Conformal Defects in Neural Network Field Theories

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In collaboration with P. Capuozzo and B. Robinson

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- Motivation
- CFTs & dCFTs: Recap
- nn-CFTs
- nn-dCFTs
- Conclusion

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Why?

Machine Learning has told us a lot about Physics, when used as a tool:

1. Find hidden symmetries in data
2. Calculate numerical approximations
3. Formulating exact results (eg. KAN)
4. Prove theorems (eg. LLMs+Lean)

Can Physics say something about ML in exchange? (*spoiler: that's the hope!*)

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Why CFTs?

1. Huge symmetry group $\rightarrow$ greatly constrains dynamics
2. Fixed-points of renormalisation group flows of QFTs (conformal manifold)
3. Appear naturally as low-energy theories of certain D-brane

Why dCFTs?

1. Extended operators are NEEDED to fully characterise a given theory
2. Some symmetries only act on these extended operators (higher-form symmetries)
3. Spectrum of local operators + extended operators = the full theory

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A CFT is specified by a set of scaling dimensions and OPE coefficients $(\Delta_i, \lambda_{ijk})$.

Under a conformal transformation *conformal primary operators* transform covariantly

$$\phi(x) \mapsto \lambda^{-\Delta}\phi(x). \tag{1}$$

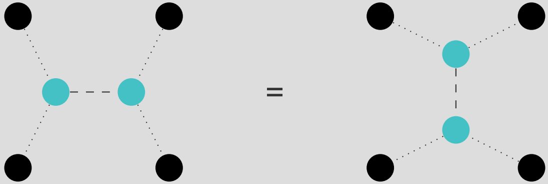
Their 2pt is completely determined by conformal symmetry (up to a constant)

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \delta_{\Delta_1,\Delta_2}\frac{c_{12}}{\|x_1 - x_2\|^{2\Delta_1}}. \tag{2}$$

The operator product expansion is convergent

$$\phi_1(x)\phi_2(y) \sim \sum_{\mathcal{O}} f_{\phi_1\phi_2\mathcal{O}} P(x-y,\partial_y)\cdot\mathcal{O}(y) \tag{3}$$

The coefficients are constrained by the crossing equations



dCFTs: a quick recap

Inserting a conformal defect means breaking the global symmetry group to its defect subgroup $SO(d+1,1) \to SO(p+1,1) \times SO(q)$.

There are now *ambient* and *defect* operators.

Defect CFT data: $(\hat{\Delta}_i, \Delta_i, \lambda_{ij}, \lambda_{ijk})$

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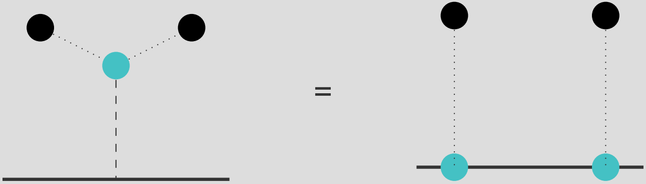
The OPE has now two channels: *ambient* and *defect*.

$$\phi(x) \sim \sum_{\hat{\mathcal{O}}} f_{\phi\hat{\mathcal{O}}} Q(x_\perp - y, \partial_y) \hat{\mathcal{O}}(y) \tag{4}$$

The coefficients, again, obey some constraints



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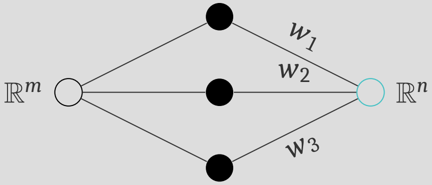
Neural Networks and UATs

Neural Network: a map $f : \mathbb{R}^m \to \mathbb{R}^n$

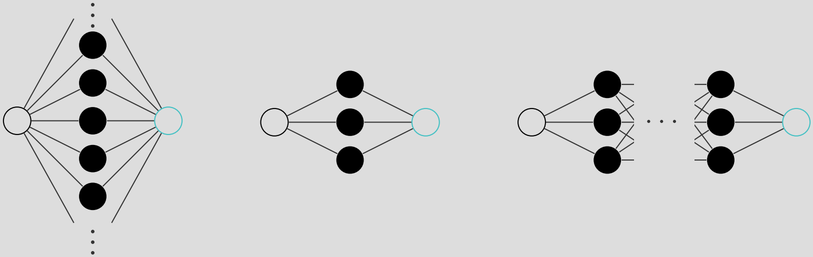
Operations: composition, addition, scalar multiplication

$f(x) = f_o\left(\sum_i w_i f_i(x)\right)$:



Neural Networks and UATs

Universal Approximation Theorems (UATs): the NNs are dense in a space of maps (in some limit).



$\hookrightarrow$ Depends on choice of basis, and choice of infinite node limit.

What is a nn-CFT? [Halverson, Naskar, Tian; 2024]

Key Idea: Approximate the functional integral using the UATs and NNs.

NNs come with a set of parameters: eg. Sigmoids $= \{\sigma(a \cdot x + b)\}$.
Integrate over those with a measure

$$\int \mathcal{D}\phi\, \mathcal{F}[\phi] \longleftrightarrow \lim_{N\to\infty} \int \prod_{i}^{N} d\theta_i P(\theta_i) \tag{5}$$

In the limit where the UAT applies, both sides are equal.

What is a nn-CFT? [Halverson, Naskar, Tian; 2024]

Conformal primary	
$\phi(x) \mapsto \lambda^{-\Delta}\phi(x)$	$\Phi(X) \mapsto \lambda^{-\Delta}\Phi(X)$
Partition function	
$Z[J] = \int \mathcal{D}\phi e^{-S[\phi]} e^{\int d^d x J(x)\phi(x)}$	$Z[J] = \int d\theta \mathcal{P}(\theta) e^{\int d^d x J(x)\Phi(x)}$
Correlators	
$\langle \phi(x)\phi(y) \rangle$	$\mathbb{E}[\Phi(x)\Phi(y)]$

Takeaway: you don't need to take the UAT limit to get a CFT!

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What is a nn-CFT? [Halverson, Naskar, Tian; 2024]

Partition function	
$Z[J] = \int \mathcal{D}\phi e^{-S[\phi]} e^{\int d^d x J(x)\phi(x)}$	$Z[J] = \int d\theta \mathcal{P}(\theta) e^{\int d^d x J(x)\Phi(x)}$

Key Idea: Symmetries of $S[\phi]$ are encoded in the probability measure $\mathcal{P}(\theta)$.

Why do we care? nn-CFTs are viable CFTs, can combine them to create new nn-CFTs, their UAT limit is a free CFT

What is a nn-CFT? [Halverson, Naskar, Tian; 2024]

In practice: Use the embedding space formalism.

$$
\begin{tikzcd}
\mathbb{R}^{D+2} \arrow[r, "\text{Wick rot.}"] & \mathbb{R}^{1,D+1} \arrow[d, "\text{Poincaré section}"] \\
& \mathbb{R}^{D}
\end{tikzcd}
$$

Ingredients: $SO(D+2)$-invariant measure $\mathcal{P}(\theta)$, nn-architecture $\Phi(X) = (X \cdot \theta)^{-\Delta}$.

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What is a nn-dCFT?

Break the symmetry group: $SO(D+2) \to SO(p+2) \times SO(q)$

Choose a measure invariant under the defect group: $\mathcal{P}(\theta)$

Define defect/ambient operators: $\varphi(X)$, $\Phi(X)$ with scaling dimensions $\hat{\Delta}$ and Δ.

$$Z[\hat{J}, J] = \int d\theta \mathcal{P}(\theta) e^{\int d^{p+2}X\, \hat{J}(X)\varphi(X)} e^{\int d^{D+2}X\, J(X)\Phi(X)} \tag{6}$$

Example 1: Monomial dCFT

Ingredients:

1. $\mathcal{P}(\theta) = \mathcal{P}_{\text{defect}}(\theta)\mathcal{P}_{\text{tangent}}(\theta)$, two centred multivariate Gaussian distributions, with 2nd moments $\hat{\mu}_2$ and $\tilde{\mu}_2$
2. $\varphi(X) = (X \bullet \theta)^{\hat{n}}$: defect nn-architecture
3. $\Phi(X) = (X \bullet \theta + X \circ \theta)^{n}$: ambient nn-architecture
4. $\hat{\Delta} = -\hat{n}$ and $\Delta = -n$ are **negative**

Example 1: One-point functions

As expected: only the ambient operators pickup a vev

Defect:

$$\mathbb{E}[\varphi(X)] \overset{P.S.}{=} 0 \tag{7a}$$

Ambient:

$$\mathbb{E}[\Phi(X)] \overset{P.S.}{=} \begin{cases} \frac{\Gamma(n+1)}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)}(\tilde{\mu}_2 - \hat{\mu}_2)^{\frac{n}{2}}(X \circ X)^{\frac{n}{2}} & n \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{7b}$$

Example 1: Two-point functions

Defect-Defect:

$$\mathbb{E}[\varphi_1(X_1)\varphi_2(X_2)] \overset{P.S.}{=} \delta_{n_1,n_2}\Gamma(n_1+1)\tilde{\mu}_2^{n_1}(X_1 \bullet X_2)^{n_1} \tag{8a}$$

Defect-Ambient:

$$\mathbb{E}[\varphi_1(X_1)\Phi_2(X_2)] \overset{P.S.}{=} \frac{\Gamma(n_2+1)}{\Gamma\left(\frac{n_2-n_1}{2}+1\right)} 2^{\frac{n_1-n_2}{2}} \tilde{\mu}_2^{n_1}(\hat{\mu}_2 - \tilde{\mu}_2)^{\frac{n_2-n_1}{2}}(X_2 \circ X_2)^{\frac{n_2-n_1}{2}}(X_1 \bullet X_2)^{n_1} \tag{8b}$$

when $n_2 \geq n_1$ and $n_1 + n_2 \in 2\mathbb{Z}$, and vanishes otherwise.
Follows the expected form for the mixed correlator of scalars [Billò et al; 2016].

Example 1: Two-point functions

Ambient-Ambient:

Define $\alpha_{11} = (\tilde{\mu}_2 - \hat{\mu}_2)X_1 \circ X_1$, $\alpha_{22} = (\tilde{\mu}_2 - \hat{\mu}_2)X_2 \circ X_2$ and $\alpha_{12} = \hat{\mu}_2 X_1 \bullet X_2 + \tilde{\mu}_2 X_1 \circ X_2$,

$$\mathbb{E}[\Phi_1(X_1)\Phi_2(X_2)] \overset{P.S.}{=} \frac{\Gamma(n_2+1)}{\Gamma(n_1+1)\Gamma\left(\frac{n_2-n_1}{2}+1\right)} \left(\frac{\alpha_{22}}{2}\right)^{\frac{n_2-n_1}{2}} \alpha_{12}^{n_1}$$
$$ {}_2F_1\left(\frac{1-n_1}{2}, -\frac{n_1}{2}; \frac{n_2-n_1}{2}+1; \frac{\alpha_{11}\alpha_{22}}{\alpha_{12}^2}\right) \tag{9}$$

when $n1 \leq n_2$ and $n_1 + n_2 \in 2\mathbb{Z}$ (similar expression for $n1 \geq n_2$).

Example 1: OPE Expansion

Since $n > 0$, the defect OPE is exact and finite

$$\begin{aligned}\Phi(X) &= (X \bullet \theta + X \circ \theta)^n = \sum_{d=0}^{n} \binom{n}{d} (X \circ \theta)^{n-d} (X \bullet \theta)^d \\ &= (X \circ \theta)^n \, 1_{\text{defect}} + n(X \circ \theta)^{n-1} \varphi_1(X) + \cdots \end{aligned} \tag{10}$$

What does this mean in terms of the NNs?

Example 2: Reciprocal dCFT

Ingredients:

1. $\mathcal{P}(\theta) = \mathcal{P}_{\text{defect}}(\theta)\mathcal{P}_{\text{tangent}}(\theta)$, two centred multivariate Gaussian distributions, with 2nd moments $\hat{\mu}_2$ and $\tilde{\mu}_2$
2. $\widehat{\varphi}(X) = (X \bullet \theta)^{-\hat{\Delta}}$: defect nn-architecture
3. $\Phi(X) = (X \bullet \theta + X \circ \theta)^{-\Delta}$: ambient nn-architecture
4. $\hat{\Delta}$ and Δ are **positive**

Example 2: Reciprocal dCFT

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Peculiarity:

$$\Phi(X) = \sum_{n=0}^{\infty} \frac{(\Delta)_n}{n!}(-1)^n \tilde{\varphi}_{-n}(X)\widehat{\varphi}_{\Delta+n}(X) \tag{11}$$

Example 2: One-point functions

When projecting down to the Poincare section,

Defect:

$$\mathbb{E}[\widehat{\varphi}(X)] = 0 \tag{12a}$$

Ambient:

$$\mathbb{E}[\Phi(X)] = 0 \tag{12b}$$

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Example 2: Two-point functions

Defect-Defect:

$$\mathbb{E}[\widehat{\varphi}_1(X_1)\widehat{\varphi}_2(X_2)] = \frac{(-1)^{\hat{\Delta}_1}}{\Gamma(\hat{\Delta}_1)} \, (\hat{\mu}_2 X_1 \bullet X_2)^{-\hat{\Delta}_1} \, \delta_{\hat{\Delta}_1,\hat{\Delta}_2} \tag{13a}$$

Ambient-Defect:

$$\mathbb{E}[\Phi(X_1)\widehat{\varphi}(X_2)] = \frac{2^{\frac{\Delta_1-\hat{\Delta}_2}{2}}(-1)^{\hat{\Delta}_2}}{\Gamma(\Delta_1)\Gamma(\frac{\hat{\Delta}_2-\Delta_1}{2}+1)} \, (\tilde{\mu}_2 X_1 \circ X_1)^{\frac{\hat{\Delta}_2-\Delta_1}{2}} \, (\hat{\mu}_2 X_1 \bullet X_2)^{\hat{\Delta}_2} \tag{13b}$$

for $\hat{\Delta}_2 \geq \Delta_1$ and zero otherwise.

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Recap

What have we done so far?

1. Extended the notion of nn-CFT to the defect case
2. Constructed simple examples of fully solvable nn-dCFTs
3. Calculated the correlators between defect and ambient insertions of scalar conformal primaries

What have we NOT done so far?

1. Took the large-width limit of these nn-dCFTs
2. Calculated correlators of spinning fields (upcoming work)
3. Considered 2d conformal symmetry (upcoming work)
4. Considered fermionic primaries (upcoming work)

Recap

Key takeaways

1. UATs give a well behaved approximation of the path integral
2. nn-CFTs and nn-dCFTs are perfectly good CFTs on their own
3. Taking the large-width limit yields a free CFT (dCFT)
4. Can combine these nn-CFTs (nn-dCFTs) together, to generate infinitely many new theories

Thank you

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