Tripartite Entanglement in $e^+e^- \to t\bar{t}Z$

Alberto Navarro

Seoultech

Based on arxiv.2606.11296 with D. Gonçalves and K. Sakurai

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Introduction

- Recent interest on interpreting polarization and spin correlations observables as a density matrix to study quantum information phenomena at colliders.
- Extensive work on bipartite systems: top-quark pairs, di-boson pairs, tau-pairs, etc.

See D. Gonçalves and DW. Kang's talks.

- Colliders offer an environment to study multipartite systems. Multipartite systems exhibit other types of separability, not present in bipartite systems.
- Few studies for multi-partite systems under certain assumptions

Sakurai, Spanowsky (2024)
Horodecki et al (2024)
Aguilar-Saavedra (2024)
Morales (2024)

- Need a general framework to study multipartite entanglement. In particular, one that can be apply to mixed quantum states.
- This talk will introduce the theoretical framework for studying tripartite entanglement in colliders. This will be applied to $e^+e^- \to t\bar{t}Z$.

Density Matrix

- In QM the state of a system is fully described by a state vector
- This is only true for the so-called "pure states", though. In general, one can have statistical mixtures of pure states, i.e., a "mixed state".
- The density matrix generalizes the state vector to pure and mixed states

$$\rho = \sum_i p_i \, |\Psi_i\rangle \, \langle\Psi_i| \; , \;\; \sum_i p_i = 1 \, , \;\; p_i \geq 0 \, . \longrightarrow$$

- Hermitian
- Unit trace
- Positive semidefinite

- In the density matrix formalism

Pure states

$$|\Psi\rangle \text{ or } \rho = |\Psi\rangle\langle\Psi|$$

$$\langle\Psi|\mathcal{O}|\Psi\rangle \text{ or } \mathrm{Tr}[\rho\mathcal{O}]$$

$$\rho^2 = \rho \quad \mathrm{Tr}[\rho^2] = 1$$

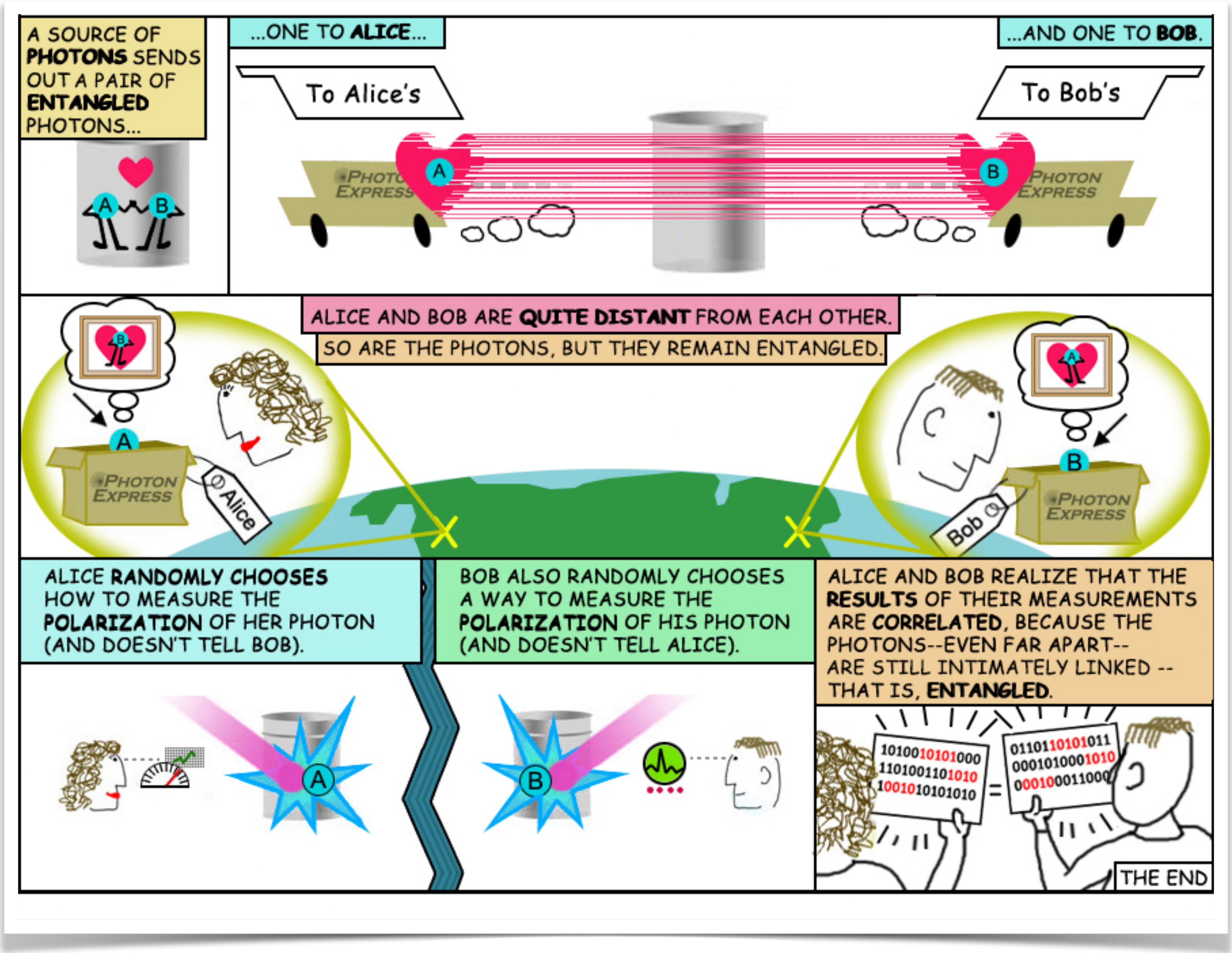
Mixed states

$$\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$$

$$\mathrm{Tr}[\rho\mathcal{O}]$$

$$\mathrm{Tr}[\rho^2] < 1$$

Entanglement



Bi-partite Separability (Entanglement)

- A system composed of two subsystems A and B is separable if it can be written as a convex sum

$$\rho^{AB}_{\rm sep} = \sum_i q_i\, \rho_i^A \otimes \rho_i^B, \qquad \rho_i^A \in \mathcal{S}(\mathcal{H}_A), \quad \rho_i^B \in \mathcal{S}(\mathcal{H}_B).$$

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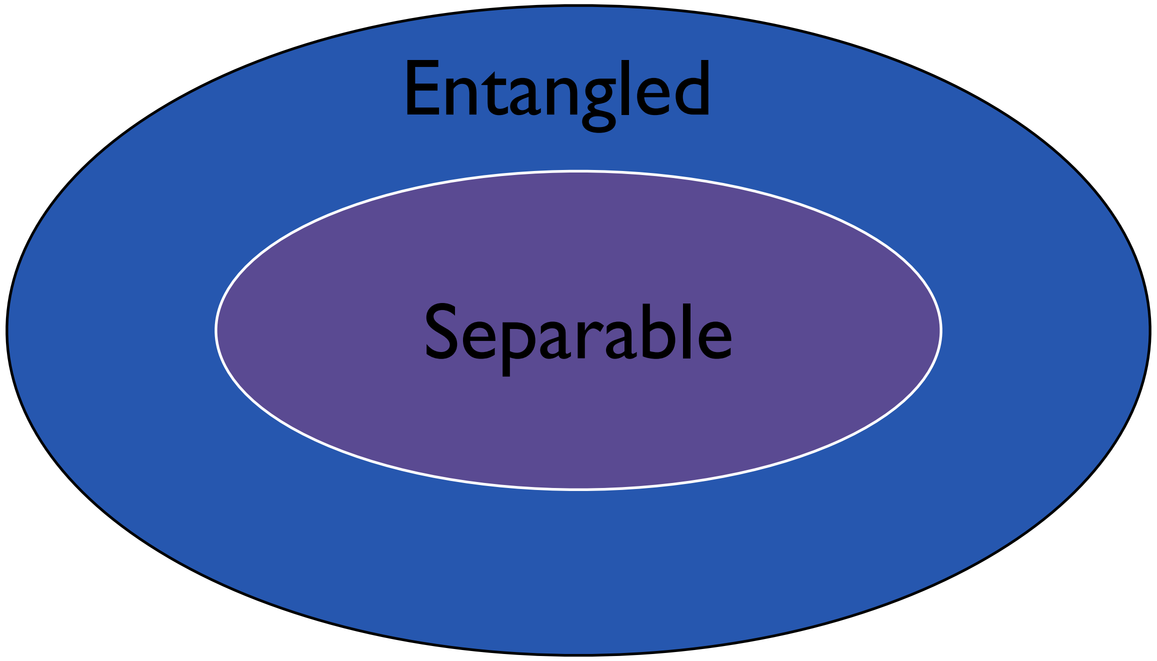

Peres-Horodecki Criterion

- If a quantum state is separable over some partition M, its partial transpose is

$$\rho^{T_M} = \sum_i q_i \, (\rho_i^M)^T \otimes \rho_i^{\overline{M}} \ , q_i \geq 0 \text{ and } \sum_i q_i = 1 \, .$$

- The presence of at least one negative eigenvalue for ρ^{T_M} implies entanglement.
- The converse also holds for systems with 2x2 and 2x3 dimensions.
- States with positive partial transpose are called PPT-states whereas states with negative partial transpose are NPT.

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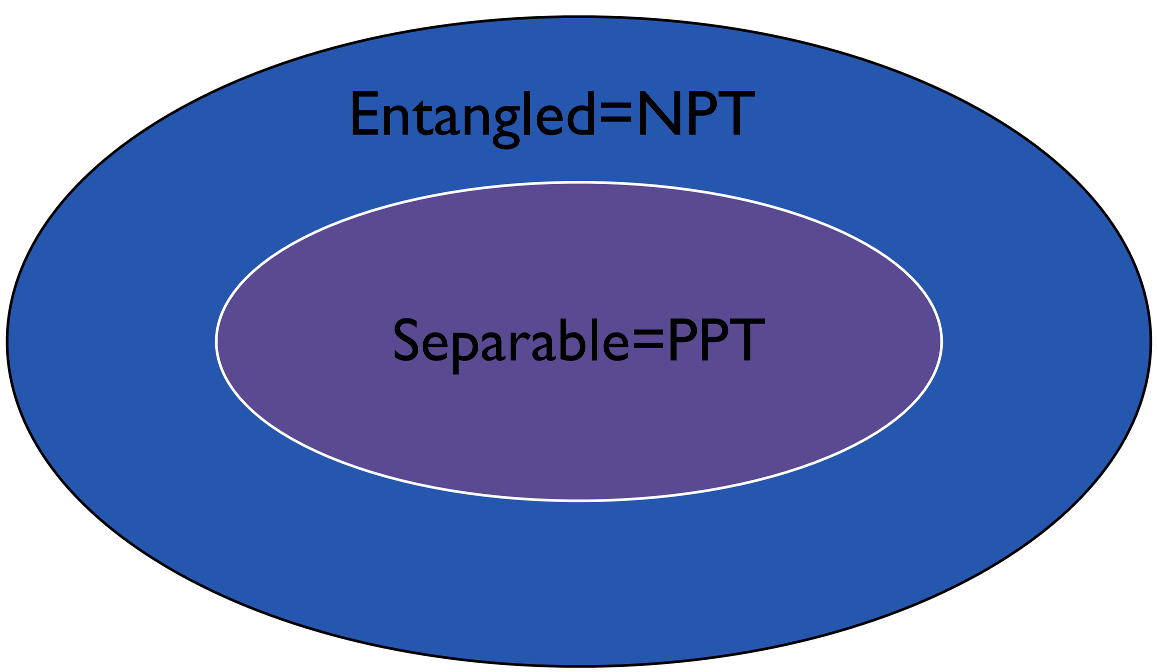

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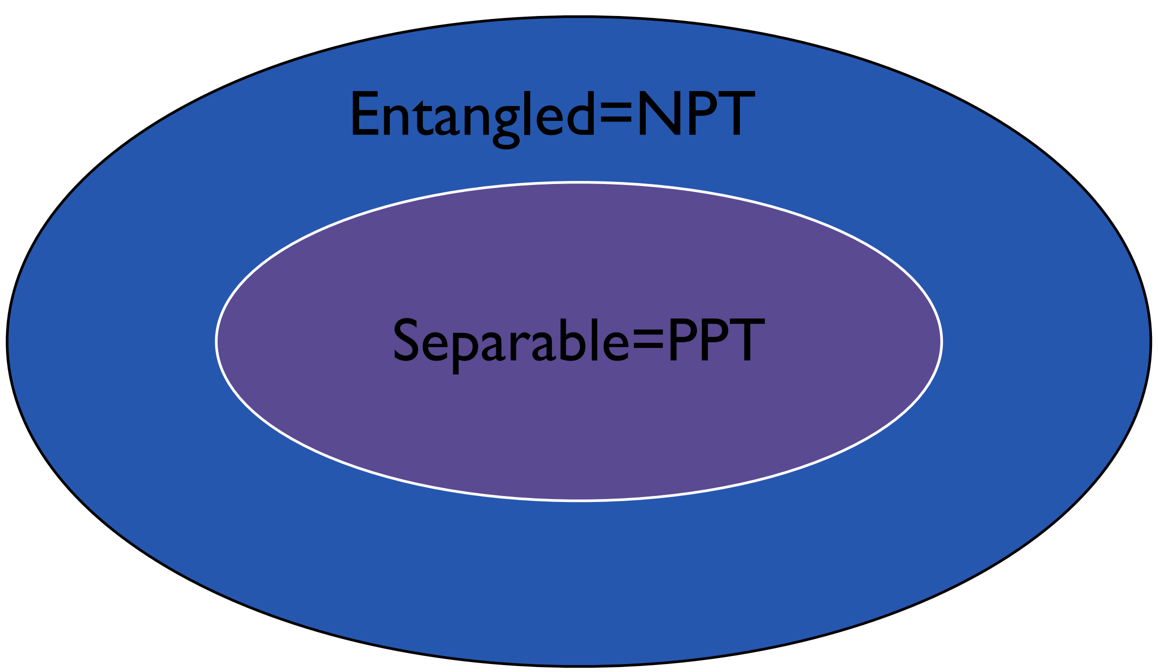
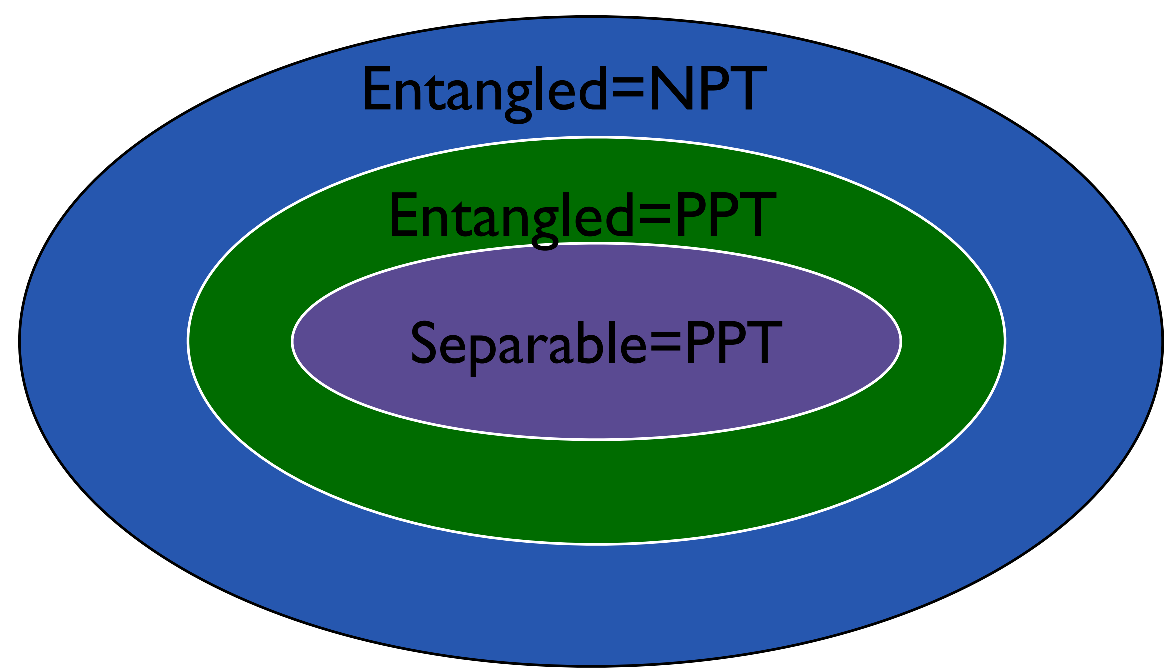


$t\bar{t} \,, tZ \,, \bar{t}Z$

$d_M \times d_{\overline{M}} > 6$



$Z|t\bar{t} \,, t|\bar{t}Z \,, \bar{t}|tZ$

Separability in Tripartite Systems

- A system composed of three parties A, B and C, is called ***fully separable*** is it admits a convex decomposition

$$\rho^{\mathrm{fs}} = \sum_i q_i \, \rho_i^A \otimes \rho_i^B \otimes \rho_i^C \,, \quad q_i \geq 0, \text{ and } \sum_i q_i = 1 \,.$$

- Fully separable states contain only classical correlations among the three subsystems.

- All entanglement measures vanish for such states.

- It called ***M-separable*** if it admits a decomposition

$$\rho^{M|\overline{M}} = \sum_i q_i \, \rho_i^M \otimes \rho_i^{\overline{M}} \,, \quad q_i \geq 0, \text{ and } \sum_i q_i = 1 \,.$$

Separability in Tripartite Systems

- A system composed of three parties A, B and C, is called ***fully separable*** is it admits a convex decomposition

$$\rho^{\text{fs}} = \sum_i a_i \, \rho_i^A \otimes \rho_i^B \otimes \rho_i^C , \quad a_i \geq 0 \quad \text{and} \quad \sum_i a_i = 1 \, .$$

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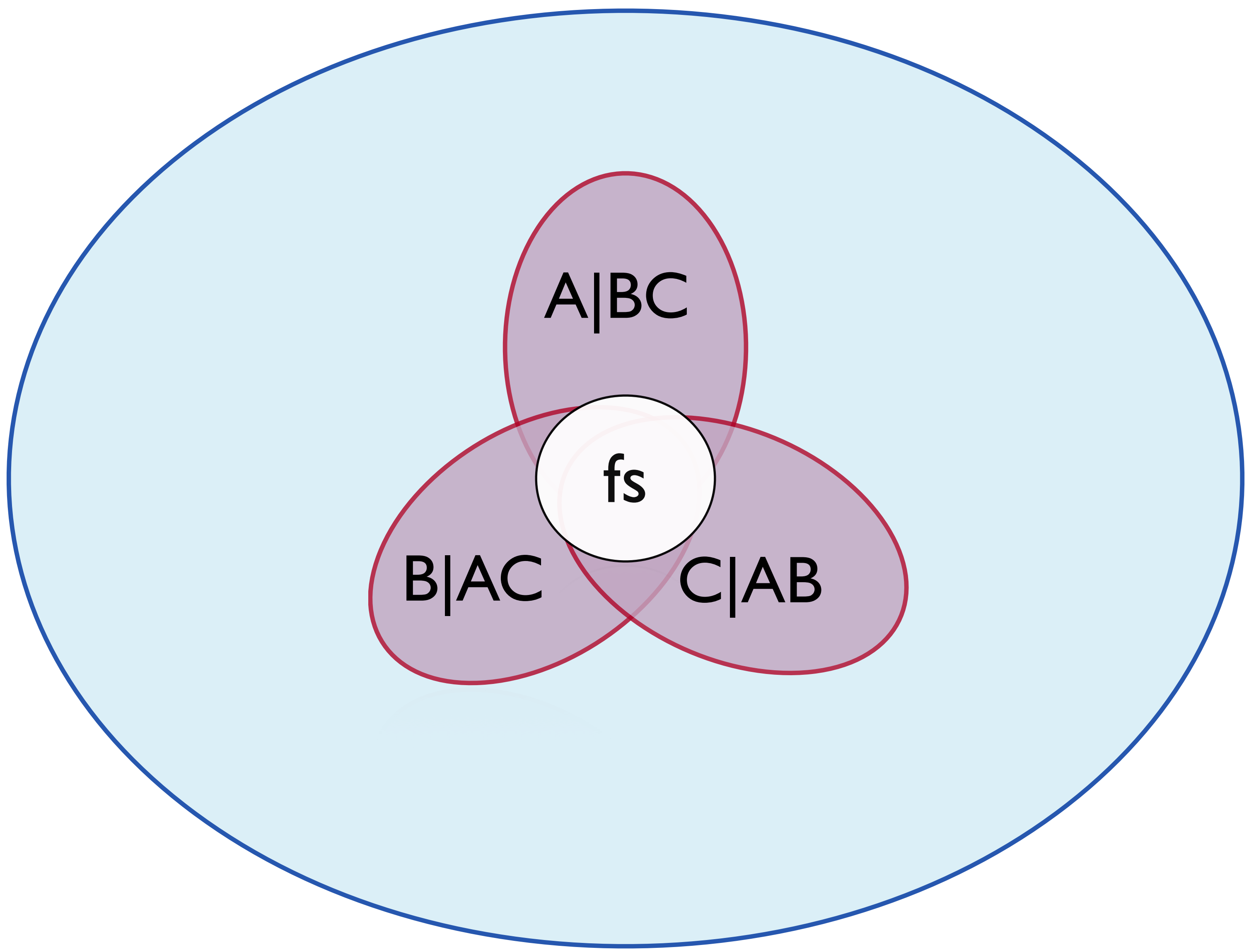
- All entanglem
- It called ***M-se***

$$\rho^M \quad \cdots \quad = 1 \, .$$



Separability in Tripartite Systems

- A tripartite state is called ***bi-separable*** if it can be expressed as a convex mixture of states that are separable across different bipartitions:

$$\rho^{\mathrm{bs}} = q_1\, \rho^{A|BC} + q_2\, \rho^{B|AC} + q_3\, \rho^{C|AB}\,,\ q_i \geq 0 \text{ and } q_1 + q_2 + q_3 = 1\,.$$

- A general mixed state may be bi-separable even if it is non-M-separable for all cuts.
- A state that is not bi-separable is called ***genuine multipartite entangled*** (GME).

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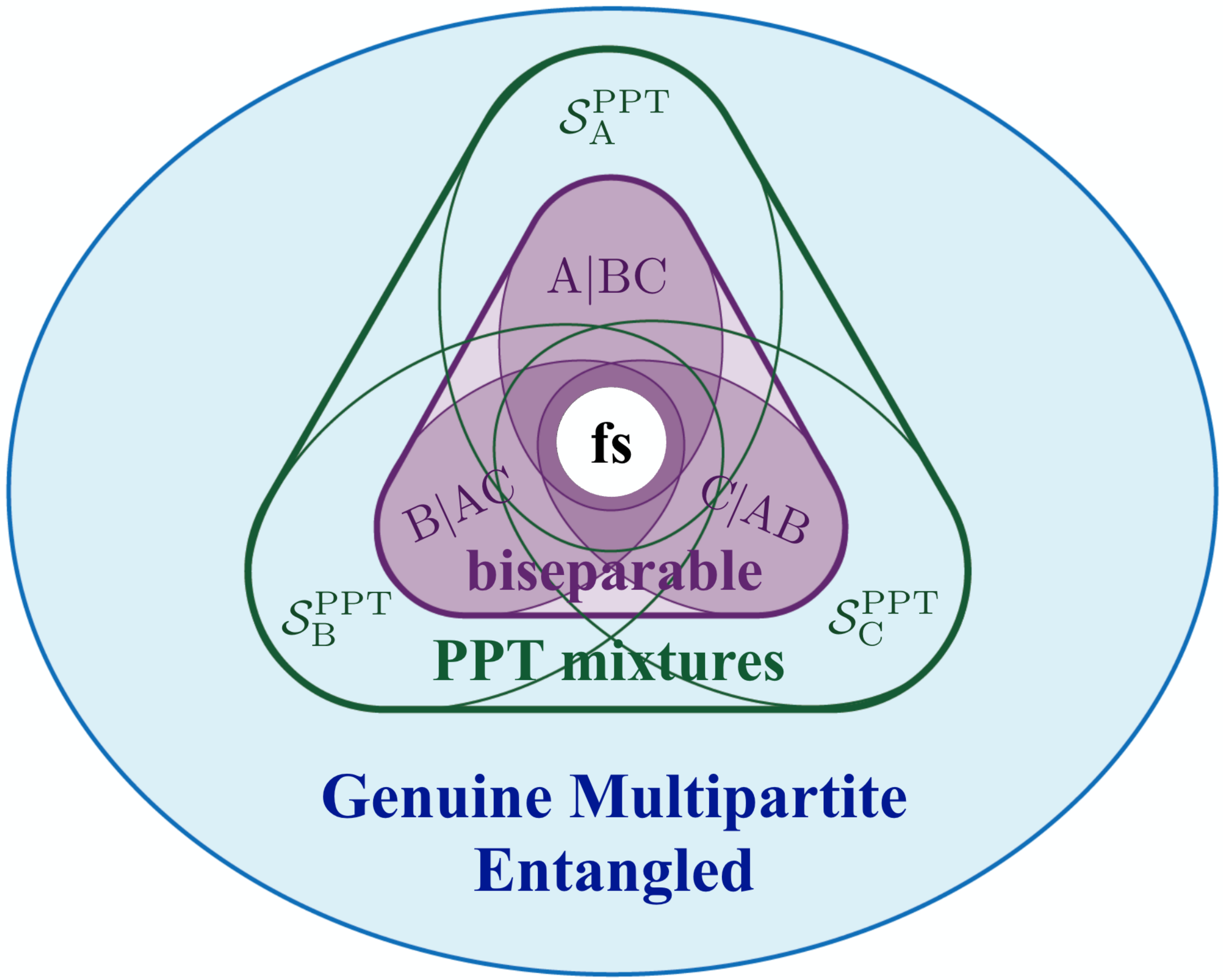

Entanglement Measures

- An entanglement measure, $E(\rho) \geq 0$. It has the property of vanishing for all separable states. Moreover, if such function is positive for all entangled states is also called ***faithful***.

- Another property is the monotonicity. An entanglement measure is called an ***entanglement monotone*** if it is non-increasing under local operations and classical communication (LOCC).

$$E(\Lambda_{\mathrm{LOCC}}(\rho)) \ \leq \ E(\rho)\,.$$

- The magnitude of the entanglement measure acquires a operational meaning: imposes a partial ordering on entangled states.

- And the convexity. An entanglement measure E is said to be **convex** if it does not increase under probabilistic mixing

$$E\left(\sum_i q_i \rho_i\right) \leq \sum_i q_i E(\rho_i)\,.$$

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Concurrence	2×2	Higher dim
Pure	Faithful Easy to compute	Faithful Easy to compute
Mixed	Faithful Easy to compute	Faithful Hard to compute

Negativity

- The Negativity for a state, ρ, is defined as $N_M(\rho) = \frac{||\rho^{T_M}||_1 - 1}{2}$. Vidal and Werner (2002)
- It can also be computed in terms of the negative eigenvalues of ρ^{T_M}

$$N_M(\rho) = \sum_i \frac{|\lambda_i| - \lambda_i}{2} = \sum_{\lambda_i<0} |\lambda_i| \,.$$

- Is positive iff the state is NPT. Vanishes for all separable states since separable states are PPT.

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- It is bounded.

$$N_M(\rho) \leq \frac{d_{\min} - 1}{2}, \qquad d_{\min} \equiv \min(d_M, d_{\overline{M}}) .$$

Negativity in the $t\bar{t}Z$ System

- Negativity can be used to test for entanglement between any two partition, A and B, after tracing out the other

$$\rho_{t\bar{t}} = \mathrm{Tr}_Z[\rho] \quad (2\times 2), \qquad \rho_{tZ} = \mathrm{Tr}_{\bar{t}}[\rho] \quad (2\times 3), \qquad \rho_{\bar{t}Z} = \mathrm{Tr}_t[\rho] \quad (2\times 3).$$

- We call this ***one-to-one entanglement***.

$$\text{(separable)} \quad 0 \leq N_{t\bar{t}},\, N_{tZ},\, N_{\bar{t}Z} \leq \frac{1}{2} \quad \text{(maximally entangled)}\,.$$

- Can also consider entanglement between one partition and the rest of the system

$$N(Z|t\bar{t}) \equiv N_Z(\rho) \;\; (3\times 4), \quad N(t|\bar{t}Z) \equiv N_t(\rho) \;\; (2\times 6), \quad N(\bar{t}|tZ) \equiv N_{\bar{t}}(\rho) \;\; (2\times 6)\,.$$

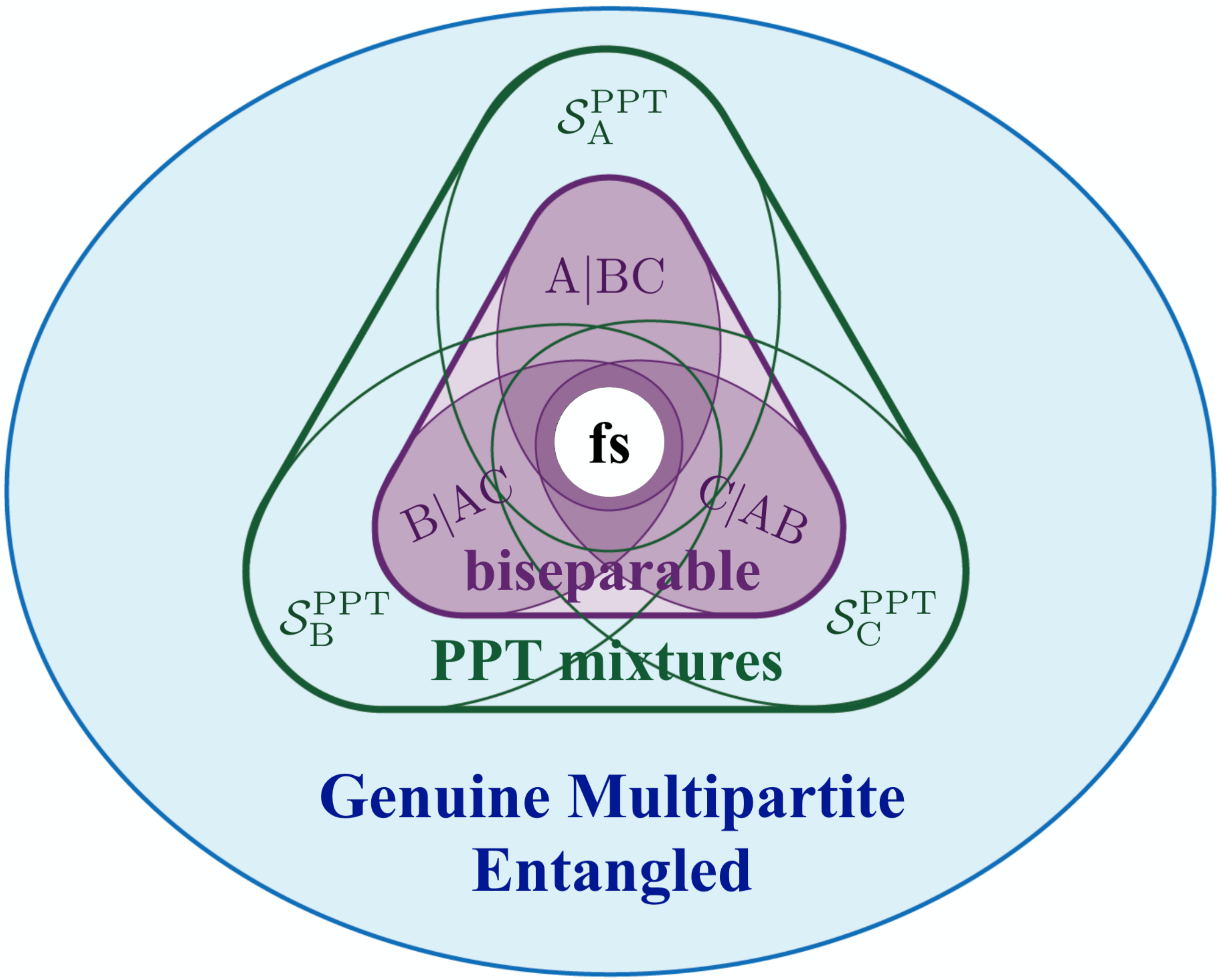
- This type of entanglement is called ***one-to-other entanglement.***

$$\text{(PPT)} \qquad \begin{array}{c} 0 \leq N(Z|t\bar{t}) \leq 1 \\ 0 \leq N(t|\bar{t}Z),\, N(\bar{t}|tZ) \leq \dfrac{1}{2} \end{array} \qquad \text{(maximally entangled)}$$

Genuine Multipartite Negativity

- First define PPT mixtures

$$\rho^{\text{PPT-mix}} = p_A\,\rho_A^{\text{PPT}} + p_B\,\rho_B^{\text{PPT}} + p_C\,\rho_C^{\text{PPT}}\,,\; p_M \geq 0 \text{ and } \sum_M p_M = 1$$



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- We want to construct a witness that detects non-PPT-mixtures. Follow the construction of Jungnitsch, Moroder and Gühne (JMG). Jungnitsch, Moroder, Gühne (2011)

- Consider a fully decomposable hermitian witness

$$\mathcal{W} = P_M + Q_M^{T_M}\,, \qquad 0 \leq P_M\,, \qquad 0 \leq Q_M \leq 1\,,$$

- To quantify how strongly a state violates the PPT mixture condition, we minimize over the set of all fully decomposable witnesses

$$\overline{\mathcal{N}}_G(\rho) \equiv \min_{\{P_M, Q_M\}} \mathrm{Tr}[\rho\,\mathcal{W}]\,.$$

- The genuine multipartite negativity (GMN) is then defined as

$$\mathcal{N}_G(\rho) \equiv -\overline{\mathcal{N}}_G(\rho) \geq 0\,.$$

- By construction

$$\rho \notin \mathcal{S}^{\text{PPT-mix}} \iff \mathcal{N}_G(\rho) > 0\,.$$

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- Consider a f

- To quantify h n, we minimize over the set

- The genuine

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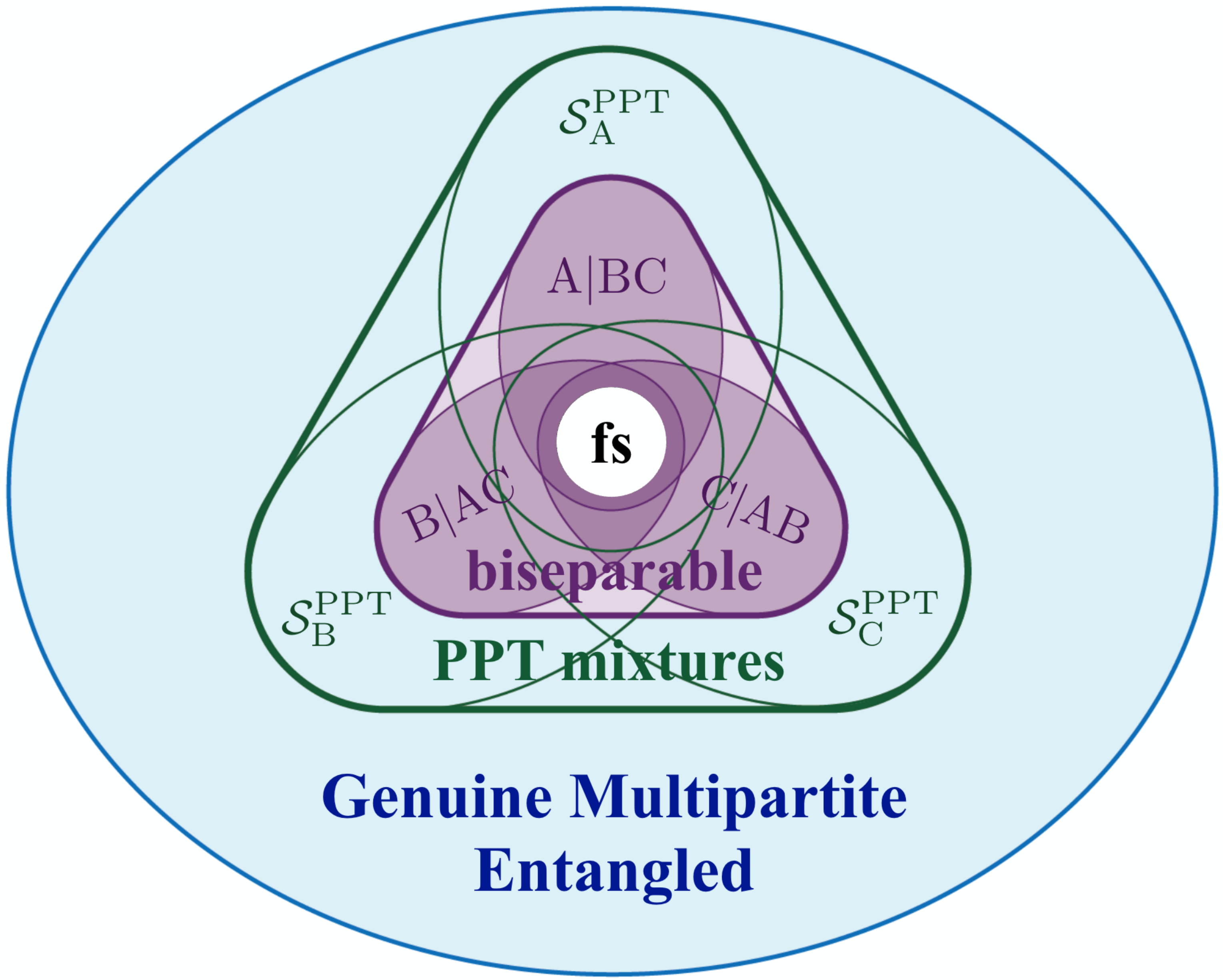

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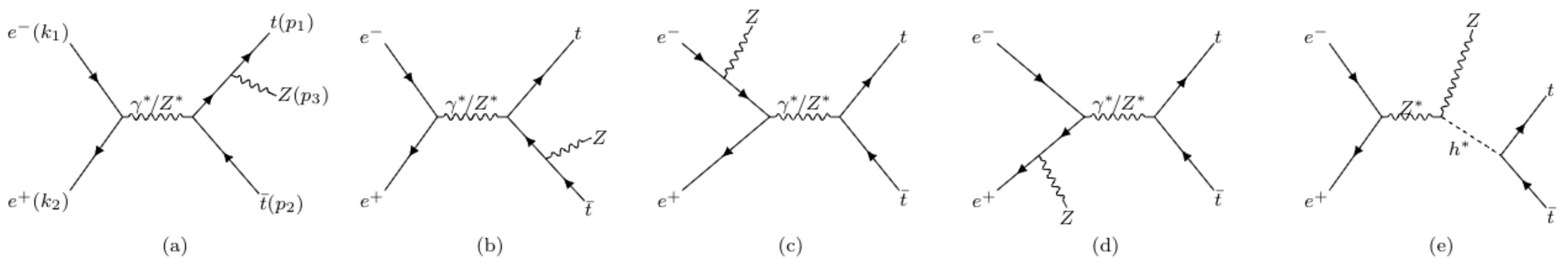
- For the $2 \times 2 \times 3$ systems

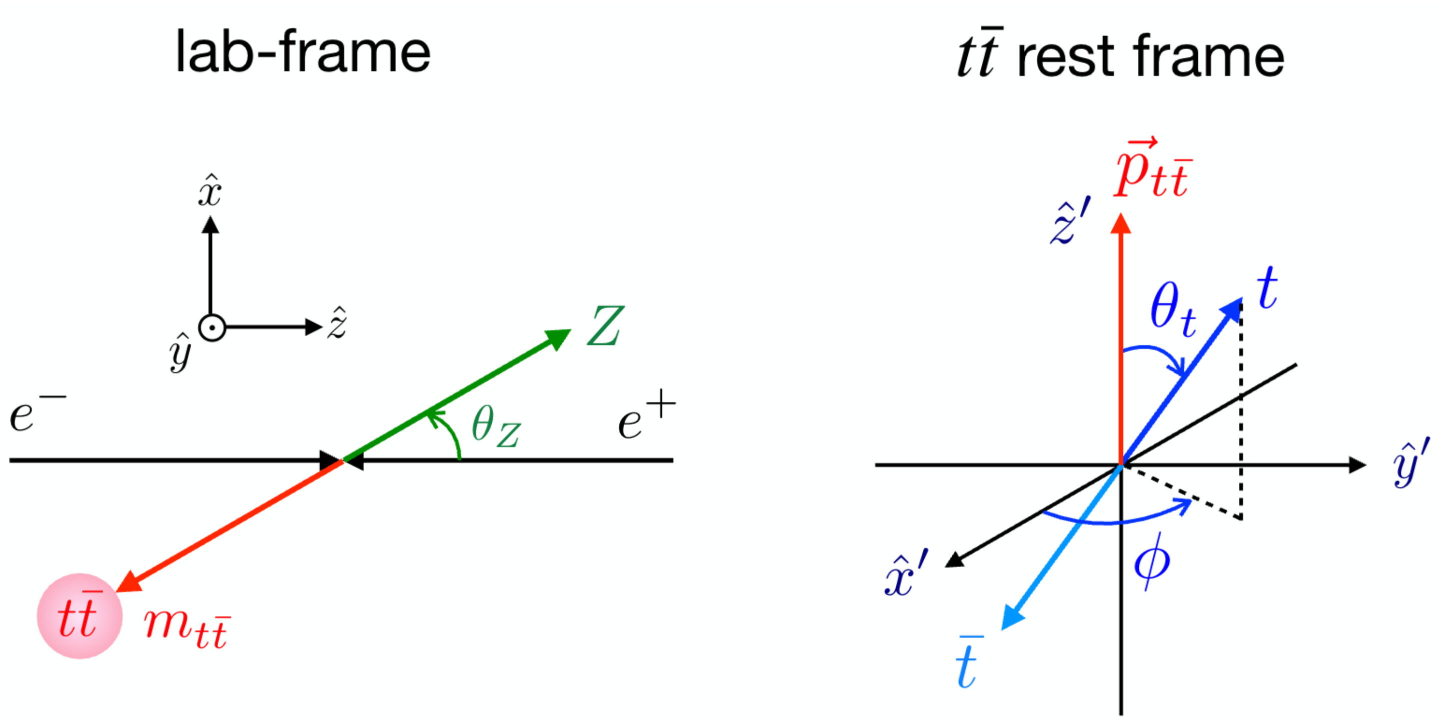
$$(\text{PPT-mixture}) \quad 0 \leq \mathcal{N}_G(\rho) \leq \frac{1}{2} \quad (\text{maximally GME})\,.$$

The $e^+e^- \to t\bar{t}Z$ System

- Consider $t\bar{t}Z$ production at electron-positron colliders.

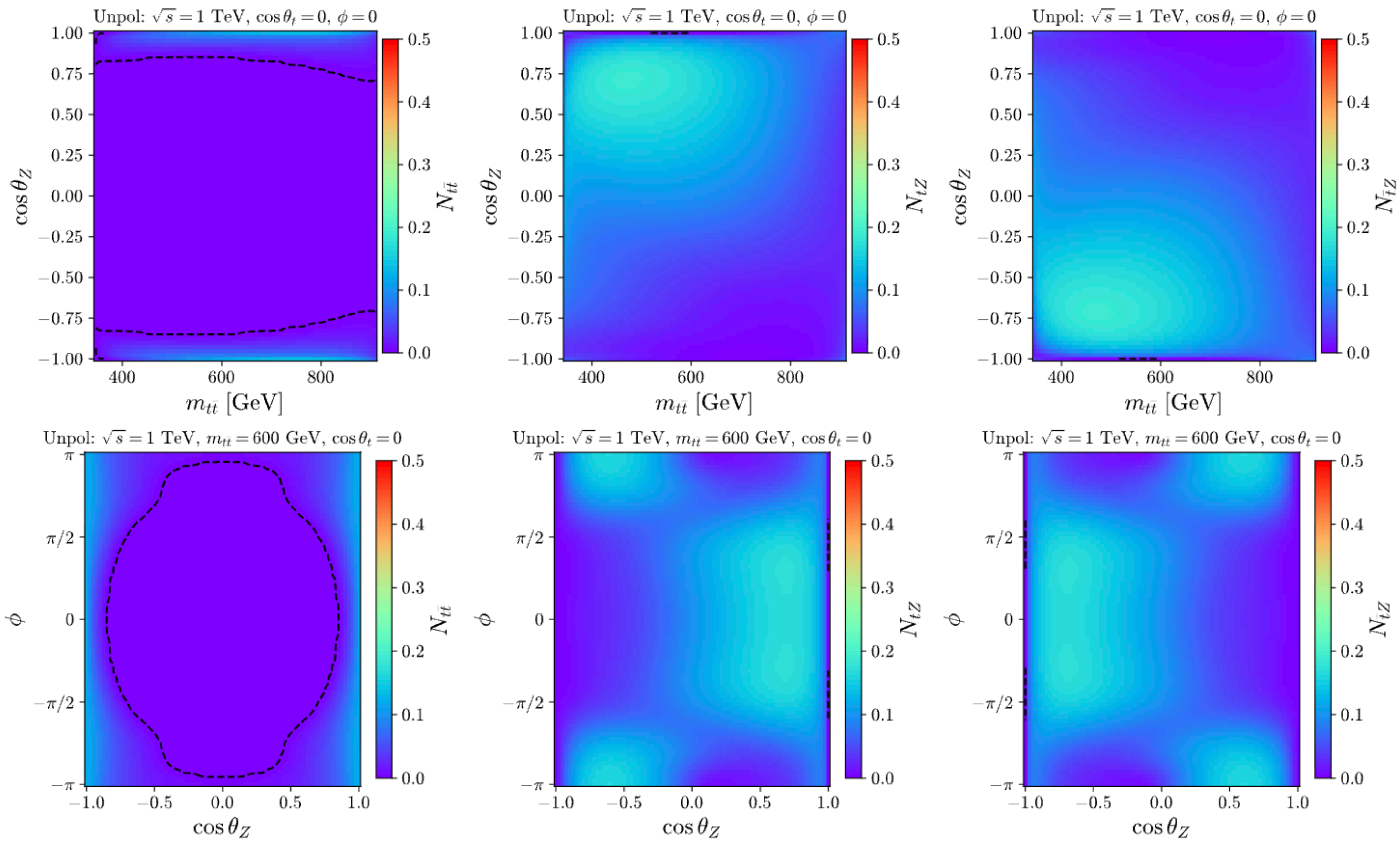


- Describe the kinematics by four observables



Entanglement Across the Phase Space

- One-to-one negativities



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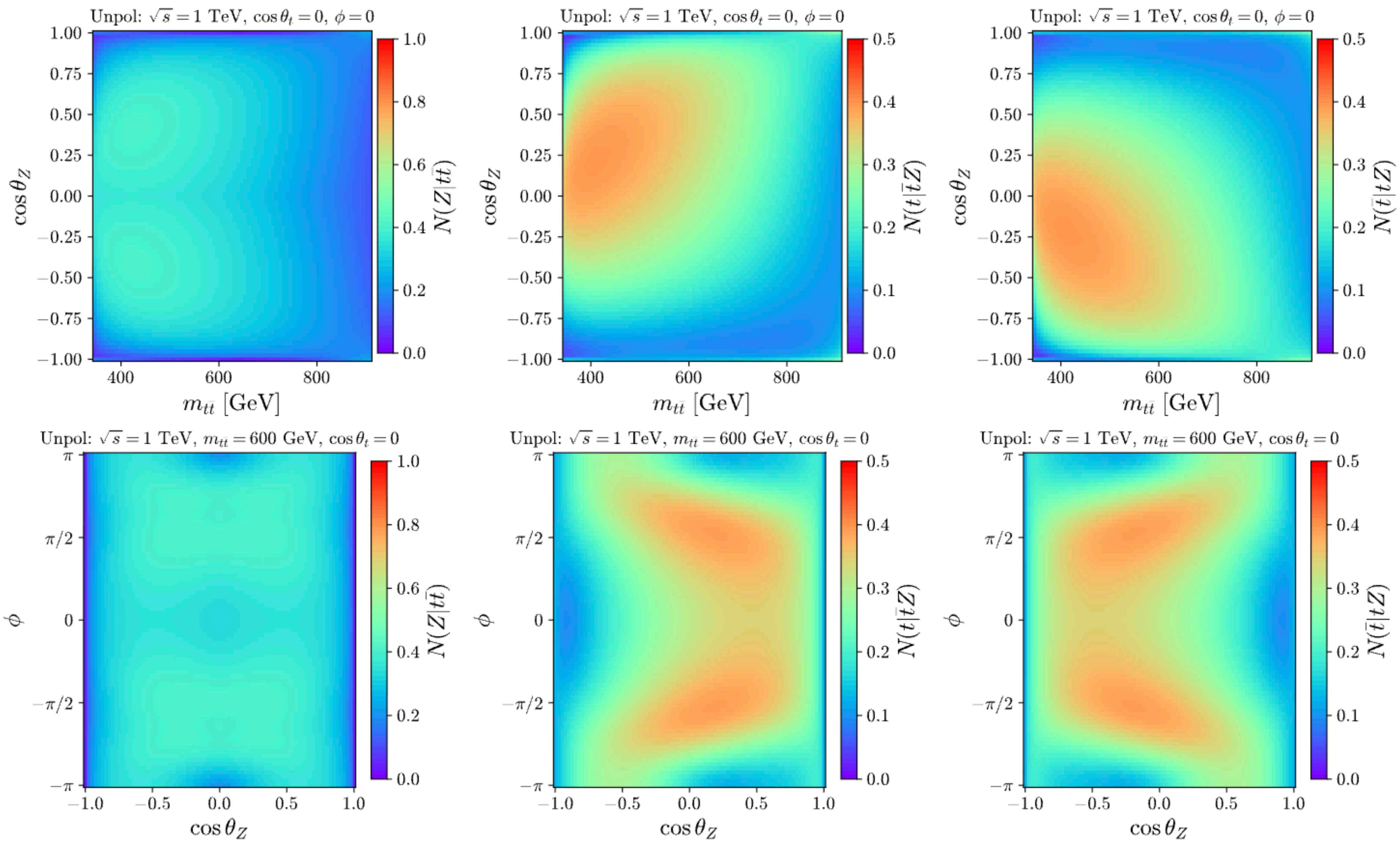

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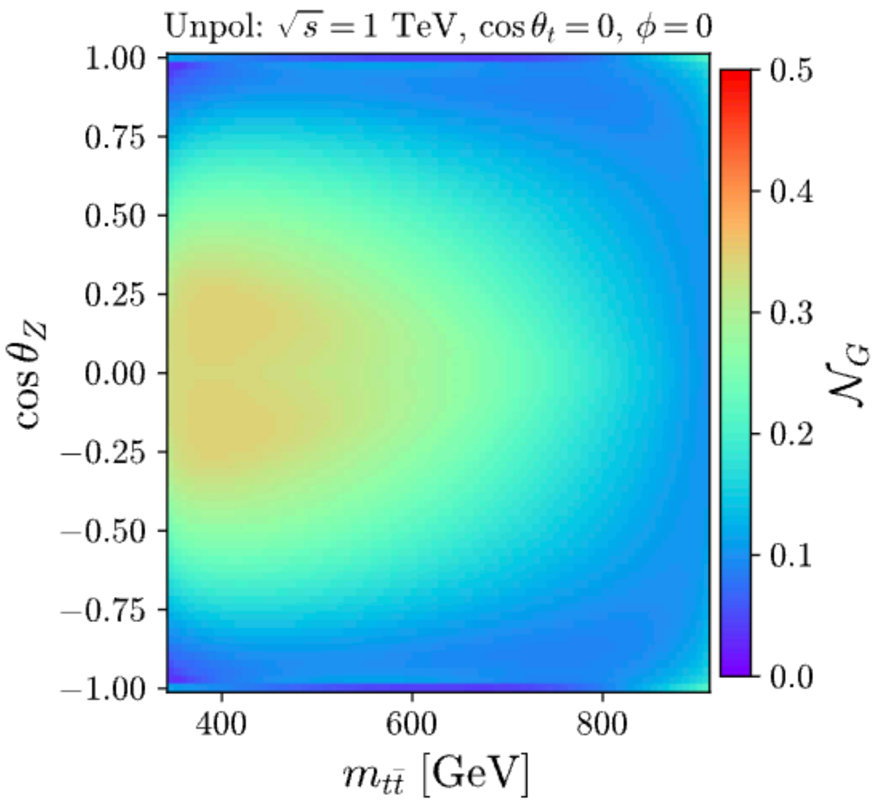
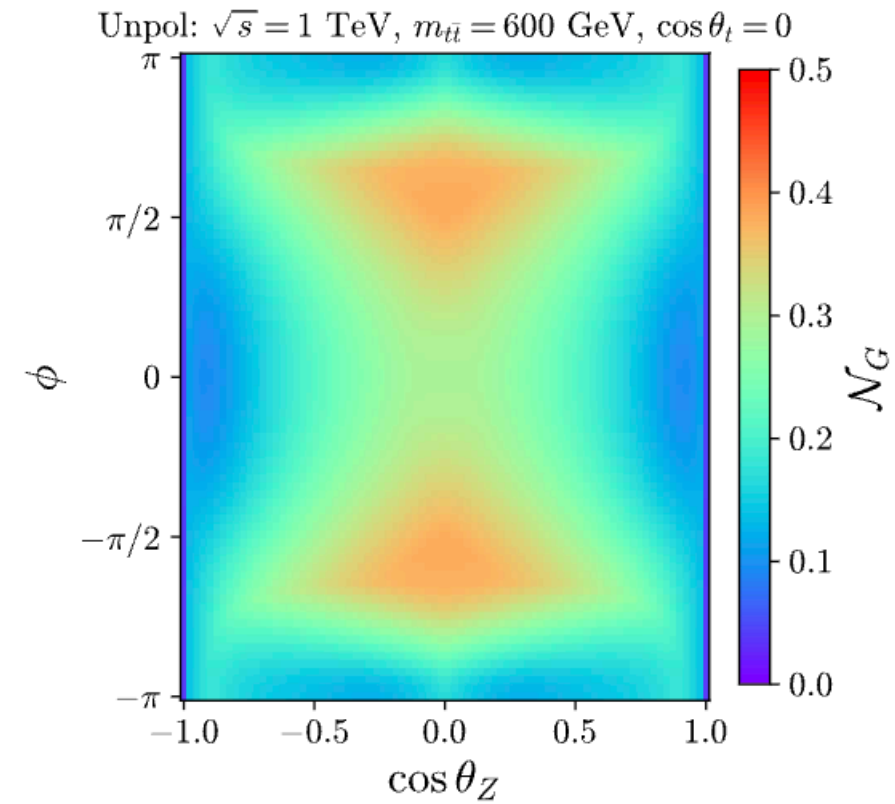
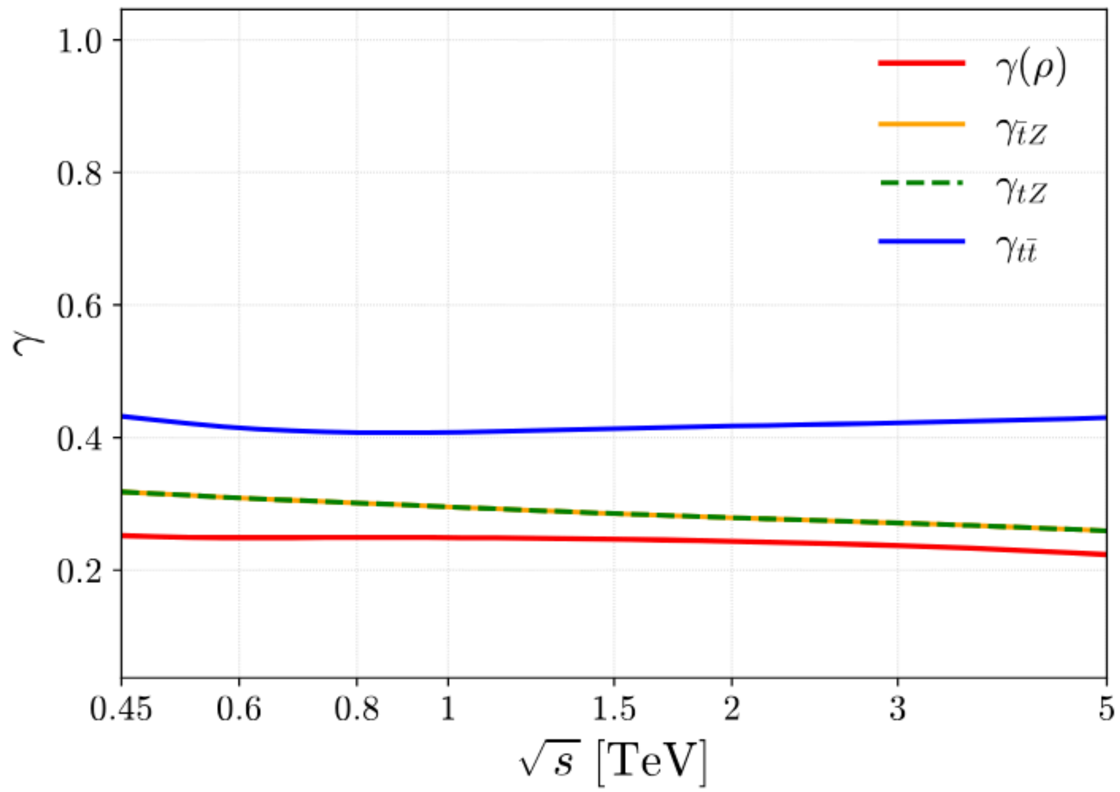
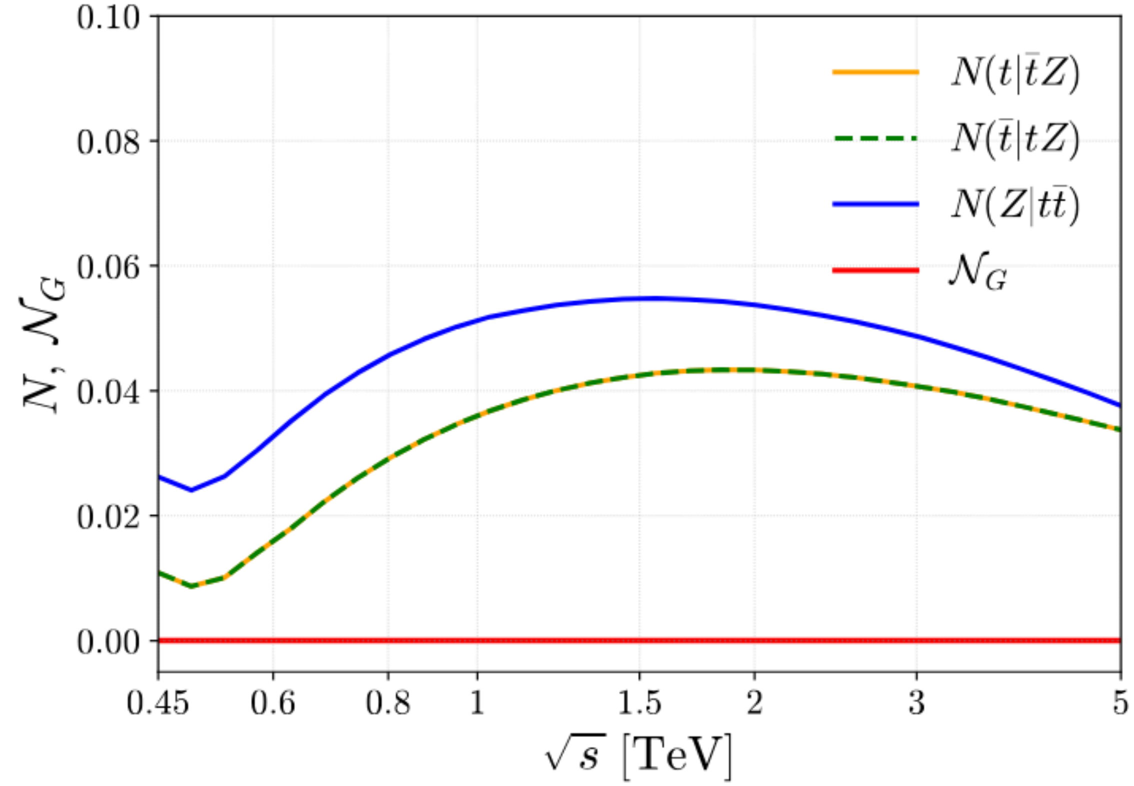




Entanglement in the Inclusive Phase Space





Predictions for Future Colliders

- Projections for probing in two semi-inclusive regions

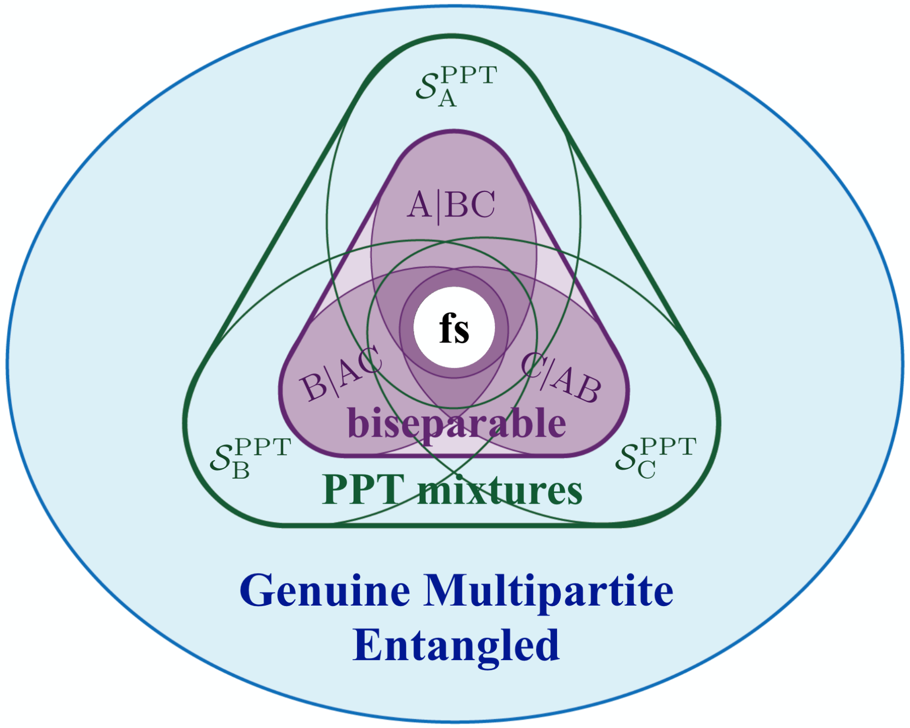
$$\Sigma_{1|2} \; : \quad |\cos\theta_Z| < 0.7 \;\; \text{and} \;\; m_{t\bar{t}} < 550 \text{ GeV} \, .$$

$$\Sigma_{\rm GME} \; : \quad |\cos\theta_Z| < 0.5 \;\; \text{and} \;\; m_{t\bar{t}} < 420 \text{ GeV} \, .$$

	$N(Z\|t\bar{t})$			$N(t\|\bar{t}Z)$			$\mathcal{N}_G$		
significance, luminosities	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$
Unpol	1.1	6.2	24.8	1.5	3.3	13.2	0.65	19.1	76.4
Pol+	1.0	7.9	31.7	1.4	4.1	16.4	0.57	24.6	98.4
Pol−	1.9	2.3	9.1	2.6	1.2	4.7	1.01	7.8	31.1

Summary

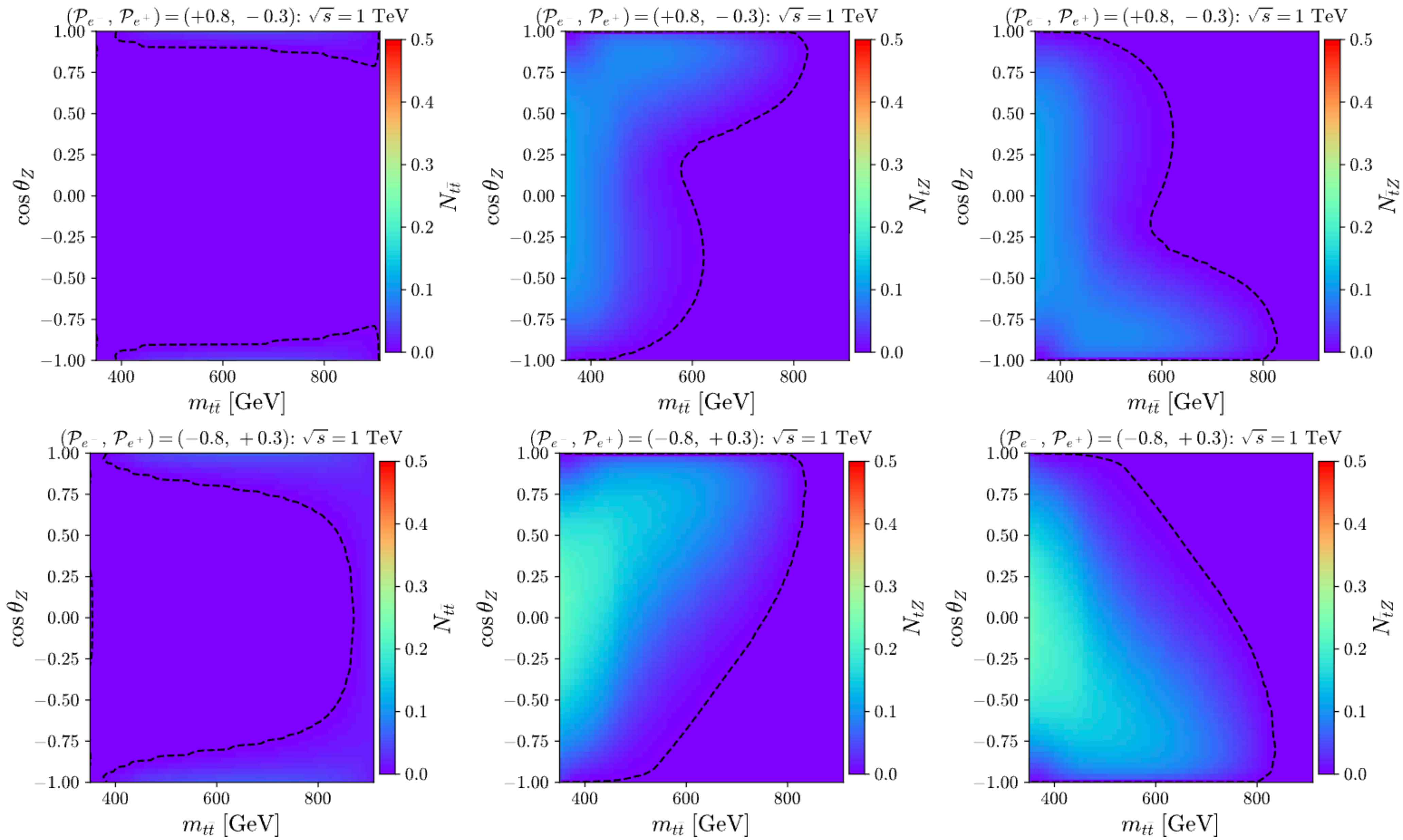
- Introduced a general framework to study multipartite entanglement in collider environments.
- Applied this formalism to $e^+e^- \to t\bar{t}Z$ and showed that bi-partite and GME is present in certain kinematic regions. Integration over the full phase space reduces the entanglement significantly. Nevertheless, one can consider partially inclusive regions.
- Possibility to extend this study to other tripartite systems or future colliders.



Thank you!

Beam Polarization Effects

- Weak dependence on beam polarizations



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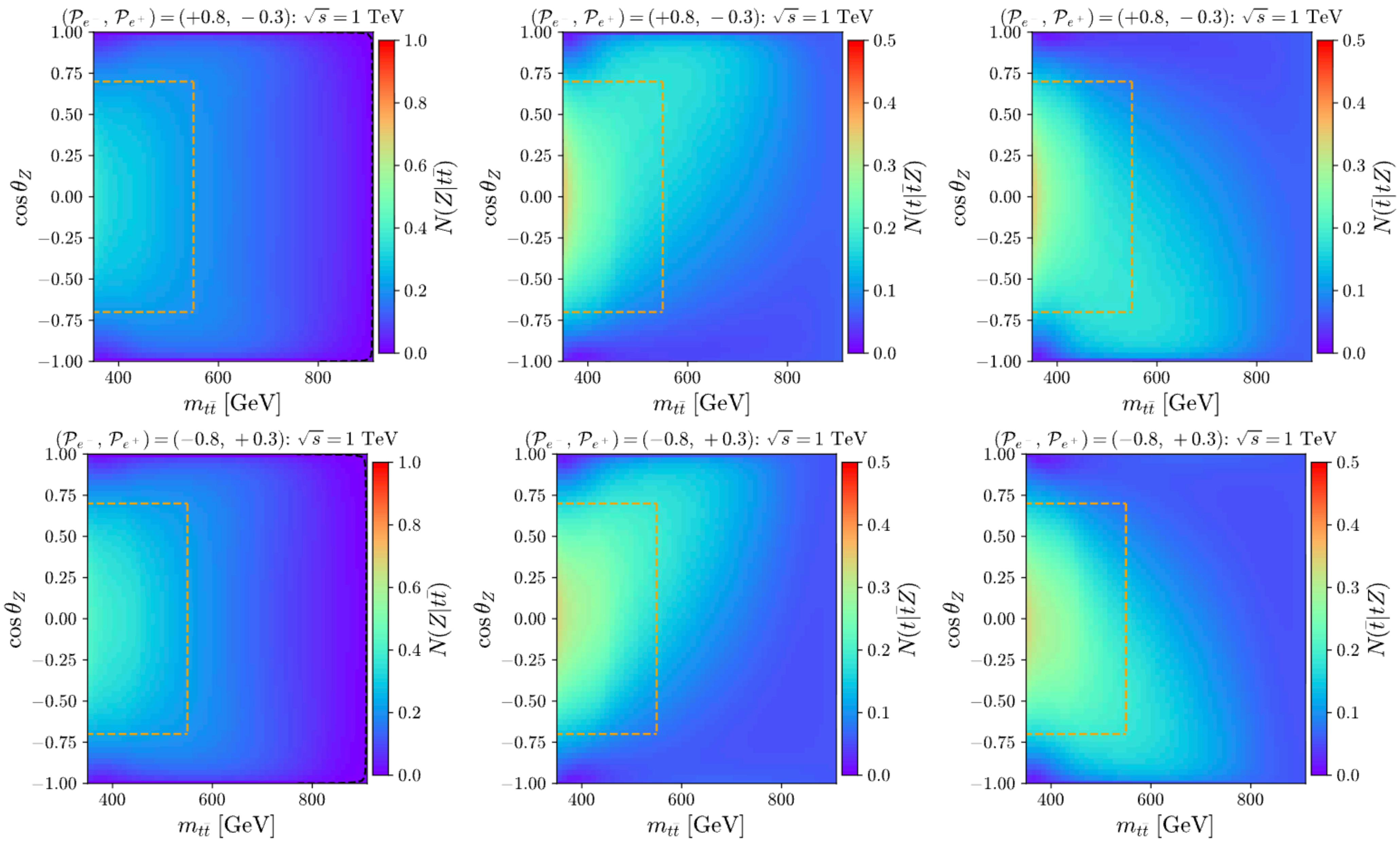

Quantum Tomography

- Density matrix parametrized in terms of the irreducible tensor operators

$$
\begin{aligned}
\rho = \frac{1}{12}\Big(& \mathbb{I}_{12} + A^1_{L_1 M_1} t^{L_1}_{M_1} \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 + A^2_{L_2 M_2} \mathbb{I}_2 \otimes t^{L_2}_{M_2} \otimes \mathbb{I}_3 + A^3_{L_3 M_3} \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes T^{L_3}_{M_3} \\
& + C^{12}_{L_1 M_1 L_2 M_2} t^{L_1}_{M_1} \otimes t^{L_2}_{M_2} \otimes \mathbb{I}_3 + C^{13}_{L_1 M_1 L_3 M_3} t^{L_1}_{M_1} \otimes \mathbb{I}_2 \otimes T^{L_3}_{M_3} + C^{23}_{L_2 M_2 L_3 M_3} \mathbb{I}_2 \otimes t^{L_2}_{M_2} \otimes T^{L_3}_{M_3} \\
& + C^{123}_{L_1 M_1 L_2 M_2 L_3 M_3} t^{L_1}_{M_1} \otimes t^{L_2}_{M_2} \otimes T^{L_3}_{M_3} \Big) ,
\end{aligned}
$$

- Full differential distribution described in term of spherical harmonics

$$
\begin{aligned}
\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3} = \frac{1}{(4\pi)^3}\Big[& 1 + A^1_{L_1 M_1} B^1_{L_1} Y^{M_1}_{L_1}(\Omega_1) + A^2_{L_1 M_1} B^2_{L_2} Y^{M_2}_{L_2}(\Omega_2) \\
& + A^3_{L_3 M_3} B^3_{L_3} Y^{M_3}_{L_3}(\Omega_3) + C^{12}_{L_1 M_1 L_2 M_2} B^1_{L_1} B^2_{L_2} Y^{M_1}_{L_1}(\Omega_1) Y^{M_2}_{L_2}(\Omega_2) \\
& + C^{13}_{L_1 M_1 L_3 M_3} B^1_{L_1} B^3_{L_3} Y^{M_1}_{L_1}(\Omega_1) Y^{M_3}_{L_3}(\Omega_3) + C^{23}_{L_2 M_2 L_3 M_3} B^2_{L_2} B^3_{L_3} Y^{M_2}_{L_2}(\Omega_2) Y^{M_3}_{L_3}(\Omega_3) \\
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\end{aligned}
$$

$L_{1,2} = 1$, $-L_{1,2} \le M_{1,2} \le L_{1,2}$, $L_3 = 1, 2$ and $-L_3 \le M_3 \le L_3$, and $B^{1,2}_1 = \frac{2\sqrt{\pi}}{3}\beta_{1,2}$, $B^3_1 = -\sqrt{2\pi}\,\eta_3$ and $B^3_2 = \sqrt{2\pi/5}\,(1 - 3\delta_3)$

- Density matrix parameters extracted using the orthogonality of the spherical harmonics

$$
\int \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3} Y^{M_i}_{L_i}(\Omega_i)^* \mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3 = \frac{B^i_{L_i}}{4\pi} A^i_{L_i M_i} , \quad i = 1, 2, 3
$$

$$
\int \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3} Y^{M_i}_{L_i}(\Omega_i)^* Y^{M_j}_{L_j}(\Omega_j)^* \mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3 = \frac{B^i_{L_i}}{4\pi}\frac{B^j_{L_j}}{4\pi} C^{ij}_{L_i M_i L_j M_j} , \quad i \neq j ,
$$

$$
\int \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3} Y^{M_1}_{L_1}(\Omega_1)^* Y^{M_2}_{L_2}(\Omega_2)^* Y^{M_3}_{L_3}(\Omega_3)^* \mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3 = \frac{B^1_{L_1}}{4\pi}\frac{B^2_{L_2}}{4\pi}\frac{B^3_{L_3}}{4\pi} C^{123}_{L_1 M_1 L_2 M_2 L_3 M_3} .
$$