# Holography and Optimal Transport

## Emergent Wasserstein spacetime from quantum states

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Based on
Koji Hashimoto and Norihiro Tanahashi
*Holography and Optimal Transport:*
*Emergent Wasserstein Spacetime in Harmonic Oscillator, SYK and Krylov Complexity*
arXiv:2604.17649 [hep-th]

# Message for this talk

**Can holographic geometry emerge from distances between quantum states?**

We show the answer is **yes** — using optimal transport as the distance measure.

## The question

- Can geometry be *reconstructed* from quantum state data?
- If so, which *distance* between states gives the radial direction?

## The proposal

**Optimal transport:**
minimum cost to move one probability distribution into another

$\Rightarrow$ use this as the distance between quantum states

## The findings

From harmonic oscillator and SYK, we extract

- an extra-dimensional coordinate
- BH-like spacetime

$\Rightarrow$ Optimal transport distance *can* serve as a holographic coordinate.

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   (a) Static geometry from state distances
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# Holography as an inverse problem

**Forward map (bulk $\to$ boundary)**

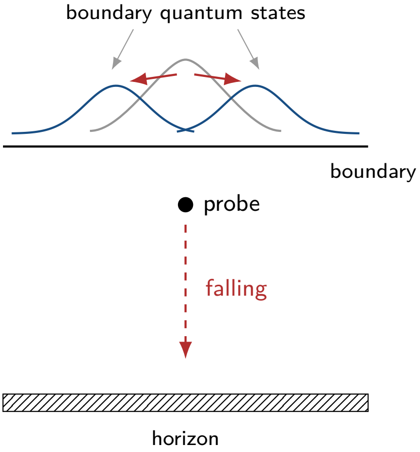
A given bulk geometry predicts boundary quantum dynamics.

**Inverse map (boundary $\to$ bulk)**

Bulk information is encoded in boundary quantum data only indirectly

**Hard part: strong dimensional reduction**

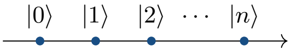
The state/distribution space of a quantum system is huge, while a classical bulk geometry is low-dimensional.
How can we extract a low-dimensional geometry from high-dimensional quantum data?



# What should a holographic distance know?

## Requirement

The holographic distance should reflect *how far apart* states are along the radial direction — not just whether they are orthogonal.




## Standard choices fall short

- **Inner product:** $\langle n|m\rangle = 0$ $(n \neq m)$ is orthogonality, not a geometric distance
- **KL divergence:** not symmetric; violates triangle inequality
- **Overlap-based distances** (e.g. Fubini–Study): blind to "where probability moves"

Orthogonality alone does not give a geometric distance:
$d(n, m) = 0$ for $n \neq m$

KL is not a metric:
$$D_{\mathrm{KL}}(P\|Q) \neq D_{\mathrm{KL}}(Q\|P)$$
$$D_{\mathrm{KL}}(P\|R) \nleq D_{\mathrm{KL}}(P\|Q) + D_{\mathrm{KL}}(Q\|R)$$

**We need a true geometric distance between distributions.**

# Three choices to realize dimensional reduction

Infinite-dim. quantum state space $\longrightarrow$ Low-dim. holographic bulk geometry

Which choices of states, representation, and distance can realize this reduction?

$$\begin{pmatrix} \text{states} \\ \text{representation} \\ \text{distance} \end{pmatrix} \Longrightarrow \underbrace{\text{bulk geometry}}_{\text{low-dim.}}$$

**Our proposal in this work:**

1. **States:** energy eigenstates and open-system trajectories.
2. **Representation:** positive distributions, especially Husimi $Q$ functions.
3. **Distance:** Wasserstein distance from optimal transport.

**Manifold hypothesis as selection principle**

The right choices make the quantum data collapse onto a simple low-dimensional manifold.

# Why optimal transport is natural here

## Bulk side

Probe motion follows an extremal path:
the result of optimizing the particle action.

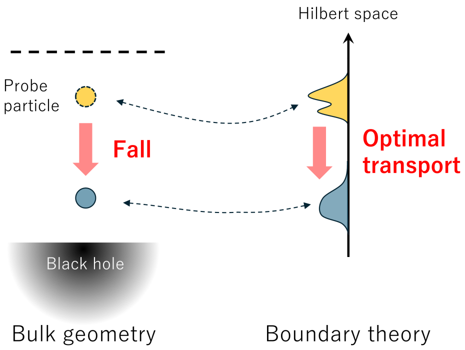
## Boundary side

The quantum state changes shape over time.
*If* it corresponds to the probe motion, the distance between states should also arise from an optimization.

## ⇒ Proposal

Wasserstein space of boundary distributions
= emergent holographic spacetime.

(Wasserstein distance = distance in optimal transport)



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# Optimal transport in various fields

A problem born in 18th-century resource allocation has become a cornerstone of modern data science.

### Economics (origins)

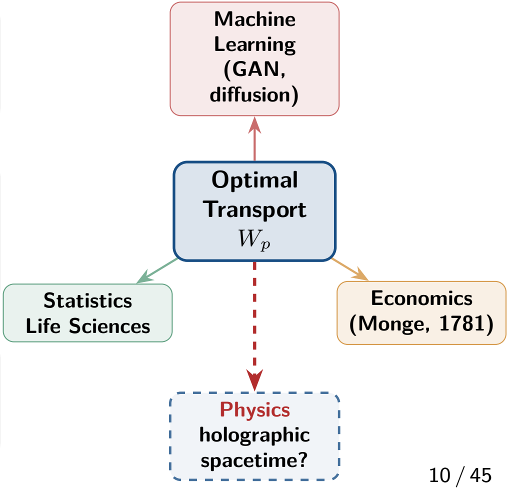
**Monge (1781):** optimal transport of earth.
**Kantorovich (1942):** coupling formulation; Nobel Prize 1975.

### Machine Learning

**Wasserstein GAN** (Arjovsky et al., 2017):
$W_1$ as training loss — more stable than KL-based GAN.
Also: diffusion models, domain adaptation.

### Statistics / Life Sciences

**Wasserstein barycenter**: average of distributions.
Single-cell RNA-seq, shape interpolation.



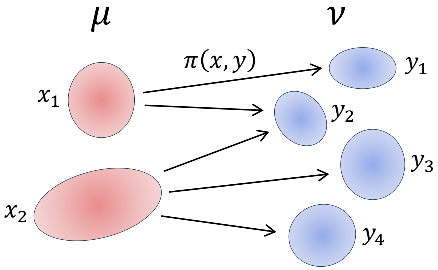
# Optimal transport basics

Think of $\mu$ and $\nu$ as two piles of sand.
*How cheaply can we reshape $\mu$ into $\nu$?*

$$W_p(\mu,\nu) = \left[\min_{\pi\in\Pi(\mu,\nu)} \int d(x,y)^p \; d\pi(x,y)\right]^{1/p}$$

- $\pi(x,y)$: **transport plan** — how much mass moves from $x$ to $y$, with all mass conserved.
- $d(x,y)$: distance between $x$ and $y$ (moving farther costs more)
- Minimize total cost over all valid plans.

$\Rightarrow$ $W_p$ is a true metric on probability distributions.




Why is the minimum cost a metric?
- *Non-negativity*: cost $\geq 0$; zero iff $\mu = \nu$.
- *Symmetry*: any plan can be run in reverse.
- *Triangle inequality*: a shortcut is never worse than a detour.

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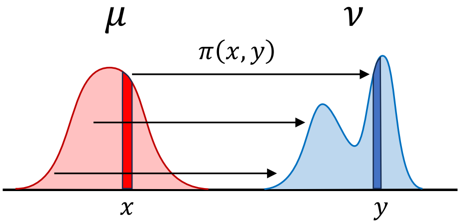


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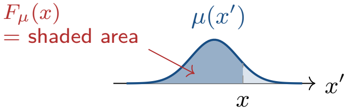
# Optimal transport basics: simplification in one-dimensional case

In one dimension, Wasserstein distance has a closed-form expression.

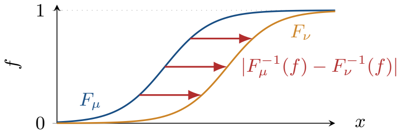
Cumulative distribution function (CDF):

$$F_\mu(x) = \int_{-\infty}^{x} \mu(x')\, dx'$$



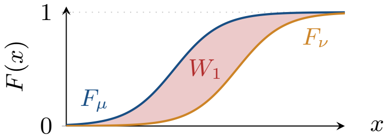

General $p$-Wasserstein formula $(F^{-1}$: inverse CDF):

$$W_p(\mu,\nu) = \left[\int_0^1 \left|F_\mu^{-1}(f) - F_\nu^{-1}(f)\right|^p df\right]^{1/p}$$




For $p = 1$ this reduces to

$$W_1(\mu,\nu) = \int_{-\infty}^{\infty} |F_\mu(x) - F_\nu(x)|\ dx.$$




$W_1$ = area between CDFs.

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# The harmonic oscillator as our test bed

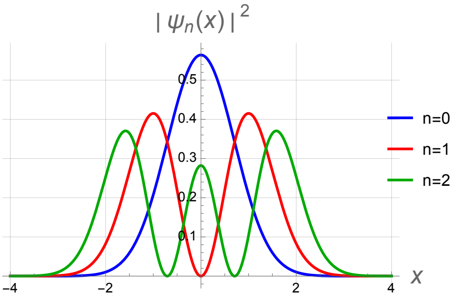
$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2, \quad E_n = \left(n + \tfrac{1}{2}\right)\omega$$

## Why the harmonic oscillator?

- Simplest quantum system with a clear energy ladder
- Represents a single mode of a quantum field
- Hilbert space is analytically tractable

## Why energy eigenstates?

In holography, the radial direction $\leftrightarrow$ energy scale.
$\Rightarrow$ States $|0\rangle, |1\rangle, |2\rangle, \ldots$ **should** form a 1D space
*if we choose the right distance.*




**Question:** Which distance on which representation makes $\{|n\rangle\}$ form a *1D space*?

# Two representations of a quantum state

## Position-space probability density

$$\rho_n(x) = |\psi_n(x)|^2 = |\langle x|n\rangle|^2$$

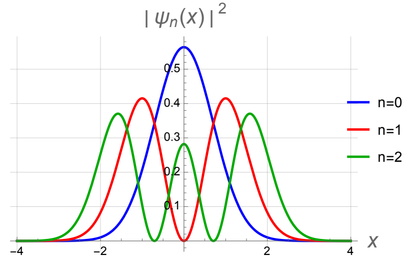
Probability of finding the particle at position $x$.
Defined on the real line.

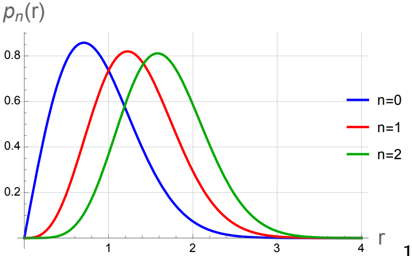
## Husimi $Q$ distribution

$$Q_n(\alpha) = \frac{1}{\pi}|\langle\alpha|n\rangle|^2$$

Overlap with coherent state $|\alpha\rangle$ at phase-space point $\alpha \in \mathbb{C}$.
Always $\geq 0$; defined on 2D phase space $(x, p)$.

**Question:** Which of these two, combined with which distance, gives an emergent 1D geometry?



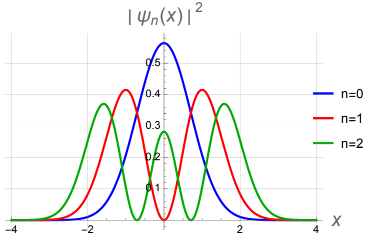
# Two representations of quantum states

## Position-space probability density

$$\rho_n(x) \equiv |\langle x|n\rangle|^2, \qquad x \in \mathbb{R}$$

$$\langle x|n\rangle = \left(\frac{m\omega}{2^{2n}(n!)^2\pi}\right)^{1/4} H_n\left(\sqrt{m\omega}\,x\right) e^{-\frac{m\omega x^2}{2}}$$

($H_n$: Hermite polynomial)

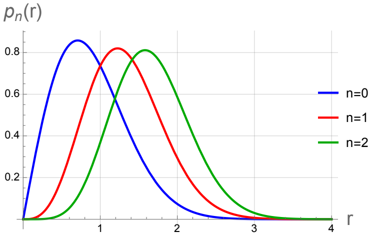



## Husimi $Q$ distribution

$$Q_n(\alpha) \equiv \frac{1}{\pi}|\langle\alpha|n\rangle|^2 = \frac{|\alpha|^{2n}}{n!\,\pi}\,e^{-|\alpha|^2}, \quad \alpha \in \mathbb{C}$$

$|\alpha\rangle$: coherent state s.t. $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ for $\hat{a} \equiv \dfrac{m\omega\hat{x} + i\hat{p}}{\sqrt{2m\omega}}$

$\alpha \in \mathbb{C}$ labels the semiclassical phase-space position of $|\alpha\rangle$



# Why Husimi distributions?

## Key property for the harmonic oscillator

$$Q_n(\alpha) \equiv \frac{1}{\pi}|\langle\alpha|n\rangle|^2 = \frac{|\alpha|^{2n}}{n!\,\pi}\, e^{-|\alpha|^2}$$

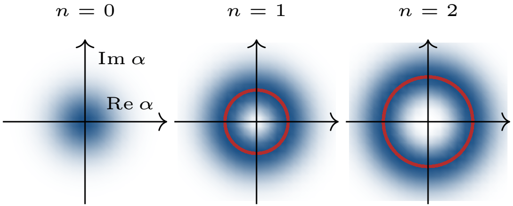
**(i)** Depends only on $r = |\alpha|$ (rotationally symmetric)
$\Rightarrow$ reduces to the 1D radial distribution

$$p_n(r) \equiv 2\pi r\, Q_n(r) = \tfrac{2}{n!}\, r^{2n+1} e^{-r^2}, \quad r \geq 0.$$

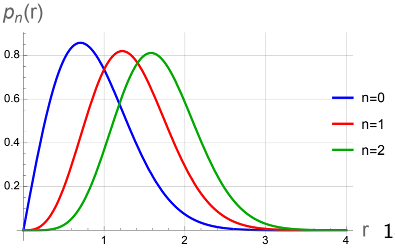
**(ii)** $p_n(r)$ has **no nodes**: single peak at $r \sim \sqrt{n}$

- Peak at $r\sim\sqrt{n}$ reflects the single classical orbit at that energy.
- No nodes: coherent-state smearing washes out quantum interference.

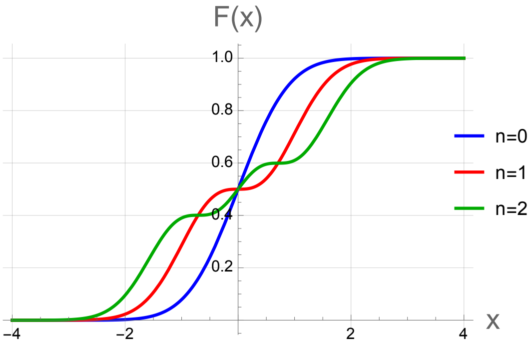
By contrast, $\rho_n(x) = |\langle x|n\rangle|^2$ has $n$ nodes:
$\Rightarrow$ its *CDFs* for different $n$ *cross* each other (next slide)



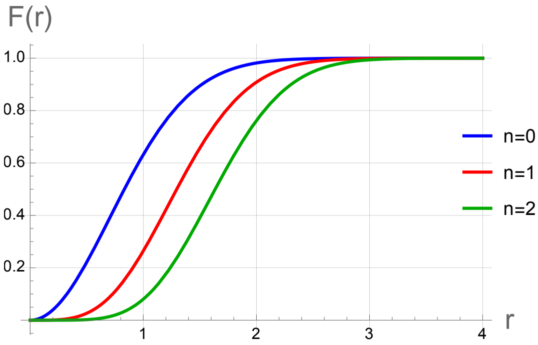

phase space $(\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha)$
$\alpha = (x + ip)/\sqrt{2}$ $(m{=}\omega{=}1)$
— classical orbit at $r{=}\sqrt{n}$



# The key visual reason: ordered CDFs



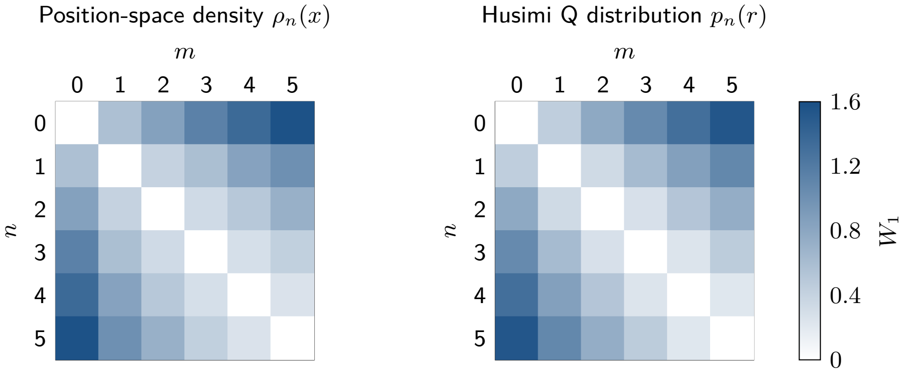

Position-space CDFs cross.




Husimi radial CDFs stay ordered.

When CDFs are ordered, areas add like lengths along a line.

# Computing $W_1$ distance matrices ($N = 6$ eigenstates)




Both matrices look similar visually — the difference is revealed by the MDS eigenvalue test.

# Embedding into Euclidean Space: Classical MDS

## Classical Multidimensional Scaling

Given $\{d_{nm}\} = \{W_p(n,m)\}$:

1. $\Delta_{nm} \equiv d_{nm}^2$
2. Gram matrix: $B \equiv -\frac{1}{2} J \Delta J$
   $(J = I - \frac{1}{N}\mathbf{1}\mathbf{1}^\top)$
3. Eigendecompose $B = V \Lambda V^\top$

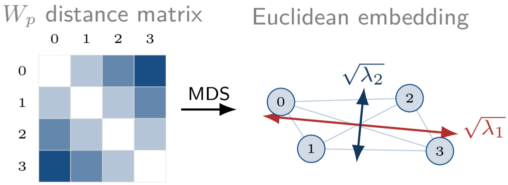
## Embedding criterion

All $\lambda_k \geq 0$: embeds in $D_{\min}$ dims.

Any $\lambda_k < 0$: no Euclidean embedding.

Min. dim: $D_{\min} \equiv \#\{k : \lambda_k > 0\} = \operatorname{rank}(B)$

Eff. dim: $D_{\text{eff}} \equiv \dfrac{\left(\sum_k \lambda_k\right)^2}{\sum_k \lambda_k^2}$




**Gram matrix** $B$ eigenvalues $\lambda_k$:

$\lambda_1 > 0,\ \lambda_2 = \cdots = 0 \quad \Rightarrow$ **1D**

$\lambda_1, \lambda_2 > 0,\ \lambda_3 = \cdots = 0 \quad \Rightarrow$ **2D**

any $\lambda_k < 0 \quad \Rightarrow$ **not Euclidean**

# Classical MDS: embedding dimension

## Classical MDS

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**Position density** $\rho_n(x)$ $(N = 6)$:

| | $\lambda_1$ | $\lambda_2$ | $\cdots$ | $\lambda_6$ | $D_{\min}$ | $D_{\text{eff}}$ |
|---|---|---|---|---|---|---|
| $p=1$ | 1.602 | 0.055 | $\cdots$ | $-0.016$ | – | – |
| $p=2$ | 1.926 | 0.018 | $\cdots$ | 0 | 5 | 1.055 |
| $p=3$ | 2.129 | 0.023 | $\cdots$ | 0 | 5 | 1.045 |
| $p=4$ | 2.267 | 0.024 | $\cdots$ | 0 | 5 | 1.036 |

**Husimi** $p_n(r)$ $(N = 6)$:

| | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\cdots$ | $D_{\min}$ | $D_{\text{eff}}$ |
|---|---|---|---|---|---|---|
| $p=1$ | 1.572 | 0. | 0. | $\cdots$ | **1** | **1.000** |
| $p=2$ | 1.574 | $3.1 \times 10^{-4}$ | $3.6 \times 10^{-7}$ | $\cdots$ | $N-1$ | 1.000 |
| $p=3$ | 1.576 | $5.3 \times 10^{-4}$ | $-6.1 \times 10^{-7}$ | $\cdots$ | – | – |
| $p=4$ | 1.578 | $6.8 \times 10^{-4}$ | $-2.0 \times 10^{-6}$ | $\cdots$ | – | – |

Only $W_1$ + Husimi: $D_{\min} = D_{\text{eff}} = 1$ *exactly*.

**Manifold hypothesis** requires $D_{\min}$ finite and independent of $N$.
Other choices: $D_{\min} = N-1$ (depends on $N$) or non-Euclidean.

$\Rightarrow W_1$ + Husimi is **uniquely selected**.

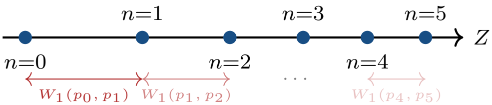
# The emergent 1D coordinate

For the harmonic oscillator Husimi distribution,

$$p_n(r) = \frac{2}{n!} r^{2n+1} e^{-r^2}, \qquad r = |\alpha|.$$

These CDFs are non-crossing ($F_n \le F_m$ for $n > m$), so $W_1 = \int_0^\infty (F_m - F_n)\, dr$.
This equals the difference of means, giving

$$W_1(p_n, p_m) = Z(n) - Z(m), \quad Z(n) = \frac{\Gamma(n+3/2)}{\Gamma(n+1)} - \frac{\sqrt{\pi}}{2}.$$

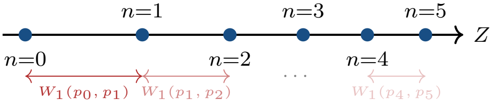



## Physical meaning

$Z(n) \sim \sqrt{n} \sim \sqrt{E_n/\omega}$ (for $n \gg 1$): the Wasserstein coordinate is an energy/radial coordinate.

# The emergent coordinate

$$W_1(p_n, p_m) = Z(n) - Z(m), \qquad Z(n) = \langle r \rangle_{p_n} - \langle r \rangle_{p_0} = \frac{\Gamma(n+3/2)}{\Gamma(n+1)} - \frac{\sqrt{\pi}}{2}$$




## Emergent geometry

Extend $n$ to a continuous coordinate $\zeta \in \mathbb{R}$ and require

$$Z(n) - Z(m) = \int_m^n \sqrt{g_{\zeta\zeta}(\zeta)}\, d\zeta \qquad \Rightarrow \qquad g_{\zeta\zeta} = \left[\frac{d}{d\zeta}\frac{\Gamma(\zeta+3/2)}{\Gamma(\zeta+1)}\right]^2 \sim \frac{1}{4\zeta} \qquad (\zeta \gg 1)$$

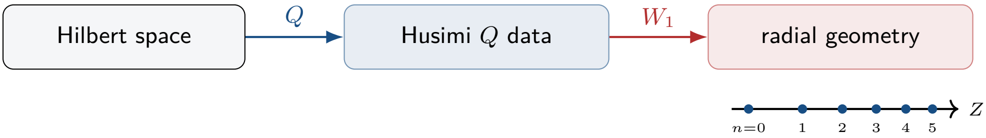
$\Rightarrow$ The Wasserstein distances define a **1D Riemannian geometry** on the state space.

# What has been achieved so far?

**Starting point**
An infinite-dimensional quantum Hilbert space.

**Selection**
Husimi $Q + W_1$: *uniquely* gives $D_{\min} = D_{\text{eff}} = 1$.

**Output**
1D energy/radial geometry; $g_{\zeta\zeta} \sim 1/(4\zeta)$.




**So far:** only a static geometry — eigenstates $|n\rangle$ are stationary; no time direction.

To get **spacetime**, we need a **trajectory** in Wasserstein space.

**Strategy:** couple the oscillator to a bath $\Rightarrow$ Lindblad dynamics $\Rightarrow$ evolving $W_1(t)$.

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# Adding time: couple the oscillator to a bath

## Lindblad master equation (zero-$T$ bath)

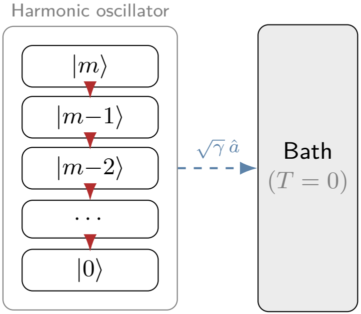
$$\frac{d\hat{\rho}}{dt} = -i\omega[\hat{n}, \hat{\rho}] + \gamma\left(\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\{\hat{n}, \hat{\rho}\}\right)$$

Jump operator $\hat{a}$: each quantum jump lowers the energy by one quantum ($|n\rangle \to |n-1\rangle$).

## Relaxation

$$\hat{\rho}(t) = \sum_k C_k(t)|k\rangle\langle k|$$

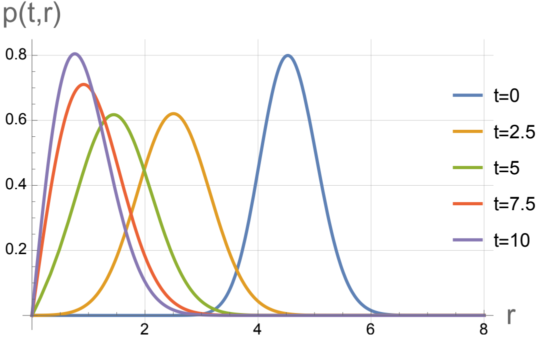
Starting from $\hat{\rho}(0) = |m\rangle\langle m|$, relaxes toward $|0\rangle\langle 0|$. Diagonal form $\Rightarrow W_1(t)$ analytically tractable.



# The distribution falls in phase space

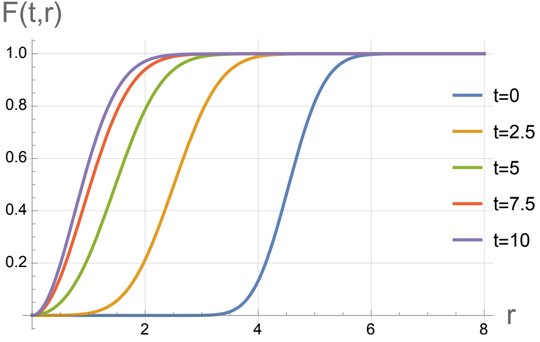
Lindblad eq.: $$\frac{d\hat\rho}{dt} = -i\omega[\hat n, \hat\rho] + \gamma\big(\hat a\hat\rho\hat a^\dagger - \tfrac12\{\hat n, \hat\rho\}\big)$$

$$\hat\rho(0) = |m\rangle\langle m| \quad\Rightarrow\quad \hat\rho(t) = \sum_k C_k(t)|k\rangle\langle k|, \quad \dot C_k = \gamma\big((k+1)C_{k+1} - k\,C_k\big)$$




Husimi distribution moves toward the ground state.

(initial state: $\hat\rho(0) = |20\rangle\langle 20|$)



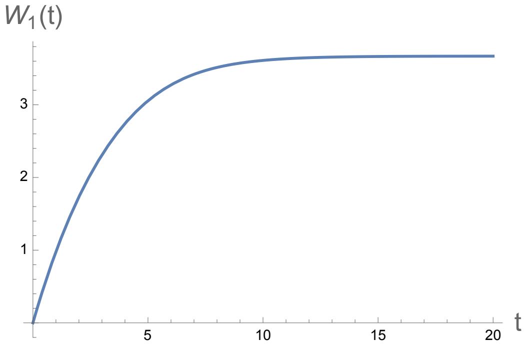

CDFs remain ordered during the motion.

# The Wasserstein trajectory

Because the CDFs stay ordered ($F_k \leq F_m$ at all $t$), $W_1$ reduces to a weighted average of coordinates $Z(k)$:

$$W_1(t) = Z(m) - \sum_{k=0}^{m} C_k(t) Z(k)$$

$$\left(C_k(t) = \binom{m}{m-k} e^{-m\gamma t}(e^{\gamma t} - 1)^{m-k}\right)$$




Two important regimes:

- **Early time** ($t \ll \gamma^{-1}$): linear growth

$$W_1(t) \sim \tfrac{1}{2}\sqrt{m}\,\gamma t$$

- **Late time** ($t \gg \gamma^{-1}$): exponential decay

$$W_1(t) = Z(m) - \tfrac{\sqrt{\pi}}{4} m\, e^{-\gamma t} + \cdots$$

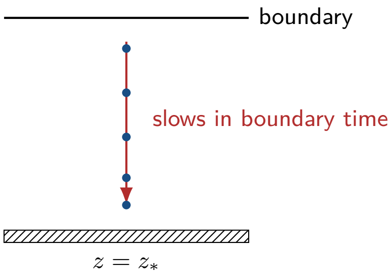
# Why this looks like a horizon

The Wasserstein distance approaches a final value exponentially.

That is exactly how a falling probe appears to approach a black-hole horizon in boundary time:

$$z_* - z(t) \propto e^{-\kappa t}.$$

Dissipation in the boundary state space maps to a particle frozen at the emergent horizon.



# Metric reconstruction from a null trajectory

**Identifications:** (i) $z = W_1(t)$ is the radial coordinate; (ii) the trajectory is **null** in the 2D ansatz

$$ds^2 = \sigma(z)\left[-f(z)dt^2 + \frac{dz^2}{f(z)}\right].$$

Null condition $f(z(t)) = dz/dt$, combined with late-time $f(z) \sim \gamma(z_* - z)$, gives

$$ds^2 \simeq \sigma(z)\left[-\gamma(z_* - z)dt^2 + \frac{dz^2}{\gamma(z_* - z)}\right].$$

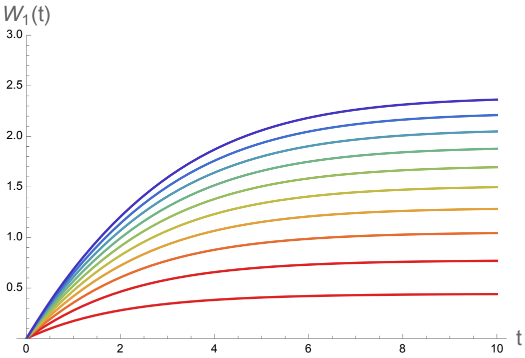
The conformal factor $\sigma(z)$ is not fixed by this analysis. Choosing $\sigma(z) = 1/z^2$ reproduces the $AdS_2$ black-hole metric.

## Emergent black-hole horizon

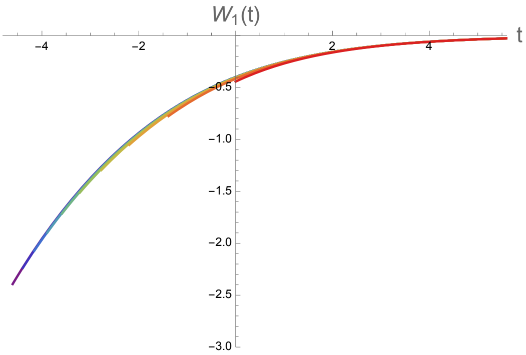
The Wasserstein spacetime has a horizon at $z = z_*$, for appropriate $\sigma$.

# A shared geometry from many initial states

**Universality check:** does the emergent geometry depend on the initial state $|m\rangle$?




$W_1(t)$ for different $m$: saturates at $z_* = Z(m)$.




After shifting $t \to t - t_0(m)$: converge onto a single master curve.

Universal trajectory $\Rightarrow$ BH-like metric $ds^2 \propto \left[-\gamma(z_* - z)dt^2 + \cdots\right]$ is a universal emergent geometry.

# Harmonic oscillator: results and scope

## Summary: what the harmonic oscillator demonstrates

- MDS dimension analysis **uniquely selects** $W_1$ + Husimi: $D_{\min} = D_{\text{eff}} = 1$ exactly
- Exact 1D coordinate $Z(n)$; emergent metric $g_{\zeta\zeta} \sim 1/(4\zeta)$
- Lindblad dynamics $\Rightarrow$ $W_1(t)$ trajectory with black-hole-like horizon at $z = z_*$
- Universal collapse across initial states $\Rightarrow$ geometry is not an artifact of one fit

**Key point:** the holographic spatial dimension $(D = 1)$ is not assumed — it is *read off* from the $W_1$ distance matrix $(D_{\min} = D_{\text{eff}} = 1)$.

## Caveats

- The conformal factor $\sigma(z)$ is undetermined — $\sigma = 1/z^2$ gives AdS$_2$, but this is an additional input.
- The null identification itself is an ansatz, not derived.
- A genuine holographic duality requires large-$N$, a gravity action, and partition function matching — none of which are present here.

$\Rightarrow$ To test whether the Wasserstein geometry is compatible with an *established* holographic dual, we turn to **SYK**.

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# Why look at SYK next?

The oscillator is analytically clean, but not a standard gravity dual.

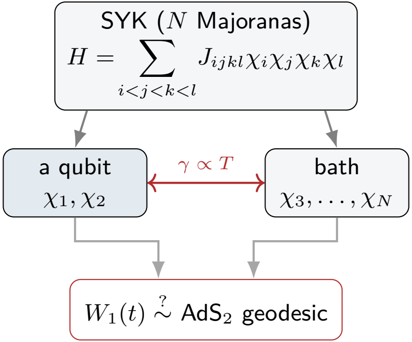
Consistency check: does the $W_1$ geometry reproduce the known $\mathrm{AdS}_2$ black-hole structure?

## SYK model

$N$ Majorana fermions in 0+1d QM, random all-to-all quartic coupling.
Low-energy large-$N$ limit is dual to JT gravity / nearly-AdS$_2$.

## Setup

Select 2 Majoranas as a qubit; the remaining $N-2$ act as a bath.
Qubit evolves by Lindblad dynamics with $\gamma \propto T$.



# A two-Majorana subsystem

## A qubit in SYK

Combine $\chi_1, \chi_2$ into a complex fermion:

$$\hat{c} = \chi_1 + i\chi_2, \quad \hat{c}^\dagger = \chi_1 - i\chi_2$$

$\Rightarrow$ fermionic algebra: $\{\hat{c}, \hat{c}^\dagger\} = 1,\ \ \hat{c}^2 = 0.$

Number operator $\hat{n} \equiv \hat{c}^\dagger \hat{c}$ has eigenvalues $n = 0, 1$

$\Rightarrow$ **qubit** $\{|0\rangle, |1\rangle\}$; remaining Majoranas = **bath**.

## Jump operators from SYK structure

Dividing $J_{ijkl}\chi_i\chi_j\chi_k\chi_l$ into subsystem $(1,2)$ and bath $(I{\geq}3)$:

$$\begin{array}{lcl} \chi_{1,2}\chi_I\chi_J\chi_K & \Rightarrow & \hat{L}_1{=}\hat{c},\ \hat{L}_2{=}\hat{c}^\dagger \\ \chi_1\chi_2\chi_I\chi_J & \Rightarrow & \hat{L}_3{=}\hat{c}^\dagger\hat{c} \end{array}$$

Bath structure of SYK *fixes* the jump operators.

## Lindblad master equation

$\hat{H}_0 = 0$ (low energy = scale-free):

$$\Rightarrow \quad \partial_t \hat{\rho} = \sum_{i=1}^{3} \gamma_i \left( \hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \tfrac{1}{2}\{\hat{L}_i^\dagger \hat{L}_i, \hat{\rho}\} \right)$$

$$\Rightarrow \quad \hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix},$$

$$\begin{cases} \partial_t \rho_{11} & = -\gamma_1 \rho_{11} + \gamma_2 \rho_{00} \\ \partial_t \rho_{00} & = \quad \gamma_1 \rho_{11} - \gamma_2 \rho_{00} \\ \partial_t \rho_{10,01} & = -\tfrac{1}{2}(\gamma_1{+}\gamma_2{+}\gamma_3)\, \rho_{10,01} \end{cases}$$

- Off-diagonal $\rho_{10}, \rho_{01}$ decohere and decay $\Rightarrow$ work with diagonal $(\rho_{11}, \rho_{00})$
- No preferred orientation $\Rightarrow \gamma_1 = \gamma_2 \equiv \gamma$

# Lindblad relaxation and Wasserstein distance

With $\gamma_1 = \gamma_2 \equiv \gamma$ and $\rho_{00} = 1 - \rho_{11}$:

$$\partial_t \rho_{11} = -\gamma \rho_{11} + \gamma \rho_{00} = \gamma(1 - 2\rho_{11})$$

Time evolution from $\hat{\rho}(0) = \begin{pmatrix} \rho_{11}(0) & 0 \\ 0 & 1 - \rho_{11}(0) \end{pmatrix}$

$$\Rightarrow \quad \begin{cases} \rho_{11}(t) = \left(\rho_{11}(0) - \frac{1}{2}\right) e^{-2\gamma t} + \frac{1}{2} \\ \rho_{00}(t) = \left(\frac{1}{2} - \rho_{11}(0)\right) e^{-2\gamma t} + \frac{1}{2} \end{cases}$$

**Coupling $\sim$ Temperature:**
$T$ is the only low-energy scale of SYK $\Rightarrow \; \gamma \propto T$.

## From density matrix to $W_p$

Identify $(\rho_{11}(t), \rho_{00}(t))$ as a probability distribution on a **two-point space**.

The $p$-Wasserstein distance then evolves as

$$W_p(t) = d \left| \rho_{11}(0) - \frac{1}{2} \right|^{1/p} \left(1 - e^{-2\gamma t}\right)^{1/p}$$

With $\gamma = cT/2$ & $p = 1$,

$$W_1(t) \propto 1 - e^{-cTt}$$

# Matching the AdS$_2$ null geodesic

## Wasserstein trajectory from SYK

From SYK qubit Lindblad dynamics:

$$W_1(t) \propto 1 - e^{-cTt}$$

## Gravity dual of SYK: AdS$_2$ BH

SYK $\leftrightarrow$ JT gravity / nearly-AdS$_2$ $(z_* = 1/4\pi T)$

$$ds^2 = \frac{1}{z^2}\left[-\frac{z_* - z}{z_*}dt^2 + \frac{z_*}{z_* - z}dz^2\right]$$

Null trajectory: $z_{\text{null}}(t) \propto 1 - e^{-4\pi T t}$

## $W_1$ matches the AdS$_2$ null trajectory

With the identification $W_1 = z$, both results have the **same functional form** (if $c = 4\pi$).

$\Rightarrow W_1$ traces the null geodesic of the known AdS$_2$ dual of SYK

It confirms that the Wasserstein approach extends to a model with a genuine holographic dual.

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# $W_1$ is a generalized Krylov complexity

$W_1(t)$ derived for the Lindblad harmonic oscillator strongly resembles **the Krylov complexity** $K(t)$:

**Wasserstein $W_1$**

$$W_1(t) = \sum_{k=0}^{m} C_k(t)\, W_1(p_k, p_m)$$

$C_k(t)$ : occupation probability of $|k\rangle$

$W_1(p_k, p_m)$ : $W_1$ distance between $|k\rangle$ & $|m\rangle$

**Krylov complexity**

$$K(t) = \sum_n n\, |\phi_n(t)|^2$$

$n$ : integer steps along Krylov chain

$|\phi_n(t)|^2$ : probability in Krylov basis, $\mathcal{O}(t) = \sum_n \phi_n(t)\mathcal{O}_n$

| | **Krylov** | **Wasserstein $W_1$** |
|---|---|---|
| "distance" | step index $n$ | $W_1(p_k, p_m)$ |
| probability weight | $\|\phi_n(t)\|^2$ | $C_k(t)$ |
| physical picture | operator spreading | distributional motion |

$W_1$ is a generalized Krylov complexity — and also a genuine metric between states.

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| | **Krylov** | **Wasserstein** $W_1$ |
|---|---|---|
| "distance" | $m - k$ | $W_1(p_k, p_m)$ |
| genuine metric? | $\times$ | $\checkmark$ |
| state-independent? | $\times$ | $\checkmark$ |

Krylov complexity *depends on initial operator* — $K(t)$ for different initial states cannot be compared.

$W_1$ is a genuine metric on states; its geometry reflects **intrinsic property of the system**.

# Why the metric structure matters for holography

To construct emergent geometry from quantum states, a genuine metric on the space of states is essential.

Krylov complexity, despite its holographic motivation, **cannot serve this role**:

- it measures spread from one fixed reference state,
- it changes when the initial state changes,
- it is not a distance between states — no triangle inequality.

$W_1$ has all the properties needed to **construct geometry**:

- genuine metric $\Rightarrow$ defines distances on state space,
- state-independent $\Rightarrow$ the geometry is a property of the system, not of one initial condition,
- dynamics-independent $\Rightarrow$ dynamics under different settings all live in the **same** $W_1$ space and can be directly compared.

**Upshot**

$W_1$ is the first complexity-type quantity from which an emergent holographic geometry can be *directly read off*

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# Summary

## What was demonstrated (harmonic oscillator + SYK)

1. **MDS analysis uniquely selects** $W_1$+Husimi: $D_{\min} = D_{\text{eff}} = 1$ exactly. The holographic spatial dimension is *read off* from state distances, not assumed.
2. **Exact emergent geometry:** 1D coordinate $Z(n)$ and Riemannian metric $g_{\zeta\zeta} \sim 1/(4\zeta)$.
3. **Lindblad dynamics gives spacetime:** $W_1(t)$ traces a null geodesic in a black-hole geometry, universally across different initial states.
4. **SYK consistency check:** $W_1(t) \propto 1 - e^{-cTt}$ matches the $AdS_2$ Schwarzschild null trajectory exactly.
5. $W_1$ **= generalized Krylov complexity,** with the additional property of being a metric — enabling state-independent holographic geometry.

## Lesson

The Wasserstein space of quantum state distributions carries the structure of holographic spacetime — coordinate, metric, and horizon — derived from state distances alone under mild assumptions

# Open questions

## Towards realistic holography

- Does the MDS analysis select $D = d+1$ for a holographic CFT$_d$? *(main open problem)*
- What is the natural state representation for non-phase-space systems, e.g. SYK at finite $N$?
- Does the universal collapse of $W_1(t)$ trajectories persist in larger models?

## Generalizing the framework

- The conformal factor $\sigma(z)$ is undetermined: can it be fixed from boundary data?
- The suitability of Husimi Q representation: how about other state families (e.g. thermal states, ...)?
- Can quantum Wasserstein distances (on density matrices directly) replace classical OT of Husimi Q?
- Any relation to bulk action (CA) or extremal volume (CV) proposals?

The proof of concept for well-controlled models is in hand.
Does it extend to a large-$N$ holographic CFT with an established gravity dual?
Any other usages of the optimal transport and Wasserstein distance in QFT?