is constant along ingoing radial null rays: it labels infalling light.

1 §4.1 A massless scalar on Schwarzschild

Test field with $\nabla^2 \psi = 0$, where $\nabla^2 \psi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi \right)$. On Schwarzschild:

$$-\frac{1}{f(r)} \frac{\partial^2 \psi}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 f(r) \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} = 0.$$

Spherically symmetric $\Rightarrow$ separate with spherical harmonics:

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y^{lm}}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y^{lm}}{\partial \phi^2} = -l(l+1) Y^{lm}.$$

Decompose:

$$\psi = \frac{\psi^{lm}}{r} Y^{lm} e^{-i\omega t}$$

The factor $1/r$ removes the first-derivative term — and leaves a clean one-dimensional problem.

1 §4.1 A Schrödinger-like equation

Separation reduces the field equation to

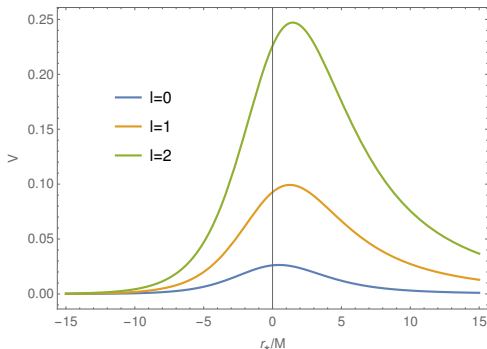
$$f(r) \frac{d}{dr} \left(f(r) \frac{d\psi_{lm}}{dr} \right) + (\omega^2 - V_l) \psi_{lm} = 0,$$

effective potential

$$V_l = \left(1 - \frac{2M}{r} \right) \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right).$$

In the tortoise coordinate, its natural form:

$$\frac{d^2 \psi_{lm}}{dr_*^2} + (\omega^2 - V_l) \psi_{lm} = 0$$



V_l versus r_* [script Fig. 4.1]. Vanishes at $r_* \rightarrow +\infty$ and at the horizon $r_* \rightarrow -\infty$ — a **potential barrier** on an infinite line.

1 §4.1 Quasinormal modes: the boundary conditions

Black-hole perturbations $\rightarrow$ a 1D Schrödinger-like problem. Unlike a real Schrödinger problem: not after energy eigenstates, but after particular *boundary conditions*.

$V_l \rightarrow 0$ at both ends, so

$$\psi(r_* \rightarrow +\infty) \sim A_+ e^{-i\omega(t+r_*)} + A_- e^{-i\omega(t-r_*)},$$

$$\psi(r_* \rightarrow -\infty) \sim B_+ e^{-i\omega(t+r_*)} + B_- e^{-i\omega(t-r_*)}.$$

$e^{-i\omega(t+r_*)}$ left-moving; $e^{-i\omega(t-r_*)}$ right-moving.

At infinity

Outgoing only — nothing comes in from infinity: $A_+ = 0$.

At the horizon

Ingoing only — nothing comes out of the horizon: $B_- = 0$.

Radiating, dissipative: energy leaves through both ends. Already a hint that the eigenfrequencies will not be real.

1 §4.1 Why nothing comes out of the horizon

Look in the coordinate $v = t + r_*$, well behaved across the horizon.

Ingoing:

$$e^{-i\omega(t+r_*)} = e^{-i\omega v}$$

perfectly regular across the horizon.

Outgoing:

$$e^{-i\omega(t-r_*)} = e^{-i\omega v + 2i\omega r_*} \sim e^{-i\omega v} \left(\frac{r}{2M} - 1 \right)^{4iM\omega}$$

singular at $r = 2M$ for generic ω .

The quasinormal mode problem

$$\psi \sim \begin{cases} e^{+i\omega r_*} & r_* \rightarrow +\infty \\ e^{-i\omega r_*} & r_* \rightarrow -\infty \end{cases}$$

— a black hole *emitting* gravitational radiation.

1 §4.1 The quasinormal mode spectrum

Solutions exist only for a **discrete set** of frequencies ω — and those frequencies are **complex**.

$\text{Im}(\omega) < 0$: generic

Mode decays — $e^{-i\omega t}$ is a damped oscillation.
System *dissipative*: energy radiated to infinity and down the horizon.

$\text{Im}(\omega) > 0$: instability

Possible in special situations; the perturbation then grows. Schwarzschild is stable; other backgrounds need not be.

Not just a scalar-field exercise

Same structure — separate, reduce to $d^2\psi/dr_*^2 + (\omega^2 - V)\psi = 0$, impose outgoing/ingoing conditions — for electromagnetic and for *gravitational* perturbations, with different potentials [script §4.3–4.4].
Gravitational: odd and even parity families, different potentials, **same** spectrum (isospectrality).

1 Lecture 1 — summary

- 1 **§1.1** Weak field on flat space. Gauge freedom $h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$; trace reversal; harmonic gauge gives $\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}/c^4$, and with it $\partial^\nu T_{\mu\nu} = 0$.
- 2 **§1.2** In vacuum: a wave equation. Residual gauge $\square \xi^\mu = 0$ takes 10 components to 2. TT gauge: h_+ , $h_\times$.
- 3 **§1.4** No local energy density for the field. Averaged over a region, $T_{\mu\nu}^{\text{GW}} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle$; flux follows.
- 4 **§2.2** Post-Newtonian: one small parameter $\epsilon \sim v/c \sim \sqrt{R_S/R}$; orders in ϵ^2 ; components entering at different powers.
- 5 **§3.1–3.2** Numerical relativity: foliate into $(\gamma_{ij}, \alpha, \beta^i)$; intrinsic and extrinsic curvature; field equations split into **constraints** and **evolution** equations.
- 6 **§4.1** Perturbed black holes: Schrödinger problem with a barrier; outgoing at infinity, ingoing at the horizon; discrete, complex quasinormal spectrum.

Still missing: what the wave does to matter (§1.2). First thing is Lecture 2 — because it is the first thing

2 Outline

- ① Lecture 1: Theory — waves, and how to compute them
- ② Lecture 2: Gravitational-wave detectors
- ③ Lecture 3: Sources and their detection

Lecture 2

Gravitational-wave detectors

Script §1.3 | §6.1 | §6.2 | §6.6 | §6.3 | §6.4

2 Lecture 2: the plan

- 1 **Effect on matter** (§1.3) — geodesic deviation; ring of test particles. What there is to detect, and the trap that comes with it.
- 2 **Detector frame** (§6.1) — proper distance in the TT frame; the wave as a Newtonian force; when that picture fails.
- 3 **Interferometers** (§6.2) — can a Michelson reach $h \sim 10^{-20}$? Optics; Michelson; Fabry–Pérot cavity and its gain factor; the FPML.
- 4 **Noise sources** (§6.6) — seismic, Newtonian, thermal, shot, radiation pressure. *Concepts and governing expressions.*
- 5 **Time-delay interferometry** (§6.3) — Doppler shift from a passing wave; LISA's one-way measurements; cancelling laser frequency noise.
- 6 **Pulsar timing** (§6.4) — timing residuals; Earth term and pulsar term.

The through-line

Items 2–6 all measure the same thing: a light travel time, or a frequency shift — the same statement

2 §1.3 Effect on matter: geodesic deviation

One free-falling particle tells us nothing — it just follows a geodesic:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0.$$

Take *two* nearby particles: $x^\mu(\tau)$ and $x^\mu(\tau) + \zeta^\mu$. Subtract; expand to first order in ζ^μ :

$$\frac{d^2 \zeta^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{d\zeta^\rho}{d\tau} + \zeta^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0.$$

With $\frac{DV^\mu}{D\tau} = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} V^\nu \frac{dx^\rho}{d\tau}$:

$$\boxed{\frac{D^2 \zeta^\mu}{D\tau^2} = -R^\mu_{\nu\rho\sigma} \zeta^\rho \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau}}$$

Geodesic deviation: relative motion of nearby particles, as a **tidal force** set by the Riemann tensor.

2 §1.3 Reducing it to the TT perturbation

At a point P on a geodesic: a coordinate transformation makes $\Gamma^\mu_{\nu\rho}(P) = 0$ — the local Lorentz frame. Particles non-relativistic, $dx^i/d\tau \ll dx^0/d\tau$:

$$\frac{d^2\zeta^i}{d\tau^2} + \zeta^j \partial_j \Gamma^i_{00} \left(\frac{dx^0}{d\tau} \right)^2 = 0.$$

At P : $\Gamma^\mu_{\nu\rho} = 0$ and $\partial_0 \Gamma^i_{0j} = 0$, so $R^i_{0j0} = \partial_j \Gamma^i_{00} - \partial_0 \Gamma^i_{0j} = \partial_j \Gamma^i_{00}$. With $dx^0/d\tau \simeq c$, $\tau = t$:

$$\ddot{\zeta}^i = -c^2 R^i_{0j0} \zeta^j.$$

Linearised Riemann is *invariant* — so evaluate it in the TT frame:

$$R^i_{0j0} = R_{i0j0} = -\frac{1}{2c^2} \ddot{h}^{\text{TT}}_{ij}.$$

Geodesic deviation in linearised theory [script Def. 1.5]

2 §1.3 Example: a ring of test particles — the + polarisation

Wave along z ; particles in the (x, y) plane at $z = 0$; origin of time such that $h_{ij}^{\text{TT}} = 0$ at $t = 0$:

$$h_{ij}^{\text{TT}} = h_+ \sin(\omega t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}.$$

Write $\zeta^i = (x_0 + \delta x(t), y_0 + \delta y(t), 0)$:

$$\delta \ddot{x} = -\frac{h_+}{2} (x_0 + \delta x) \omega^2 \sin \omega t, \quad \delta \ddot{y} = +\frac{h_+}{2} (y_0 + \delta y) \omega^2 \sin \omega t.$$

Small displacements, $\delta x \ll x_0$, $\delta y \ll y_0$ — simplify and integrate:

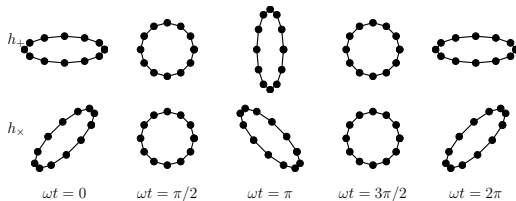
$$\delta x(t) = \frac{h_+}{2} x_0 \sin \omega t, \quad \delta y(t) = -\frac{h_+}{2} y_0 \sin \omega t$$

Displacement $\propto$ *unperturbed separation* x_0 . The effect is a **strain**, not a displacement — which is

2 §1.3 The $\times$ polarisation, and the ring

Analogously, for the cross polarisation:

$$\delta x(t) = \frac{h_{\times}}{2} y_0 \sin \omega t, \quad \delta y(t) = \frac{h_{\times}}{2} x_0 \sin \omega t.$$



Ring of test particles under $+$ (top) and $\times$ (bottom) [script Fig. 1.1].

Two things to read off

Polarisations rotated by 45° — not 90° as for light: the signature of a [spin 2](#) field. And the pattern is

2 §1.3 The trap: does the wave move the mirror, or stretch the ruler?

That derivation used the (x, y, z) frame of a local observer. Now take the **TT frame**.

Coordinates do not move. Two masses at rest at $\tau = 0$: $dx^i/d\tau = 0$, $dx^0/d\tau = c$, and the deviation equation gives

$$\ddot{\zeta}^i = 0.$$

Coordinate separation ζ^i unchanged. TT coordinates are *comoving with free-falling masses*.

But proper distance does. Masses at $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$, from the line element:

$$s = \int_{x_1}^{x_2} dx \sqrt{1 + h_+ \cos \omega t} \approx L \left(1 + \frac{1}{2} h_+ \cos \omega t \right).$$

The resolution

Coordinates are labels. *Proper distance* is physical — and so is the light travel time between the masses, which is what an interferometer measures. Where §6.1 starts, next slide.

2 §6.1.1 Proper distance in the TT frame

Two TT-frame events at $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$; coordinate distance $x_2 - x_1 = L$, *constant* in coordinate time. Physics is in the **proper distance**:

$$s = \int ds = \int_{x_1}^{x_2} dx \sqrt{1 + h_+ \cos \omega t} \approx L \left(1 + \frac{1}{2} h_+ \cos \omega t \right).$$

General directions:

$$s = \sqrt{L^2 + h_{ij}(t)L^i L^j},$$

L^i the spatial separation, L the coordinate distance.

The observable

Proper distance expands and shrinks periodically. *Light travel time* between two test masses follows the proper distance — so the wave leaves an imprint on a photon's round trip.

An interferometer measures that imprint, as a difference between its two arms.

2 §6.1.2 The proper detector frame

Alternative route. Detector much smaller than the gravitational wavelength $\Rightarrow$ the whole detector sits in a near local-Lorentz (freely falling) frame, even with the wave present:

$$ds^2 \approx -dt^2 + \delta_{ij} dx^i dx^j + \mathcal{O}\left(\frac{x^i x^j}{L_B^2}\right),$$

L_B the typical variation scale of the metric.

Two free-falling masses separated by ζ^i ; geodesic deviation written out:

$$\frac{d^2 \zeta^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} + \zeta^\sigma \Gamma^\mu_{\nu\rho,\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0.$$

Two simplifications: masses non-relativistic, $dx^i/d\tau$ negligible against $dx^0/d\tau$; and in a near local-Lorentz frame the $\Gamma^\mu_{\nu\rho}$ term is negligible. Hence

$$\frac{d^2 \zeta^i}{d\tau^2} + \zeta^j \Gamma^i_{00,j} \left(\frac{dx^0}{d\tau}\right)^2 = 0.$$

2 §6.1.2 The wave as a Newtonian force

Local-Lorentz frame: $R^i{}_{0j0} = \Gamma^i{}_{00,j} - \Gamma^i{}_{0j,0} = \Gamma^i{}_{00,j}$, so

$$\frac{d^2 \zeta^i}{d\tau^2} + R^i{}_{0j0} \zeta^j \left(\frac{dx^0}{d\tau} \right)^2 = 0 \quad \xrightarrow{dx^0/d\tau \approx 1, \tau \approx t} \quad \ddot{\zeta}^i = -R^i{}_{0j0} \zeta^j.$$

Riemann is gauge invariant — evaluate in TT gauge, $R^i{}_{0j0} = -\frac{1}{2} \ddot{h}_{ij}^{\text{TT}}$:

Geodesic deviation in the proper detector frame

$$\ddot{\zeta}^i = \frac{1}{2} \ddot{h}_{ij}^{\text{TT}} \zeta^j$$

Why this is worth the trouble

In a near local-Lorentz frame, the wave resembles a **Newtonian force**. So the detector can be treated with ordinary classical mechanics: suspended mirrors, springs, resonances, forces.

The same equation as §1.2 — now read as a statement about a laboratory, not about idealized free

2 §6.1.2 When the Newtonian picture fails

Caution. Geodesic deviation holds only for ζ small against the metric's variation scale — for a wave, the reduced wavelength $\bar{\lambda} = \lambda/2\pi$. So:

$$L \ll \bar{\lambda}$$

L the detector length scale.

	$\bar{\lambda}$	L	
LIGO / Virgo	$\approx 10^5 \text{ m}$	$\approx 10^3 \text{ m}$	valid
LISA	$\approx 10^{10} \text{ m}$	$\approx 10^9 \text{ m}$	not valid

Consequence

LISA needs a fully relativistic treatment. Hence §6.3: space-based detection through the *Doppler shift* of

2 §6.2 Can an interferometer do this at all?

Current detectors are interferometers. Can they reach $h \sim 10^{-20}$?

Simple Michelson. Measurable length change $\sim$ wavelength of the light:

$$h = \frac{\delta L}{L} \sim \frac{\lambda_{\text{laser}}}{L} \sim \frac{10^{-6} \text{ m}}{10^3 \text{ m}} = 10^{-9}.$$

Eleven orders of magnitude short

Nowhere near sufficient. Two improvements close the gap.

- 1 **Longer optical path.** More round trips, more accumulated phase.
- 2 **Better fringe measurement.** Photon counting is statistical; statistics improve with N .

2 §6.2 Improvement 1: a longer optical path

Fold the light: *effective* arm length L_{eff} exceeds the physical arm length.

But there is a ceiling

L_{eff} can only be as long as a gravitational wavelength λ_{GW} . Beyond that, the wave reverses sign while the light is still in the arm — accumulated phase starts to *cancel*.

$$h \sim \frac{\delta L}{L_{\text{eff}}} \sim \frac{\lambda_{\text{laser}}}{\lambda_{\text{GW}}} \sim \frac{10^{-6} \text{ m}}{10^6 \text{ m}} = 10^{-12}.$$

Three orders gained. Eight to go.

A 100 Hz wave: $\lambda_{\text{GW}} = c/f \sim 3 \times 10^6 \text{ m}$. With 4 km arms — a few hundred round trips. Exactly what the

2 §6.2 Improvement 2: measuring the fringe

Photon arrival is Poisson $\Rightarrow$ count fluctuation $\sqrt{N}$. The fringe can be split far more finely than one wavelength:

$$\delta L \sim \frac{N_{\text{photon}}^{1/2}}{N_{\text{photon}}} \lambda_{\text{laser}}.$$

Photons per gravitational-wave period:

$$N_{\text{photon}} = \frac{P_{\text{laser}}}{hc/\lambda_{\text{laser}}} \tau = \frac{P_{\text{laser}}}{hcf_{\text{GW}}/\lambda_{\text{laser}}}.$$

$P_{\text{laser}} = 1 \text{ W}$, $\lambda_{\text{laser}} = 1 \mu\text{m}$, $f_{\text{GW}} \sim 300 \text{ Hz}$ gives $N_{\text{photon}} \sim 10^{16}$:

$$h \sim \frac{\delta L}{L_{\text{eff}}} \sim \frac{N^{-1/2} \lambda_{\text{laser}}}{\lambda_{\text{GW}}} \sim \frac{10^{-8} \times 10^{-6} \text{ m}}{10^6 \text{ m}} = 10^{-20}.$$

That is the target

The design of a gravitational-wave interferometer in two instructions made into hardware: [fold the light](#)

2 §6.2.1 Recap of optics

Monochromatic beam along z :

$$E(t, z) \propto \Re \left\{ E^+ e^{i(\omega t - kz)} + E^- e^{i(\omega t + kz)} \right\},$$

E^+ , E^- the complex fields travelling in $\pm z$. After a distance d : a phase e^{-ikd} .

At a mirror — part reflected, part transmitted:

$$r = \frac{E_R}{E_I}, \quad t = \frac{E_T}{E_I}, \quad \text{lossless:} \quad 1 = |r|^2 + |t|^2.$$

One side of the mirror only. From the other side: r' , t' . For simple mirrors with real r , t :

$$r = -r', \quad t = t'.$$

That sign is not cosmetic — it is what makes one Michelson port dark while the other is bright.

2 §6.2.2 The Michelson interferometer

Beam splitter: a mirror with $r_{\text{BS}} = t_{\text{BS}} = 1/\sqrt{2}$. Light split equally into arms X, Y of mean lengths l_1, l_2 , ending on mirrors r_1, r_2 .

Two beams leave — back towards the laser (**symmetric** port) and orthogonal to it (**anti-symmetric** port):

$$E_{\text{sym}} = r_{\text{BS}} r_2 E_2 e^{-2ikl_2} + t_{\text{BS}} r_1 E_1 e^{-2ikl_1} = \frac{1}{2} E_I \left[r_2 e^{-2ikl_2} + r_1 e^{-2ikl_1} \right],$$

$$E_{\text{ant}} = t_{\text{BS}} r_2 E_2 e^{-2ikl_2} - r_{\text{BS}} r_1 E_1 e^{-2ikl_1} = \frac{1}{2} E_I \left[r_2 e^{-2ikl_2} - r_1 e^{-2ikl_1} \right].$$

Mean length $\bar{l} = \frac{1}{2}(l_1 + l_2)$, difference $\delta = l_1 - l_2$:

$$E_{\text{sym}} = \frac{1}{2} E_I e^{-2ik\bar{l}} \left[r_2 e^{ik\delta} + r_1 e^{-ik\delta} \right], \quad E_{\text{ant}} = \frac{1}{2} E_I e^{-2ik\bar{l}} \left[r_2 e^{ik\delta} - r_1 e^{-ik\delta} \right].$$

Anti-symmetric port responds to the *difference* δ — the quadrupolar signature — and is blind to common-mode motion.

2 §6.2.3 The Fabry–Pérot cavity

Michelson alone is not enough. To extend the effective arm length: a **Fabry–Pérot cavity** — two mirrors, *input test mass* (ITM) and *end test mass* (ETM), separated by L .

E_I enters through the ITM; E_+ circulates towards the ETM, returns as $E_- = E_+ e^{-2ikL}$. At the ITM:

$$E_+ = t_{\text{ITM}} E_I - r_{\text{ITM}} E_- = t_{\text{ITM}} E_I - r_{\text{ITM}} e^{-2ikL} E_+,$$

so

$$E_+ = \frac{t_{\text{ITM}}}{1 + r_{\text{ITM}} e^{-2ikL}} E_I$$

Resonance

Tune $L = L_{\text{res}}$ with $e^{-2ikL_{\text{res}}} = -1$:

$$E_+ = \frac{t_{\text{ITM}}}{1 - r_{\text{ITM}}} E_I.$$

2 §6.2.3 The cavity as an effective mirror

Field reflected by the cavity, i.e. transmitted back out of the ITM:

$$E_R = r_{\text{ITM}}E_I + t_{\text{ITM}}e^{-2ikL}E_+ = \left[r_{\text{ITM}} + \frac{t_{\text{ITM}}^2 e^{-2ikL}}{1 + r_{\text{ITM}}e^{-2ikL}} \right] E_I = \frac{r_{\text{ITM}} + e^{-2ikL}}{1 + r_{\text{ITM}}e^{-2ikL}} E_I.$$

So the cavity has an **effective reflectivity**

$$r_{\text{FP}} = \frac{r_{\text{ITM}} + e^{-2ikL}}{1 + r_{\text{ITM}}e^{-2ikL}}.$$

At resonance: $r_{\text{FP}} = -1$

Perfectly reflective. Which is exactly what lets a Fabry–Pérot cavity serve as an effective mirror inside a Michelson.

The compound object behaves like a mirror — but one whose phase response to a length change is

2 §6.2.3 The gain factor

Cavity length changes by ΔL , so $L = L_{\text{res}} + \Delta L$:

$$\begin{aligned} r_{\text{FP}}(L_{\text{res}} + \Delta L) &\simeq r_{\text{FP}}(L_{\text{res}}) + \left. \frac{dr_{\text{FP}}}{dL} \right|_{L_{\text{res}}} \Delta L \\ &= -1 + (-2ik\Delta L) \left[\frac{e^{-2ikL}(1 - r_{\text{ITM}}^2)}{(1 + r_{\text{ITM}}e^{-2ikL})^2} \right]_{L_{\text{res}}} = -1 + \frac{1 + r_{\text{ITM}}}{1 - r_{\text{ITM}}} 2ik\Delta L \\ &\simeq -\exp \left[-2ik \left(\frac{1 + r_{\text{ITM}}}{1 - r_{\text{ITM}}} \right) \Delta L \right]. \end{aligned}$$

Gain factor

$$G_{\text{FP}} = \frac{1 + r_{\text{ITM}}}{1 - r_{\text{ITM}}}$$

— phase gain over an ordinary Michelson.

Derivation of $G_{\text{FP}} > 100$ for $L_{\text{res}} = 15 \text{ km}$ and $r_{\text{ITM}} = 0.9999999999$ with $G_{\text{FP}} = 100$ for $L_{\text{res}} = 15 \text{ km}$ and $r_{\text{ITM}} = 0.9999999999$

2 §6.2.4 The Fabry–Pérot Michelson interferometer

Replace each Michelson end mirror by a cavity. Each arm is now a compound mirror of complex, frequency-dependent reflectivity r_{FP} . With $r_1 = r_{\text{FP}}(L_1)$, $r_2 = r_{\text{FP}}(L_2)$:

$$r_{\text{FPMI}}(L_1, L_2) = \frac{1}{2}e^{-2ik\bar{l}} \left[r_{\text{FP}}(L_2)e^{ik\delta} + r_{\text{FP}}(L_1)e^{-ik\delta} \right],$$

$$t_{\text{FPMI}}(L_1, L_2) = \frac{1}{2}e^{-2ik\bar{l}} \left[r_{\text{FP}}(L_2)e^{ik\delta} - r_{\text{FP}}(L_1)e^{-ik\delta} \right],$$

$\bar{l} = (l_1 + l_2)/2$, $\delta = l_2 - l_1$. At resonance, $r_{\text{FP}} = -1$ in both arms:

$$r_{\text{FPMI, res}} = -e^{-2ik\bar{l}} \cos(k\delta), \quad t_{\text{FPMI, res}} = -ie^{-2ik\bar{l}} \sin(k\delta).$$

The dark fringe

$\delta = 0 \Rightarrow t_{\text{FPMI}} = 0$: **no light out of the anti-symmetric port**. The operating point — signal on a black background.

2 §6.2.4 The linear response

A wave — or an environmental disturbance — gives small deviations ΔL_1 , ΔL_2 from resonance. To first order:

$$r_{\text{FPMI}} \approx r_{\text{FPMI, res}} + \frac{1}{2} e^{-2ik\bar{l}} G_{\text{arm}} \left[2ik\Delta L_2 e^{ik\delta} + 2ik\Delta L_1 e^{-ik\delta} \right],$$
$$t_{\text{FPMI}} \approx t_{\text{FPMI, res}} + \frac{1}{2} e^{-2ik\bar{l}} G_{\text{arm}} \left[2ik\Delta L_2 e^{ik\delta} - 2ik\Delta L_1 e^{-ik\delta} \right],$$

with $G_{\text{arm}} = G_{\text{FP}}$ the gain factor.

Read the structure

- ▶ Symmetric port $\rightarrow$ the *sum* $\Delta L_1 + \Delta L_2$: common mode.
- ▶ Anti-symmetric port $\rightarrow$ the *difference* $\Delta L_2 - \Delta L_1$. The wave, being quadrupolar, is precisely a differential signal.
- ▶ Laser frequency and intensity noise: to first order, common.

2 §6.6 Noise sources: the concepts

Among the most sensitive instruments ever built — and limited by noise. What follows is an *overview*: what each source is, where it dominates, what it scales with.

Source	Where it bites	Physical origin
Seismic	lowest frequencies	ground motion, coupled mechanically
Newtonian (gravity gradient)	$\lesssim 10\text{--}20\text{ Hz}$	density fluctuations, coupled <i>gravitationally</i>
Thermal	$\sim 10\text{--}1000\text{ Hz}$	dissipation in suspensions and coatings
Shot noise	high frequencies	discreteness of photons
Radiation pressure	low frequencies	photon momentum kicking the mirrors

One organising idea

The last two share an origin — the quantum nature of the light — and pull in opposite directions with

2 §6.6.1 Seismic noise

Vibrations in the Earth's crust. Sources: natural seismic activity (earthquakes, ocean waves); human activity (traffic, machinery); environment (wind, thermal fluctuations).

The tool: the displacement power spectrum

$S_d(f)$, in m^2/Hz — power of seismic displacement per unit frequency.

- ▶ Spans below 1 Hz to several tens of Hz.
- ▶ Amplitude generally *decreases* with frequency.
- ▶ Low frequencies: a power law, typically $\propto 1/f^n$, $n > 0$.

The good news

Couples *mechanically* — so it can be fought mechanically. Multi-stage pendulums, each stage suppressing

2 §6.6.2 Newtonian noise: the one you cannot shield

Gravity gradient noise. *Not* vibrations mechanically moving the test masses, but **time-varying gravitational forces** from fluctuations in nearby mass distributions — seismic waves in the crust, atmospheric density perturbations.

Why it is fundamental

Couples to the test masses *through gravity*. No mechanical or seismic isolation can shield it. Limits ground-based interferometers below $\sim 10\text{--}20$ Hz.

A density fluctuation $\delta\rho(\mathbf{x}')$ induces

$$\frac{d^2\mathbf{x}}{dt^2} = G \int \frac{\delta\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x',$$

and in the frequency domain, displacement along the arm:

$$\tilde{x}(f) = -\frac{G}{4\pi^2 c^2} \int \frac{\delta\tilde{\rho}(f)}{r^2} \sin\theta \cos\phi dV.$$

2 §6.6.2 Newtonian noise: the spectrum

Density fluctuations coherent within regions of size $(\lambda/2)^3$, $\lambda = v_s/f$, v_s the speed of sound:

$$S_x(f) \approx \left(\frac{G}{(2\pi f)^2} \right)^2 S_\rho(f) \left(\frac{\lambda}{2} \right)^3 \int_{r > \lambda/4} \frac{1}{r^4} \sin^2 \theta \cos^2 \phi \, dV = \frac{4\pi^3}{3} \left(\frac{G v_s}{(2\pi f)^4} \right)^2 S_\rho(f),$$

$S_\rho(f)$ the power spectral density of environmental density fluctuations.

Atmosphere. Pressure $\rightarrow$ density; adiabatic:

$$\frac{\delta \rho}{\rho} = \frac{1}{\gamma} \frac{\delta p}{p}, \quad \gamma \approx 1.4.$$

Ground. Seismic motion $\rightarrow$ local density variation; one horizontal component:

$$\frac{\delta \rho}{\rho} = \frac{\delta X}{\lambda/2},$$

$\lambda/2$ the coherently moving piece of ground.

Note the f^{-8} in S_x : a steep rise towards low frequency. The main argument for putting third-generation detectors *underground*.

2 §6.6.3 Thermal noise and the fluctuation–dissipation theorem

Dominates the most sensitive band, $\sim 10\text{--}1000\text{ Hz}$. Origin: microscopic thermal motion of atoms, appearing as macroscopic mechanical fluctuation. It arises from *dissipation*, and the quantitative link is the *fluctuation–dissipation theorem*:

$$S_x(f) = \frac{4k_B T}{(2\pi f)^2} \operatorname{Re}[Y(f)], \quad Y(f) = \frac{\tilde{v}(f)}{\tilde{F}(f)} = i2\pi f \frac{\tilde{x}(f)}{\tilde{F}(f)},$$

$Y(f)$ the mechanical admittance.

Undamped mass on a spring, $m\ddot{x} + kx = F$:

$$\frac{\tilde{x}(f)}{\tilde{F}(f)} = \frac{1}{k - m(2\pi f)^2}, \quad Y(f) = i2\pi f \cdot \frac{1}{k - m(2\pi f)^2}.$$

Here $\operatorname{Re}[Y] = 0$: *no dissipation, no noise*. All thermal noise comes from the imaginary part of the spring

2 §6.6.3 The loss angle, and the two thermal noises

Real materials dissipate through internal friction. Model: a complex spring constant

$$k \rightarrow k(1 + i\phi),$$

$\phi \ll 1$ the **loss angle**. Then $m\ddot{x} + k(1 + i\phi)x = F$, so

$$\frac{\tilde{x}(f)}{\tilde{F}(f)} = \frac{1}{k(1 + i\phi) - m(2\pi f)^2}, \quad Y(f) = i2\pi f \cdot \frac{1}{k(1 + i\phi) - m(2\pi f)^2}.$$

Suspension thermal noise

Test masses hang as pendulums, isolating them from the ground. The suspensions themselves fluctuate — internal friction in the wires.

Coating thermal noise

Mirrors fluctuate — mechanical loss in substrate and dielectric coatings. Fused silica substrates are low-loss; the *coatings* dominate.

Two levers, directly: lower T , or lower ϕ . Both are being pursued — cryogenics, better coating materials.

2 §6.6.4 Shot noise

A fundamental *quantum* noise: discreteness of photons. It limits the precision on the light intensity — hence on the phase shift caused by a wave.

Photodetector, angular frequency ω_L , time T , N_γ photons: $E = N_\gamma \hbar \omega_L$, so $P = N_\gamma \hbar \omega_L / T$. Poisson $\Rightarrow \Delta N_\gamma = \sqrt{N_\gamma}$, so

$$(\Delta P)_{\text{shot}} = \frac{\Delta N_\gamma \hbar \omega_L}{T} = \frac{\sqrt{N_\gamma} \hbar \omega_L}{T}.$$

Substitute $N_\gamma = PT / \hbar \omega_L$:

$$\boxed{(\Delta P)_{\text{shot}} = \left(\frac{\hbar \omega_L P}{T} \right)^{1/2}}$$

Relative fluctuation falls as $P^{-1/2}$ — more power, better phase resolution. Improvement 2 of §6.2, seen from the noise side.

2 §6.6.5 Radiation pressure

Each reflected photon transfers momentum $2p = 2\hbar\omega_L/c$. Beam power P gives average force $F = 2P/c$; the shot-noise fluctuation in P makes it fluctuate:

$$\Delta F = \frac{2}{c}\Delta P = \frac{2}{c}\left(\frac{\hbar\omega_L P}{T}\right)^{1/2}.$$

Uncorrelated photon arrivals (white noise) $\Rightarrow$ single-sided spectral density

$$S_F^{1/2}(f) = \frac{2\sqrt{2\hbar\omega_L P}}{c}.$$

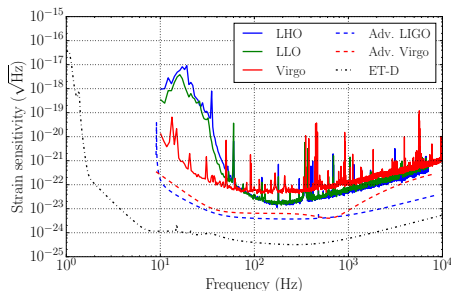
Free mass M : $F = M\ddot{x}$ gives $\tilde{F}(f) = -M(2\pi f)^2 \tilde{x}(f)$, so

$$S_x^{1/2}(f) = \frac{S_F^{1/2}(f)}{M(2\pi f)^2} = \frac{2\sqrt{2\hbar\omega_L P}}{Mc(2\pi f)^2}$$

The tension

Shot noise $\propto P^{-1/2}$; radiation pressure $\propto P^{1/2}$ and $\propto f^{-2}$. So: an optimal power at each frequency.

2 §6.6 The noise budget, assembled



Strain sensitivity $\sqrt{S_n(f)}$ [script Fig. 7.2]:
first-generation LIGO and Virgo (solid), Advanced
LIGO and Advanced Virgo design (dashed). Einstein

Reading the curve, left to right:

- ▶ *seismic wall* at the lowest frequencies, where isolation runs out;
- ▶ Newtonian noise underneath it — no isolation removes it;
- ▶ broad thermal-noise region through the most sensitive band;
- ▶ rising *shot-noise* tail, $\propto \sqrt{f}$ in strain.

Best sensitivity: a bucket around 100 Hz — not coincidentally, where stellar-mass binaries merge.
 $\sqrt{S_n(f)}$ in $1/\sqrt{\text{Hz}}$; definition in §7.1.

2 §6.3.1 A different observable: the Doppler shift

Another route: measure the **frequency shift** of light transmitted from one object to another. A passing wave alters that shift.

Objects A and B; B at the origin, A in the x - z plane. Signal from A to B along $\hat{\mathbf{p}}$, at angle θ to the propagation direction $\hat{\mathbf{n}} = \mathbf{e}_3$. Plane wave along z , plus polarisation only:

$$h_{11} = -h_{22} = h_+(t - z/c).$$

Null geodesic, tangent $k^\mu = dx^\mu/d\lambda$. To leading order in h_+ :

$$k^\mu = \nu \left[1, -\left(1 - \frac{1}{2}h_+\right) c \sin \theta, 0, -c \cos \theta \right],$$

$\nu = dt/d\lambda$ the frequency seen by a stationary observer.

The natural language for LISA, where §6.1 said the geodesic-deviation picture fails: never moving mirrors, only light frequencies

2 §6.3.1 Integrating along the light ray

Geodesic equation $dk^\mu/d\lambda + \Gamma_{\alpha\beta}^\mu k^\alpha k^\beta = 0$, time component, with $\Gamma_{11}^0 = \frac{1}{2c^2} \partial h_+ / \partial t$:

$$\frac{d\nu}{d\lambda} = -\Gamma_{11}^0 k^1 k^1 = -\frac{1}{2c^2} \frac{\partial h_+}{\partial t} \left[\nu \left(1 - \frac{1}{2} h_+ \right) c \sin \theta \right]^2 = -\frac{1}{2c^2} \frac{\partial h_+}{\partial t} \nu^2 \sin^2 \theta + \mathcal{O}(h_+^2).$$

Partial derivative as a total derivative along the ray:

$$\frac{d \ln \nu}{d\lambda} = -\frac{d}{d\lambda} \left[\frac{1}{2} (1 - \cos \theta) h_+(t - z/c) \right],$$

which integrates from emission at A to reception at B:

$$\ln \left(\frac{\nu_A}{\nu_B} \right) = \frac{1}{2} (1 - \cos \theta) [h_+(x_B^\alpha) - h_+(x_A^\alpha)] + \mathcal{O}(h^2).$$

Depends only on h_+ at the two endpoints — not on the history along the ray. That is what makes the next slides possible.

2 §6.3.1 The one-way Doppler shift

With $\ln(\nu_A/\nu_B) = \ln(1+z) \simeq z$:

$$z = \frac{1}{2}(1 - \cos \theta) [h_+(x_B^\alpha) - h_+(x_A^\alpha)] + \mathcal{O}(h_+^2)$$

Arbitrary polarisation and wave direction $\hat{\mathbf{n}}$:

$$z = \frac{\nu_{\text{em}} - \nu_{\text{rec}}}{\nu_{\text{rec}}} = \frac{1}{2} \frac{\hat{p}^i \hat{p}^j}{1 + \hat{\mathbf{n}} \cdot \hat{\mathbf{p}}} \left[h_{ij}^{\text{TT}}(t_{\text{rec}}, \vec{x}_{\text{rec}}) - h_{ij}^{\text{TT}}(t_{\text{em}}, \vec{x}_{\text{em}}) \right].$$

Two endpoints, two terms

A *difference* of the metric perturbation at reception and at emission. Everything left in this lecture — time-delay interferometry, pulsar timing — is built on this one expression.

$\hat{n}^i \hat{n}^j$: sensitive only to the polarisation component along the line of sight. $(1 + \hat{\mathbf{n}} \cdot \hat{\mathbf{p}})^{-1}$: the usual beaming

2 §6.3.2 LISA and its one-way measurements

LISA: three spacecraft, nearly equilateral triangle, arm lengths $L \sim 5 \times 10^9$ m, trailing Earth in a heliocentric orbit. Continuous laser exchange between all pairs.

Positions $\vec{r}_1, \vec{r}_2, \vec{r}_3$:

$$\hat{r}_i = \frac{\vec{r}_i}{\ell}, \quad \hat{p}_{ij} = \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|},$$

ℓ the barycentre-to-spacecraft distance.

Each spacecraft measures the frequency shift from the other two — **six** one-way fractional Doppler measurements:

$$z_{21}(t), \quad z_{31}(t), \quad z_{12}(t), \quad z_{32}(t), \quad z_{13}(t), \quad z_{23}(t),$$

$z_{ij}(t)$ measured at spacecraft j , from a signal sent by i .

No beam splitter, no fringe. The “interferometer” is assembled afterwards, in software, from these six time

2 §6.3.2 Signal and laser noise in one measurement

Gravitational-wave contribution

$$z_{21}^{\text{GW}}(t) = \frac{1}{2} \hat{p}_{21}^i \hat{p}_{21}^j \frac{h_{ij}\left(t - \frac{\ell}{c} \hat{r}_1 \cdot \hat{n}\right) - h_{ij}\left(t - \frac{\ell}{c} \hat{r}_2 \cdot \hat{n} - \frac{L_{21}}{c}\right)}{1 - \hat{p}_{21} \cdot \hat{n}}$$

$\hat{n}$ the propagation direction. The two-endpoint formula again — endpoints at the two spacecraft.

Laser noise contribution

Light on spacecraft i at $\nu_i = \nu_0 [1 + C_i(t)]$, $C_i(t)$ the fractional frequency noise:

$$z_{21}^{\text{laser}}(t) = C_2(t - L_{21}/c) - C_1(t).$$

Laser frequency noise sits many orders of magnitude above the signal. But in a difference such as $z_{31}(t) - z_{21}(t) = C_3(t - L_{31}/c) - C_2(t - L_{21}/c)$, the common $C_1(t)$ **cancels**. Seed of the whole idea.

2 §6.3.2 First-generation TDI

TDI cancels the overwhelming laser frequency noise: specific linear combinations of the one-way measurements, with appropriate *time delays*. Two families.

Sagnac-like variables

$$\begin{aligned}\alpha(t) = & z_{31}(t) - z_{21}(t) + z_{23}(t - L_{13}/c) - z_{32}(t - L_{12}/c) \\ & + z_{12}(t - (L_{13} + L_{23})/c) - z_{13}(t - (L_{12} + L_{23})/c)\end{aligned}$$

plus cyclic permutations, $\beta(t)$ and $\gamma(t)$.

Michelson-like variables

$$\begin{aligned}X(t) = & z_{31}(t) - z_{21}(t) + z_{13}(t - L_{13}/c) - z_{12}(t - L_{12}/c) \\ & - z_{31}(t - 2L_{12}/c) + z_{21}(t - 2L_{13}/c) \\ & - z_{13}(t - (2L_{12} + L_{13})/c) + z_{12}(t - (2L_{13} + L_{12})/c)\end{aligned}$$

2 §6.3.2 Why a second generation is needed

The assumption behind first-generation TDI

α, β, γ and X, Y, Z assume a **static, equilateral** triangle with *constant* arm lengths.

Reality: the spacecraft orbit the Sun. The constellation slowly *rotates* and *flexes*, so arm lengths vary. Symmetry broken; first-generation combinations leave residual laser noise on real data.

Second-generation TDI

Extra time-delay terms, accounting for time-dependent arm lengths and for the **non-commutativity** of the delay operators. Restores the cancellation in a rotating, flexing configuration — essential for the real LISA.

Non-commutativity is the crux: with time-varying delays, “delay by L_{12} then by L_{13} ” differs from the reverse. That residual is what second-generation TDI cancels.

2 §6.4 Pulsar timing

Pulsars: highly magnetised, rotating neutron stars, beaming from their magnetic poles. Rotationally stable
⇒ precise cosmic clocks.

Pulsar timing monitors pulse arrival times and looks for deviations — *timing residuals* — from astrophysical or gravitational effects, a passing wave among them.

Pulse phase, Taylor expanded around a reference epoch T_0 :

$$\phi_{\text{model}}(T) = \phi_0 + (T - T_0)\omega_0 + \frac{1}{2}(T - T_0)^2\dot{\omega}_0,$$

ϕ_0 the phase at T_0 , ω_0 the angular rotation frequency, $\dot{\omega}_0$ the spin-down rate.

Timing residual

$$R(t) = \frac{\phi_{\text{model}}(T) - \phi_{\text{obs}}(T)}{2\pi\nu}, \quad \nu = \omega_0/2\pi.$$

2 §6.4 The residual from a gravitational wave

Residual from the Doppler-induced redshift: the *integral* of the shift,

$$R(t) = \int_{t_0}^t z(t') dt',$$

with $z(t)$ the same two-endpoint formula as in §6.3:

$$z(t) = \frac{\nu_{\text{pulsar}} - \nu_{\text{Earth}}(t)}{\nu_{\text{Earth}}} = \frac{1}{2} \frac{\hat{p}^i \hat{p}^j}{1 + \hat{p} \cdot \hat{n}} \left[h_{ij}^{\text{Earth}}(t) - h_{ij}^{\text{pulsar}}(t_p) \right],$$

$\hat{p}$ from Earth to pulsar, $\hat{n}$ the propagation direction, pulsar term at the retarded time $t_p = t - D/c$.

Same physics, different baseline

§6.3: 5×10^9 m between two spacecraft. Here: the *Earth–pulsar distance* D , of order kiloparsecs. Longer baseline $\Rightarrow$ lower frequency — the nanohertz band.

2 §6.4 The Earth term and the pulsar term

The redshift separates into two contributions.

Earth term $h_{ij}^{\text{Earth}}(t)$

Instantaneous effect at reception. Common to all pulsars observed at a given time.

Pulsar term $h_{ij}^{\text{pulsar}}(t_p)$

Effect at the pulsar, retarded by D/c . Distances and sky positions differ, so it is uncorrelated across pulsars — an additional noise-like component.

The correlated Earth terms are the key to detection with a pulsar timing array.

One pulsar cannot do it — its residuals are dominated by its own timing noise. An *array* of tens of millisecond pulsars can: the gravitational-wave contribution correlates between pairs in a way that depends only on their angular separation.

2 Lecture 2 — summary

- 1 **§1.3** Geodesic deviation, $\zeta^i = \frac{1}{2}\ddot{h}_{ij}^{\text{TT}}\zeta^j$: a quadrupolar *strain*, displacement $\propto$ unperturbed separation, polarisations at 45° — a spin-2 signature.
- 2 **§6.1** TT frame: coordinate separations do not change; *proper distance* does, $s \approx L(1 + \frac{1}{2}h_+ \cos \omega t)$, and with it the light travel time. Proper detector frame, while $L \ll \bar{\lambda}$: the wave acts as a Newtonian tidal force.
- 3 **§6.2** Bare Michelson: $h \sim 10^{-9}$. Folding the light gains three orders (capped by λ_{GW}); Poisson statistics on $N_\gamma \sim 10^{16}$ photons gains eight more. In hardware: Fabry–Pérot arm cavities, $G_{\text{FP}} = (1 + r_{\text{ITM}})/(1 - r_{\text{ITM}}) > 100$, Michelson on the dark fringe.
- 4 **§6.6** Seismic (mechanically isolable); Newtonian (*not* isolable); thermal, via fluctuation–dissipation and the loss angle; and two quantum noises — shot ($\propto P^{-1/2}$) against radiation pressure ($\propto P^{+1/2}$).
- 5 **§6.3** The wave shifts light frequency by a *difference* of h_{ij} at emission and reception. LISA: six one-way shifts, interferometer built in software, TDI cancelling laser frequency noise.
- 6 **§6.4** Same formula, kiloparsec baseline. Timing residuals: common Earth term is signal, uncorrelated pulsar term is noise.

3 Outline

- ① Lecture 1: Theory — waves, and how to compute them
- ② Lecture 2: Gravitational-wave detectors
- ③ Lecture 3: Sources and their detection

Lecture 3

Gravitational-wave sources and their detection

Script §5.1–5.3 | §7.1–7.5

3 Lecture 3: the plan

- 1 **Generation** (§5.1) — retarded solution; multipole expansion; why the leading emission is quadrupolar.
- 2 **Two sources** (§5.2–5.3) — circular binary, rotating rigid body. *Set-up and end results.*
- 3 **The data** (§7.1) — detector tensor and beam patterns; noise as a stationary Gaussian process; its power spectral density.
- 4 **Matched filtering** (§7.2) — why a buried signal is recoverable; optimal filter; noise-weighted inner product; covering the parameter space.
- 5 **Poorly modelled signals** (§7.3) — excess power, and the price of not knowing the waveform.
- 6 **Inference** (§7.4) — inductive logic; Bayes; parameter estimation; hypothesis testing. *Concepts only.*
- 7 **Bayesian inference for gravitational waves** (§7.5) — likelihood and posterior; maximum likelihood; Fisher matrix; sampling.

3 §5.1.1 Solving the field equations with a Green's function

Back to §1.1:

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$

Linear $\Rightarrow$ solve with a Green's function, $\square_x G(x - x') = \delta^4(x - x')$:

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' G(x - x') T_{\mu\nu}(x').$$

Boundary condition **no incoming radiation from infinity** selects the *retarded* Green's function:

$$G(x - x') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \delta(x_{\text{ret}}^0 - x'^0), \quad t_{\text{ret}} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}.$$

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\right)$$

3 §5.1.3 The multipole expansion: the far-field limit

§5.1.2, quoted: how to get into the TT gauge

Outside the source, act with the projector $\Lambda_{ijkl}(\hat{\mathbf{n}}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$, built from $P_{ij} = \delta_{ij} - n_in_j$: idempotent, transverse in every index, traceless on both index pairs. Then $h_{ij}^{\text{TT}} = \Lambda_{ijkl}h_{kl}$ is *equivalent* to a gauge transformation into TT gauge. Used here as a black box.

Now take $r \gg d$, the source size, and expand $|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + \mathcal{O}(d^2/r)$:

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ijkl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}' \right).$$

Fourier-expand T_{kl} . Typical source: power only up to some ω_s ; non-relativistic means $\omega_s d \ll c$. With $|\mathbf{x}'| \lesssim d$, the dominant frequencies satisfy $\frac{\omega}{c} \mathbf{x}' \cdot \hat{\mathbf{n}} \lesssim \frac{\omega_s d}{c} \ll 1$.

So $\omega \mathbf{x}' \cdot \hat{\mathbf{n}}/c$ is a good expansion parameter.

3 §5.1.3 Expanding, and the stress multipole moments

Time domain:

$$T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\right) = T_{kl}(t - r/c, \mathbf{x}') + \frac{x'^i n^i}{c} \partial_0 T_{kl} + \frac{1}{2c^2} x'^i x'^j n^i n^j \partial_0^2 T_{kl} + \dots$$

Multipole moments of the stress tensor [script Def. 5.1]:

$$S^{ij} = \int d^3x T^{ij}, \quad S^{ij,k} = \int d^3x T^{ij} x^k, \quad S^{ij,kl} = \int d^3x T^{ij} x^k x^l, \quad \dots$$

Multipole expansion of the perturbation [script Def. 5.2]

$$h_{ij}^{\text{TT}} = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ijkl}(\hat{\mathbf{n}}) \left[S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp} + \dots \right]_{\text{ret}}$$

An expansion in v/c : each extra moment brings $x^m \sim \mathcal{O}(d)$, each time derivative $\mathcal{O}(\omega_s)$; with the $1/c$, that is $\mathcal{O}(\omega_s d/c)$.

3 §5.1.3 From stresses to masses

Moments of the *stresses* T^{ij} are hard to determine. Far more intuitive: moments of the *mass density* $c^{-2}T^{00}$ and the *momentum density* $c^{-1}T^{0i}$.

Mass moments [script Def. 5.3]

$$\begin{aligned}M &= \frac{1}{c^2} \int d^3x T^{00}, \\M^i &= \frac{1}{c^2} \int d^3x T^{00} x^i, \\M^{ij} &= \frac{1}{c^2} \int d^3x T^{00} x^i x^j, \dots\end{aligned}$$

Momentum moments [script Def. 5.4]

$$\begin{aligned}P^i &= \frac{1}{c} \int d^3x T^{0i}, \\P^{ij} &= \frac{1}{c} \int d^3x T^{0i} x^j, \\P^{ijk} &= \frac{1}{c} \int d^3x T^{0i} x^j x^k, \dots\end{aligned}$$

Stress moments re-expressed in these. The leading term is the key step.

3 §5.1.3 The conservation law does the work

Start from S^{ij} . Use $\partial_k x^i = \delta_k^i$; integrate by parts (boundary term vanishes, $T^{kl} \rightarrow 0$ at infinity); apply $\partial_\mu T^{\mu\nu} = 0$ from §1.1.3:

$$S^{ij} = \int d^3x (\partial_k x^i)(\partial_l x^j) T^{kl} = - \int d^3x x^i (\partial_l x^j) \partial_k T^{kl} = \int d^3x x^i (\partial_l x^j) \partial_0 T^{0l}.$$

Same two moves again:

$$\begin{aligned} S^{ij} &= - \int d^3x x^i x^j \partial_l \partial_0 T^{0l} - \int d^3x \delta_l^i x^j \partial_0 T^{0l} = \int d^3x x^i x^j \partial_0^2 T^{00} + \int d^3x x^j \partial_k T^{ki} \\ &= \frac{1}{c^2} \int d^3x x^i x^j \ddot{T}^{00} - \int d^3x T^{ij} = \ddot{M}^{ij} - S^{ij}. \end{aligned}$$

$$S^{ij} = \frac{1}{2} \ddot{M}^{ij}$$

A stress integral has become a *second time derivative of a mass moment* — the one calculation that turns §1.1.3 into the quadrupole formula.

3 §5.1.4 The quadrupole formula

Leading order in v/c :

$$\left[h_{ij}^{\text{T}}(t, \mathbf{x})\right]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^4} \Lambda_{ijkl}(\hat{\mathbf{n}}) \ddot{M}^{kl}(t - r/c).$$

Mass quadrupole radiation. Λ_{ijkl} makes $\ddot{M}^{kl}$ traceless, so replace M^{kl} by

$$Q^{ij} \equiv M^{ij} - \frac{1}{3} \delta^{ij} M_{kk} \xrightarrow{\text{lowest order in } v/c} \int d^3x \rho(t, \mathbf{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij}\right)$$

— the quadrupole tensor of Newtonian theory.

Quadrupole formula [script Def. 5.5]

$$\left[h_{ij}^{\text{T}}(t, \mathbf{x})\right]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^4} \ddot{Q}_{ij}^{\text{TT}}(t - r/c), \quad Q_{ij}^{\text{TT}} = \Lambda_{ijkl}(\mathbf{n}) Q_{ij}.$$

The prefactor $G/c^4 \sim 10^{-45} \text{ s}^2 \text{ kg}^{-1} \text{ m}^{-1}$ is why this is hard.

3 §5.1.4 No monopole, no dipole

Leading contribution: the mass *quadrupole*. Monopole or dipole would need time derivatives of the mass monopole M or the momentum dipole P^i . But

$$\dot{M} = \frac{1}{c} \int d^3x \partial_0 T^{00} = -\frac{1}{c} \int d^3x \partial_i T^{0i} = 0,$$

— $T^{ij} \rightarrow 0$ at infinity, so the total divergence gives a vanishing boundary term. Similarly $\dot{P}^i = 0$.

The physical statement

Total mass and momentum are **conserved** — and that is exactly what forbids monopole and dipole radiation.

Compare electromagnetism: charge conservation kills the monopole, but the dipole survives. EM radiation therefore starts one order earlier.

M^i can always be set to zero by putting the origin at the centre of mass. The argument here is linearised; the result holds in the full non-linear theory too.

3 §5.1.4 Example: a quadrupolar wave along z

Set $\hat{n} = \hat{z}$ — no loss of generality, provided the *orientation of the source* stays arbitrary. Then P_{ij} projects onto the (x, y) plane, and for any 3×3 matrix A :

$$\Lambda_{ijkl} A_{kl} = (PAP)_{ij} - \frac{1}{2} P_{ij} \text{Tr}(PA) = \begin{pmatrix} (A_{11} - A_{22})/2 & A_{12} & 0 \\ A_{21} & -(A_{11} - A_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}.$$

Compare with $h_{ij}^{\text{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$ and read off:

$$h_+ = \frac{1}{r} \frac{G}{c^4} (\ddot{M}_{11} - \ddot{M}_{22}), \quad h_\times = \frac{2}{r} \frac{G}{c^4} \ddot{M}_{12},$$

both at the retarded time $t - r/c$.

Two lines of matrix algebra — and a formula applicable to any source. Used twice in the next ten minutes.

3 §5.1.4 A wave from a general direction

Generic direction $\hat{n}$ at angles (θ, ϕ) : rotate the mass quadrupole with

$$R = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}.$$

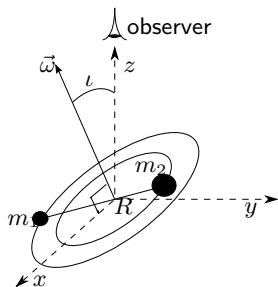
Rotating $\ddot{M}_{ij}$ explicitly:

$$\begin{aligned} h_+ &= \frac{G}{rc^4} \left[\ddot{M}_{11} (\cos^2 \phi - \sin^2 \phi \cos^2 \theta) + \ddot{M}_{22} (\sin^2 \phi - \cos^2 \phi \cos^2 \theta) \right. \\ &\quad \left. - \ddot{M}_{33} \sin^2 \theta - \ddot{M}_{12} \sin 2\phi (1 + \cos^2 \theta) + \ddot{M}_{13} \sin \phi \sin 2\theta + \ddot{M}_{23} \cos \phi \sin 2\theta \right] \\ h_\times &= \frac{G}{rc^4} \left[(\ddot{M}_{11} - \ddot{M}_{22}) \sin 2\phi \cos \theta + 2\ddot{M}_{12} \cos 2\phi \cos \theta - 2\ddot{M}_{13} \cos \phi \sin \theta + 2\ddot{M}_{23} \sin \phi \sin \theta \right]. \end{aligned}$$

And the radiated power

Quadrupole formula + energy flux of §1.4 = the Einstein quadrupole luminosity formula, $P_{\text{quad}} = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle$.

3 §5.2 Binary systems: the set-up



[script Fig. 5.1]

Masses m_1, m_2 ; circular orbit, angular velocity ω , about the common centre of mass. Separation R ; distance to observer r .

Coordinates: **observer along z** ; orbital plane intersects the x axis; origin at the centre of mass. Inclination ι between the orbital normal and the observer.

Positions:

$$\vec{y}_1(t) = \frac{\mu}{m_1} R \hat{e}(t), \quad \vec{y}_2(t) = -\frac{\mu}{m_2} R \hat{e}(t),$$

$\mu = m_1 m_2 / (m_1 + m_2)$, and

$$\hat{e}(t) = (\cos \omega t, \cos \iota \sin \omega t, \sin \iota \sin \omega t).$$

The rest is an application of the two boxed results of §5.1: compute $M^{ij}(t)$, differentiate twice, read off h_+ and $h_\times$.

3 §5.2.1 The quadrupole moment, and the waveform

Point masses:

$$M^{ij}(t) = \int d^3\vec{y} T^{00}(t, \vec{y}) y^i y^j \approx \int d^3\vec{y} [m_1 \delta(\vec{y} - \vec{y}_1) + m_2 \delta(\vec{y} - \vec{y}_2)] y^i y^j$$

$$\approx \mu R^2 \begin{pmatrix} \cos^2 \omega t & \cos \iota \cos \omega t \sin \omega t & \sin \iota \cos \omega t \sin \omega t \\ \cos \iota \cos \omega t \sin \omega t & \cos^2 \iota \sin^2 \omega t & \cos \iota \sin \iota \sin^2 \omega t \\ \sin \iota \cos \omega t \sin \omega t & \cos \iota \sin \iota \sin^2 \omega t & \sin^2 \iota \sin^2 \omega t \end{pmatrix}.$$

Insert into $h_+ = \frac{G}{rc^4}(\ddot{M}_{11} - \ddot{M}_{22})$, $h_\times = \frac{2G}{rc^4}\ddot{M}_{12}$. Up to a phase $\pi/2$ (choice of time origin):

$$h_+ = \frac{4\mu R^2 \omega^2}{r} \frac{1 + \cos^2 \iota}{2} \cos(2\omega t_{\text{ret}}), \quad h_\times = \frac{4\mu R^2 \omega^2}{r} \cos \iota \sin(2\omega t_{\text{ret}}).$$

Read the inclination dependence

Face-on ($\iota = 0$): equal amplitudes, 90° out of phase — circular polarisation, and the loudest case.
Edge-on ($\iota = \pi/2$): $h_\times$ vanishes, h_+ halved — linear polarisation, the quietest case.

3 §5.2.1 Kepler, and the chirp mass

Masses far enough apart for the Newtonian centripetal force. For m_1 :
 $m_1(R\mu/m_1)^2\omega^2/(R\mu/m_1) = m_1m_2/R^2$, and similarly for m_2 . Solve for R ; define the **chirp mass**

$$\mathcal{M}_c = \frac{(m_1m_2)^{3/5}}{(m_1 + m_2)^{1/5}}.$$

$$h_+ = \frac{4\mathcal{M}_c^{5/3}\omega^{2/3}}{r} \frac{1 + \cos^2 \iota}{2} \cos(2\omega t), \quad h_\times = \frac{4\mathcal{M}_c^{5/3}\omega^{2/3}}{r} \cos \iota \sin(2\omega t).$$

What this says

At leading order the amplitude depends on the two masses only through the *single* combination $\mathcal{M}_c$.
Hence: chirp mass best measured; individual masses much harder to separate.

3 §5.2.1 Twice the orbital frequency, and the chirp

Radiating at 2ω

M^{ij} is invariant under $y^i \rightarrow -y^i$ — the configuration repeats twice per orbit. Beyond the dominant quadrupole: harmonics $\omega_n = n\omega$.

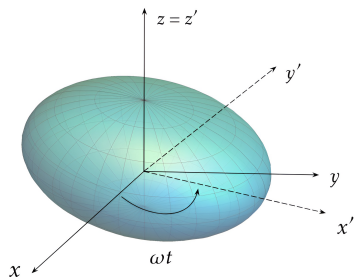
The chirp

Expressions above: monochromatic, constant amplitude. Reality: orbital energy carried off by the waves (§1.4), separation shrinks, angular velocity *increases*. Larger amplitude, higher frequency — a **chirp**, which gives $\mathcal{M}_c$ its name.

The radiated power

$P_{\text{quad}} = \frac{32}{5} \frac{c^5}{G} \left(\frac{GM_c \omega}{c^3} \right)^{10/3}$. Equate to $-dE_{\text{orbit}}/dt$ with $E_{\text{orbit}} = -\frac{1}{2}(G\omega)^{2/3} \mathcal{M}_c^{5/3}$ for the inspiral rate; via $\dot{T}/T = -\frac{3}{2} \dot{E}_{\text{orbit}}/E_{\text{orbit}}$ it predicts the orbital decay of PSR B1913+16 — the Hulse–Taylor test [script Ex. 5.5].

3 §5.3 A rotating rigid body: the set-up



A rigid body rotating about a principal axis — in particular, an isolated neutron star. Any rigid body is characterised by its inertia tensor

$$I^{ij} = \int d^3x \rho(\vec{x}) (r^2 \delta^{ij} - x^i x^j).$$

Triaxial ellipsoid, principal moments I_1, I_2, I_3 in the body frame (x', y', z') , rotating about z' at ω . Observer along $z = z'$ in the fixed frame. Frames related by

$$\mathcal{R}_{ij} = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij}, \quad I = \mathcal{R}^T I' \mathcal{R}.$$

[script Fig. 5.2]

M^{ij} differs from I^{ij} only by a sign and the trace term. And $\text{Tr } I' = I_1 + I_2 + I_3$ is constant, so $M_{ij} = -I_{ij} + c_{ij}$ with c_{ij} constant — and constants drop out under two time derivatives.

3 §5.3.1 The result: mountains on neutron stars

Carry out the rotation, insert into the same two formulae:

$$h_+ = \frac{4\omega^2}{r}(I_1 - I_2) \cos(2\omega t_{\text{ret}}), \quad h_\times = \frac{4\omega^2}{r}(I_1 - I_2) \sin(2\omega t_{\text{ret}}).$$

Monochromatic, constant amplitude, and — as for the binary — at *twice* the rotation frequency. Emission **only if $I_1 \neq I_2$** : a perfectly axisymmetric star rotating about its symmetry axis radiates nothing.

With the **ellipticity** $\epsilon = (I_1 - I_2)/I_3$:

$$h_+ = \frac{4\epsilon I_3 \omega^2}{r} \cos(2\omega t_{\text{ret}}), \quad h_\times = \frac{4\epsilon I_3 \omega^2}{r} \sin(2\omega t_{\text{ret}}), \quad P_{\text{quad}} = \frac{32}{5} \frac{G}{c^5} \epsilon^2 I_3^2 \omega^6.$$

How small is ϵ ?

Typical neutron star: $\epsilon \lesssim 10^{-7}$ — a “mountain” about **1 mm** high. Faster rotation, stronger signal; but as energy is radiated the rotation frequency decreases.

3 §7.1.1 What a detector measures: the detector tensor

Interferometers are sensitive to a difference of two distances — the **strain**. Arms along unit vectors u^i, v^i :

$$h(t) = \frac{1}{2} (h_{ij} u^i u^j - h_{ij} v^i v^j) = D^{ij} h_{ij}(t), \quad D^{ij} = \frac{1}{2} (u^i u^j - v^i v^j),$$

D^{ij} the **detector tensor**.

Linear in $h_+, h_\times$, so equivalently

$$h(t) = F_+ h_+(t) + F_\times h_\times(t)$$

with $F_{+,\times}$ the **beam pattern functions**.

A single detector cannot do much

One number per sample: a linear combination of the two polarisations, weighted by sky position and orientation. Sky position, distance, inclination and polarisation angle are all entangled in $F_+, F_\times$. Untangling them needs a *network*.

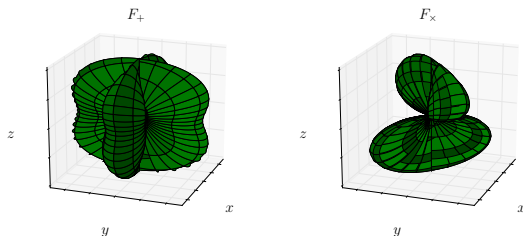
3 §7.1.1 The beam pattern functions

Detector frame (x, y, z) versus source frame (x', y', z') . Rotate by ψ about z' (aligning x', y' with the axes of the polarisation ellipse), then θ about x' , ϕ about z' , with (θ, ϕ) the propagation direction in the detector frame.

Perpendicular arms:

$$F_+^{(90^\circ)} = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \cos 2\psi - \cos \theta \sin 2\phi \sin 2\psi,$$

$$F_\times^{(90^\circ)} = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \sin 2\psi + \cos \theta \sin 2\phi \cos 2\psi.$$



F_+ (left), $F_\times$ (right), 90° arms [script Fig. 7.1]: broad response, four blind directions.

3 §7.1.1 Arms at 60° , and the triangle

Arms at 60° rather than 90° : an overall factor.

$$F_+^{(60^\circ)} = \frac{\sqrt{3}}{2} \left[\frac{1}{2}(1 + \cos^2 \theta) \cos 2\phi \cos 2\psi - \cos \theta \sin 2\phi \sin 2\psi \right],$$

$$F_\times^{(60^\circ)} = \frac{\sqrt{3}}{2} \left[\frac{1}{2}(1 + \cos^2 \theta) \cos 2\phi \sin 2\psi + \cos \theta \sin 2\phi \cos 2\psi \right].$$

The cost

Less sensitive, by $\sqrt{3}/2$.

The benefit

Three of them fit into one equilateral triangle at a single site. Underground, that saves enormously on tunnelling.

And one more thing: for a co-located triangle the three detector tensors sum to zero, so the summed response vanishes identically — a **null stream**, sky-position independent, a powerful veto against instrumental artefacts. The Einstein Telescope design [script Ex. 7.1].

3 §7.1.2 The noise

A real detector sees wave *and* noise:

$$s(t) = n(t) + h(t).$$

So understanding the noise matters as much as understanding the signal.

Take $n(t)$ a **Gaussian stochastic process** — uniquely characterised by mean and autocorrelation. Without loss of generality,

$$\langle n(t) \rangle = 0,$$

brackets an ensemble average. In practice there is only *one* realisation of the detector: assume **ergodicity**, replace ensemble average by time average.

Autocorrelation

$$R(\tau) \equiv \langle n(t + \tau)n(t) \rangle,$$

typically decaying to zero as $|\tau| \rightarrow \infty$ — the noise at t says little about the noise at $t + \tau$.

Two idealisations here: *stationary* and *Gaussian*. Real data is neither — much of the practical work in §7.2 is about coping with that.

3 §7.1.2 The power spectral density

One-sided power spectral density:

$$\frac{1}{2} S_n(f) \equiv \int_{-\infty}^{\infty} d\tau R(\tau) e^{i2\pi f\tau}.$$

$R(\tau)$ real $\Rightarrow S_n(-f) = S_n^*(f)$; time-translation invariance $\Rightarrow S_n(f)$ real. Hence $S_n(-f) = S_n(f)$.

Invert, evaluate at $\tau = 0$:

$$R(0) = \langle n^2(t) \rangle = \int_0^{\infty} df S_n(f).$$

Integrated power spectral density = variance of the noise.

Equivalently, in the frequency domain: $\langle \tilde{n}^*(f) \tilde{n}(f') \rangle = \delta(f - f') \frac{1}{2} S_n(f)$.

$\sqrt{S_n(f)}$: the *amplitude spectral density*, in $1/\sqrt{\text{Hz}}$ — what was plotted in §6.6. Noise per unit bandwidth, and exactly the weight in the optimal filter.

3 §7.2 Why a buried signal is recoverable at all

Naive expectation: $h(t)$ detectable only if $|n(t)| < |h(t)|$. Ground-based detectors sit firmly at $|n(t)| \gg |h(t)|$.

Know the shape of $h(t)$; average $s(t)h(t)$ over a time T :

$$\frac{1}{T} \int_0^T dt s(t)h(t) = \frac{1}{T} \int_0^T dt h^2(t) + \frac{1}{T} \int_0^T dt n(t)h(t).$$

First term: positive definite, grows as T :

$$\frac{1}{T} \int_0^T dt h^2(t) \sim h_0^2.$$

Second term: n, h uncorrelated — grows like $T^{1/2}$, a random walk:

$$\frac{1}{T} \int_0^T dt n(t)h(t) \sim \sqrt{\frac{\tau_0}{T}} n_0 h_0.$$

The criterion

Not necessary that $|n| < |h|$. Enough that

$$h_0 > \sqrt{\frac{\tau_0}{T}} n_0.$$

3 §7.2 How much this buys you

Integrate over a period long compared with τ_0 — the noise averages away.

Compact binary

$f \sim 100 \text{ Hz}$, $\tau_0 \sim 10^{-2} \text{ s}$, $T \sim 100 \text{ s}$:

$$\sqrt{\tau_0/T} \sim 10^{-2}.$$

Detectable at a **factor 100** below the noise amplitude.

Millisecond pulsar

$\tau_0 \sim 10^{-3} \text{ s}$, $T \sim 10^7 \text{ s}$ (one year):

$$\sqrt{\tau_0/T} \sim 10^{-5}.$$

Sensitive a **factor 10^5** below the noise amplitude.

Price: you must know the shape of $h(t)$ — for a continuous wave, coherently, for a year.

That price is what §7.3 is about: what to do when you do not know the shape.

3 §7.2 The optimal filter

Optimise the argument. Filtered statistic $\hat{s} = \int_{-\infty}^{\infty} dt s(t) K(t)$; signal-to-noise ratio $\rho = S/N$, with S the expectation of $\hat{s}$ with a signal, N its root-mean-square without one. Since $\langle n(t) \rangle = 0$:

$$S = \int dt \langle s(t) \rangle K(t) = \int dt h(t) K(t) = \int df \tilde{h}(f) \tilde{K}^*(f),$$

and for $h = 0$, using the definition of S_n :

$$N^2 = \langle \hat{s}^2 \rangle = \iint dt dt' K(t) K(t') \langle n(t) n(t') \rangle = \frac{1}{2} \int df S_n(f) |\tilde{K}(f)|^2.$$

$$\rho = \frac{\int_{-\infty}^{\infty} df \tilde{h}(f) \tilde{K}^*(f)}{\sqrt{\int_{-\infty}^{\infty} df \frac{1}{2} S_n(f) |\tilde{K}(f)|^2}}$$

3 §7.2 The inner product, and the Wiener filter

Noise-weighted inner product:

$$(a | b) = \Re \int_{-\infty}^{\infty} df \frac{\tilde{a}^*(f) \tilde{b}(f)}{\frac{1}{2} S_n(f)}$$

With $\tilde{u}(f) = \frac{1}{2} S_n(f) \tilde{K}(f)$: $\rho = (u | h) / \sqrt{(u | u)}$.

A geometry problem

A normalised projection — maximised when u points *in the same direction* as h . Hence the optimal filter, the [Wiener filter](#):

$$\tilde{K}(f) \propto \frac{\tilde{h}(f)}{S_n(f)}, \quad \text{giving} \quad \rho = \sqrt{(h | h)}.$$

Read the filter: weight each frequency by the expected signal there, and *divide by the noise* there. It depends on both the signal sought and the noise — so both must be understood.

3 §7.2 Networks, and the detection statistic

Noise uncorrelated between detectors $\Rightarrow$ network signal-to-noise adds in quadrature:

$$\rho_{\text{net}}^2 = \sum_i \rho_i^2.$$

Detection statistic: the **likelihood ratio**

$$\Lambda(H_1, s) = \frac{p(s | H_1)}{p(s | H_0)},$$

H_1 signal + noise, H_0 noise only. Gaussian noise:

$$p(s | H_0) \propto e^{-(s | s)/2}, \quad p(s | H_1) \propto e^{-(s-h | s-h)/2},$$

so

$$\Lambda(H_1, s) = e^{(s | h)} e^{(h | h)/2}.$$

Monotonic in $(s | h)$ — the matched filter. “Matched filtering” and “likelihood-ratio testing under Gaussian noise” are the same thing.

3 §7.2 What the derivation assumed, and what is true

The derivation assumed signal shape and noise properties *known*. Three problems.

1. Characterising the noise is not trivial

Neither stationary nor Gaussian, in general. And an inherent ambiguity: is a signal being absorbed into the noise characterisation?

2. We rarely have the exact signal

For most realistic sources the waveform is only an approximation — which is what §2.2, §3 and §4.1 of Lecture 1 were for.

3. The filter bank is discrete

Many parameters. Discretising the filters gives a signal-to-noise ratio *lower* than the exact Wiener filter.

Item 3 is a well-posed geometry problem — next slide.

3 §7.2.1 Covering the parameter space

Matched filtering needs the signal shape *a priori*. A compact binary coalescence takes at least fifteen parameters — so: consider all of them, and maximise the SNR over them.

Symbol	Parameter
m_1, m_2	component masses
$\mathbf{S}_1, \mathbf{S}_2$	spins
θ, ϕ	sky position
ι, ψ	orientation
D	distance
ϕ_c	coalescence phase
t_c	time of coalescence

Most extrinsic parameters need no search. Sky position, orientation and luminosity distance enter only the *amplitude* at the detector — provided they stay constant through the evolution.

Except for misaligned spins. Inclination and polarisation angles then become time dependent, modify the signal non-trivially, and must be searched over.

Aligned spins: those angles are constant, degenerate with sky position and distance. What remains is masses and spins — at most **four** parameters for non-precessing systems.

[script Tab. 7.1]

3 §7.3 When there is no reliable model

Compact binaries: accurate models, so matched filtering is statistically *optimal*. But some important sources have no reliable waveform model — **poorly modelled** or **unmodelled** sources.

Examples

Supernovae; magnetar bursts; other cataclysmic events with complicated astrophysics and possible asymmetries, where simulations are limited or give diverse outcomes.

Why matched filtering fails here

It needs an accurate prediction of the signal's *phase evolution*. Get the phase wrong by half a cycle at the wrong moment, and the integral cancels instead of accumulating.

Partial information may still be available — approximate duration, frequency range, time of arrival. Enough for a more general detection strategy.

3 §7.3 The excess-power statistic

The idea: no precise template. Assume instead that a signal shows up as a statistically significant *increase in power* in some time–frequency window.

Let the space of possible signals be spanned by N orthonormal basis functions $\{\tilde{e}_i(t)\}_{i=1}^N$ — wavelets, sine-Gaussians, localised in time and frequency, covering where a signal is expected. Orthonormal under the noise-weighted inner product:

$$(\tilde{e}_i, \tilde{e}_j) = \delta_{ij}.$$

Any signal in that space:

$$h(t) = \sum_{i=1}^N c_i \tilde{e}_i(t).$$

The move

Knowing the c_i would let us reconstruct the signal and matched-filter. We do not — so treat them as **unknown parameters**, and maximise over them.

3 §7.3 Maximising over the coefficients

Likelihood ratio for a signal in $s(t)$; take the logarithm, insert the signal model:

$\ln \Lambda(\{c_i\}, s) = \sum_{i=1}^N \left[c_i (s, \tilde{e}_i) - \frac{1}{2} c_i^2 \right]$. Quadratic in the c_i , maximising analytically at $c_{i,\max} = (s | \tilde{e}_i)$.

Substitute back:

Excess-power statistic

$$E = 2 \ln \Lambda(\{c_{i,\max}\} | s) = \sum_{i=1}^N (s | \tilde{e}_i)^2$$

Sum of squared projections onto each basis function — how much of the data lies in the signal subspace.

Maximum-likelihood waveform for free: $\hat{h}(t) = \sum_i (s, \tilde{e}_i) \tilde{e}_i(t)$.

What counts as a significant excess

Gaussian noise: E is χ^2 with N degrees of freedom — and for a window of duration T and bandwidth F , $N = 2TF$. With a signal in the space: non-central χ^2 , non-centrality $\rho^2 = (h, h)$.

Detection needs $\rho \gtrsim (2N)^{1/4}$, against $\rho \gtrsim$ a few for matched filtering. That $N^{1/4}$ is the price of not knowing the waveform.

3 §7.4 Inference: deductive and inductive logic

Bayesian inference: a consistent framework for **inductive logic**, or *plausible reasoning* — the branch of formal logic that evaluates statements which are not logically certain. Contrast **deductive logic**, which draws conclusions *from* certain statements.

Central object: a **proposition** — subject plus predicate, asserting its (un)truth. For instance “hypothesis H is correct”, or “quantity x has the value 3”.

Boolean operations

- ▶ conjunction AB or $A \wedge B$ — both true;
- ▶ disjunction $A + B$ or $A \vee B$ — at least one true;
- ▶ denial $\bar{A}$ or $\neg A$ — A false;
- ▶ implication $A \rightarrow B$; bi-implication $A \leftrightarrow B$, i.e. $A = B$.

The extension

Not just *true* or *false*, but a **degree of certainty** — a plausibility. Always dependent on what is taken as known: a cloudy sky makes rain more plausible. Hence the *conditional plausibility*

$$A|B \quad \text{“}A \text{ given } B\text{”}.$$

3 §7.4.2 Cox's desiderata, and the rules that follow

Desiderata (Jaynes's formulation)

- 1 Plausibilities are real numbers.
- 2 Qualitative correspondence with common sense.
- 3 **Consistency**: a conclusion reachable in more than one way must come out the same every way.

Axioms, $w(A)$ the plausibility of A :

- 1 degrees of belief can be *ordered*, hence mapped onto real numbers;
- 2 $w(A) = f[w(\bar{A})]$ for some f ;
- 3 $w(AB) = g[w(A|B), w(B)]$ for some g .

Not a philosophical preference — a theorem. *Any* scheme satisfying the desiderata is isomorphic to probability theory.

Cox's result

Under these axioms, the rules of plausible reasoning *coincide with the laws of probability*. With w from 0 (impossible) to 1 (certain):

$$p(A|I) + p(\bar{A}|I) = 1 \quad (\text{sum rule})$$

$$p(A, B|I) = p(A|B, I) p(B|I) \quad (\text{product rule})$$

I : all relevant background information, held before any plausibility is assigned.

3 §7.4.2 Bayes' theorem and marginalisation

Two results follow from the sum and product rules. These are the two we use.

Bayes' theorem

$$p(A|B, I) = \frac{p(B|A, I) p(A|I)}{p(B|I)}$$

Marginalisation

$p(A|I) = p(A, B|I) + p(A, \bar{B}|I)$. For a mutually exclusive, exhaustive set $\{B_k\}$ with $\sum_k p(B_k|I) = 1$:

$p(A|I) = \sum_k p(A, B_k|I)$. Continuum limit:

$$p(A) = \int_{x_{\min}}^{x_{\max}} \text{pdf}(A, x) dx.$$

On the continuum: for a continuous variable there are arbitrarily many propositions " $x = \alpha$ ", so only $p(x_1 \leq x \leq x_2|I) = \int_{x_1}^{x_2} \text{pdf}(x)dx$ has useful meaning. Probabilities are dimensionless; *densities* carry the inverse dimension of x . Both written p — read from context.

3 §7.4.3 Parameter estimation

A model H describing the data d given the parameter θ : a **generative** model. Plausibility of the data is then the **likelihood** $p(d|\theta, H, I)$. We can usually assign that — but we are exposed only to d , and want the **posterior** $p(\theta|d, H, I)$. Bayes connects them:

$$\underbrace{p(\theta|d, H, I)}_{\text{posterior}} = \frac{\overbrace{p(d|\theta, H, I)}^{\text{likelihood}} \overbrace{p(\theta|H, I)}^{\text{prior}}}{\underbrace{p(d|H, I)}_{\text{evidence}}} \propto p(d|\theta, H, I) p(\theta|H, I).$$

- ▶ **Prior** $p(\theta|H, I)$ — state of knowledge *before* the experiment: the expected interval, or the expected distribution.
- ▶ **Evidence** $p(d|H, I)$ — independent of θ , so it drops out of parameter estimation. Crucial once the *model* is the object of interest.

Inference = likelihood weighted by prior. In line with desideratum 3.

3 §7.4.4 Hypothesis testing

Parameter estimation needs a known generative model. Often there is none — normal, or Lorentzian? Comparing models is **hypothesis testing** (model comparison, model selection). Apply Bayes to the models themselves; take the ratio:

Odds ratio

$$O_Y^X \equiv \frac{p(X|d, I)}{p(Y|d, I)} = \underbrace{\frac{p(d|X, I)}{p(d|Y, I)}}_{\text{Bayes factor } B_Y^X} \times \underbrace{\frac{p(X|I)}{p(Y|I)}}_{\text{prior odds}}$$

— the factors $p(d|I)$ cancelling.

Marginalise Bayes over θ : $p(d|H, I)$ is θ -independent and the posterior is normalised, so the evidence is the fully marginalised, prior-weighted likelihood — the **marginal likelihood**:

$$p(d|H, I) = \int p(d|\theta, H, I) p(\theta|H, I) d\theta.$$

$O_Y^X > 1$ favours X ; $= 1$ indifference; < 1 favours Y . The *magnitude* says by how much: $O_Y^X = 5$ means X five times more likely.

3 §7.5 Bayesian inference for gravitational waves

Why: accurate component masses reduce false alarm rates when comparing near-simultaneous triggers across detectors — and once a detection is made, we want every bit of physical information about the source.

Idealised data again: stationary *and* Gaussian, so the distribution over noise realisations is

$$P(n) = \mathcal{N} \exp \left[-\frac{1}{2} \int_{-\infty}^{\infty} df \frac{|\tilde{n}(f)|^2}{\frac{1}{2} S_n(f)} \right] = \mathcal{N} \exp [- (n | n) / 2],$$

the variance of the Fourier mode at f being $\frac{1}{2} S_n(f)$ — exactly §7.1.2.

The likelihood

With $s(t) = n(t) + h(t; \theta)$, and $\theta = \{\theta_1, \dots, \theta_N\}$ a priori unknown:

$$P(s|\theta) = \mathcal{N} \exp [- (s - h(\theta) | s - h(\theta)) / 2].$$

3 §7.5 The posterior

Bayes:

$$P(\theta|s) = \frac{P(s|\theta) P(\theta)}{P(s)}.$$

s is given, so $P(s)$ goes into an overall prefactor.

On the prior

$P(\theta)$ carries prior knowledge and assumptions. Neutron star masses: sharply peaked around $1.35 M_{\odot}$. Distances, isotropic sources: $p^{(0)}(r)dr \propto r dr$ in the galactic disk, $\propto r^2 dr$ extragalactic. No prior assumptions at all: a **uniform prior**, absorbed into the normalisation.

Uniform prior, and $\exp[-(s|s)/2]$ absorbed into $\mathcal{N}$ too:

$$P(\theta|s) = \mathcal{N} \exp \left[(h(\theta) | s) - \frac{1}{2} (h(\theta) | h(\theta)) \right]$$

Precisely the likelihood ratio of §7.2 — now read as a distribution over parameters, not as a detection statistic.

3 §7.5.1 Maximum likelihood, geometrically

For error estimates: find the peak of $P(\theta|s)$. Flat prior $\Rightarrow$ its location $\hat{\theta}_{\text{ML}}$ is the **maximum likelihood estimator**, and maximising $P(\theta|s)$ is the same as maximising $P(s|\theta)$.

An elegant picture

The waveform family $h(t; \theta)$ is a **manifold**. Noise takes us *off* it, giving $s(t)$. Maximising $P(s|\theta) =$ minimising $(s - h(\theta) | s - h(\theta))$ — the squared *distance* between data and waveform, in the metric of the noise-weighted inner product.

$\hat{\theta}_{\text{ML}}$: the point on the signal manifold closest to the data.

$(\cdot | \cdot)$ is genuinely a metric — positive definite, symmetric, bilinear. Parameter estimation becomes differential geometry on the waveform manifold, which is also how template banks are laid out.

3 §7.5.1 Maximum likelihood also maximises the SNR

Take $\log P(\theta|s) = (h(\theta) | s) - \frac{1}{2} (h(\theta) | h(\theta))$; split off an amplitude, $h(t; \theta) = \mathcal{A}h_0(t; \theta')$, $\theta = \{\mathcal{A}, \theta'\}$:

$$\log P(\theta|s) = \mathcal{A} (h_0(\theta') | s) - \frac{\mathcal{A}^2}{2} (h_0(\theta') | h_0(\theta')).$$

$\partial \log P / \partial \mathcal{A} = 0$ gives $\hat{\mathcal{A}}_{\text{ML}} = (h_0(\theta') | s) / (h_0(\theta') | h_0(\theta'))$. Substitute back:

$$\max_{\mathcal{A}} \log P(\theta|s) = \frac{1}{2} \frac{(h_0(\theta') | s)^2}{(h_0(\theta') | h_0(\theta'))} \implies \max_{\theta} \log P(\theta|s) = \frac{1}{2} \max_{\theta} \frac{(h(\theta) | s)^2}{(h(\theta) | h(\theta))}.$$

The $\hat{\theta}_{\text{ML}}$ maximising the posterior also maximise the [signal-to-noise ratio](#).

So the search of §7.2 and the inference of §7.5 point at the same place. The search reports a number; the inference reports a distribution.

3 §7.5.1 The Fisher information matrix

Large SNR $\Rightarrow$ small uncertainties. Expand the exponent about $\hat{\theta}_{\text{ML}}$, with $\theta = \hat{\theta}_{\text{ML}} + \delta\theta$:

$$h(t; \theta) = h(t; \hat{\theta}_{\text{ML}}) + \partial_i h(t; \hat{\theta}_{\text{ML}}) \delta\theta^i.$$

Zeroth order goes into $\mathcal{N}$; linear terms vanish, since $\hat{\theta}_{\text{ML}}$ extremises the exponent. Leading non-trivial contribution is quadratic:

$$P(\delta\theta) = \mathcal{N} \exp \left[-\frac{1}{2} \Gamma_{ij} \delta\theta^i \delta\theta^j \right], \quad \Gamma_{ij} = (\partial_i \partial_j h | h - s) + (\partial_i h | \partial_j h).$$

First term: $h - s = -n$, and at large SNR $|h|$ dominates $|n|$. So:

Fisher information matrix

$$\Gamma_{ij} = (\partial_i h | \partial_j h)$$

The posterior is now a Gaussian. All measurement uncertainty is encoded in one matrix of inner products of waveform derivatives — computable without running a sampler.

3 §7.5.1 Covariances and correlations

One can show

$$\langle \delta\theta^i \delta\theta^j \rangle = (\Gamma^{-1})^{ij} \equiv \Sigma^{ij},$$

$\langle \dots \rangle$ the average with respect to $P(\delta\theta)$. Σ is the **covariance matrix**. Diagonal entries give the root-mean-square, i.e. 1σ , uncertainties:

$$\Delta\theta^i \equiv \sqrt{\langle (\delta\theta^i)^2 \rangle} = \sqrt{\Sigma^{ii}} \quad (\text{no summation}).$$

Off-diagonal entries give covariances, usually quoted as **correlation coefficients**

$$c^{ij} \equiv \frac{\Sigma^{ij}}{\sqrt{\Sigma^{ii}\Sigma^{jj}}} \in [-1, 1].$$

Reading c^{ij}

Near zero: parameters essentially independent, as far as the measurement goes.

Near ± 1 : strongly (anti-)correlated, hard to disentangle — though some *combination* may still be measured accurately.

The chirp mass is the textbook example of such a combination; m_1 and m_2 separately are strongly correlated.

3 §7.5.2 Monte Carlo methods

The Fisher matrix gives one piece of information — variances. For the complete picture we want the full density $P(x)$. Two obstacles:

- ▶ Often only an *unnormalised* $\tilde{P}(x)$ is available,

$$P(x) = \frac{\tilde{P}(x)}{Z},$$

with Z unknown.

- ▶ Even with Z known: evaluating $P(x)$ across a high-dimensional space is prohibitive. Gravitational-wave parameter spaces have fifteen dimensions or more.

The Monte Carlo idea

Use random numbers to generate **samples** $\{x^{(r)}\}$ from $\tilde{P}(x)$. Means, credible intervals, marginal distributions — all estimated from the samples. Marginalisation becomes *throwing away columns*.

Z is exactly the evidence of §7.4.4. Methods that compute it while sampling — nested sampling — give

3 §7.5.3 Markov chain Monte Carlo

MCMC samples $P(x)$ with an object that transitions from one point of parameter space to another. Characterised by an initial distribution $p^{(0)}(x)$ and a transition probability $T(x', x)$:

$$p^{(t+1)}(x') = \int dx T(x', x) p^{(t)}(x).$$

Two required properties.

1. $P(x)$ invariant

$$\pi(x') = \int dx T(x', x) \pi(x)$$

or, sufficiently, **detailed balance**:

$$T(x_a, x_b)P(x_b) = T(x_b, x_a)P(x_a).$$

2. Chain ergodic

Tends to an invariant distribution regardless of where it started:

$$p^{(t)}(x) \rightarrow \Lambda(x) \text{ as } t \rightarrow \infty, \text{ any } p^{(0)}(x).$$

3 §7.5.4 Metropolis–Hastings sampling

Sample a hard target using a distribution we *can* sample. Draw a candidate x' from the proposal $Q(x; x^{(t)})$; compute the **acceptance**

$$a = \frac{P(x')}{P(x^{(t)})} \frac{Q(x^{(t)}; x')}{Q(x'; x^{(t)})} \xrightarrow{\text{symmetric } Q} a = \frac{P(x')}{P(x^{(t)})}.$$

Then

$$x^{(t+1)} = \begin{cases} x' & \text{if } a \geq 1, \\ x' \text{ with probability } a, & x^{(t)} \text{ with probability } 1 - a & \text{otherwise.} \end{cases}$$

Disadvantage

Correlated samples. Keep every n th iteration, n proposal steps exceeding the autocorrelation length.

Advantage

The proposal need not match the target. Generically applicable — posterior, prior, anything.

a depends on P only through a *ratio* — so the unknown Z cancels. The whole trick.

3 Lecture 3 — summary

- 1 **§5.1** Retarded solution, projected into TT gauge with $\Lambda_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$, expanded in multipoles. Conservation of $T^{\mu\nu}$ turns stress moments into mass moments, $S^{ij} = \frac{1}{2}\dot{M}^{ij}$, giving $h_{ij}^{\text{TT}} = \frac{2G}{rc^4}\ddot{Q}_{ij}^{\text{TT}}$ — and no monopole or dipole.
- 2 **§5.2–5.3** Circular binary: radiates at 2ω , amplitude set by the chirp mass $\mathcal{M}_c$, energy loss makes it chirp. Rotating body: radiates only if $I_1 \neq I_2$, amplitude $\propto \epsilon I_3 \omega^2$, $\epsilon \lesssim 10^{-7}$ — a millimetre mountain.
- 3 **§7.1** $h(t) = D^{ij}h_{ij} = F_+h_+ + F_\times h_\times$. Noise a stationary Gaussian process, characterised by $S_n(f)$, whose integral is the variance.
- 4 **§7.2** A signal 10^2 – 10^5 below the noise is recoverable if its shape is known. Wiener filter $\tilde{K} \propto \tilde{h}/S_n$ gives $\rho = \sqrt{(h|h)}$; matched filtering *is* the Gaussian likelihood-ratio test.
- 5 **§7.3** No reliable model: project onto a subspace of candidate signals. Excess power $E = \sum_i (s|\tilde{e}_i)^2$ is χ^2 with $N = 2TF$ degrees of freedom, so detection needs $\rho \gtrsim (2N)^{1/4}$.
- 6 **§7.4** Cox: consistent plausible reasoning *is* probability theory. Bayes gives parameter estimation; the odds ratio — Bayes factor times prior odds — gives model comparison.
- 7 **§7.5** Gaussian noise gives $P(\theta|s) \propto \exp[(h|s) - \frac{1}{2}(h|h)]$. Maximum likelihood: closest point on the waveform manifold, and it maximises the SNR. Fisher matrix $\Gamma_{\alpha\beta} = (\partial_\alpha h|\partial_\beta h)$ gives errors and correlations:

3 The three lectures in one slide

Script	What it establishes	What it is for
§1.1–1.2	$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}/c^4$; two polarisations	everything else
§1.4	$T_{\mu\nu}^{\text{GW}}$ and the radiated flux	radiation reaction; the chirp
§2.2	$\epsilon \sim v/c$, orders counted in ϵ^2	the inspiral waveform
§3.1–3.2	3 + 1: constraints and evolution	the merger waveform
§4.1	a barrier problem, complex eigenfrequencies	the ringdown waveform
§1.3	geodesic deviation; a quadrupolar strain	what there is to detect
§6.1	proper distance; the wave as a tidal force	why an interferometer works
§6.2	Michelson + Fabry–Pérot, $G_{\text{FP}} > 100$	LIGO, Virgo, KAGRA
§6.6	five noise sources, and their scalings	the sensitivity curve
§6.3–6.4	the wave as a Doppler shift of light	LISA; pulsar timing arrays
§5.1–5.3	the quadrupole formula, and two sources	what we are looking for
§7.1–7.2	$S_n(f)$, the inner product, the Wiener filter	detection
§7.3	excess power, $\rho \gtrsim (2N)^{1/4}$	unmodelled searches
§7.4–7.5	Bayes; Fisher; MCMC	measuring the source

3 Where to read more

The script these lectures follow

Gravitational Waves, Tjonnje G. F. Li & Francesco Cireddu, KU Leuven. Chapters 1–7, with exercises. Everything quoted here is derived there in full.

The textbooks the script draws on

- ▶ §1: Maggiore, *Gravitational Waves* Vol. 1, Ch. 1; Schutz, *A First Course in General Relativity*, Ch. 9; Carroll, *Spacetime and Geometry*, Ch. 7; Creighton & Anderson, Ch. 3.
- ▶ §4: Maggiore Vol. 2, Ch. 12; Berti, Cardoso & Starinets, *Class. Quantum Grav.* **26** (2009) 163001.
- ▶ §6–7: Creighton & Anderson, *Gravitational-Wave Physics and Astronomy*; Jaynes, *Probability Theory: The Logic of Science* for §7.4.

And the data

The Gravitational Wave Open Science Center, gwosc.org, publishes the strain data for every event. The script's exercises are built on it: estimate a PSD, compute a matched-filter SNR time series, and run a parameter-estimation

Thank you

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