AI methods for the analytic computation of Feynman integrals

SIMIS-APCTP meetig
August 27th , 2026

Yang Zhang and Yuanche Liu
University of Science and Technology of China

Outline

Analytic Feynman integral frontier — Yang Zhang

Example 1: 2loop 6point Feynman integrals

Example 2: 3loop 5point Feynman integrals

"Learning" Feynman Integrals — Yuanche Liu

AI method 1: Machine Learning for Symbol Letters

AI method 2: AI agent for finding pure integral basis

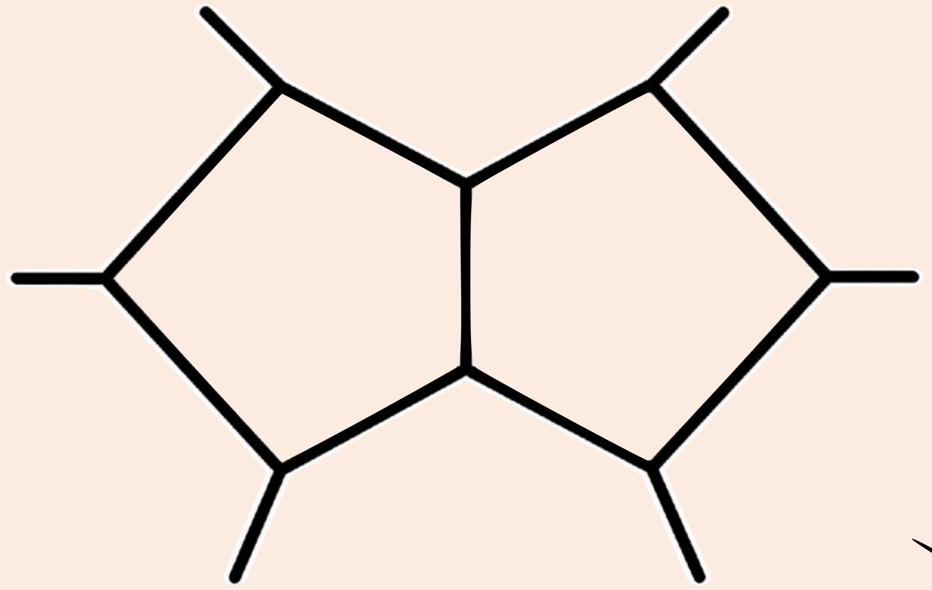
Based on

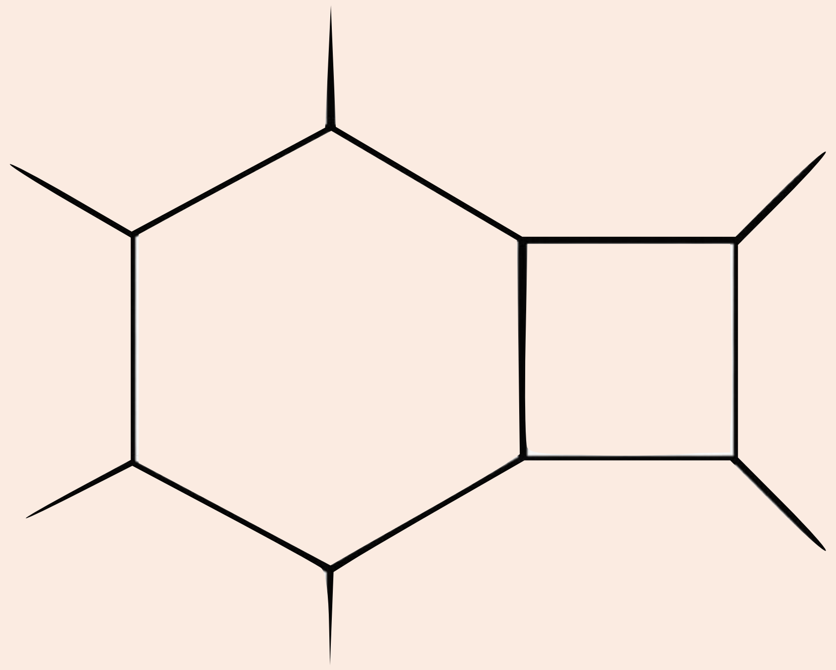
Feynman integral Integral evaluation

Henn

Matijašić

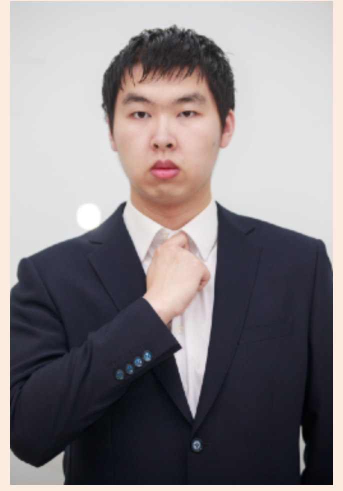
Miczajka





✓ analytic computation of all 2loop 6point planar massless integrals

Peraro



Liu

Xu

Yang

Henn, Matijašić, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

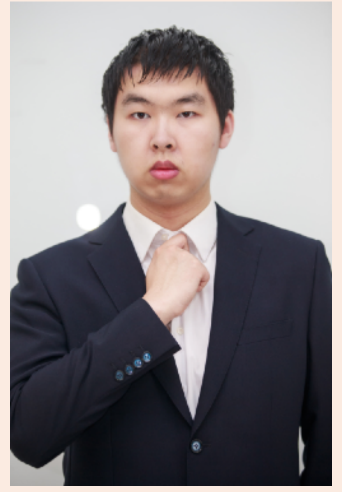
Henn, Matijašić, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

Liu, Matijasic, Peraro, Xu, Yang, YZ, arXiv. 2603.16831, JHEP 07 (2026) 223

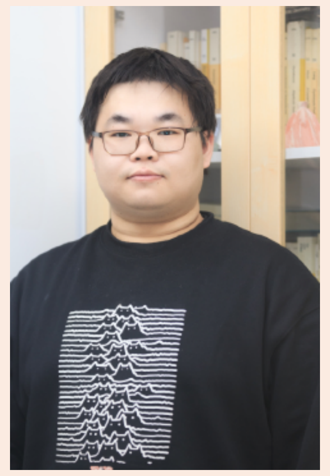
Based on

Feynman integral Integral evaluation

Xu



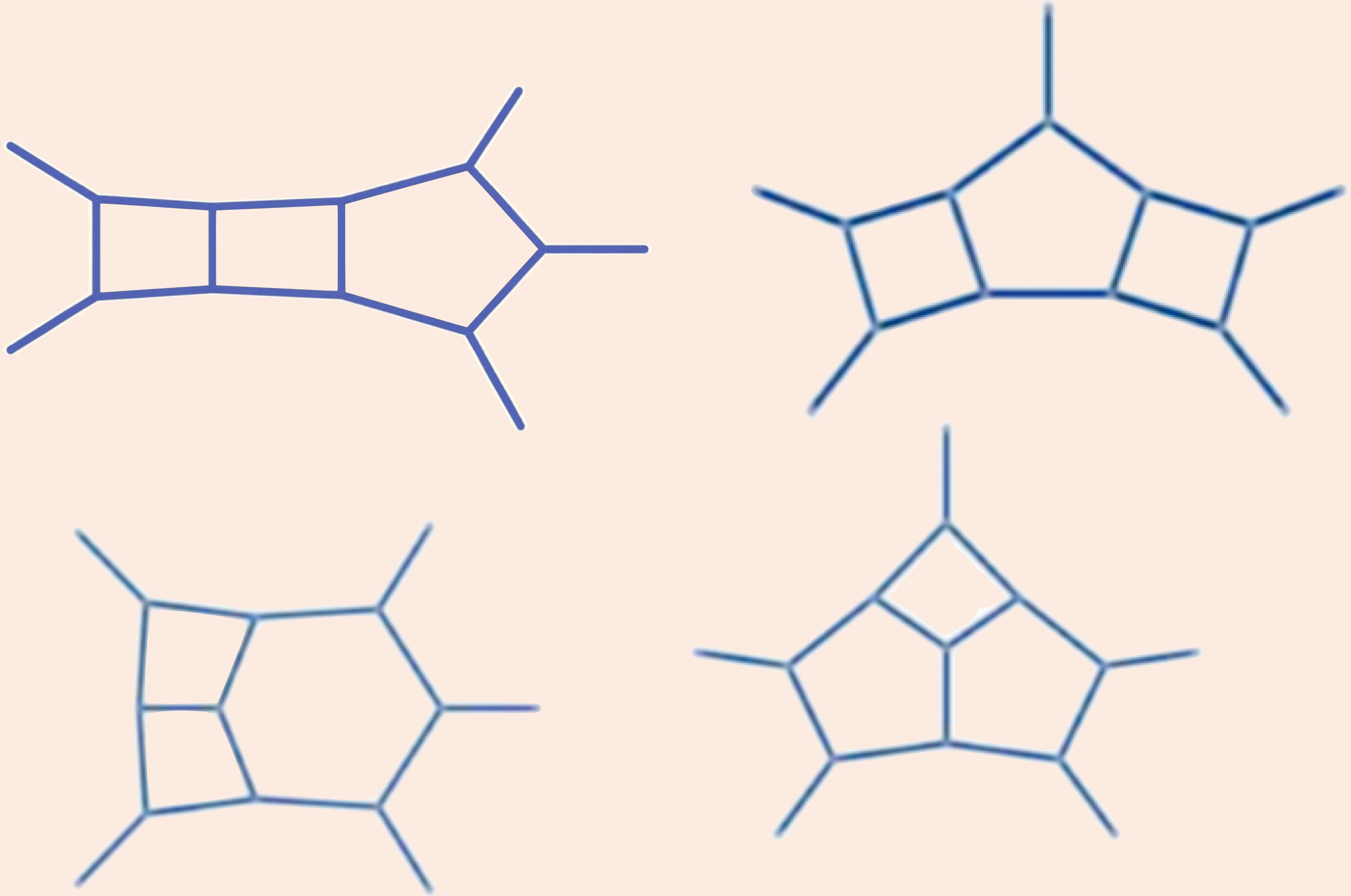
Liu



Wu

Xu

Wu



Chicherin

Zhang

Matijašić

Miczajka

✓ analytic computation of all 3loop 5point planar massless integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 (Editors' Suggestion)

Chicherin, Wu, Wu, Xu, Zhang, YZ, arXiv: 2512.17330, Phys. Rev. Lett. accepted

Based on package development

NeatIBP 1.0, a package generating small-size integration-by-parts relations for Feynman integrals
Wu, Boehm, Ma, Xu, YZ, Comput. Phys. Commun. 295 (2024), 108999

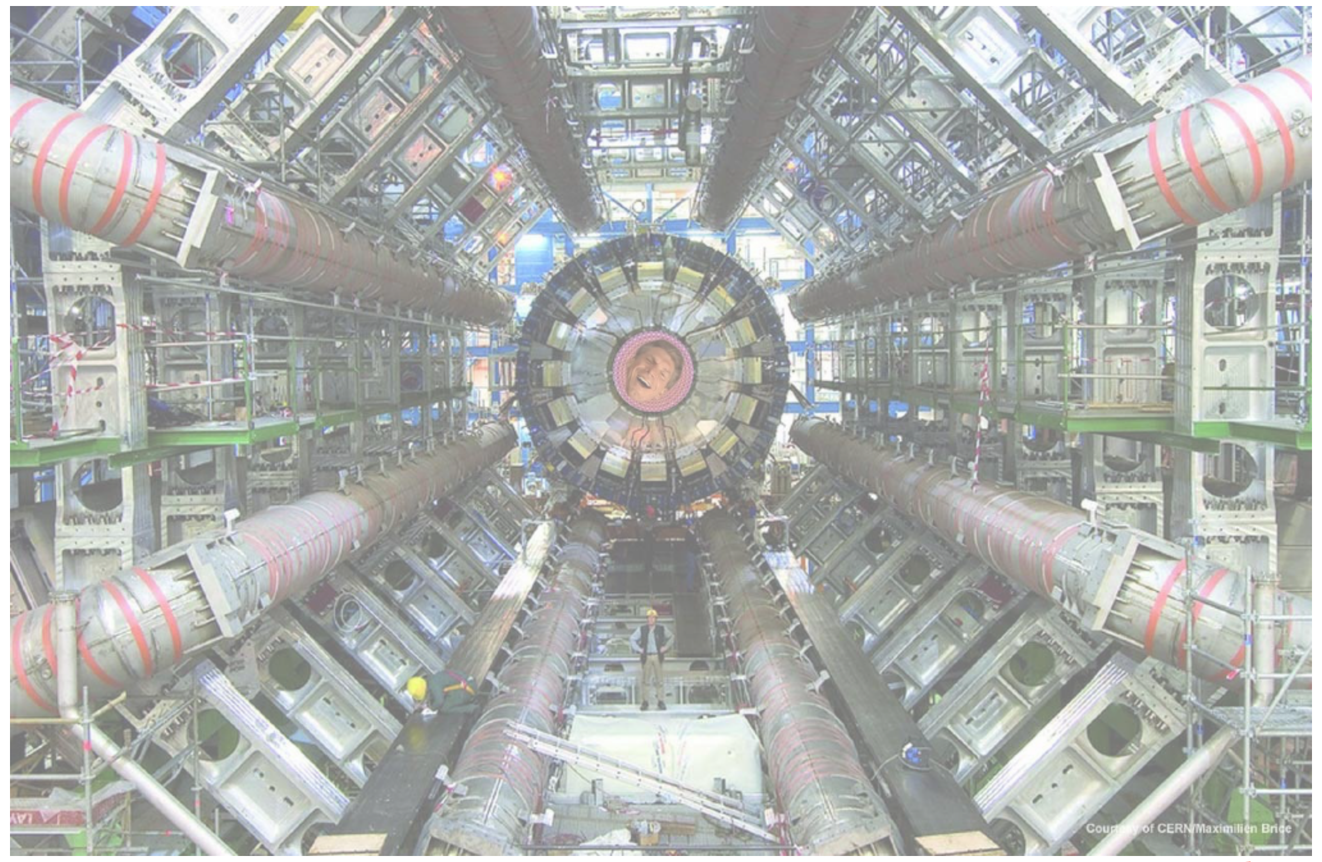
Performing integration-by-parts reductions using **NeatIBP 1.1** + Kira
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

pfd-parallel, *a Singular/GPI-Space package for massively parallel multivariate partial fractioning*
Bendle, Boehm, Heymann Ma, Rahn, Ristau, Wittmann, Wu, Xu, YZ, Comput.Phys.Commun. 294 (2024) 108942

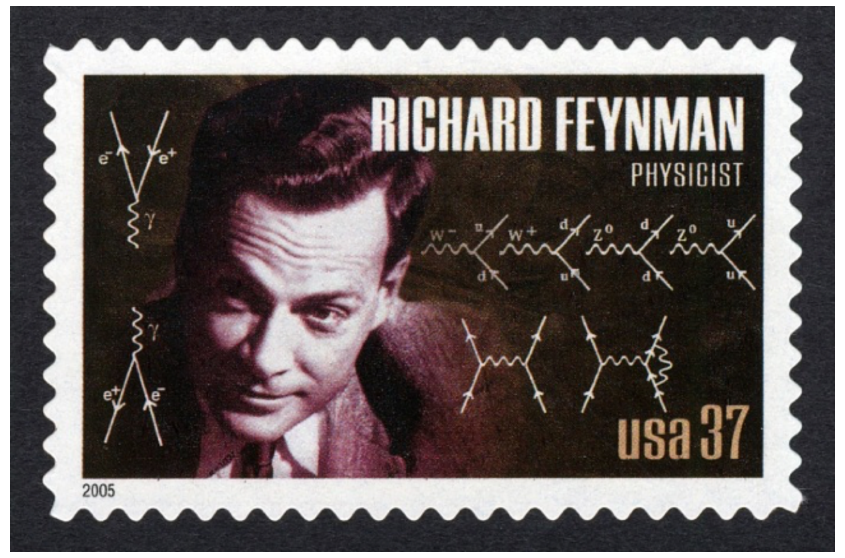
CHESS: CHEbyshev pSeudo-Spectral transport for Feynman integral differential equations

Liu, YZ, arXiv: 2606.26691

Introduction



(CERN art image for supergravity)



Feynman integrals

Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

for instance,
Driesse, Jakobsen , Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch
Nature 641 (2025) 8063, 603-607

Formal theories

N=8 supergravity UV finiteness

Gravitational wave template computations

Why *analytic* multi-loop Feynman integrals?

- Historically, analytic results may lead to a lot of phenomenology applications
- Theoretical aspects of quantum field theory

  for example: 2loop spacelike splitting amplitude, factorization violation

  Henn, Ma, Xu, Yan, YZ, Zhu, Phys.Rev.D 112 (2025) 7, 076003

  Buccioni, Fang, Yan arXiv: 2603.27123

- Quantum field theory computation of gravitational wave
- Function space bootstrap for physical observable

  2loop hexagonal Wilson loop + Lagrangian Carrôlo, Chicherin, Henn, Yang, YZ, JHEP 07 (2025) 214

  2loop six-point QCD amplitudes, weight4, Carrôlo, Chicherin, Henn, Yang, YZ, *Phys.Rev.Lett.* 136 (2026) 18, 181602

  3loop pentagonal Wilson loop + Lagrangian Chicherin, Henn, Xu, Zhang, YZ arxiv: 2512.17881

Analytic Feynman integrals to phenomenology applications

Two-loop Five-point Massless integrals

Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia, 2019
Abreu, Dixon, Herrmann, Page, Zeng, 2019

$pp \to \gamma\gamma\gamma$ Abreu, Page, Pascual, Sonikov 2020
Chawdhry, Czakon, Mitov, Poncelet 2021
Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov 2023

$pp \to \gamma\gamma j$ Agarwal, Buccioni, von Manteuffel, Tancredi 2021
Chawdhry, Czakon, Mitov, Poncelet 2021
Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia 2021
Buccioni, Chen, Feng, Gehrmann, Huss, Marcoli 2025

$pp \to \gamma jj$ Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia 2023

$pp \to jjj$ Abreu, Febres Cordero, Ita, Page, Sotnikov 2021
De Laurentis, Ita, Klinkert, Sotnikov 2023
Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi 2023
De Laurentis, Ita, Sotnikov 2023

… …

Two-loop Five-point integrals with one external mass

Abreu, Ita, Moriello, Page, Tschernow, 2020
Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 2022
Abreu, Ita, Page, Tschernow 2022
Chicherin, Sotnikov, Zoia 2022
Abreu, Chicherin, Page, Sotnikov, Tschernow, Zoia 2024

$pp \to Wbb$ Badger, Hartanto, Zoia 2021 Hartanto, Poncelet, Popescu, Zoia 2022

$pp \to Wjj$ Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 2022
De Laurentis, Ita, Page, Sotnikov 2025

$pp \to Hbb$ Badger, Hartanto, Krys, Zoia 2021
Badger, Hartanto, Poncelet, Wu, *YZ*, Zoia 2024

$pp \to W\gamma j$ Badger, Hartanto, Krys, Zoia 2022

$pp \to W/Zbb$ Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini 2022
Mazzitelli, Sotnikov, Wiesemann 2024

$pp \to W\gamma\gamma$ Badger, Hartanto, Wu, *YZ*, Zoia 2024

$pp \to ttH$ Wang, Xie, Yang, Ye 2024

$pp \to tt\gamma$ Wang, Yang 2025

…

adapt Matteo Becchetti's slide in HOCtools-II-2025

Analytic Feynman integral and the methodology

Status of analytic Feynman integrals

with dimensional regulation

Planar massless → nonplanar massless → massive

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results		?	?
Three-loop	some results		?	?	?
Four-loop	some results	?	?	?	?

Status of analytic Feynman integrals

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Planar massless → nonplanar massless → massive

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Frontier

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with dimensional regulation

Planar massless → nonplanar massless → massive

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results		?	?
Three-loop	some results		?	?	?
Four-loop	some results	?	?	?	?

Frontier

Liu, Matijasic, Miczajka, Xu, Xu, YZ,
Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion
Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv Phys. Rev. Lett. accepted

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, PRL. 135, 031601
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027
Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056
Abreu, Monni, Page, Usovitsch JHEP 06 (2025) 112

Feynman integrals: analytic functions

in terms of

Goncharov polylogarithm function

$$G(\mathbf{0}_k;z)=\frac{1}{k!}(\log z)^k,\quad G(a_1,\dots,a_k;z)=\int_0^z\frac{dt}{t-a_1}G(a_2,\dots a_k;t)$$

arguments related to Letters, algebraic function of kinematics

well studied function with Hopf algebra structure

elliptic polylogarithm

$$\widetilde{\Gamma}\left(\begin{matrix} n_1, & \dots, & n_r \\ a_1, & \dots, & a_r\end{matrix};\xi\middle|\tau\right):=\int_0^\xi dt\,g^{(n_1)}(t-a_1\mid\tau)\,\widetilde{\Gamma}\left(\begin{matrix} n_2, & \dots, & n_r \\ a_2, & \dots, & a_r\end{matrix};t\middle|\tau\right),\qquad \widetilde{\Gamma}(;\xi|\tau)=1.$$

hyperelliptic polylogarithm

$$\widetilde{\Gamma}\left(\begin{matrix} i_{n_1}, & \dots, & i_{n_k} \\ x_1, & \dots, & x_k\end{matrix};z\middle|G\right):=\int_{z_0}^z\omega_{i_{n_1}}(t,x_1\mid G)\,\widetilde{\Gamma}\left(\begin{matrix} i_{n_2}, & \dots, & i_{n_k} \\ x_2, & \dots, & x_k\end{matrix};t\middle|G\right),\qquad \widetilde{\Gamma}(;z|G):=1.$$

G is Schottky group uniformizing the hyperelliptic curve

Calabi-Yau periods

$$\Pi_\alpha(z)=\int_{\Gamma_\alpha(z)}\Omega(z),\qquad \Gamma_\alpha(z)\in H_n(X_z,\mathbb{Z}).$$

...

Baune, Broedel, Im, Lisitsyn, Zerbini
J. Phys. A: Math. Theor. 57 445202 (2024)

Canonical Differential Equation

Henn 2013

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$

Existence (beyond the polylogarithm) and general algorithm to obtain canonical differential equation

ϵ-collaboration Phys.Rev.Lett. 136 (2026) 24, 241602

Symbol letters (alphabet) searching

Effortless, Matijasic, Miczajka to appear

https://github.com/antonela-matijasic/Effortless

BaikovLetter, Jiang, Liu, Xu, Yang, Phys.Lett.B 864 (2025) 139443

SOFIA Correia, Giroux, Mizera 2503.16601

…

iterative integration reorganization (to one-fold integral)

Liu, Matijasic, Miczajka, Xu, Xu, YZ, *Phys.Rev.D* 112 (2025) 1, 016021

numerical solution

DiffExp, Hidding Comput.Phys.Commun. 269 (2021) 108125

Line, Prisco, Ronca, Tramontano JHEP07(2025)219

CHESS, Liu YZ, arXiv: 2606.26691

…

$$I^{(0)} = B^{(0)}$$

$$I^{(1)} = B^{(1)} + \int_\gamma (d\tilde{A}) I^{(0)}$$

$$I^{(2)} = B^{(2)} + \int_\gamma (d\tilde{A}) I^{(1)}$$

$$\vdots$$

$$I^{(n)} = B^{(n)} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

Pure Integral Basis ?
Symbol letters ?

for Next-to-Next-to-leading order production of 4 jets, 4 photons …

2loop 6point Feynman integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett. 135 (2025) 3, 031601*

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

Liu, Matijasic, Peraro, Xu, Yang, YZ*, arXiv. 2603.16831, JHEP 07 (2026) 223*

Abreu, Monni, Page, Usovitsch, *JHEP 06 (2025) 112*

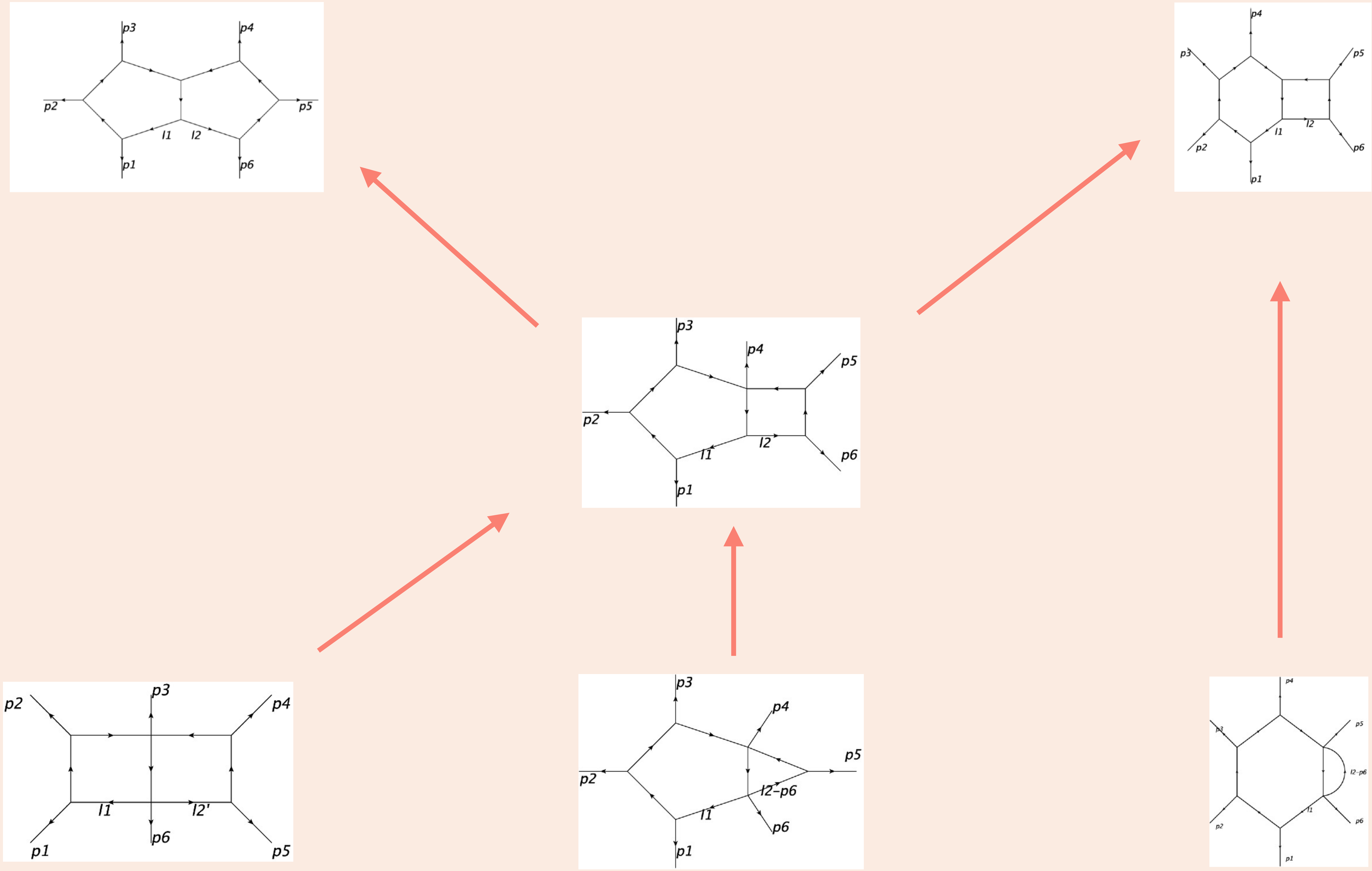
All planar massless 2loop 6point integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett. 135 (2025) 3, 031601*

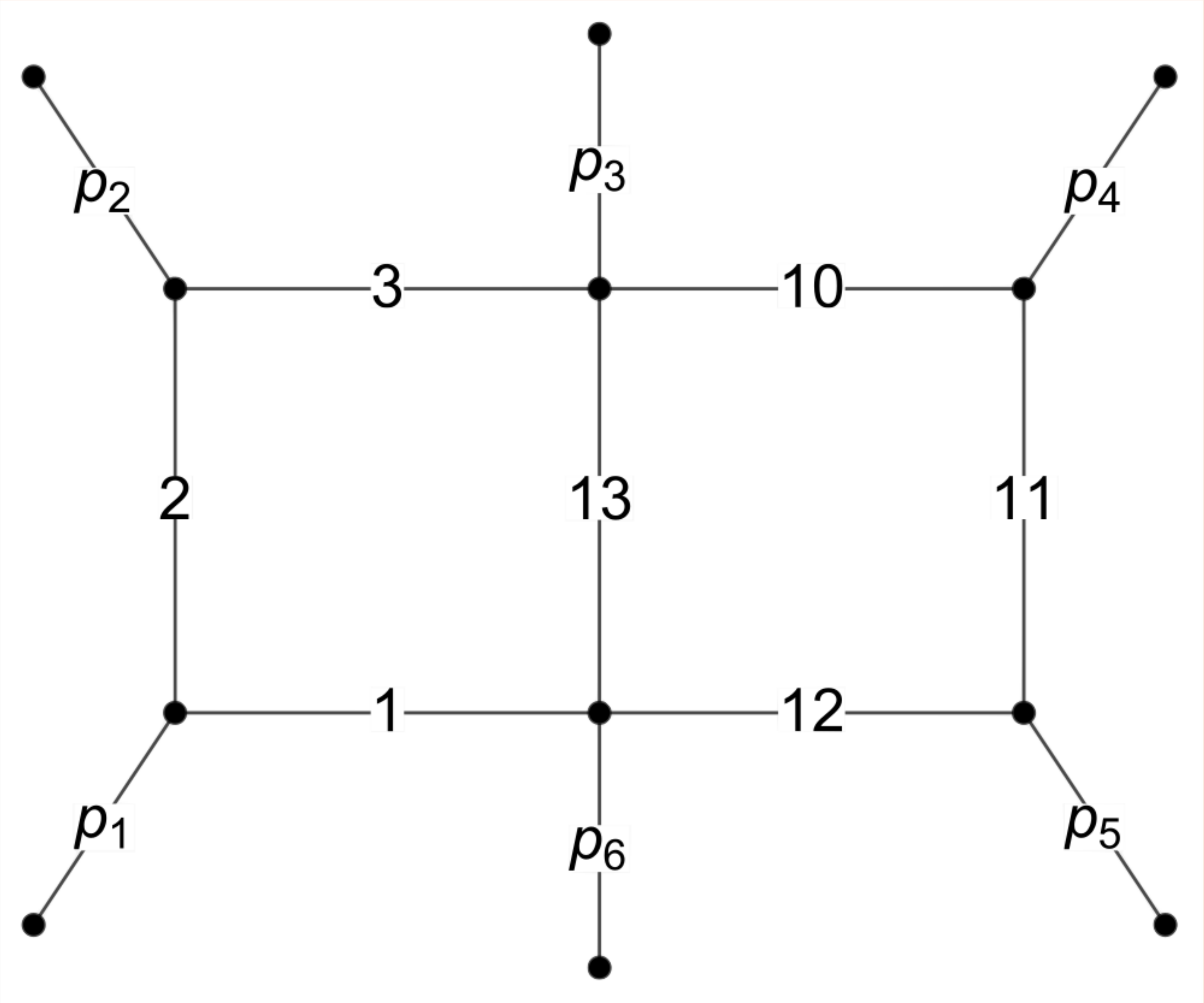
Abreu, Monni, Page, Usovitsch, JHEP 06 (2025) 112

267
pure integrals

202
pure integrals



Uniformly transcendental (UT or pure) basis determination



$$I_{\mathrm{db},i} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}}\,, \quad i = 1, \ldots, 7$$

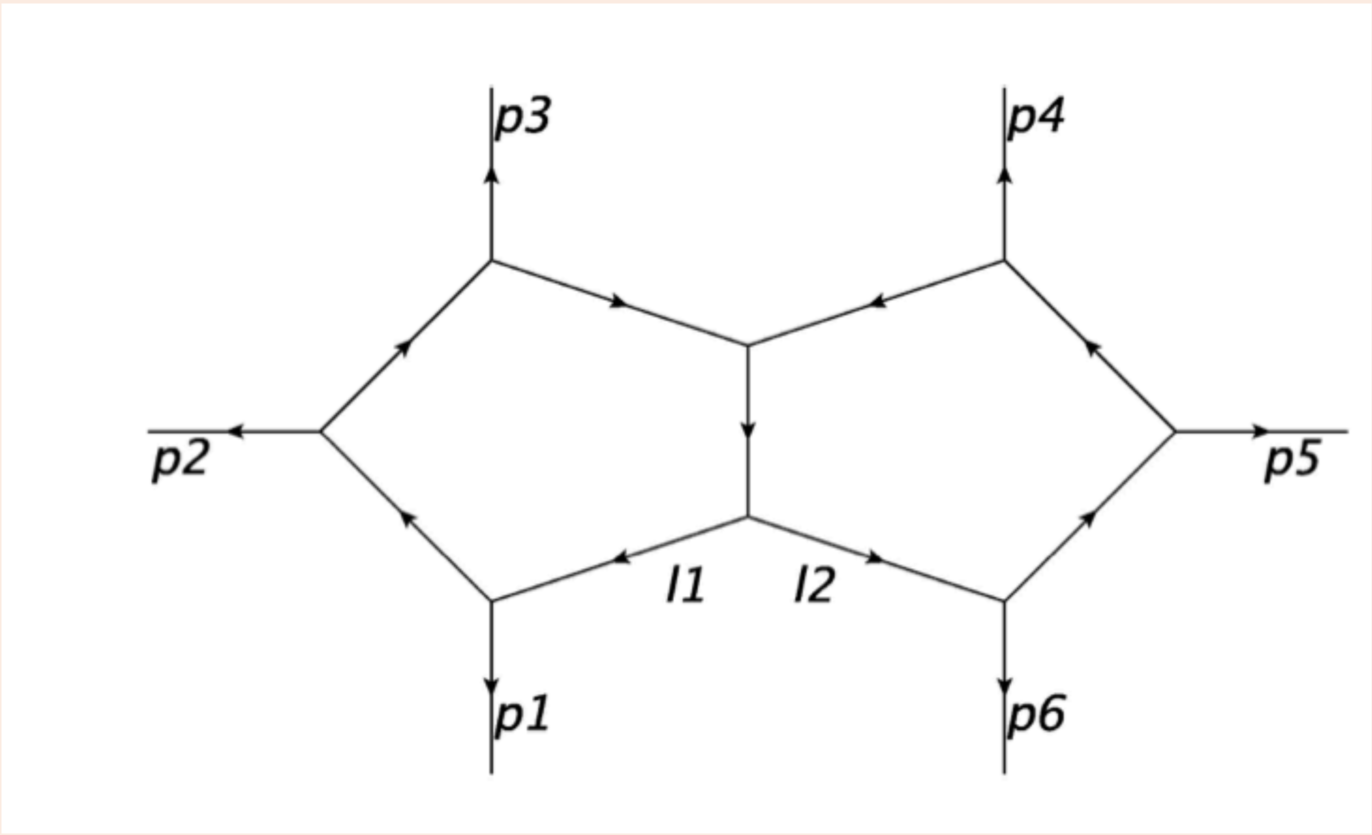
$$\begin{aligned}
N_1 &= -s_{12} s_{45} s_{156}\,,\\
N_2 &= -s_{12} s_{45} (l_1 + p_5 + p_6)^2\,,\\
N_3 &= \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix},\\
N_4 &= \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2 - p_6 & p_5 & p_4 & p_1 + p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix},\\
N_5 &= -\frac{1}{4} \frac{\epsilon_{1245}}{G(1,2,5,6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix},\\
N_6 &= \frac{1}{8} G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2 - p_6 & p_4 & p_5 \end{pmatrix} + \frac{D_2 D_{11} (s_{123} + s_{126})}{8}\,,\\
N_7 &= -\frac{1}{2\epsilon} \frac{\Delta_6}{G(1,2,4,5) D_{13}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}.
\end{aligned}$$

Chiral numerator
(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)
/ Gram determinant
correspondence

$$\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

see also Abreu, Monni, Page, Usovitsch *JHEP 06 (2025) 112* , *a systematic way of finding pure basis*

2loop 6point top sector, pure integrals



Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

5 MIs (this sector)
267 MIs (whole family)

UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

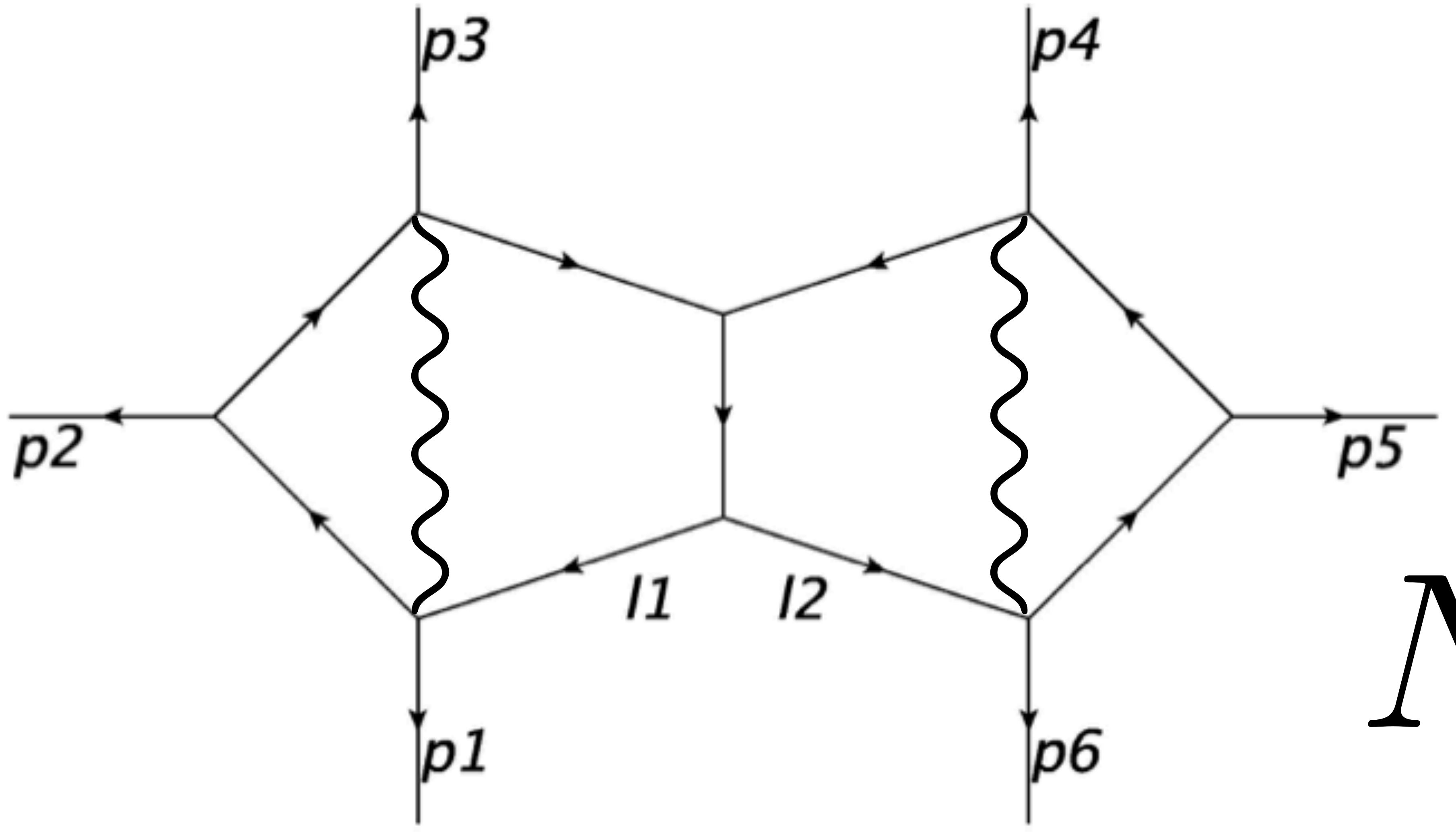
$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5 \mu_{12}}{D_1 \dots D_9}$$

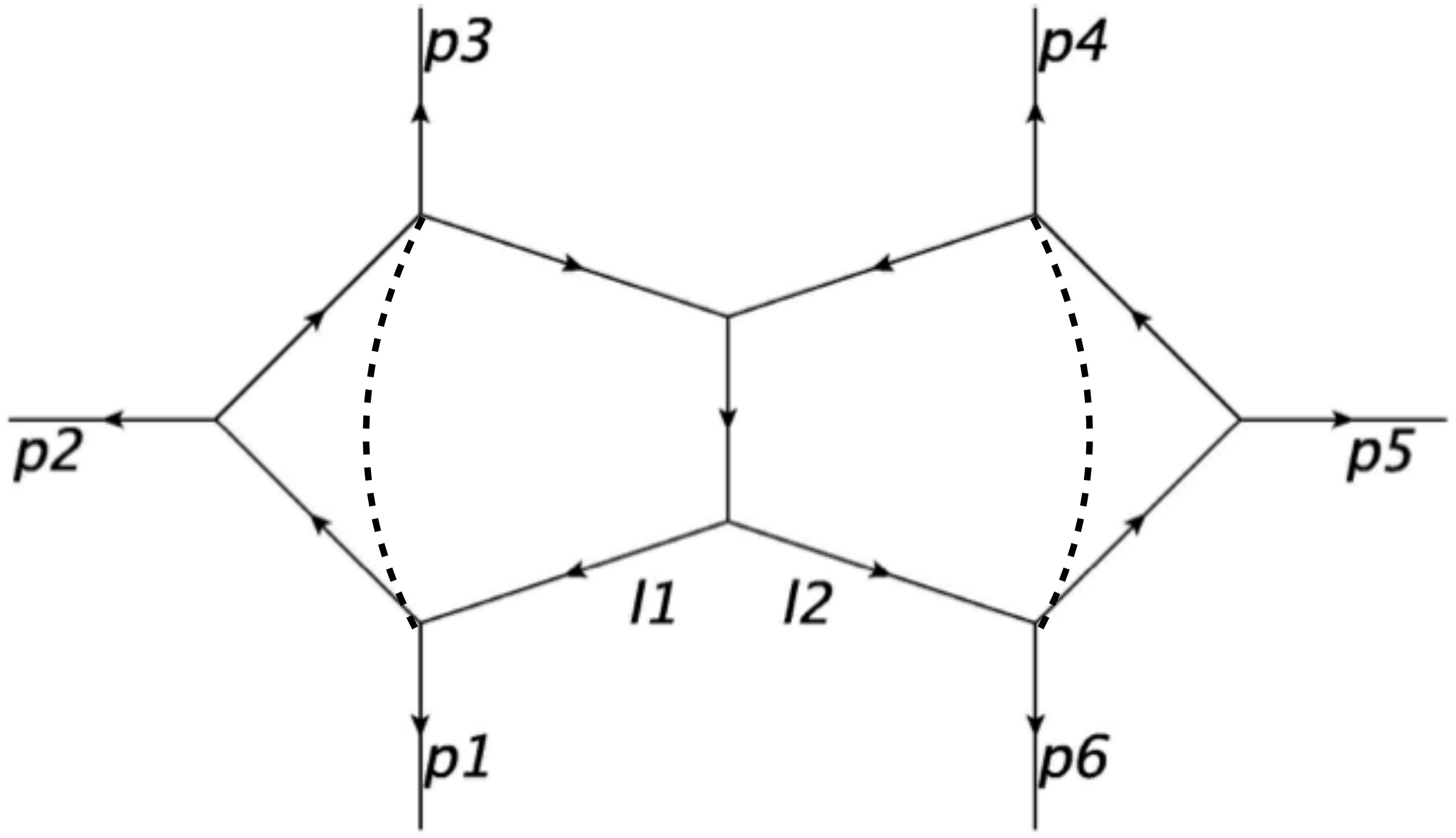
N_1, N_2, N_3 and N_4 are chiral numerators

The weight-4 parts of those integrals known from DCI studies



N_1

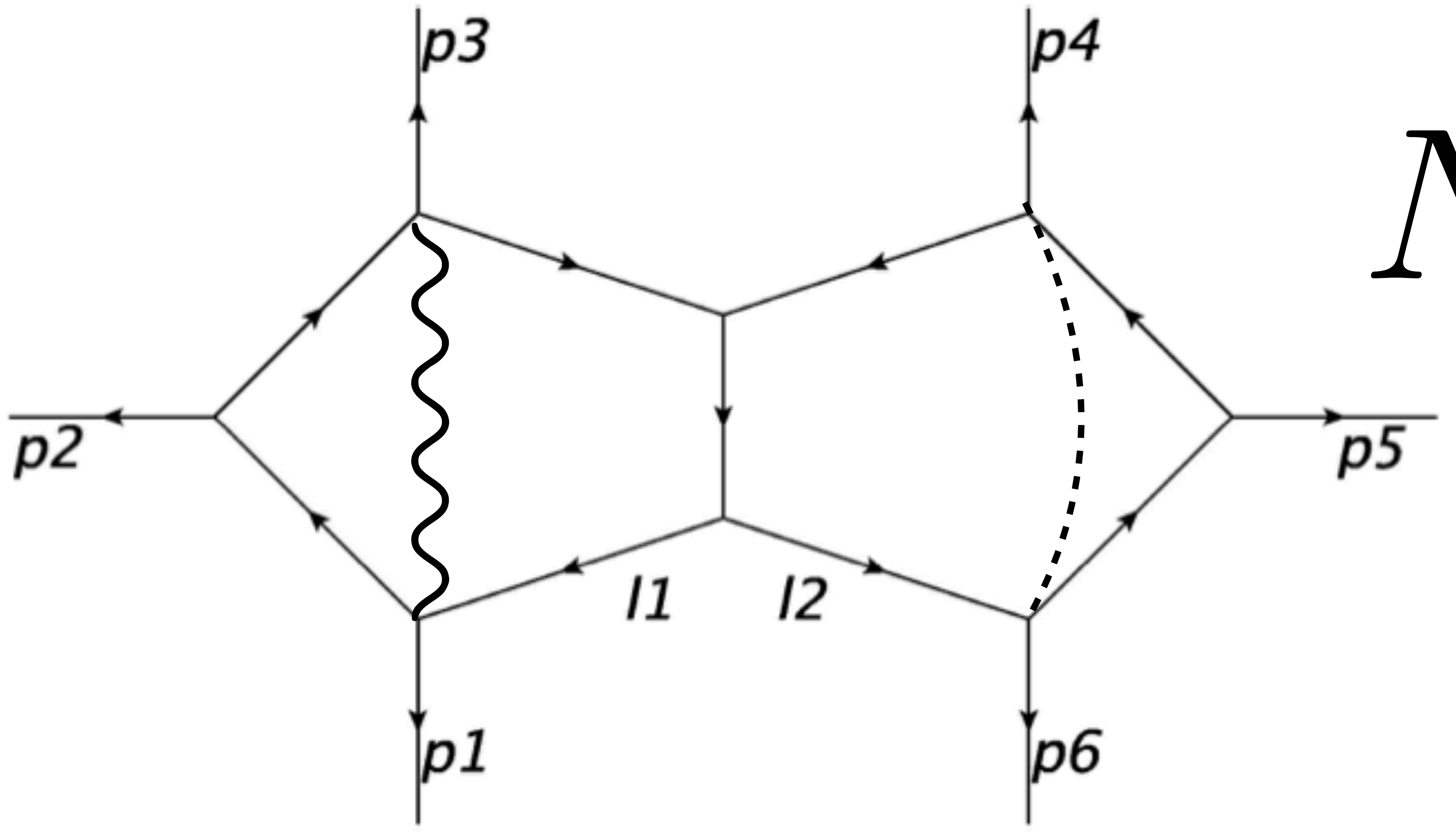
N_4



$$ut_2 = I[N_2 - N_3] = -2\tilde{\Omega}_{\text{odd}} + O(\epsilon)\,,$$
$$ut_5 = I[N_1 + N_4] = 2\Omega_{\text{even}} + O(\epsilon)$$

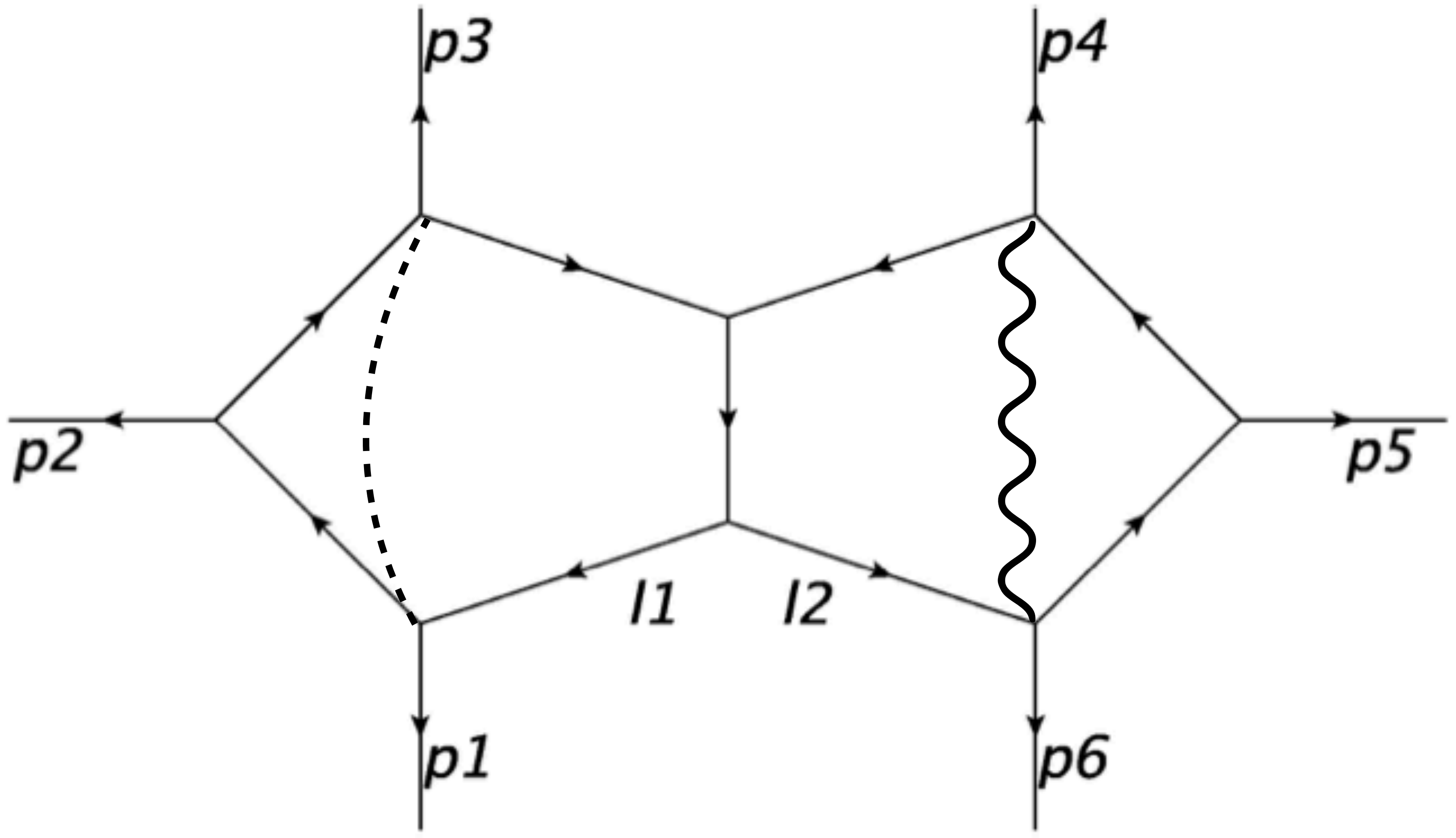
chiral-numerator integrals are finite and analytically calculated to weight-4 long time ago

Dixon, Drummond, Henn 2011

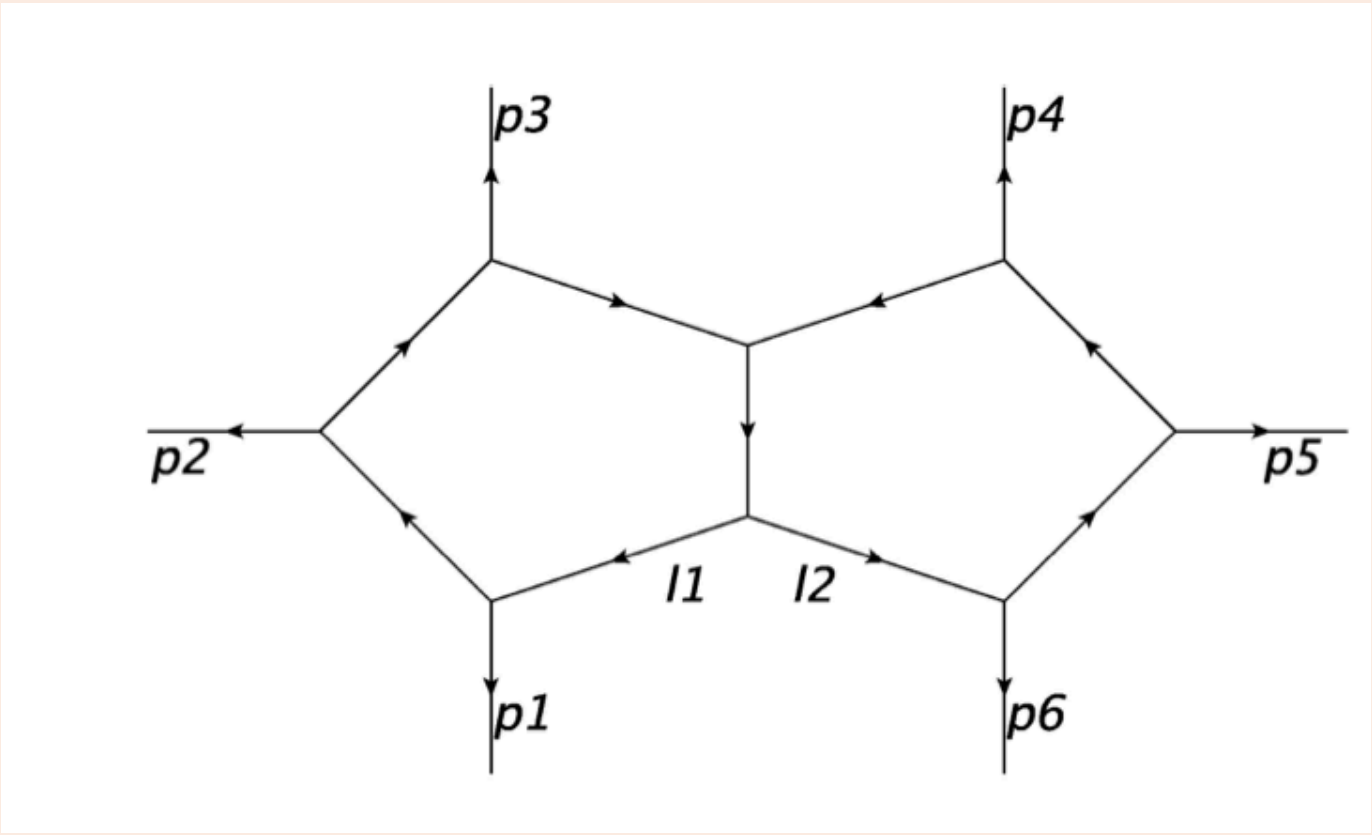


N_2

N_3



2loop 6point top sector, pure integrals



Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

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$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}}\frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}}\frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1\dots D_9}$$

$$I_3^{\text{DP-a}} = F_3\int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}}\frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}}\frac{\mu_{12}}{D_1\dots D_9} \longrightarrow \text{evanescent}$$

$$I_4^{\text{DP-a}} = F_4\epsilon^2\int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}}\frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}}\frac{1}{D_1\dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}}\frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}}\frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5\mu_{12}}{D_1\dots D_9}$$

Abreu, Monni, Page, Usovitsch *JHEP 06 (2025) 112*

Those integrals provide no independent weight-4 functions …
not needed for two-loop amplitudes to the $O(\epsilon^0)$ order

This sector has no contribution
to divergent and finite part of two-loop
gauge theory amplitudes

Alphabet, 2loop 6point massless planar integrals, up to weight 4

Even letter $F(s)$ a polynomial in Mandelstam variables
or homogeneously linear in square roots

a Feynman integral's even letters are connected to Landau singularity

Odd letter $\dfrac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

"square roots": ϵ_{ijkl}, Δ_6, , $\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

pseudo
scalar

leading
singularity
hexagon

Källin function
from massive triangle
diagrams

More
complicated
letter $\dfrac{P(s)-\sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s)+\sqrt{Q_1(s)}\sqrt{Q_2(s)}}$

Matijasic, Miczajka, to appear

Effortless
https://github.com/antonela-matijasic/Effortless

Even letter, Odd letter and the more complicated …

Cluster algebra structure

Bossinger, Drummond, Glew, Gürdoğan, arXiv: 2507.01015
Pokraka, Spradlin, Volovich, Weng, JHEP 03 (2026) 120

156 Even letters

$$s_{12}, \quad s_{123}$$
$$s_{12} - s_{123}$$
$$\dots$$
$$-s_{12}s_{45} + s_{123}s_{345}$$
$$\dots$$
$$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl}$$

79 Odd letters

$$\frac{s_{12}+s_{34}-s_{56}-\sqrt{\lambda(s_{12},s_{34},s_{56})}}{s_{12}+s_{34}-s_{56}+\sqrt{\lambda(s_{12},s_{34},s_{56})}}$$
$$\dots$$
$$\frac{s_{12}s_{23}-s_{23}s_{34}+s_{23}s_{56}+s_{34}s_{123}-s_{234}(s_{12}+s_{123})-\epsilon_{1234}}{s_{12}s_{23}-s_{23}s_{34}+s_{23}s_{56}+s_{34}s_{123}-s_{234}(s_{12}+s_{123})+\epsilon_{1234}},$$
$$\dots \quad \dots$$
$$\frac{-s_{12}s_{45}s_{234}+s_{34}s_{61}s_{123}+s_{345}(-s_{23}s_{56}+s_{123}s_{234})-\Delta_6}{-s_{12}s_{45}s_{234}+s_{34}s_{61}s_{123}+s_{345}(-s_{23}s_{56}+s_{123}s_{234})+\Delta_6}$$

245 letters

up to the weight four

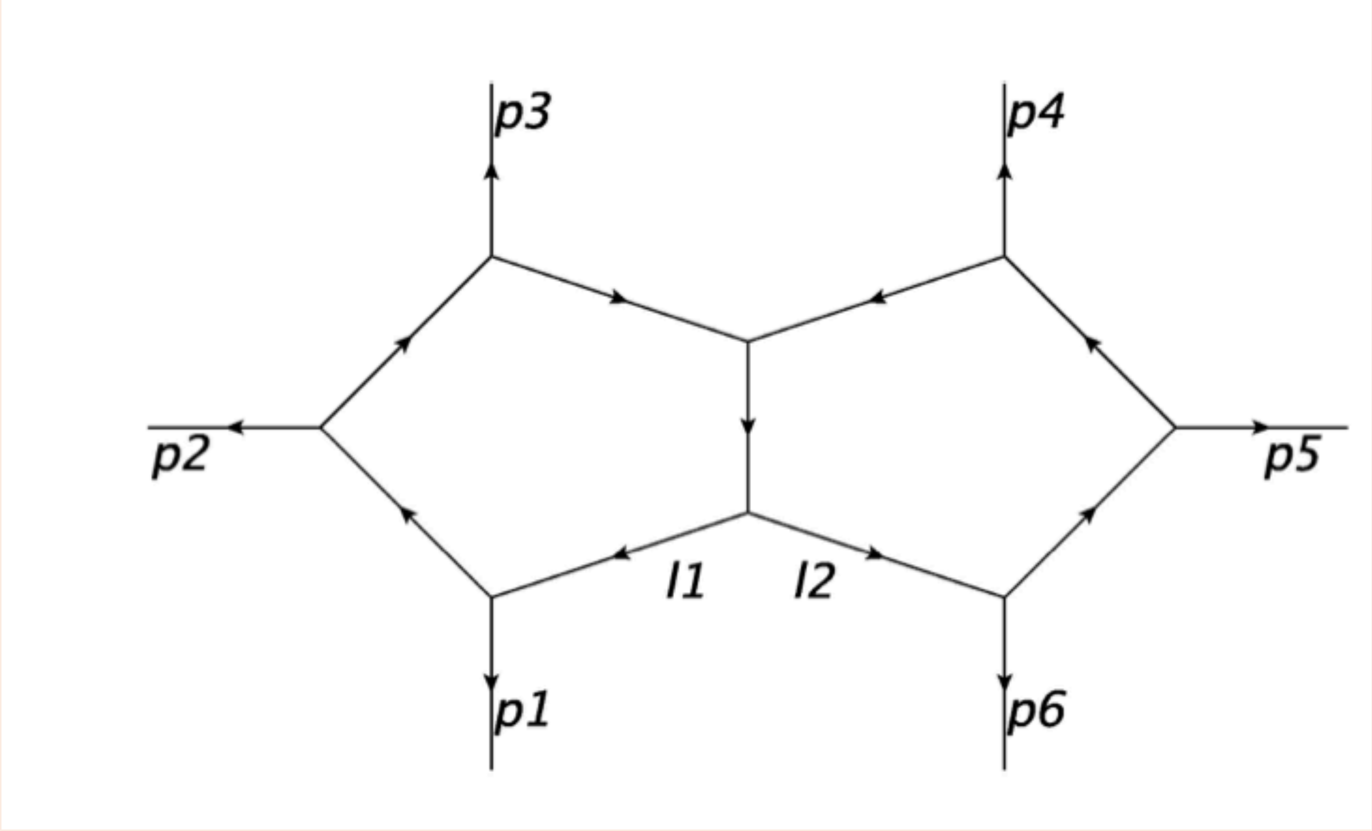
10 More complicated letters

$$\frac{P-\epsilon_{1234}\sqrt{\lambda(s_{12},s_{34},s_{56})}}{P+\epsilon_{1234}\sqrt{\lambda(s_{12},s_{34},s_{56})}} \qquad \dots$$

Then the canonical DE is obtained analytically after ~50 times of numeric IBP running

Two-loop Six-point integrals to higher orders of ϵ

Liu, Matijasic, Peraro, Yang, Xu, YZ, arXiv. 2603.16831

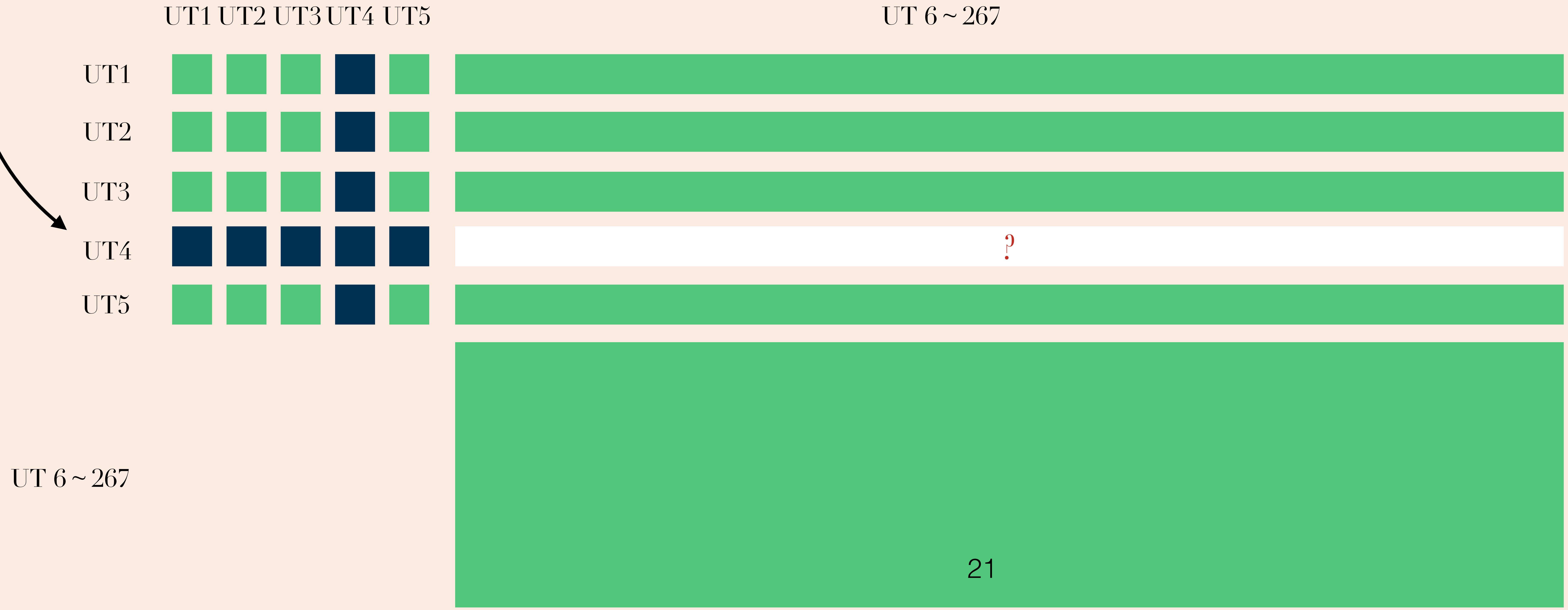


6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

$$F_4 = \frac{1}{8}\left(G(1,2,3,6)s_{45}^2 + G(1,2,3,5)s_{46}^2 + G(1,2,3,4)s_{56}^2 \right.$$
$$\left. -2\epsilon_{1235}\epsilon_{1236}s_{45}s_{46} - 2\epsilon_{1234}\epsilon_{1236}s_{45}s_{56} - 2\epsilon_{1234}\epsilon_{1235}s_{46}s_{56}\right)^{1/2},$$
$$= \frac{1}{8}\sqrt{\lambda(s_{45}\epsilon_{1236}, s_{46}\epsilon_{1235}, s_{56}\epsilon_{1234})}$$

a new square root
not rationalised
by momentum twistor

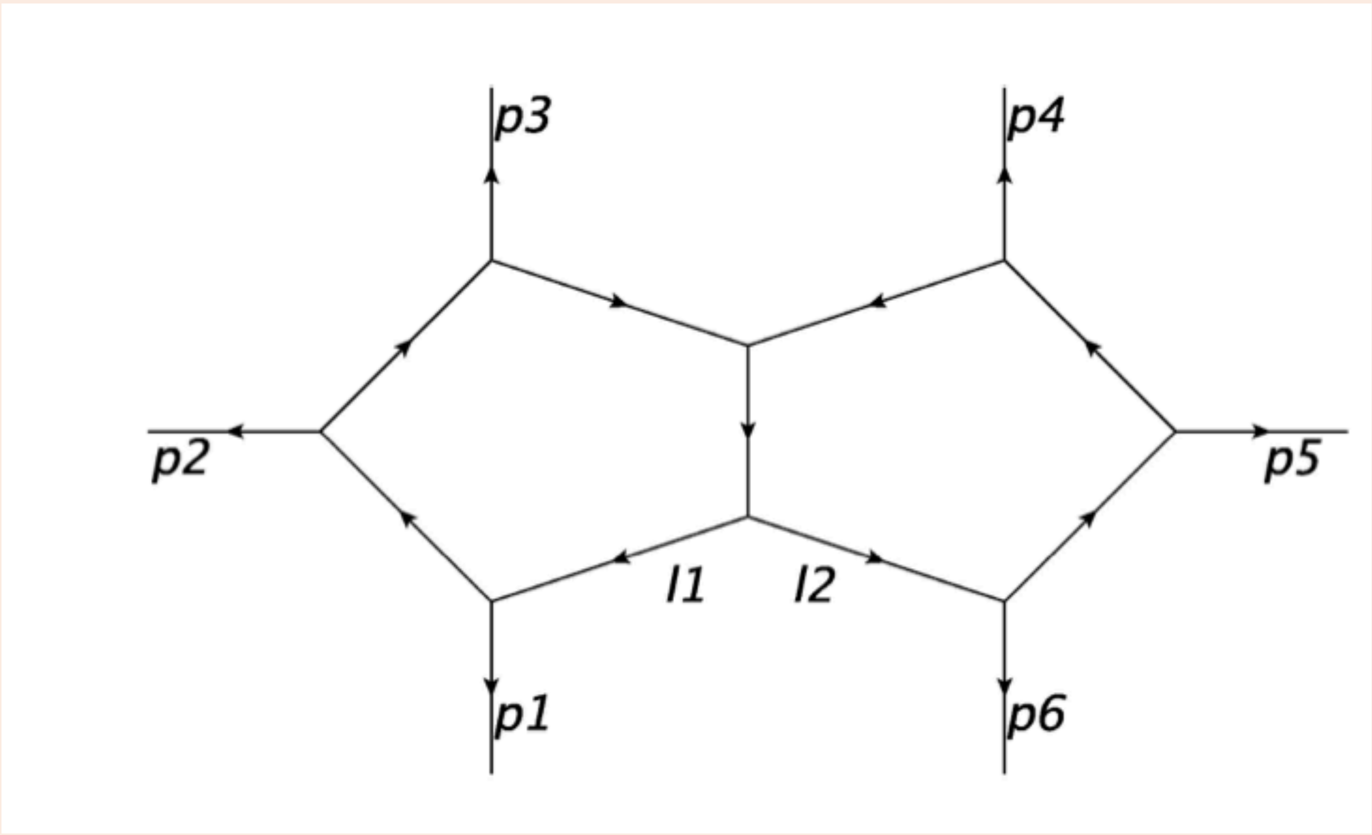


$$\frac{F_4 + R_i}{F_4 - R_i}, \quad i = 1, \dots 5$$

18 more letters
from the diagonal block

Two-loop Six-point integrals to higher orders of ϵ

Liu, Matijasic, Peraro, Yang, Xu, YZ, arXiv. 2603.16831

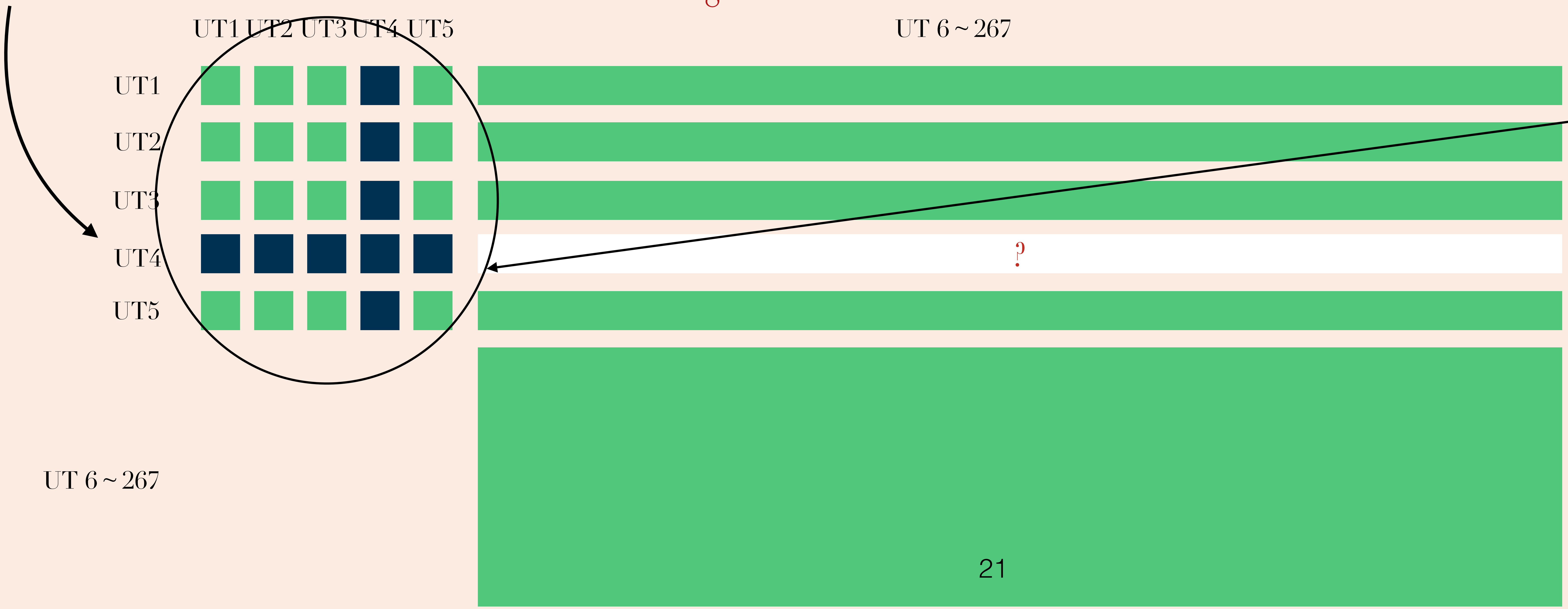


6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4\epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

$$F_4 = \frac{1}{8}\Big(G(1,2,3,6)s_{45}^2 + G(1,2,3,5)s_{46}^2 + G(1,2,3,4)s_{56}^2$$
$$-2\epsilon_{1235}\epsilon_{1236}s_{45}s_{46} - 2\epsilon_{1234}\epsilon_{1236}s_{45}s_{56} - 2\epsilon_{1234}\epsilon_{1235}s_{46}s_{56}\Big)^{1/2},$$
$$= \frac{1}{8}\sqrt{\lambda(s_{45}\epsilon_{1236}, s_{46}\epsilon_{1235}, s_{56}\epsilon_{1234})}$$

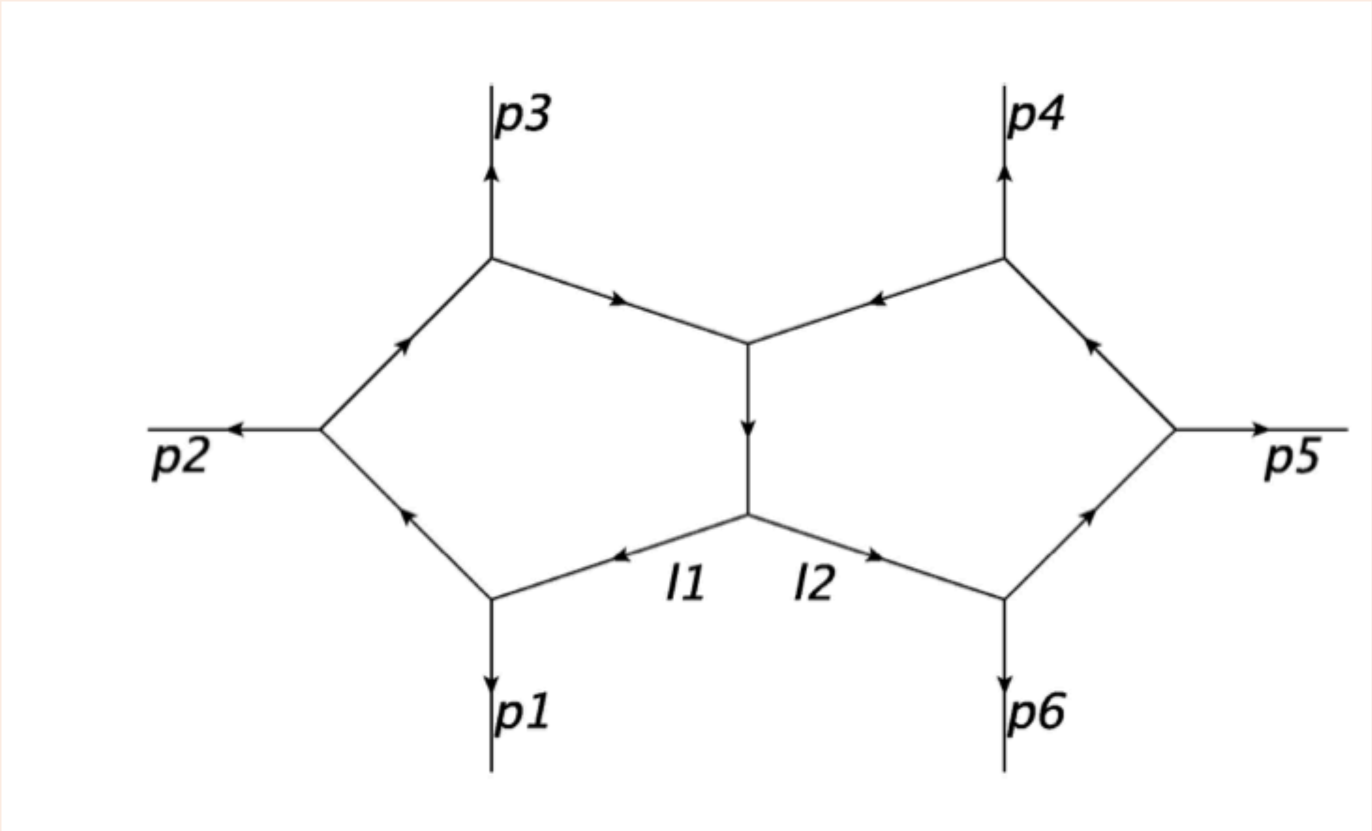
a new square root
not rationalised
by momentum twistor



$$\frac{F_4 + R_i}{F_4 - R_i}, \quad i = 1, \dots 5$$

18 more letters
from the diagonal block

x-point integrals to higher orders of ϵ



6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

Calico: Bertolini, Fontana, Peraro

FiniteFlow: Perao

Two-loop Six-point integrals to higher orders of ϵ

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

Liu, Matijasic, Peraro, Yang, Xu, YZ, arXiv: 2603.16831

closed under the
dihedral group action

$$245 + 18 + 6 = 269$$ letters in total

up to weight-4

2loop 6point massless planar integrals
calculated to weight-six

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

Chebyshev pseudospectral transport,
to get weight-6 numerics in one minute

Boundary Values

Euclidean region $X_0: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \{-1,-1,-1,-1,-1,-1,-1,-1,-1\}$

Boundary values at the symmetric Euclidean point, are obtained up to weight 4: Zeta values and polylogarithm of roots of unity

$$\epsilon^4 I_{\mathrm{db},1}(X_0) = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$
$$\epsilon^4 I_{\mathrm{db},2}(X_0) = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$
$$I_{\mathrm{db},3}(X_0) = I_{\mathrm{db},4}(X_0) = I_{\mathrm{db},5}(X_0) = 0,$$
$$\epsilon^4 I_{\mathrm{db},6}(X_0) = -\left(\frac{\pi^4}{540} + \frac{4}{3}\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,$$
$$\epsilon^4 I_{\mathrm{db},7}(X_0) = 0.$$

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

from the ordinary differential equation
spurious pole asymptotic analysis

Physical region, $12 \to 3456$

obtained with AMFlow (about 3 days), ~200 digits, to weight 6

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \left\{\frac{325}{16}, -\frac{181}{16}, \frac{65}{8}, \frac{9}{4}, -\frac{19}{8}, 7, -\frac{109}{16}, \frac{115}{8}\right\}$$

Solution of canonical DE, two-loop six-point integrals

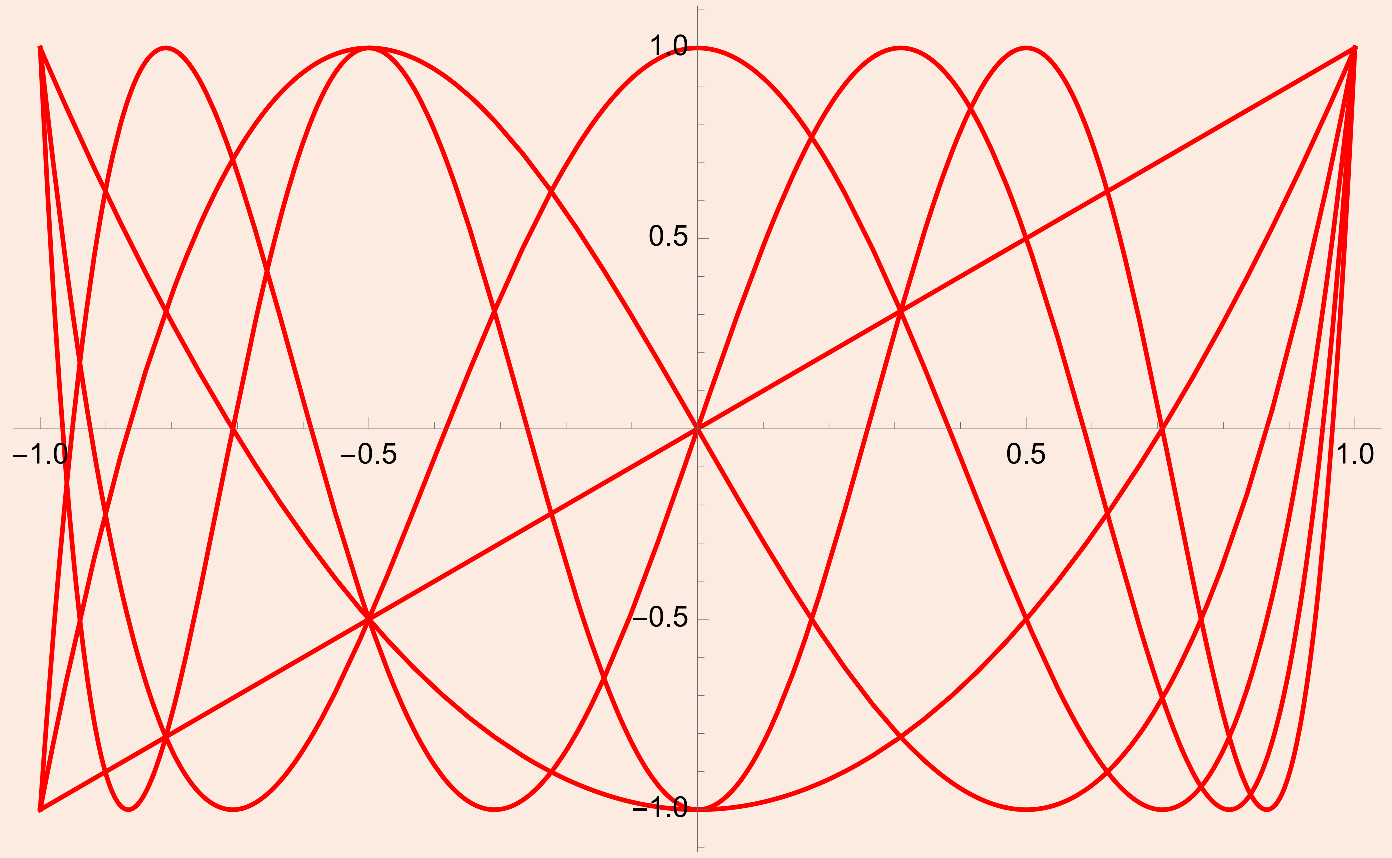
weight-1, 2 — logarithm and classical poly-logarithm

weight-3, 4 — one-fold integration

$$\vec{I}^{(4)} = \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{I}^{(3)}(\vec{x}_0) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1} \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_2} \vec{f}^{(2)}(t_2)$$

$$= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{f}^{(2)}(t) \right).$$

CHEbyshev pSeudo-Spectral transportation (CHESS) Liu and YZ, arXiv:2606.26691



approximate the Feynman integrals by Chebyshev polynomial expansion

An efficient way to solve (canonical) differential equation numerically

Physical region, up to weight 6, all 267 integrals in double pentagon family

Chebyshev Nodes	absolute error	time	RAM
12	$\sim 7.58 \times 10^{-16}$	18.09 s	2.9 GB
24	$\sim 1.36 \times 10^{-31}$	144.76 s	26.1 GB

for Next-to-Next-to-Next-to-leading order

production of 3 jets, 3 photons … with 2loop 6point Feynman integrals

3loop 5point Feynman integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ

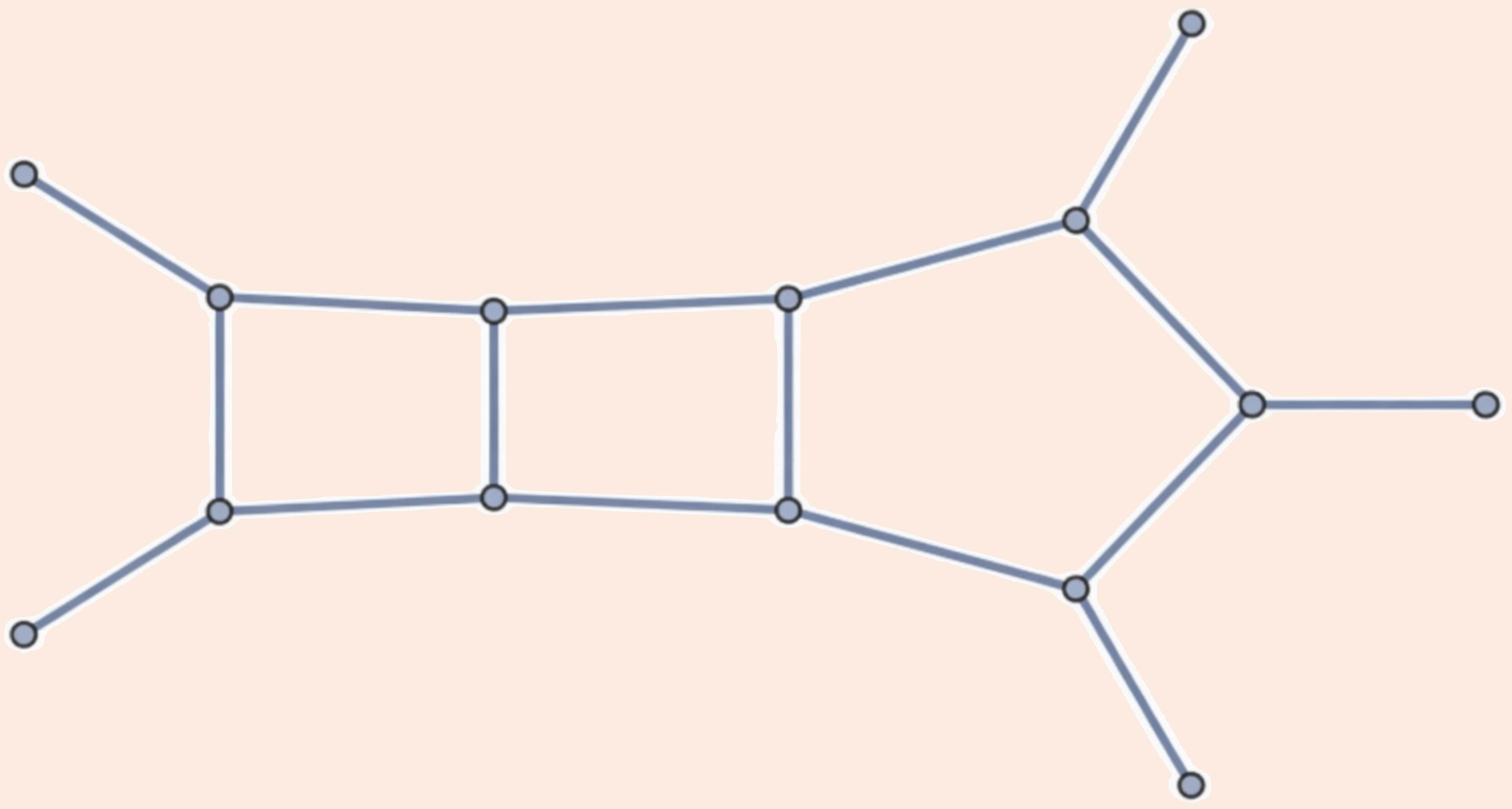
Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

Chicherin, Wu, Wu, Xu, Zhang, YZ

arXiv Phys. Rev. Lett. accepted

3loop 5point planar family

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021
Editors' suggestion



UT basis found!

Baikov analysis
Gram determinant

5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

Canonical differential equation

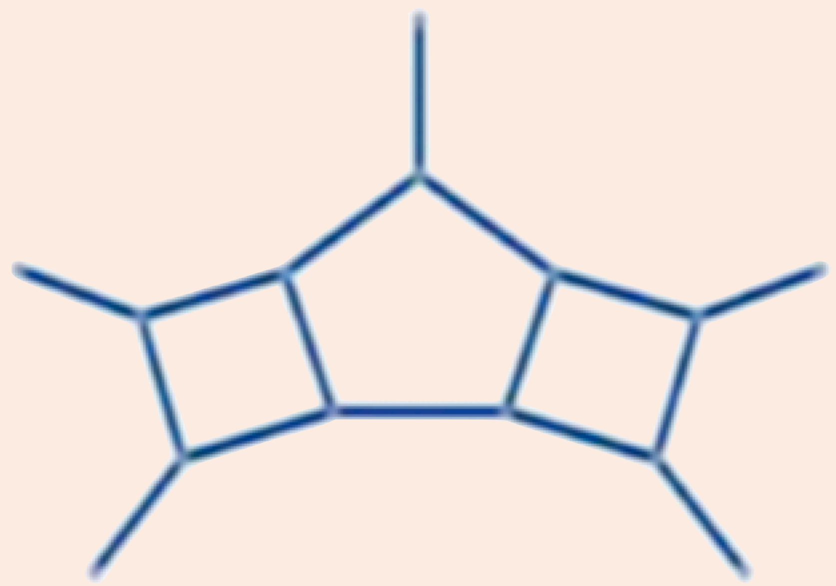
We use NeatIBP 1.0 to derive the differential equation

~100 million IBPs → 85000 IBPs

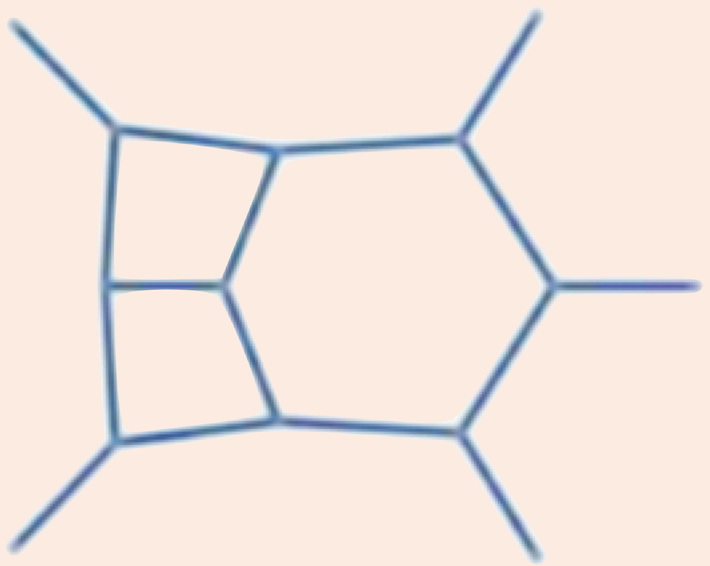
weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration
It takes 2 minutes on a laptop to get 10 digits from our analytic solution

Complete result of all 3loop 5point planar massless integrals

Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv Phys. Rev. Lett. accepted



367 MIs



431 MIs

734 MIs

56 letters …

IBP reduction is extremely complicated, we have to use the latest NeatIBP 1.1 version

NeatIBP 1.1: Syzygy + Spanning Cuts

Most complicated IBP system with cuts, consists of $\sim 10^5$ equations, ~100 cuts

Complete result of all 3loop 5point planar massless integrals

Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv Phys. Rev. Lett. accepted

pure integral basis determination

top sector has 19 MIs

loop-by-loop dlog integrand generation, dimensional shift

module lift method

$$\mathcal{N} = \sum_I c_I ISP^{\alpha_I}$$

$\sum_I c_I \text{ residue}_j\left(\text{ISP}^{\alpha_I}\right) = r_j,$

c_I polynomial in Mandelstam variables, r_j rational numbers

$$\begin{aligned}
\mathcal{N}_4 &= \frac{\varepsilon(s_{12}-s_{45})}{(1+2\varepsilon)\epsilon_5} G(\ell_2,\ell_3,k_1,k_2,k_3,k_4),\\
\mathcal{N}_{13} &= s_{12}s_{34}s_{45}(\ell_2-k_1)^2(\ell_3-k_{123})^2,\\
\mathcal{N}_{17} &= s_{12}s_{45}(\ell_3+k_5)^2\left((\ell_2-k_{123})^2(\ell_3-k_{12})^2 - (\ell_3-k_{123})^2(\ell_2-k_{12})^2\right),\\
\mathcal{N}_{19} &= s_{12}s_{15}s_{45}\left((\ell_2-k_{123})^2(\ell_3-k_{12})^2 - (\ell_3-k_{123})^2(\ell_2-k_{12})^2\right).
\end{aligned}$$

3loop 5point integrals, symbology

Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv Phys. Rev. Lett. accepted

From the canonical differential equation

We confirmed the 3loop 5point letters from Chicherin, Henn, Trnka, Zhang *JHEP* 04 (2025) 022

56 Letters

$$\mathbb{A}_{\mathrm{P}}^{\text{3-loop}} = \underbrace{\mathbb{A}_{\mathrm{P}}^{\text{2-loop}}}_{\text{26 Letters}} \cup \{\widehat{W}_i\}_{i=1}^{5} \cup \{\widetilde{W}_i\}_{i=16}^{20} \cup \{\widetilde{W}_i\}_{i=41}^{55} \cup \{\widetilde{W}_i\}_{i=76}^{80}$$

$$\widehat{W}_1 = s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{15}\,,$$

$$\widetilde{W}_{16} = \Delta_4^{(1)}\,,$$

$$\widetilde{W}_i = \frac{q_i - \sqrt{\Delta_4^{(1)}}}{q_i + \sqrt{\Delta_4^{(1)}}},\quad i \in \{41, 46, 51\}\,,$$

$$\widetilde{W}_{76} = \frac{q_{76} - \epsilon_5\sqrt{\Delta_4^{(1)}}}{q_{76} + \epsilon_5\sqrt{\Delta_4^{(1)}}}\,,$$

$$\begin{aligned}\Delta_4^{(1)} = {} & s_{15}^2 s_{12}^2 + s_{23}^2 s_{12}^2 - 2s_{15}s_{23}s_{12}^2 - 2s_{23}^2 s_{34}s_{12} \\ & + 2s_{15}s_{23}s_{34}s_{12} + 2s_{15}s_{34}s_{45}s_{12} + 2s_{23}s_{34}s_{45}s_{12} \\ & + s_{23}^2 s_{34}^2 + s_{34}^2 s_{45}^2 - 2s_{23}s_{34}^2 s_{45},.\end{aligned}$$

Symbols

weight	1	2	3	4	5	6
3loop 5point Symbols	5	20	76	285	1000	2220

small number of symbols
good news for bootstrap

Solution of canonical DE, three-loop five-point integrals

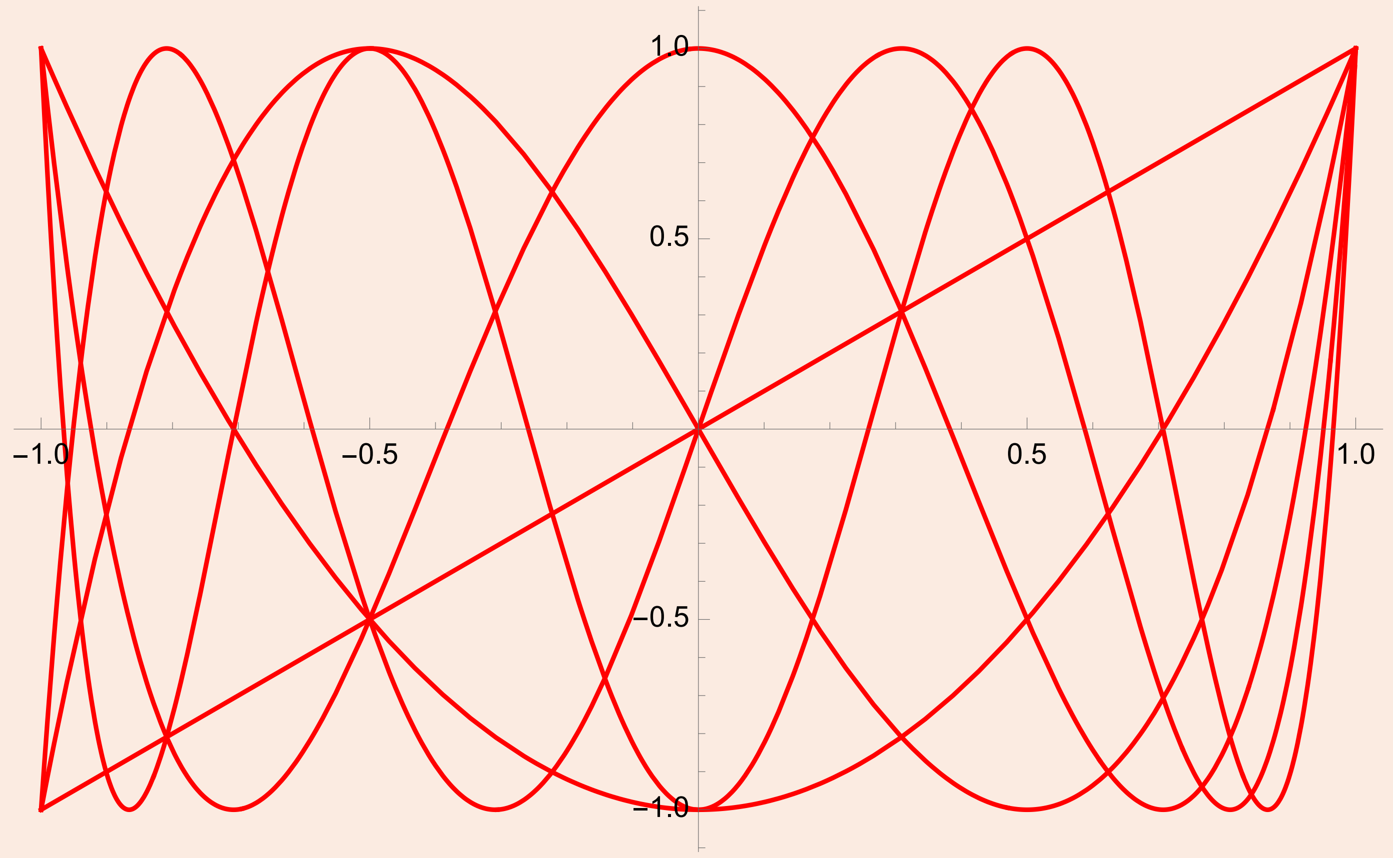
weight-1, 2, 3 — logarithm and classical poly-logarithm

weight-3, 4, 6 — one-fold integration

$$\begin{aligned}\mathbf{I}^{(n+3)}(x) =& \mathbf{I}^{(n+3)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n+2)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{A}(1) - \tilde{A}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n+1)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{A}(t) - \tilde{A}(1)\right)\tilde{A}(t)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n)}(t)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{B}(1) - \tilde{B}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n)}(t)\mathrm{d}t.\end{aligned}$$

$$\mathrm{d}\tilde{B} = (\mathrm{d}\tilde{A})\tilde{A},$$

CHEbyshev pSeudo-Spectral transportation (CHESS)



On a MacBook Pro with a 14-core M4 Pro,
we obtain the values with 10 significant digits of the 734 pure integrals,
in about 39 seconds at the following generic point in the Euclidean region.

Summary (or half time)

Analytic computation of

- ✓ all 2loop 6point planar massless integrals
- ✓ all 3loop 5point planar massless integrals

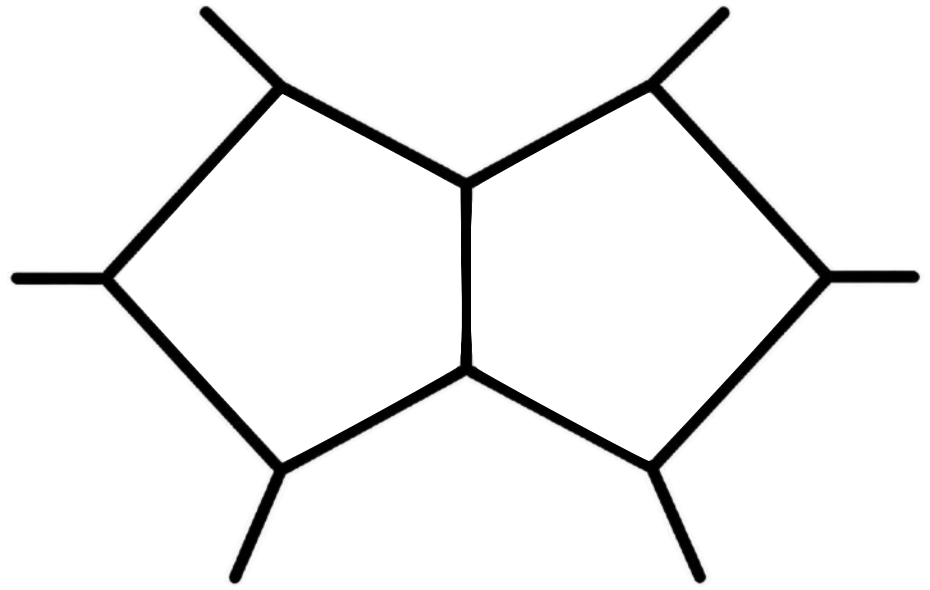
NeatIBP, a powerful package for cutting-edge IBP reduction for precision physics

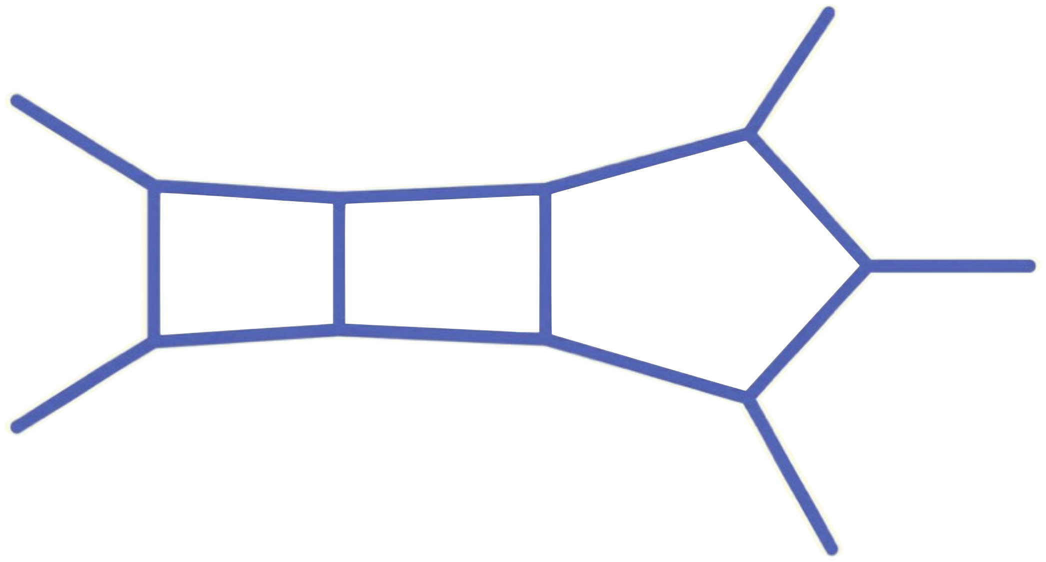
Still difficult

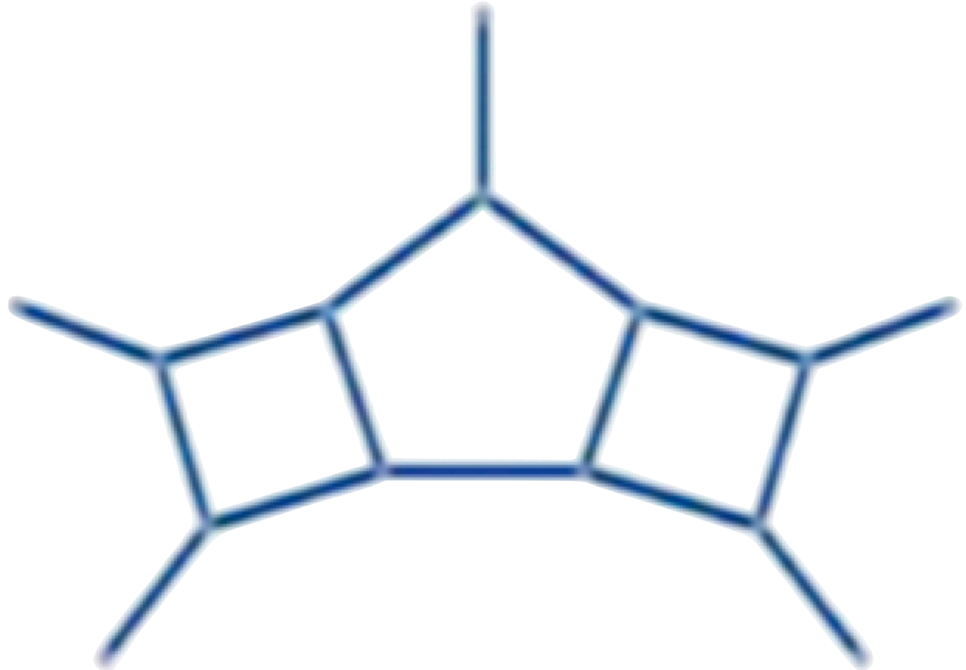
Pure Integral basis determination
Symbol Letter (alphabet) searching

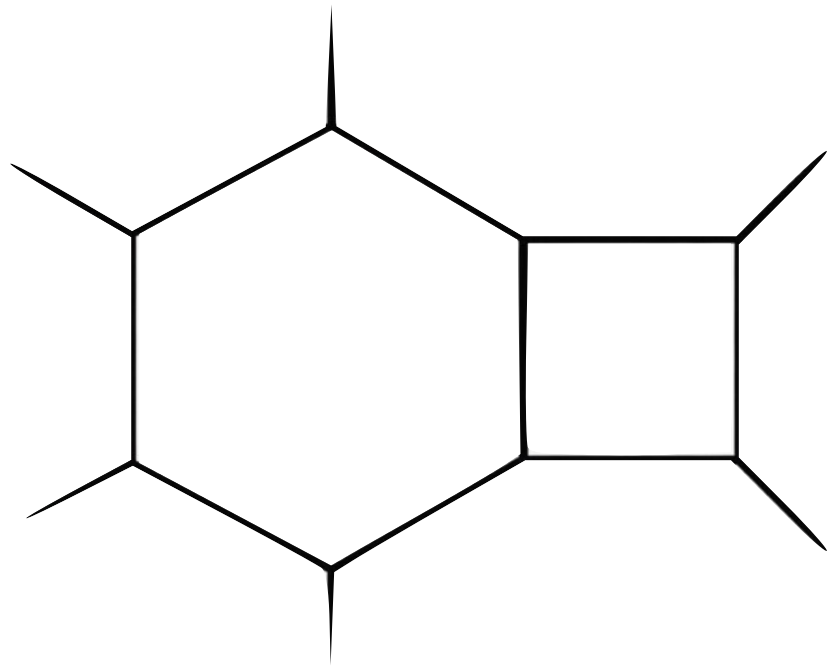
Refer to Yuanche Liu's talk

We wish that a lot more applications based on
2loop 6point, 3loop 5point integrals would emerge.









Multi-loop multi-leg Feynman integrals
are no longer that difficult!