Basic Introduction to General Relativity and Gravitational Waves

From Geometry to Gravitational Waves

Sheng-Hong Lai

Department of Physics, Chung Yuan Christian University
Center for High Energy Physics
@2026 Gravitational Wave Open Data Workshop in Taiwan

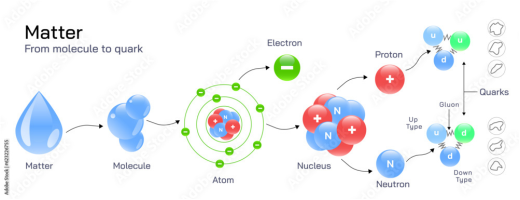
April 18, 2026

Outline

- Fundamental forces and the idea of gravity
- Newtonian gravity and General Relativity
- Geometry, curvature, parallel transport, and holonomy
- Einstein equations and the Schwarzschild solution
- Exact gravitational-wave solutions and polarizations

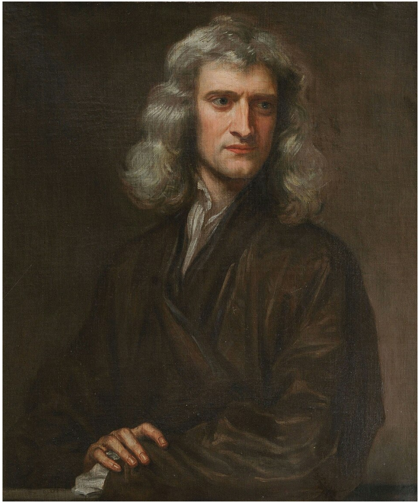
The Four Fundamental Forces

- **Strong**: binds quarks into hadrons; provides most of the proton/neutron mass; short range ($\sim 10^{-15}\,\mathrm{m}$).
- **Electromagnetism**: acts on electric charge; long range ($1/r^2$); governs atoms, chemistry, and light.
- **Weak**: responsible for β decay and neutrino interactions; very short range ($\sim 10^{-18}\,\mathrm{m}$); violates parity.
- **Gravity**: universal coupling to mass-energy; weakest but long range; dominates large-scale structure, black holes, and cosmology.



Newtonian Gravity

- **Isaac Newton** (1642–1727); *Philosophiæ Naturalis Principia Mathematica* (1687).
- Gravity as a **force** (action at a distance) with **instantaneous** propagation in the classical theory.
- Universal law:
$$F = G\frac{m_1 m_2}{r^2}.$$
- Excellent approximation for weak fields and low velocities, but:
  - cannot explain Mercury's anomalous perihelion precession,
  - conflicts with special relativity (no finite propagation speed).

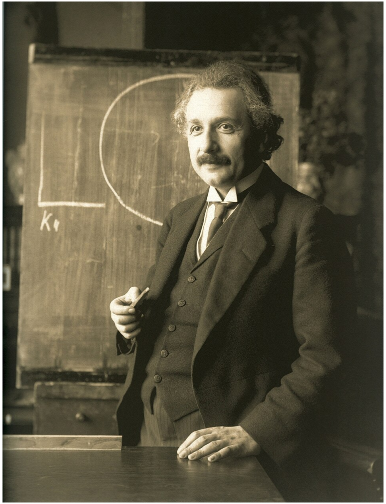


Einstein's Gravity (General Relativity)

- **Albert Einstein** (1879–1955); General Relativity (1915; final field equations in Nov. 1915).
- Gravity as **spacetime curvature** rather than a force.
- **Rubber-sheet picture**: mass-energy deforms the "sheet" of spacetime; free-fall follows the curved geometry (geodesics).
- Mathematical language: **Riemannian geometry** (metric, connection, curvature).
- Field equation:

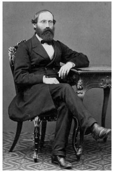
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

- In this picture, **gravitational waves** are small *ripples* of the sheet that propagate outward.
- Predicts black holes and gravitational waves.



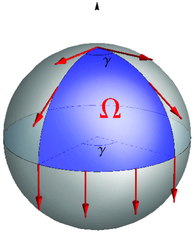
Geometry: Euclid, Gauss, and Riemann

- **Euclid** He is the first mathematician to systematically study mathematics and geometry through an explicit axiomatic framework.
- In **Stoicheia** (Greek: *Stoicheia*, "Elements"), we can clearly see this special style: starting from definitions, common notions, and postulates, then building propositions step by step. The standard language for lengths, angles, and parallel lines.
- **Gauss** introduced the idea that curvature can be *intrinsic*: it can be determined by measurements made entirely on a surface.
- **Riemann** generalized geometry to curved spaces in any dimension using a metric $g_{\mu\nu}$, paving the way for General Relativity.
- These geometric ideas are widely used in geodesy, mapping, navigation, and modern physics.

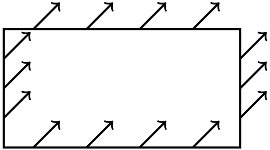


Curved Space and Curvature

- A vector at each point lives in the local *tangent space*, so comparing directions at different points requires a rule. 平行移動 → 曲率
- **Parallel transport** provides such a rule: it moves the vector along a path while keeping it as parallel as possible according to the local geometry.
- In a **flat** space, parallel transport around any closed loop returns the vector unchanged.
- In a **curved** space, the same closed loop generally produces a *net rotation*; this path-dependent rotation is an operational signature of curvature (holonomy).



Parallel transport in a curved space.

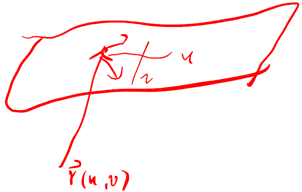


Flat plane: no rotation.

Curved Space and Curvature

Definition of a 2D Surface in 3D Space

The 2D surface S in 3D can be expressed as a position vector:



$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

Tangent vector along the u-curve:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

切向量

Tangent vector along the v-curve:

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

Curved Space and Curvature

The First Fundamental Form

The squared length of this infinitesimal element, ds^2, defines the First Fundamental Form:

$$I = d\mathbf{r} \cdot d\mathbf{r} = (\mathbf{r}_u\, du + \mathbf{r}_v\, dv) \cdot (\mathbf{r}_u\, du + \mathbf{r}_v\, dv)$$

By expanding this dot product, we obtain the classic quadratic differential form:

$$I = E\, du^2 + 2F\, du\, dv + G\, dv^2$$

The metric is defined by

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

長度.

長度$^2 = x^2 + y^2$

畢氏定理

For a 2D surface with coordinates (u, v),

$$g_{\mu\nu} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

where

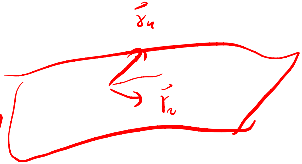
$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \qquad F = \mathbf{r}_u \cdot \mathbf{r}_v, \qquad G = \mathbf{r}_v \cdot \mathbf{r}_v$$

Curved Space and Curvature

Gauss Formulas

The Gauss formulas describe how the local tangent basis changes along the surface by splitting the second derivative into tangential and normal parts:

$$\mathbf{r}_{ij} = \sum_{k=1}^{2} \Gamma_{ij}^{k}\, \mathbf{r}_k + L_{ij}\, \mathbf{n}$$



The Christoffel symbols are determined by the metric $g_{\mu\nu}$:

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

- $\Gamma_{ij}^{k}\, \mathbf{r}_k$: tangential part, describing in-surface change.
- This encodes intrinsic geometry and underlies parallel transport.
- $L_{ij}\, \mathbf{n}$: normal part, describing bending in 3D space.

Weingarten Equations

The normal vector also changes as we move on the surface:

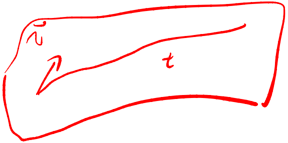
$$\partial_i \mathbf{n} = -L_{ik} g^{kj} \mathbf{r}_j$$

- Gauss formulas: L_{ij} shows how the tangent plane bends toward the normal direction.
- Weingarten equations: L_{ij} shows how the normal vector tilts toward tangent directions.
- Together, they link intrinsic geometry (metric) and extrinsic geometry (bending).

Curved Space and Curvature

Parallel Transport: Setup and Equation

$$\mathbf{V}(t) = V^i(t)\,\mathbf{r}_i, \qquad \frac{d\mathbf{r}_i}{dt} = \mathbf{r}_{ij}\frac{dx^j}{dt}$$



$$\frac{d\mathbf{V}}{dt} = \frac{dV^i}{dt}\mathbf{r}_i + V^i\mathbf{r}_{ij}\frac{dx^j}{dt}$$

Using the Gauss formula

$$\mathbf{r}_{ij} = \Gamma^k_{ij}\,\mathbf{r}_k + L_{ij}\,\mathbf{n}$$

gives

$$\frac{d\mathbf{V}}{dt} = \left(\frac{dV^k}{dt} + \Gamma^k_{ij}V^i\frac{dx^j}{dt}\right)\mathbf{r}_k + \left(L_{ij}V^i\frac{dx^j}{dt}\right)\mathbf{n}$$

For intrinsic parallel transport, the tangential part vanishes:

平移方程式

$$\frac{dV^k}{dt} + \Gamma^k_{ij}V^i\frac{dx^j}{dt} = 0$$

平移的向量 V

x: 平移的路徑

Curved Space and Curvature

Geodesics as Autoparallel Trajectories 1. The parallel transport equation

$$\frac{dV^i}{dt} + \Gamma^i_{jk} V^j \frac{dx^k}{dt} = 0$$

- An arbitrary vector V^i is carried along a curve $x^i(t)$ while remaining covariantly constant.
- Physical meaning: how do I carry an arrow so that it keeps the same local direction?

測地線方程.

2. The geodesic equation

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

距離最短的路徑

- It gives the trajectory of a freely falling particle.
- It is the straightest possible path through curved space.

3. Core conclusion Let the arbitrary vector be the curve's own velocity:

$$V^i \equiv \frac{dx^i}{dt}$$

Then the parallel transport equation becomes exactly the geodesic equation.

autoparallel trajectory $\Longleftrightarrow$ geodesic

Interpretation

In General Relativity, which uses a torsion-free connection, the path of shortest distance and the path of straightest direction are the same trajectory.

From Parallel Transport to a Surface Integral

平移一圈的結果

$$dV^k = -\Gamma^k_{lj} V^l \, dx^j, \qquad \Delta V^k = \oint_{\partial\Sigma} -\Gamma^k_{lj} V^l \, dx^j$$

向量的增量　平移的向量

For a 1-form $A_j \, dx^j$,

$$\oint_{\partial\Sigma} A_j \, dx^j = \frac{1}{2} \int_\Sigma (\partial_m A_j - \partial_j A_m) \, dx^m \wedge dx^j$$

長　面

Let $A_j = -\Gamma^k_{lj} V^l$. Then

$$\Delta V^k = -\frac{1}{2} \int_\Sigma \left[\partial_m (\Gamma^k_{lj} V^l) - \partial_j (\Gamma^k_{lm} V^l) \right] dx^m \wedge dx^j$$

Emergence of the Riemann Tensor Since V^l is parallel transported,

$$\partial_m V^l = -\Gamma^l_{im} V^i$$

Hence

$$\partial_m(\Gamma^k_{lj} V^l) = (\partial_m \Gamma^k_{lj}) V^l - \Gamma^k_{ij}\Gamma^i_{lm} V^l$$

$$\partial_j(\Gamma^k_{lm} V^l) = (\partial_j \Gamma^k_{lm}) V^l - \Gamma^k_{im}\Gamma^i_{lj} V^l$$

Subtracting gives

$$\partial_m(\Gamma^k_{lj} V^l) - \partial_j(\Gamma^k_{lm} V^l) = R^k{}_{lmj} V^l$$

where

$$R^k{}_{lmj} \equiv \partial_m \Gamma^k_{lj} - \partial_j \Gamma^k_{lm} + \Gamma^k_{im}\Gamma^i_{lj} - \Gamma^k_{ij}\Gamma^i_{lm}$$

Holonomy Measures Curvature Therefore,

曲率 向量

$$\Delta V^k = -\frac{1}{2}\int_\Sigma R^k{}_{lmj} V^l \, dx^m \wedge dx^j$$

向量的變化 面積

For an infinitesimal loop,

$$\Delta A^{mj} = \int_\Sigma dx^m \wedge dx^j$$

so the final local relation is

$$\Delta V^k = -\frac{1}{2} R^k{}_{lmj} V^l \, \Delta A^{mj}$$

- Holonomy around a loop measures the curvature enclosed by the loop.
- Curvature is the obstruction to returning with the same direction.

From Geometry to Holonomy – General Theory vs. Sphere S^2

Concept	**General Definition (Manifold M)**	**Example: 2-Sphere S^2 (Radius R)**
1. Metric Tensor	$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$	$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta\, d\phi^2$
2. Connection Matrix (ω)	${\omega^i}_j = \Gamma^i_{jk}\, dx^k$ (Packages Christoffel symbols into a 1-form matrix) Matrix expansion: $\omega = \begin{pmatrix} \Gamma^\theta_{\theta\theta} d\theta + \Gamma^\theta_{\theta\phi} d\phi & \Gamma^\theta_{\phi\theta} d\theta + \Gamma^\theta_{\phi\phi} d\phi \\ \Gamma^\phi_{\theta\theta} d\theta + \Gamma^\phi_{\theta\phi} d\phi & \Gamma^\phi_{\phi\theta} d\theta + \Gamma^\phi_{\phi\phi} d\phi \end{pmatrix}$	Resulting matrix: $\omega = \begin{pmatrix} 0 & -\sin\theta\cos\theta\, d\phi \\ \cot\theta\, d\phi & \cot\theta\, d\theta \end{pmatrix}$
3. Parallel Transport Eq.	$dV = -\omega V$ (Equivalent to $\frac{dV^k}{dt} + \Gamma^k_{ij} V^i \frac{dx^j}{dt} = 0$)	$\begin{pmatrix} dV^\theta \\ dV^\phi \end{pmatrix} = -\begin{pmatrix} 0 & -\sin\theta\cos\theta\, d\phi \\ \cot\theta\, d\phi & \cot\theta\, d\theta \end{pmatrix} \begin{pmatrix} V^\theta \\ V^\phi \end{pmatrix}$
4. Curvature Matrix (Ω) & Riemann Tensor	$\Omega = d\omega + \omega \wedge \omega$ ${\Omega^i}_j = \frac{1}{2} {R^i}_{jkl}\, dx^k \wedge dx^l$	$\Omega = \begin{pmatrix} 0 & \sin^2\theta\, d\theta \wedge d\phi \\ -d\theta \wedge d\phi & 0 \end{pmatrix}$ (Shows ${R^\theta}_{\phi\theta\phi} = \sin^2\theta$ and ${R^\phi}_{\theta\theta\phi} = -1$)
5. Holonomy Generator	Integral of the 2-form: $\Theta = \iint_\Sigma \hat{\Omega}$ (Integrated over surface Σ in an orthonormal frame to avoid coordinate singularities)	For 1/8 of the sphere (equator $\to$ pole $\to$ equator): $\Theta = \begin{pmatrix} 0 & \pi/2 \\ -\pi/2 & 0 \end{pmatrix}$
6. Tangent Vector Rotation	$R_{\text{turn}} = \exp(\Theta)$ $V_{\text{final}} = R_{\text{turn}} V_{\text{initial}}$ (Matrix exponential maps the Lie algebra generator to the Lie group rotation matrix)	$R_{\text{turn}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ A V_{initial} pointing North turns into a V_{final} pointing East, exactly a 90° rotation.

Ricci Tensor and Ricci Scalar
Contracting the Riemann tensor gives the Ricci tensor:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

- $R^{\rho}{}_{\sigma\mu\nu}$ describes the full local tidal and curvature information.
- $R_{\mu\nu}$ measures how a bundle of nearby geodesics focuses or defocuses.
- Physically, it captures local volume deformation of freely falling matter.

Contracting once more with the inverse metric gives the Ricci scalar:

$$R = g^{\mu\nu} R_{\mu\nu}$$

- R is a scalar measure of the total local curvature.

The Einstein Tensor

Matter is described by the stress-energy tensor $T_{\mu\nu}$, which obeys

$$\nabla^{\mu} T_{\mu\nu} = 0$$

So the geometric side of gravity must also be divergence-free. The correct combination is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

- $G_{\mu\nu}$ is called the Einstein tensor.
- By the Bianchi identity, it satisfies
$$\nabla^{\mu} G_{\mu\nu} \equiv 0$$
- This makes it the natural geometric partner of $T_{\mu\nu}$.

場方程

Einstein Field Equations

Equating geometry with matter gives

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

With a cosmological constant,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Left side: geometry and curvature of spacetime.
- Right side: matter, energy, momentum, and pressure.
- This is the core dynamical law of General Relativity.

Einstein's General Relativity

Geometry, Matter, and Motion

A useful summary is Wheeler's slogan:

Spacetime tells matter how to move; matter tells spacetime how to curve.

- $T_{\mu\nu}$ determines the curvature content entering the field equations.
- Solving for $g_{\mu\nu}$ determines the connection Γ.
- The metric and connection then determine geodesics and parallel transport.
- In this way, matter and geometry consistently determine one another.

The geodesic equation

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

不隨時間變 → 球對稱

1. The Static Spherically Symmetric Metric

We begin with the most general ansatz for a static, spherically symmetric spacetime. To accommodate SI units for the classical limit, define

$$(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$$

$$ds^2 = -e^{2\alpha(r)}(c\,dt)^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

The covariant metric tensor and its inverse are diagonal:

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\alpha} & 0 & 0 & 0 \\ 0 & e^{2\beta} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\alpha} & 0 & 0 & 0 \\ 0 & e^{-2\beta} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & (r\sin\theta)^{-2} \end{pmatrix}$$

2. The Connection Matrices $\boldsymbol{\Gamma}_\mu$

Using the metric definition, the non-zero Christoffel symbols are arranged into connection matrices (row = upper index, column = lower index), with $\alpha' = d\alpha/dr$ and $\beta' = d\beta/dr$.

$\boldsymbol{\Gamma}_t$: $\Gamma^t_{tr} = \alpha'$, $\Gamma^r_{tt} = \alpha' e^{2(\alpha-\beta)}$.

$$\boldsymbol{\Gamma}_t = \begin{pmatrix} 0 & \alpha' & 0 & 0 \\ \alpha' e^{2(\alpha-\beta)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{\Gamma}_r$: $\Gamma^t_{rt} = \alpha'$, $\Gamma^r_{rr} = \beta'$, $\Gamma^\theta_{r\theta} = 1/r$, $\Gamma^\phi_{r\phi} = 1/r$.

$$\boldsymbol{\Gamma}_r = \begin{pmatrix} \alpha' & 0 & 0 & 0 \\ 0 & \beta' & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1/r \end{pmatrix}$$

$\boldsymbol{\Gamma}_\theta$: $\Gamma^r_{\theta\theta} = -re^{-2\beta}$, $\Gamma^\theta_{\theta r} = 1/r$, $\Gamma^\phi_{\theta\phi} = \cot\theta$.

$$\boldsymbol{\Gamma}_\theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -re^{-2\beta} & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & \cot\theta \end{pmatrix}$$

$\boldsymbol{\Gamma}_\phi$: $\Gamma^r_{\phi\phi} = -r\sin^2\theta\, e^{-2\beta}$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^\phi_{\phi r} = 1/r$, $\Gamma^\phi_{\phi\theta} = \cot\theta$.

$$\boldsymbol{\Gamma}_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\, e^{-2\beta} \\ 0 & 0 & 0 & -\sin\theta\cos\theta \\ 0 & 1/r & \cot\theta & 0 \end{pmatrix}$$

3. The Six Independent Riemann Curvature Matrices

$$\mathbf{R}_{\mu\nu} = \partial_\mu \mathbf{\Gamma}_\nu - \partial_\nu \mathbf{\Gamma}_\mu + [\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu]$$

with $A \equiv \alpha'' + (\alpha')^2 - \alpha'\beta'$.

$$\mathbf{R}_{tr} = \begin{pmatrix} 0 & -A & 0 & 0 \\ -Ae^{2(\alpha-\beta)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{t\theta} = \begin{pmatrix} 0 & 0 & -r\alpha' e^{-2\beta} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\alpha'}{r} e^{2(\alpha-\beta)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{t\phi} = \begin{pmatrix} 0 & 0 & 0 & -r\alpha' \sin^2\theta \, e^{-2\beta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\alpha'}{r} e^{2(\alpha-\beta)} & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{r\theta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r\beta' e^{-2\beta} & 0 \\ 0 & -\frac{\beta'}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{r\phi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r\beta' \sin^2\theta \, e^{-2\beta} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta'}{r} & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{\theta\phi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin^2\theta \, (1 - e^{-2\beta}) \\ 0 & 0 & 1 - e^{-2\beta} & 0 \end{pmatrix}$$

Schwarzschild solution

4. Explicit Contraction to the Ricci Tensor $R_{\mu\nu}$
The Ricci tensor is the trace of the Riemann matrices:

$$R_{\mu\nu} = (\mathbf{R}_{\lambda\nu})^{\lambda}{}_{\mu}$$

Using the anti-symmetry property $\mathbf{R}_{\lambda\nu} = -\mathbf{R}_{\nu\lambda}$:
A. Time-Time component:

$$R_{tt} = -(\mathbf{R}_{tr})^{r}{}_{t} - (\mathbf{R}_{t\theta})^{\theta}{}_{t} - (\mathbf{R}_{t\phi})^{\phi}{}_{t}$$

$$-(\mathbf{R}_{tr})^{r}{}_{t} = (\alpha'' + (\alpha')^2 - \alpha'\beta')e^{2(\alpha-\beta)}, \qquad -(\mathbf{R}_{t\theta})^{\theta}{}_{t} = \frac{\alpha'}{r}e^{2(\alpha-\beta)}, \qquad -(\mathbf{R}_{t\phi})^{\phi}{}_{t} = \frac{\alpha'}{r}e^{2(\alpha-\beta)}$$

$$R_{tt} = e^{2(\alpha-\beta)}\left[\alpha'' + (\alpha')^2 - \alpha'\beta' + \frac{2\alpha'}{r}\right]$$

B. Radial-Radial component:

$$R_{rr} = (\mathbf{R}_{tr})^{t}{}_{r} - (\mathbf{R}_{r\theta})^{\theta}{}_{r} - (\mathbf{R}_{r\phi})^{\phi}{}_{r}$$

$$(\mathbf{R}_{tr})^{t}{}_{r} = -\alpha'' - (\alpha')^2 + \alpha'\beta', \qquad -(\mathbf{R}_{r\theta})^{\theta}{}_{r} = \frac{\beta'}{r}, \qquad -(\mathbf{R}_{r\phi})^{\phi}{}_{r} = \frac{\beta'}{r}$$

$$R_{rr} = -\alpha'' - (\alpha')^2 + \alpha'\beta' + \frac{2\beta'}{r}$$

4. Explicit Contraction to the Ricci Tensor $R_{\mu\nu}$ (continued)

C. Polar component:

$$R_{\theta\theta} = (\mathbf{R}_{t\theta})^t{}_\theta + (\mathbf{R}_{r\theta})^r{}_\theta - (\mathbf{R}_{\theta\phi})^\phi{}_\theta$$

$$(\mathbf{R}_{t\theta})^t{}_\theta = -r\alpha' e^{-2\beta}, \qquad (\mathbf{R}_{r\theta})^r{}_\theta = r\beta' e^{-2\beta}, \qquad -(\mathbf{R}_{\theta\phi})^\phi{}_\theta = -(1 - e^{-2\beta})$$

$$R_{\theta\theta} = 1 - e^{-2\beta}\left[1 + r(\alpha' - \beta')\right]$$

D. Azimuthal component:

$$R_{\phi\phi} = (\mathbf{R}_{t\phi})^t{}_\phi + (\mathbf{R}_{r\phi})^r{}_\phi + (\mathbf{R}_{\theta\phi})^\theta{}_\phi$$

$$(\mathbf{R}_{t\phi})^t{}_\phi = -r\alpha' \sin^2\theta\, e^{-2\beta}, \qquad (\mathbf{R}_{r\phi})^r{}_\phi = r\beta' \sin^2\theta\, e^{-2\beta}, \qquad (\mathbf{R}_{\theta\phi})^\theta{}_\phi = -\sin^2\theta\,(1 - e^{-2\beta})$$

$$R_{\phi\phi} = \sin^2\theta \left(1 - e^{-2\beta}\left[1 + r(\alpha' - \beta')\right]\right) = R_{\theta\theta}\sin^2\theta$$

Schwarzschild solution

5. Solving the Vacuum Einstein Equations $R_{\mu\nu} = 0$
In the vacuum region outside the mass:
Step A: Time-Radial symmetry

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0$$

cancels the non-linear terms entirely:

$$\frac{2}{r}(\alpha' + \beta') = 0 \quad \Rightarrow \quad \alpha(r) = -\beta(r) + C_1$$

To match flat Minkowski space at $r \to \infty$, $C_1 = 0$, so

$$\alpha = -\beta$$

Step B: Solving for the metric function Substitute $\alpha = -\beta$ into $R_{\theta\theta} = 0$:

$$1 - e^{2\alpha}(1 + 2r\alpha') = 0$$

Let $f(r) = e^{2\alpha}$. Then

$$1 - \frac{d}{dr}(rf) = 0$$

Integrating gives $rf = r + K$, hence

$$e^{2\alpha} = 1 + \frac{K}{r}$$

6. The Newtonian Limit (Deriving K via SI Units)
For a weak field and a slow-moving particle ($v \ll c$), the geodesic equation reduces to

$$\frac{d^2 r}{dt^2} \approx -c^2 \Gamma^r_{tt}$$

From $\mathbf{\Gamma}_t$, we have

$$\Gamma^r_{tt} = \alpha' e^{2(\alpha-\beta)} \approx \alpha'$$

so the relativistic acceleration is

$$a_{\mathrm{rel}} = -c^2 \frac{d\alpha}{dr}$$

Newtonian gravity gives

$$a_{\mathrm{newton}} = -\frac{GM}{r^2}$$

Equating them:

$$-c^2 \frac{d\alpha}{dr} = -\frac{GM}{r^2} \quad \Rightarrow \quad \frac{d\alpha}{dr} = \frac{GM}{c^2 r^2}$$

Integrating with flatness at infinity gives

$$\alpha(r) = -\frac{GM}{rc^2}$$

Hence, using $e^{2\alpha} \approx 1 + 2\alpha$,

$$g_{tt} \approx -c^2 \left(1 - \frac{2GM}{rc^2}\right)$$

Schwarzschild solution

7. The Final Schwarzschild Metric

Substituting

$$e^{2\alpha} = 1 - \frac{2GM}{rc^2}, \qquad e^{2\beta} = \left(1 - \frac{2GM}{rc^2}\right)^{-1}$$

back into the original ansatz yields the exact geometry of a non-rotating black hole:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

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事件視界

event horizon.

Gravitational Waves solution

Why Light-Cone Coordinates Simplify Waves

1. Operator factorization

$$\Box\psi = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2}\right)\psi = 0$$

$$-\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\psi = 0$$

With $u = t - z$ and $v = t + z$,

$$-4\frac{\partial^2\psi}{\partial u \partial v} = 0, \qquad \psi(u, v) = f(u) + g(v)$$

2. "Freezing" a traveling wave For $\psi = \sin(z - t)$, switching to $u = z - t$ gives

$$\psi(u) = \sin(u)$$

- In light-cone coordinates, the moving wave becomes a static profile.
- This is the key intuition behind Brinkmann coordinates.

Brinkmann Metric and Inverse Metric

In Brinkmann coordinates (u, v, x, y),

$$ds^2 = -2\,du\,dv + H(u,x,y)\,du^2 + dx^2 + dy^2$$

The covariant metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} H & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and its inverse is

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -H & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\det(g) = -1$, so $\sqrt{-g} = 1$.
- The key geometric fact is $g^{uu} = 0$.

Gravitational Waves solution

Christoffel Symbols in Brinkmann Coordinates
Using

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

and noting that only $g_{uu} = H$ varies, the nonzero symbols are

$$\Gamma^{v}_{uu} = -\frac{1}{2} H_{,u}, \qquad \Gamma^{v}_{ux} = \Gamma^{v}_{xu} = -\frac{1}{2} H_{,x}, \qquad \Gamma^{v}_{uy} = \Gamma^{v}_{yu} = -\frac{1}{2} H_{,y}$$

$$\Gamma^{x}_{uu} = -\frac{1}{2} H_{,x}, \qquad \Gamma^{y}_{uu} = -\frac{1}{2} H_{,y}$$

Connection Matrices $\mathbf{\Gamma}_\mu$ Writing $\Gamma^\lambda_{\mu\rho}$ as matrices with row index λ and column index ρ:

$$\mathbf{\Gamma}_u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2}H_{,u} & 0 & -\frac{1}{2}H_{,x} & -\frac{1}{2}H_{,y} \\ -\frac{1}{2}H_{,x} & 0 & 0 & 0 \\ -\frac{1}{2}H_{,y} & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{\Gamma}_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2}H_{,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{\Gamma}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2}H_{,y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\mathbf{\Gamma}_v = \mathbf{0}$.

Riemann Curvature Matrices

Define

$$\mathbf{R}_{\mu\nu} = \partial_\mu \mathbf{\Gamma}_\nu - \partial_\nu \mathbf{\Gamma}_\mu + [\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu]$$

In Brinkmann geometry, all commutators vanish:

$$[\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu] = 0$$

Hence the curvature matrices are purely linear in derivatives. The nonzero ones are

$$\mathbf{R}_{ux} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}H_{,xx} & \frac{1}{2}H_{,xy} \\ \frac{1}{2}H_{,xx} & 0 & 0 & 0 \\ \frac{1}{2}H_{,xy} & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{uy} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}H_{,xy} & \frac{1}{2}H_{,yy} \\ \frac{1}{2}H_{,xy} & 0 & 0 & 0 \\ \frac{1}{2}H_{,yy} & 0 & 0 & 0 \end{pmatrix}$$

Gravitational Waves solution

Ricci Tensor and Einstein Field Equations
The Ricci tensor is the trace of the curvature matrices $\mathbf{R}_{\lambda\nu}$. In Brinkmann geometry, the only non-zero component is R_{uu}:

$$R_{uu} = \mathrm{Tr}(\mathbf{R}_{\lambda u}) = (\mathbf{R}_{xu})^x{}_u + (\mathbf{R}_{yu})^y{}_u = -\frac{1}{2}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right)$$

According to the vacuum Einstein field equations,

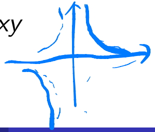
$$R_{\mu\nu} = 0$$

we obtain the transverse Laplace equation:

$$\nabla_\perp^2 H = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$$

Exact Plane-Wave Solution: $+$ and $\times$ Modes
The quadratic harmonic polynomial solution is

$$H(u,x,y) = A(u)(x^2 - y^2) + 2B(u)xy$$



Gravitational Waves solution

From the Brinkmann Profile to Geodesic Deviation

$$H(u,x,y) = A(u)(x^2 - y^2) + 2B(u)xy$$

We first compute the first derivatives:

$$\partial_x H = 2A(u)x + 2B(u)y$$

$$\partial_y H = -2A(u)y + 2B(u)x$$

The second derivatives give the four transverse Riemann components:

$$R^x{}_{uxu} = -\frac{1}{2}\partial_x\partial_x H = -A(u)$$

$$R^y{}_{uyu} = -\frac{1}{2}\partial_y\partial_y H = A(u)$$

$$R^x{}_{uyu} = -\frac{1}{2}\partial_y\partial_x H = -B(u)$$

$$R^y{}_{uxu} = -\frac{1}{2}\partial_x\partial_y H = -B(u)$$

Hence the tidal matrix is

$$\mathbf{E} = \begin{pmatrix} R^x{}_{uxu} & R^x{}_{uyu} \\ R^y{}_{uxu} & R^y{}_{uyu} \end{pmatrix} = \begin{pmatrix} -A(u) & -B(u) \\ -B(u) & A(u) \end{pmatrix}$$

The geodesic deviation equation 潮汐

$$\frac{D^2\xi^\mu}{d\tau^2} = -R^\mu{}_{\nu\rho\sigma} T^\nu \xi^\rho T^\sigma$$

simplifies, for $T^\mu = \delta^\mu_u$ and $\xi^i = (x,y)$, to

$$\frac{d^2\xi^i}{du^2} = -R^i{}_{uju}\,\xi^j$$

or in matrix form

$$\begin{pmatrix} d^2x/du^2 \\ d^2y/du^2 \end{pmatrix} = -\begin{pmatrix} -A(u) & -B(u) \\ -B(u) & A(u) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

so the final equations are

$$\frac{d^2x}{du^2} = A(u)x + B(u)y$$

$$\frac{d^2y}{du^2} = B(u)x - A(u)y$$

Thank You

Questions?