Detection of Gravitational Waves and Multi-Messenger Astronomy

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General Relativity: Einstein Equation

Einstein Equation

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

- $g_{\mu\nu}$: metric tensor,
- $R_{\mu\nu}$: Ricci tensor,
- $R = g^{\mu\nu} R_{\mu\nu}$: Ricci scalar,
- $T_{\mu\nu}$: stress-energy tensor.

Measure of interval in spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

General Relativity: Linearized Theory

Perturbation of metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1.$$

Linearized Einstein Equation

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu \bar{h}_{\mu\rho} - \partial^\sigma\partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$

- $h = \eta^{\mu\nu} h_{\mu\nu}$,
- $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$.

Harmonic Gauge and Transverse-Traceless Gauge

Harmonic Gauge

$$\partial^\nu \bar{h}_{\mu\nu} = 0.$$

Linearized Einstein Equation

$$\Box \bar{h}_{\mu\nu} = 0,$$

$T_{\mu\nu} = 0$ if we observe the GW far away from the source.

Transverse-Traceless Gauge

$$h^{0\mu}, \quad h^i_i = 0, \partial^j h_{ij} = 0,$$

which can only be chosen away from the source.

Solution of the Gravitational Wave

We can choose the propagation direction along $\hat{z}$, with the wave vector:

$$k^{\mu} = (\omega/c, \mathbf{k}),$$

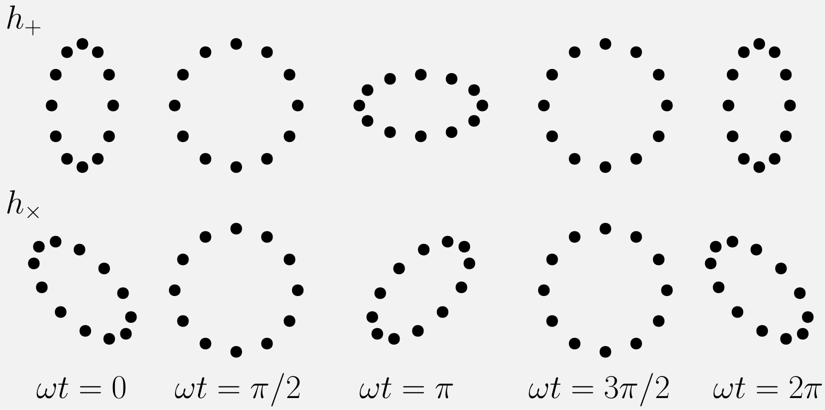
and the solution of GW:

$$h^{\mathrm{TT}}_{\mu\nu}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos\left[\omega(t - z/c)\right].$$

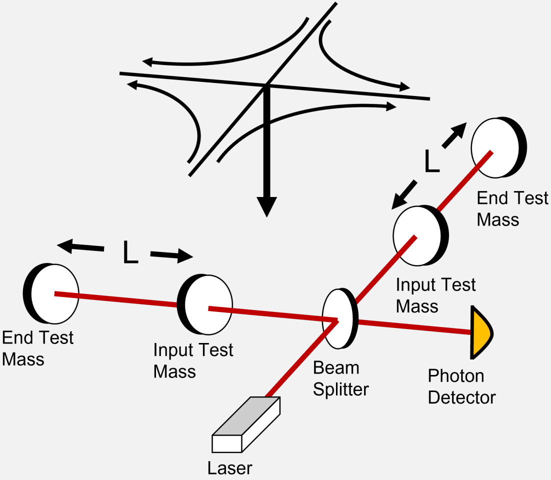
The we have the perturbed metric tensor:

$$\begin{aligned} ds^2 = & -c^2 dt^2 + dz^2 + \{1 + h_{+} \cos[\omega(t - z/c)]\} \, dx^2 \\ & - \{1 + h_{+} \cos[\omega(t - z/c)]\} \, dy^2 + 2h_{\times} \cos[\omega(t - z/c)] dx dy. \end{aligned}$$

Gravitational Wave Interacting with Test Masses



Gravitational Wave Detector



Gravitational Wave Detector

$$0 = -c^2 dt^2 + (1 + h_+)\, dx^2, \quad \to cdt \simeq \left(1 + \frac{h_+}{2}\right) |dx| \,.$$

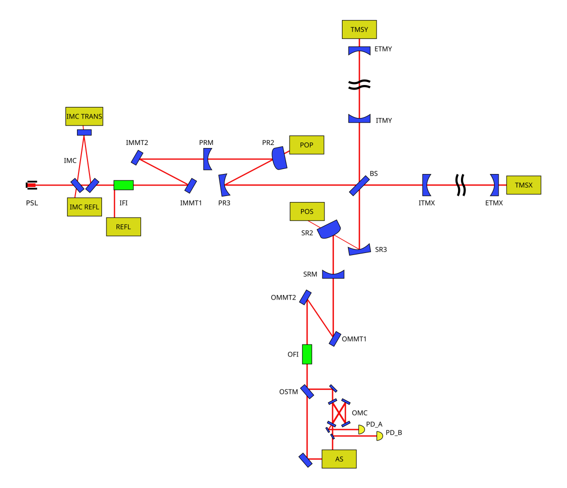
$$\Delta T_x = \frac{h_+ L_x}{2c}, \quad \Delta L_x = \frac{h_+ L_x}{2}.$$

$$0 = -c^2 dt^2 + (1 - h_+)\, dy^2, \quad \to cdt \simeq \left(1 - \frac{h_+}{2}\right) |dy| \,.$$

$$\Delta T_y = -\frac{h_+ L_y}{2c}, \quad \Delta L_y = -\frac{h_+ L_y}{2}.$$

$$\Delta L = \Delta L_x - \Delta L_y = h_+ \cdot \frac{L_x + L_y}{2} \equiv h_+ L.$$

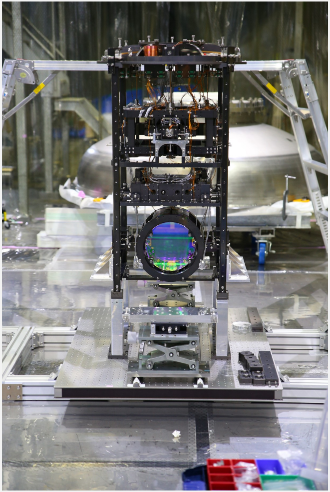
Gravitational Wave Detector

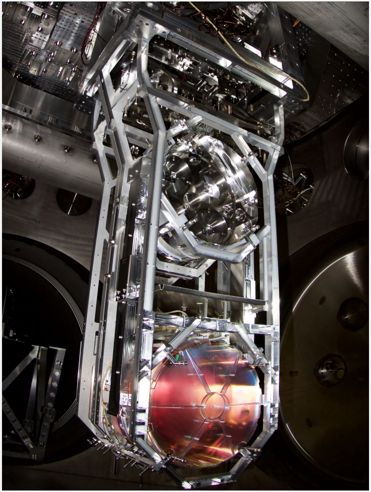


Gravitational Wave Detector

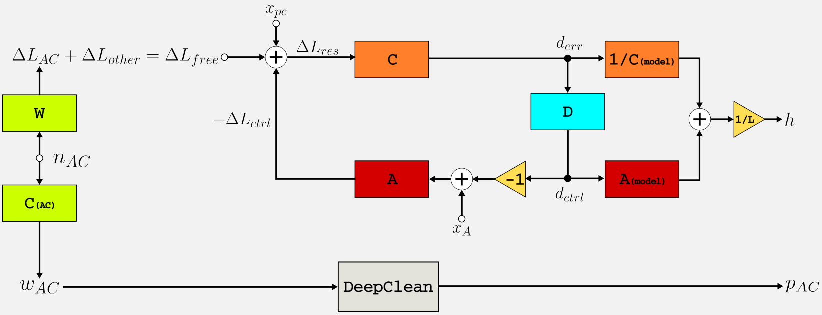
- Michelson Interferometer: https://www.youtube.com/watch?v=tQ_teIUb3tE
- Michelson Interferometer: https://javalab.org/en/michelson_interferometer_en
- Fabry-Perot Optical Cavity: https://ccahilla.github.io/fabryperot.html
- Gravitational Wave Detector: https://www.youtube.com/watch?v=h_FbHipV3No
- Optics of Gravitational Wave Detector: https://www.youtube.com/watch?v=X7RJHxeCuIY

Test Mass Mirror

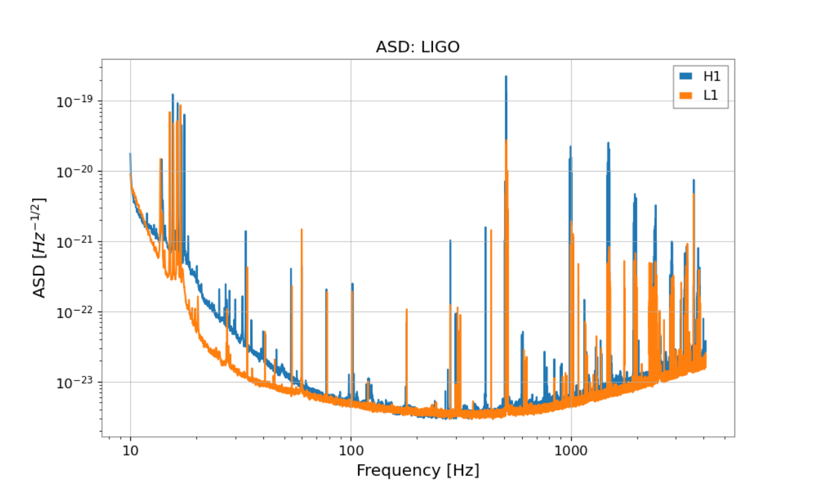




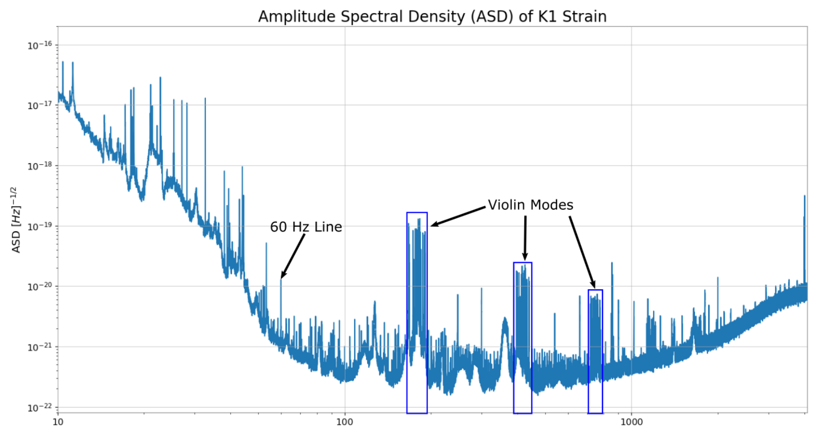
Calibration of GW Data



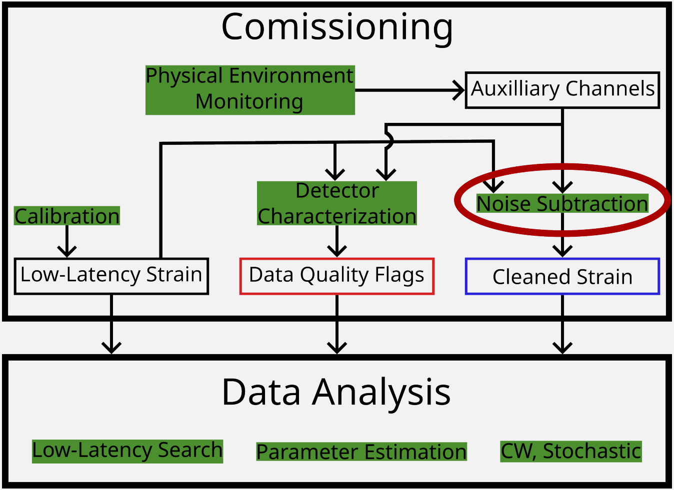
Amplitude Spectral Density



Amplitude Spectral Density



Working Groups

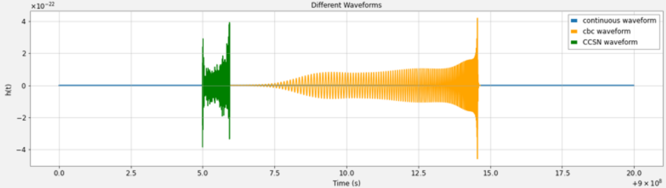


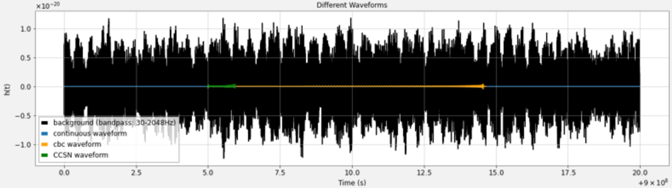
Sources of Gravitational Waves

- Compact Binary Coalescence:
  - Black Hole-Black Hole Mergers,
  - Neutron Star-Neutron Star Mergers,
  - Black Hole-Neutron Star Mergers.
- Bursts:
  - Core-collapse Supernovae, Fast Radio Burst, Gamma Ray Burst, etc.
- Continuous Waves:
  - Spinning Neutron Stars,etc.
- Stochastic Background:
  - Cosmological background, Astrophysical background, etc.

Gravitational Waveforms and Background Noise

GW strain: $h(t) = \frac{\Delta L(t)}{L} \sim 10^{-22}$.

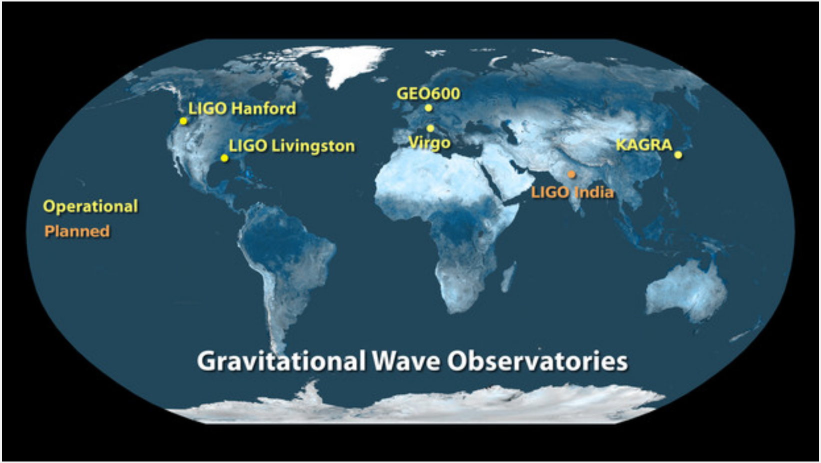




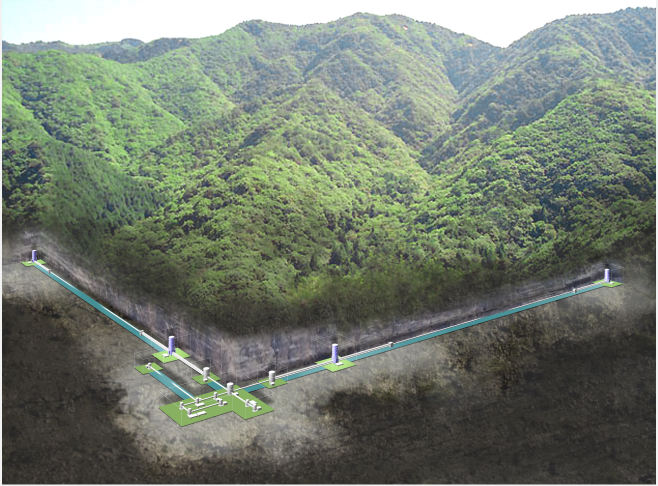
Gravitational Wave Detectors

- Ground based Interferometer
  - GEO600 (2010)
  - Advanced LIGO (2015, 2G)
  - Advanced Virgo (2016, 2G)
  - KAGRA (2019, 2G)
  - LIGO India (2023, 2G)
  - Einstein Telescope (2030s, 3G)
  - Cosmic Explorer (2030s, 3G)
- Space based Interferometer
  - DECIGO (2027)
  - Taiji (2030s)
  - LISA (2034)
  - TianQin (2035)

Gravitational Wave Detectors



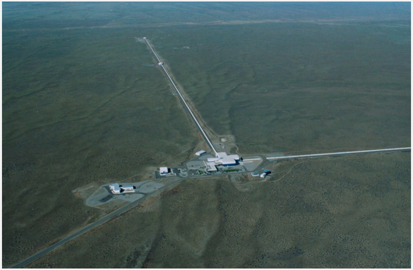
KAGRA



Virgo

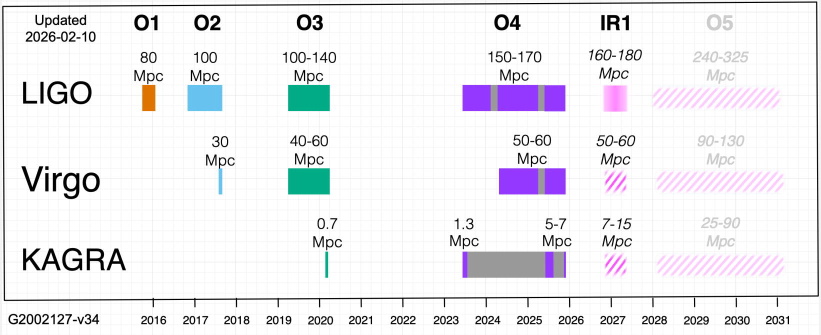
LIGO

Hanford

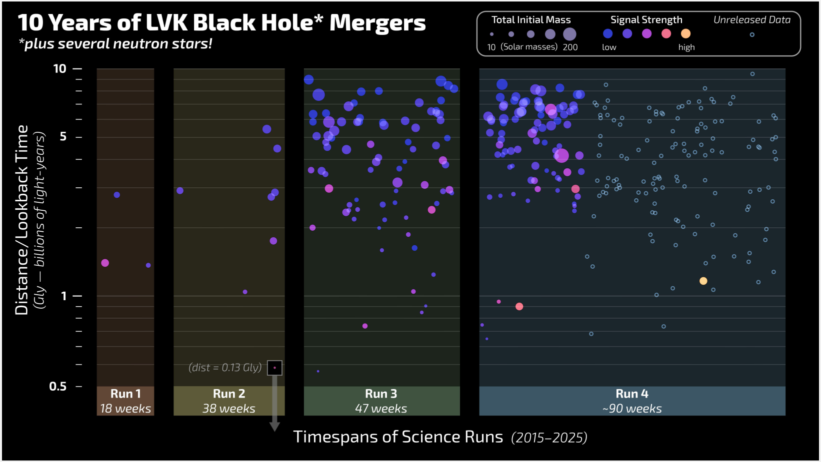


Livingston

Observation Runs

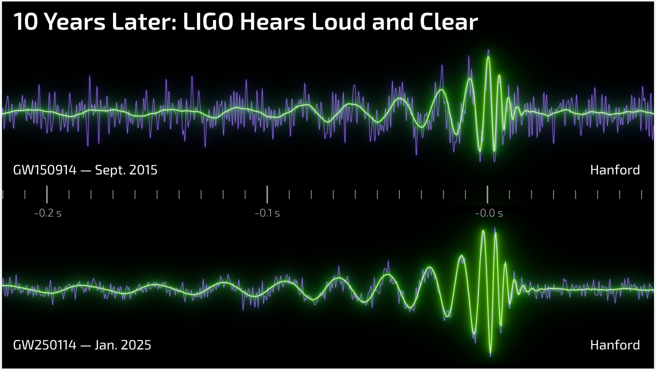


Detected CBC Events

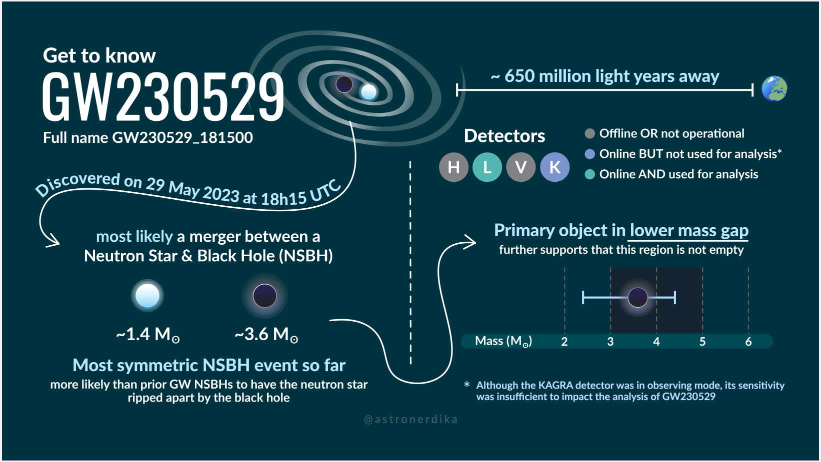


Detected CBC Events

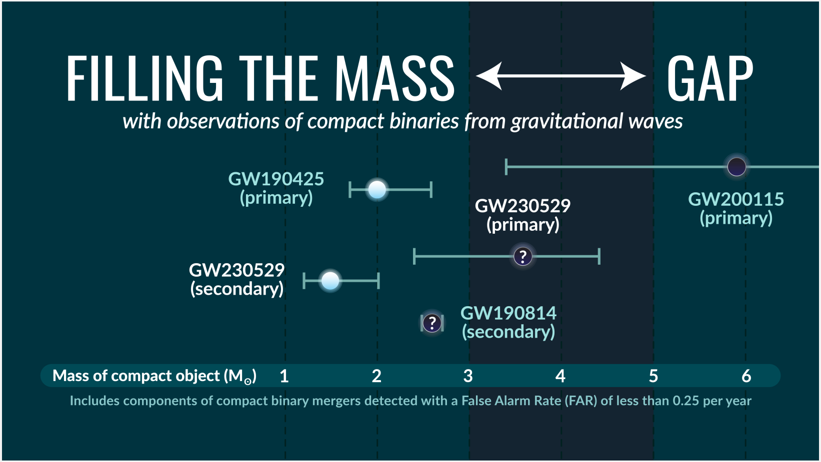
- BBH event: GW150914
- BBH event: GW250114: Simulation
- BBH event: GW250114: Ringing



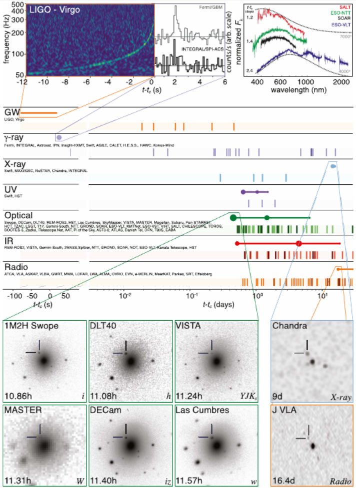
Detected CBC Events



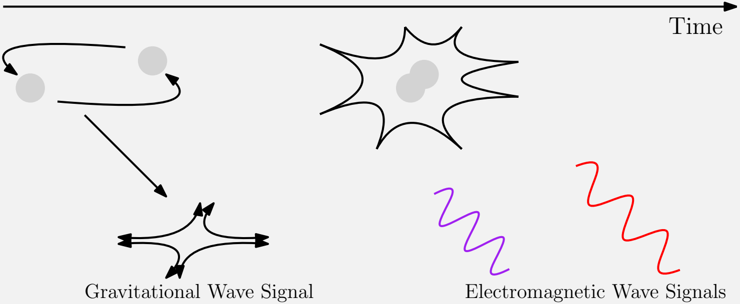
Detected CBC Events



GW170817


LIGO - Virgo
frequency (Hz)
t-t_c (s)
Fermi/GBM
INTEGRAL/SPI-ACS
counts/s (arb. scale)
normalized F_λ
SALT
ESO-NTT
SOAR
ESO-VLT
wavelength (nm)
GW
γ-ray
X-ray
UV
Optical
IR
Radio
t-t_c (days)
1M2H Swope
10.86h
DLT40
11.08h
VISTA
11.24h
Chandra
9d
X-ray
MASTER
11.31h
DECam
11.40h
Las Cumbres
11.57h
J VLA
16.4d
Radio

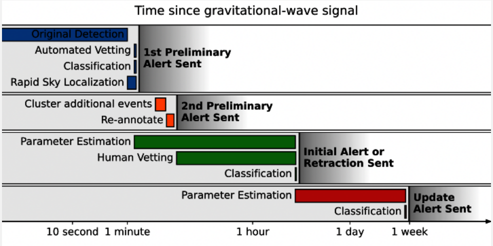
Multi-Messenger Astronomy (MMA)



Sky localization of the "Known GW Sources" from their GW signals and send out alerts in low latency for the EM telescopes to capture the follow-up EM wave signals.

Low-Latency Detection and Alert

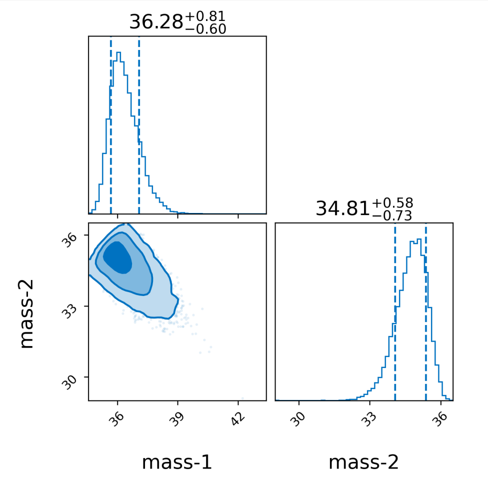
1. Detection of GW events (BNS, FRB, etc.) by Matched Filtering, Coherence Search or other detection methods.
2. Rapid Sky Localization and Human Vetting and sending initial alert.
3. Parameter Estimation and update alert sent.



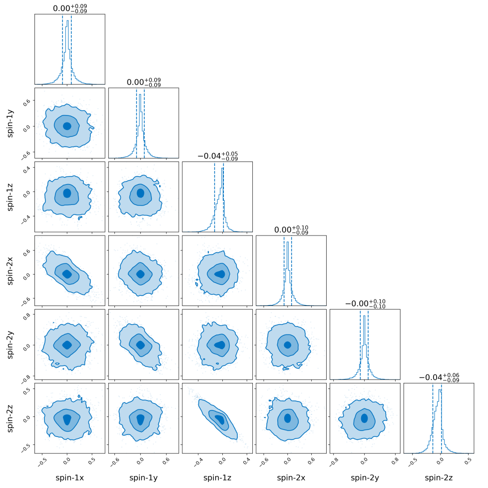
Parameter Estimation of Gravitational Waves Signal

- There are models describing gravitational wave form the coalescence binary compact objects: IMRPhenom, SEOBNR, SpinTaylor, IMRSpinPrecEOB...
- From the strain data of an event, we estimate the most probable model and the parameters.
- Intrinsic Parameters: $m_1, m_2, a_1, a_2, \theta_1, \theta_2, \delta\phi, \phi_{jl}$.
- Extrinsic Parameters: $ra, dec, \theta_{jn}, \psi, d_L, \phi_c, t_c$.

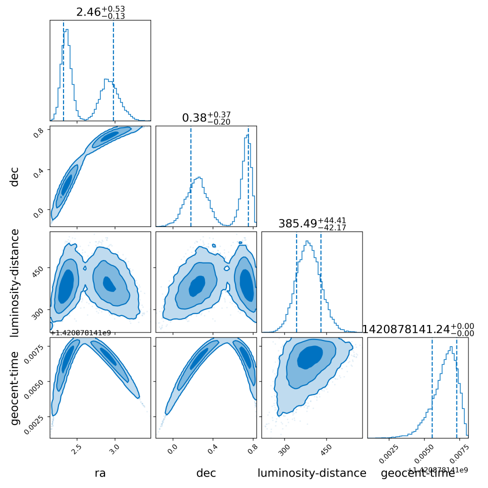
Parameter Estimation of GW250114



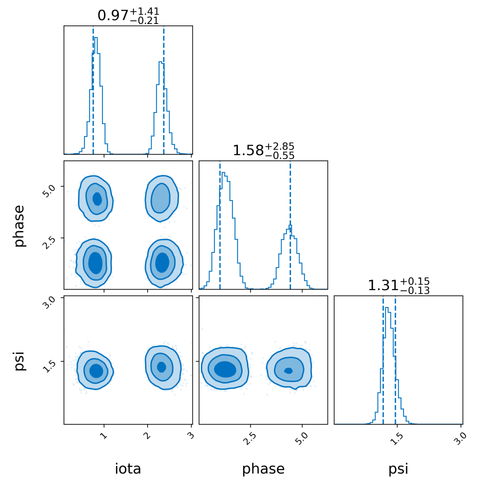
Parameter Estimation of GW250114



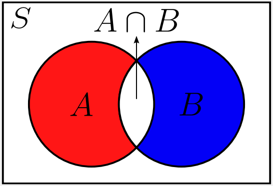
Parameter Estimation of GW250114



Parameter Estimation of GW250114

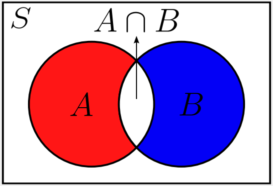


Conditional Probability



- $P(A)$: Probability of event A.
- $P(A \cap B)$: Probability of event A and event B occur simultaneously.
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$: Probability of event A given event B has occured.

Conditional Probability



$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A)$$

$$\Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Bayes' Theorem

Bayes' Theorem

$$p(\theta_i|d, H) = \frac{p(d|\theta_i, H)p(\theta_i|H)}{p(d|H)}.$$

- H: model,
- θ_i: parameters for the model H,
- d: observed data,
- $p(\theta_i|H)$: prior,
- $p(d|\theta_i, H)$: likelihood,
- $p(\theta_i|d, H)$: posterior,
- $p(d|H)$: evidence.

Bayes' Theorem

Marginalization

$$p(\theta_1|d,H) = \int_{\theta_2^{\text{min}}}^{\theta_2^{\text{max}}} \cdots \int_{\theta_N^{\text{min}}}^{\theta_N^{\text{max}}} p(\theta_1,\cdots,\theta_N|d,H)d\theta_2\cdots d\theta_N.$$

Evidence

$$p(d|H) = \int_{\theta_1^{\text{min}}}^{\theta_1^{\text{max}}} \cdots \int_{\theta_N^{\text{min}}}^{\theta_N^{\text{max}}} p(\theta_1,\cdots,\theta_N|d,H)d\theta_1\cdots d\theta_N.$$

Evaluating Posterior

Bayes' Theorem

$$p(\theta_i|d,H) = \frac{p(d|\theta_i,H)p(\theta_i|H)}{p(d|H)} = \frac{L(\theta_i)\cdot\pi(\theta_i)}{Z}.$$

- We don't know the normalization constant Z.
- Use Markov Chain Monte Carlo algorithms to generate samples from $L(\theta_i)\cdot\pi(\theta_i)$ in the parameter space.

$$L(\theta_i) = \mathcal{N}\exp\left\{-\frac{1}{2}\Big(s-h(\theta_i)|s-h(\theta_i)\Big)\right\},$$

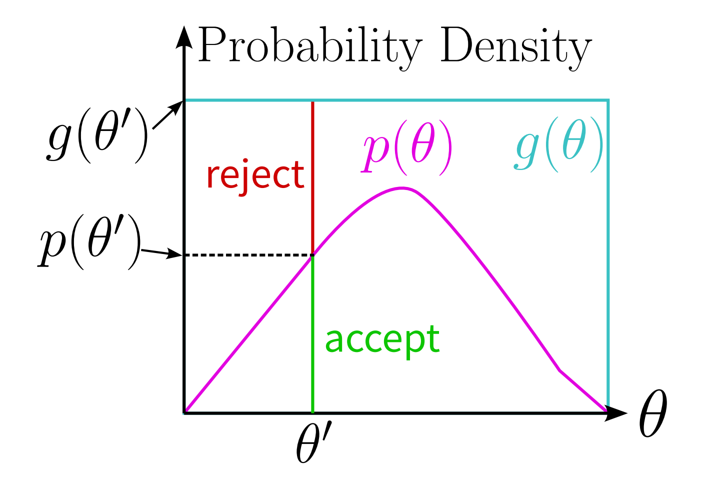
where

$$(A|B) = 4\mathrm{Re}\int_0^\infty \tilde{A}^*(f)[S_n^{-1}(f)]\tilde{B}(f)df.$$

Rejection Sampling

1. Draw a sample θ' from a probability distribution $g(\theta)$ which is easy to be sampled.
2. Calculate the value of the target probability distribution $p(\theta')$.
3. Calculate the weight $w = p(\theta')/g(\theta')$.
4. Draw a random number u from the uniform distribution in $[0, 1]$.
5. If $w > u$, append θ' to a set of samples, otherwise reject θ' and repeat from the first step.

Rejection Sampling



Markov Chain Monte Carlo Algorithm

Generate a Markov Chain:

$$\left\{\theta_i^{(0)} \to \theta_i^{(1)} \to \theta_i^{(2)} \to \cdots \to \theta_i^{(N)}\right\},$$

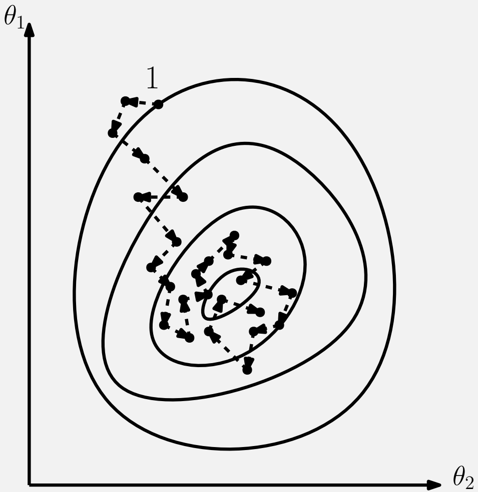
The steps are stochastic and determined by the probabilities $T(\theta_i, \theta_i')$ associated with the trasition $\theta_i \to \theta_i'$.

- $T(\theta_i, \theta_i') \geq 0$.
- The posterior is an invariant distribution of the chain: $p(\theta_i'|d, H) = \int p(\theta_i|d, H) T(\theta_i, \theta_i') d\theta_i$.
- Detailed Balance: $p(\theta_i|d, H) T(\theta_i, \theta_i') = p(\theta_i'|d, H) T(\theta_i', \theta_i)$
- Ergodicity:
    - $\exists n$ such that $T^n(\theta_i, \theta_i') > 0$ for all θ_i, θ_i',
    - $\exists \theta_i$ such that $T(\theta_i, \theta_i) > 0$.

Metropolis-Hasting Sampling

1. Starting at a random $\theta_i^{(0)}$ in the parameter space.
2. An update proposal θ_i' of $\theta_i^{(k)}$ is generated by sampling form a known proposal distribution $\tilde{p}(\theta_i')$ (e.g. Gaussian distribution centered at $\theta_i^{(k)}$).
3. Acceptance: $A(\theta_i^{(k)}, \theta_i') = \min\left(1, \frac{p(\theta_i')}{p(\theta_i^{(k)})}\frac{\tilde{p}(\theta_i^{(k)})}{\tilde{p}(\theta_i')}\right)$.
4. If $A(\theta_i^{(k)}, \theta_i') \geq 1$ (accepted): record $\theta_i^{(k+1)} = \theta_i'$, else:
   - $\theta_i^{(k+1)} = \theta_i'$ with the probability: $\frac{p(\theta_i')}{p(\theta_i^{(k)})}\frac{\tilde{p}(\theta_i^{(k)})}{\tilde{p}(\theta_i')}$.
   - $\theta_i^{(k+1)} = \theta_i^{(k)}$ with the probability: $1 - \frac{p(\theta_i')}{p(\theta_i^{(k)})}\frac{\tilde{p}(\theta_i^{(k)})}{\tilde{p}(\theta_i')}$.

Metropolis-Hasting Sampling



Metropolis-Hasting Sampling

- Now the Markov chain we get: $\left\{\theta_i^{(0)} \rightarrow \cdots \rightarrow \theta_i^{(N)}\right\}$ can be considered as a set of samples from the correction of the proposal distribution $\tilde{p}(\theta_i)$ to the posterior distribution $p(\theta_i|d,H)$.
- Now Drawing the histogram plot of $\left\{\theta_i^{(0)}, \cdots, \theta_i^{(N)}\right\}$, we can see the distribution proportional to the posterior and find out the most probable parameters.

Nested Sampling

- MCMC methods: Generates samples proportional to the the posterior.
- Nested Sampling: Simultaneously estimates the evidence and the posterior.

Pros of Nested Sampling:

- well-defined stopping criteria for terminating sampling,
- generating a sequence of independent samples,
- flexibility to sample from complex, multi-modal distributions,
- the ability to derive how statistical and sampling uncertainties impact results from a single run,
- being trivially parallelizable.

Nested Sampling

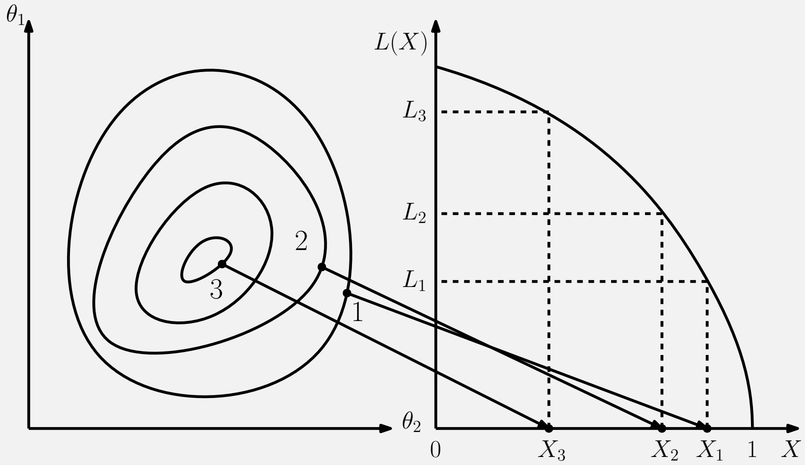
Prior Mass

$$X(\lambda) = \int_{\theta : L(\theta) > \lambda} \pi(\theta) d\theta.$$

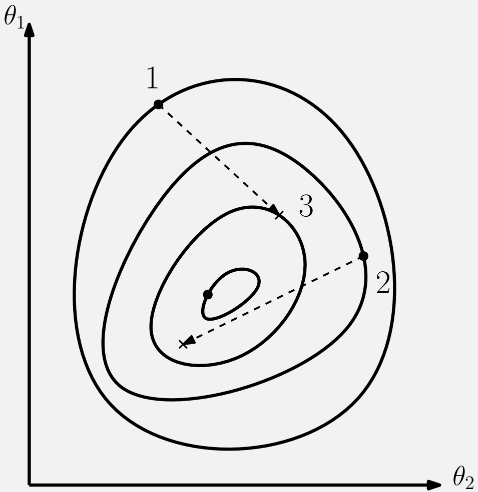
Evidence

$$Z = \int_0^1 L(X) dX, \quad \text{where} \quad L(X(\lambda)) = \lambda.$$

Nested Sampling



Nested Sampling



Nested Sampling

1. Sample N live points $\{\theta^{(1)} \cdots \theta^{(N)}\}$ from prior $\pi(\theta)$.
2. While not termination condition:
   1. record live point (i) with the lowest L_i as L_k,
   2. assign $X_k = t_k X_{k-1}$ where t_k from $P(t_k) = N t_k^{N-1}$,
   3. replace point (i) with sample from $\pi(\theta)$ subject to $L_i > L_k$.
3. Estimate evidence Z by integrating $\{L_k, X_k\}$.

Using Bilby for Parameter Estimation

1. Get strain data,
2. Estimate Power Spectral Density,
3. Determine Waveform model,
4. Set the prior distributions of the estimated parameters,
5. Set likelihood,
6. Set the sampler,
7. Run the sampler!
8. Plot the results.

Thank you