

Feynman integral relations from infrared singularities

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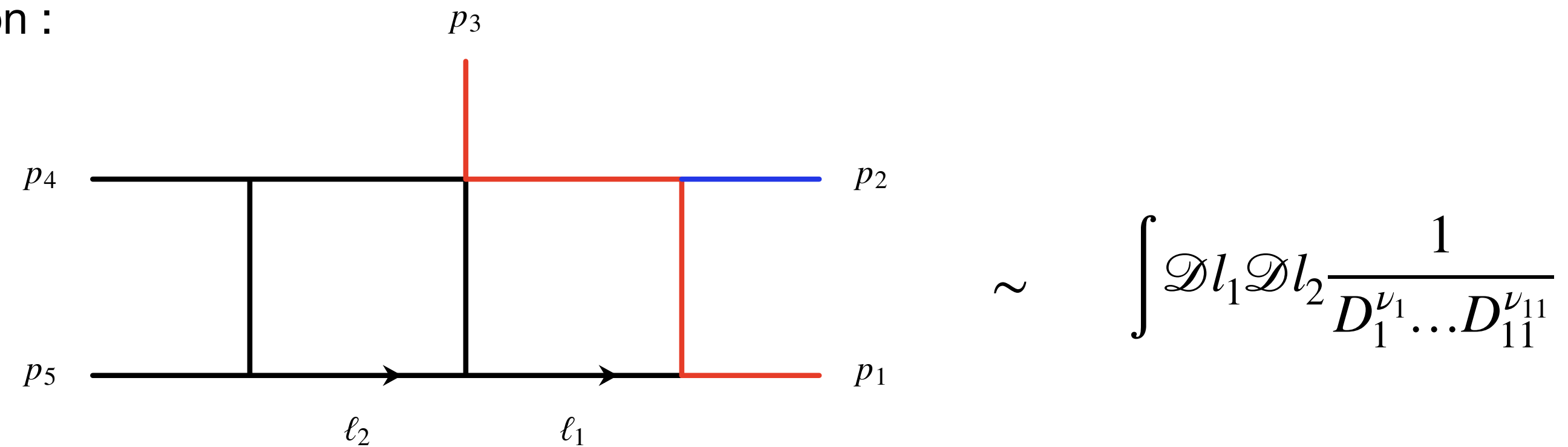
Outline of the talk

- IBPs and syzygy equations
- Critical syzygies
- Landau decomposition
- Examples

IBPs

Feynman integrals in dimensional regularization :

$$I(\underline{\nu}) = \int \frac{\mathcal{D}l_i}{\prod_j D_j^{\nu_j}}, \quad \nu_j \in \mathbb{Z}$$



Integration by parts (IBPs) :

$$\int \mathcal{D}l_i \frac{\partial}{\partial l_i^\mu} \left[v^\mu \frac{1}{\prod_j D_j^{\nu_j}} \right] = 0, \quad v^\mu = \begin{cases} p_i^\mu & \text{external momenta} \\ l_i^\mu & \text{loop momenta} \end{cases}$$

[Chetyrkin, Tkachov]

■ Linear relations for different choices of v^μ and ν_j

■ “Laporta algorithm” $\longrightarrow$ Solve a large and sparse system of equations

[Laporta]

Most well-known and successful method

■ Many public implementations [FIRE, Kira, Reduze, LiteRed]

IBPs

Increasing the number of loops and legs, or adding more scales, the complexity of the system and of its coefficients grows

- Huge intermediate expressions

- Expensive symbolic linear algebra

One of the main bottlenecks in cutting-edge computations

- Very large IBPs tables

Different approaches :

- Reconstruction from finite fields samples

[Von Manteuffel, Schabinger; Peraro]

- Intersection theory

[Mizera, Mastrolia]

- Syzygy equations

[Gluza, Kajda, Kosower; Böhm, Georgoudis, Larsen, Schulze, Zhang; Ita]

Focus of the talk

IBPs in Baikov representation

Dimensionally regulated l-loop Feynman integral in Baikov representation :

$$I_{\Gamma}(\mathcal{N}, \underline{\nu}) = \frac{c(d)}{G(p_1, \dots, p_E)} \int_{\mathcal{C}} d^n \underline{z} \left[B(\underline{z})^{\gamma} \frac{\mathcal{N}}{\prod_e z_e^{\nu_e}} \right] \quad \gamma = (d - l - E - 1)/2 \quad \underline{z} = \{z_e : e \in Props(\Gamma)\} \cup \{z_i : i \in ISPs(\Gamma)\}$$

$$B(\underline{z}) = G(\underline{l}, p_1, \dots, p_E)$$

IBPs in Baikov representation :

$$0 = \int_{\mathcal{C}} d^n \underline{z} \left[\partial_k \left(B^{\gamma} \frac{a_k}{\prod_j z_j^{\nu_j}} \right) \right]$$

$$0 = \int_{\mathcal{C}} d^n \underline{z} \frac{B^{\gamma}}{\prod_j z_j^{\nu_j}} \left[\sum_k \left(\partial_k a_k + \boxed{\gamma a_k \frac{\partial_k B}{B}} - \boxed{\nu_k \frac{a_k}{z_k}} \right) \right]$$

■ Dimension shifting term :

■ Raised propagator power term :

$$B^{\gamma-1} \iff d \rightarrow d - 2$$

$$\frac{1}{z_k^{\nu_k}} \frac{1}{z_k} = \frac{1}{z_k^{\nu_k+1}}$$

Syzygy equations

- No dimension shifting term :

$$\sum_k a_k \partial_k B = a_0 B$$

- No raised propagator powers term :

$$a_e = \bar{a}_e z_e$$

Polynomials solving these constraints form a Syzygy module

A syzygy is an equation of the form :

$$\sum_j \boxed{g_j(\underline{z})} \boxed{a_j(\underline{z})} = 0$$

- Set of known polynomials
- Set of unknown polynomials

- We can put the second constraint into the first and solve it
- We can combine solutions of the two constraints

$$0 = a_0 B + \sum_{i \in ISPs(\Gamma)} a_i \partial_i B + \sum_{e \in Props(\Gamma)} \bar{a}_e z_e \partial_e B$$

Unknown polynomials $\{a_0, a_i, \bar{a}_e\}$ depend on all Baikov variables [\[Larsen,Zhang\]](#)

We consider syzygies of the form :

$$0 = a_0 B + \sum_{i \in ISPs(\Gamma)} a_i \partial_i B + \sum_{e \in Props(\Gamma)} \tilde{a}_e z_e B + \sum_{e \in Props(\Gamma)} \bar{a}_e z_e \partial_e B$$

a_0 depends on ISPs

Integral relations from syzygy equations

$$0 = a_0 B + \sum_{i \in \text{ISPs}(\Gamma)} a_i \partial_i B + \sum_{e \in \text{Props}(\Gamma)} \tilde{a}_e z_e B + \sum_{e \in \text{Props}(\Gamma)} \bar{a}_e z_e \partial_e B \quad \text{Syzygy equation}$$

$a_0, a_i, \tilde{a}_e, \bar{a}_e$ are polynomials in $\underline{z}$ variables $\longrightarrow \text{Syz}(\Gamma)$: set of syzygy solutions (R-module)

From syzygies we can construct integral relations, each syzygy leads to an IBP :

$$\int_{\mathcal{C}} d^N z \frac{B^\gamma}{\prod_{e \in \text{props}} z_e^{\nu_e}} \left[\underbrace{a_0 + \sum_{e \in \text{props}} \tilde{a}_e z_e - \frac{1}{\gamma} \left(\sum_{i \in \text{ISPs}} \partial_i a_i + \sum_{e \in \text{props}} (z_e \partial_e \bar{a}_e - (\nu_e - 1) \bar{a}_e) \right)}_{S_\Gamma(\vec{a}, \vec{\nu})} \right] = 0$$

Surface term constructed from syzygies $S_\Gamma \in \text{Surface}(\Gamma)$

IBPs can be controlled by syzygy equations [Gluza, Kajda, Kosower; Böhm, Georgoudis, Larsen, Schulze, Zhang; Ita]

Critical syzygies

Constructing $\text{Syz}(\Gamma)$ is challenging $\longrightarrow$ Algebraic geometry methods to identify a subset of $\text{Syz}(\Gamma)$

Surface terms on the maximal cut in the large- ϵ limit : $\lim_{\epsilon \rightarrow \infty} S_\Gamma(\vec{a}, \vec{\nu}) \Big|_{z_e=0} = \boxed{a_0}$ Limit studied in intersection theory
[Mizera; Mizera, Pokraka]

$$0 = \boxed{a_0} B + \sum_{i \in \text{ISPs}(\Gamma)} a_i \partial_i B + \sum_{e \in \text{Props}(\Gamma)} \tilde{a}_e z_e B + \sum_{e \in \text{Props}(\Gamma)} \bar{a}_e z_e \partial_e B$$

$$J_{\text{syz}}^\Gamma = \langle \partial_i B \rangle + \langle z_e B, z_e \partial_e B \rangle \longrightarrow A_0^\Gamma = \{p \in R \mid pB \in J_{\text{syz}}^\Gamma\} = J_{\text{syz}}^\Gamma : \langle B \rangle \quad \text{Ideal quotient}$$

“Saturation hypothesis”

One-to-one correspondence between A_0^Γ and surface terms, up to terms that vanish on the cut of Γ

$$\text{CSyz}(\Gamma) \simeq A_0^\Gamma / J_{\text{cut}}^\Gamma \quad \text{Module of critical syzygies}$$

$$\boxed{\text{CSyz}(\Gamma) = \text{Syz}(\Gamma) / \sim}$$

Landau decomposition

Main result :

$$CSyz(\Gamma) = \sum_{i=0}^{n_{IR}} FittSyz(U_i^\Gamma)$$

Landau decomposition

Critical syzygies can be generated component by component from the maximal-cut Landau locus

We consider U_{Landau}^Γ : for fixed kinematics it is a variety in the Baikov variables only [Landau; Caron-Huot,Correia,Giroux; Correia,Giroux,Mizera]

For the two-loop five-point topologies studied, the maximal-cut Landau locus decomposes as :

$$U_{Landau}^\Gamma = V(\boxed{J_{4d}} + \boxed{J_{cut}^\Gamma}) \cup \left[\bigcup_{i=1}^{n_{IR}} \boxed{U_i^\Gamma} \right] = U_0^\Gamma \cup \left[\bigcup_{i=1}^{n_{IR}} U_i^\Gamma \right]$$

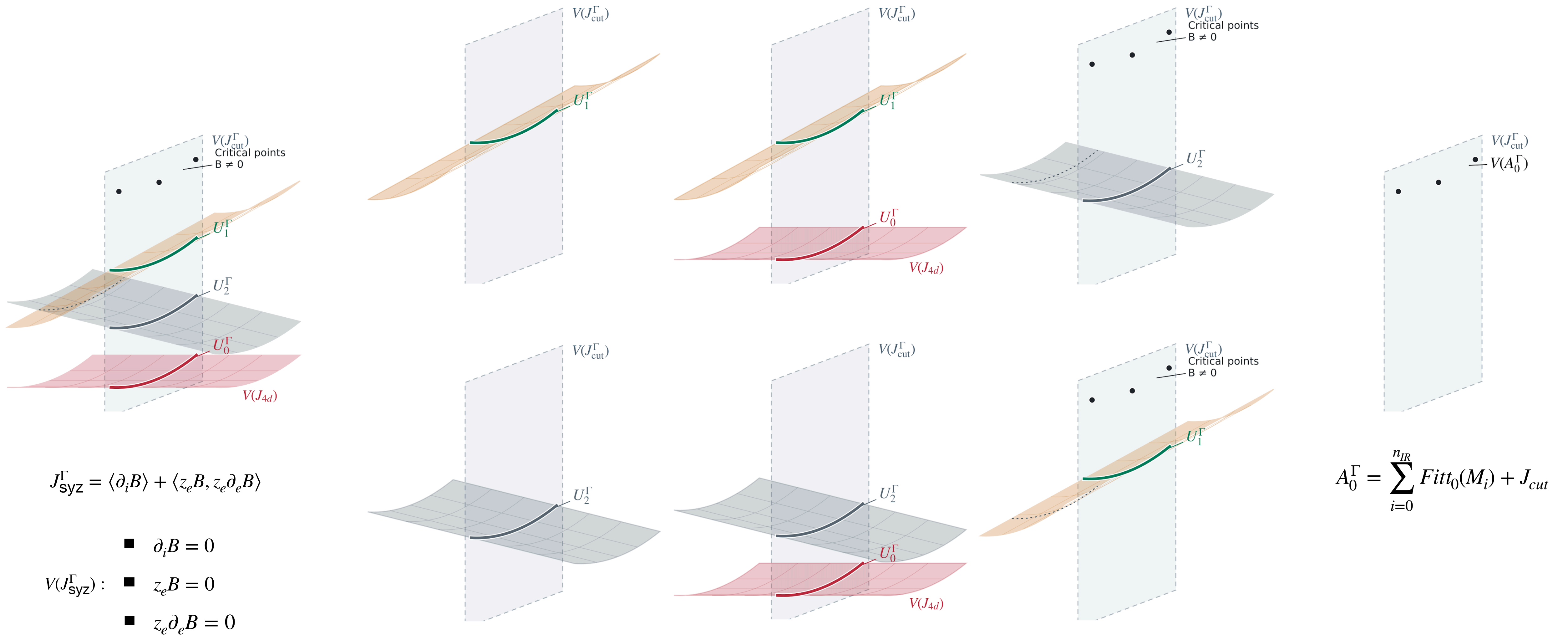
■ $J_{4d} = \langle \mu_{11}, \mu_{22}, \mu_{12} \rangle$

■ $J_{cut}^\Gamma = \langle z_e \rangle$

■ Associated with soft/collinear singularities

$\longleftarrow \mu_{ij} = \frac{G \left(\begin{smallmatrix} \bar{l}_i & p_1 p_2 p_3 p_4 \\ \bar{l}_j & p_1 p_2 p_3 p_4 \end{smallmatrix} \right)}{G(p_1 p_2 p_3 p_4)}$

Landau decomposition



Building F_i matrix

$$A_0^\Gamma = \sum_{i=0}^{n_{IR}} \text{Fitt}_0(M_i) + J_{cut} \quad \text{We relate an abstract definition of ideal quotient to a determinantal construction}$$

We construct a matrix F_i which satisfies :

$$F_i \begin{pmatrix} -1 \\ \vec{v}_i \end{pmatrix} = 0$$

- Columns are not independent
- Maximal minors vanish

Syzygies are obtained from the maximal minors of F_i

$$M_i = K_i / J_{syz}^\Gamma, \quad \text{where} \quad K_i = \langle v_{i,1}, \dots, v_{i,n} \rangle \quad \text{and} \quad J_{syz}^\Gamma = \langle b_1, \dots, b_m \rangle$$

$$\blacksquare \sum_{k=1}^n \rho_k^{(\alpha)} v_{i,k} = 0, \quad \alpha = 1, \dots, s \quad \longrightarrow \quad R_i = \begin{pmatrix} \rho_1^{(1)} & \dots & \rho_n^{(1)} \\ \vdots & & \vdots \\ \rho_1^{(s)} & \dots & \rho_n^{(s)} \end{pmatrix} \in \text{Mat}_{s \times n}(R) \quad \text{Such that : } R_i v_i = 0 \quad \text{Syzygies between } v_{i,k}$$

$$\blacksquare b_j = \sum_k (C_i)_{jk} v_{i,k} \quad \text{This defines a polynomial matrix } C_i$$

$$\blacksquare B = \sum_k c_{i,k} v_{i,k}$$

Syzygies from F_i

$$F_i = \begin{pmatrix} B & \underline{c}_i^T \\ \underline{b} & C_i \\ \underline{0} & R_i \end{pmatrix} \longrightarrow F_i \begin{pmatrix} -1 \\ \underline{v}_i \end{pmatrix} = 0$$

- First block : $-B + \underline{c}_i^T \cdot \underline{v}_i = -B + B = 0$
- Second block : $-\underline{b} + C_i \cdot \underline{v}_i = -\underline{b} + \underline{b} = 0$
- Third block : $R_i \cdot \underline{v}_i = 0$

Syzygies follow from the vanishing of the maximal minors of F_i :

$$0 = \det \begin{pmatrix} B & \vec{c}_i^T \\ f_{j_1,0} & f_{j_1,1} \cdots f_{j_1,n} \\ \vdots & \vdots \\ f_{j_n,0} & f_{j_n,1} \cdots f_{j_n,n} \end{pmatrix} \longrightarrow 0 = B \det \begin{pmatrix} f_{j_1,1} & \cdots & f_{j_1,n} \\ \vdots & & \vdots \\ f_{j_n,1} & \cdots & f_{j_n,n} \end{pmatrix} + \sum_{\alpha=1}^n (-1)^\alpha f_{j_\alpha,0} \det \begin{pmatrix} c_{i,1} & \cdots & c_{i,n} \\ f_{j_1,1} & \cdots & f_{j_1,n} \\ \vdots & & \vdots \\ \widehat{f_{j_\alpha,1} \cdots f_{j_\alpha,n}} & & \vdots \\ \vdots & & \vdots \\ f_{j_n,1} & \cdots & f_{j_n,n} \end{pmatrix} \quad \text{Syzygy}$$

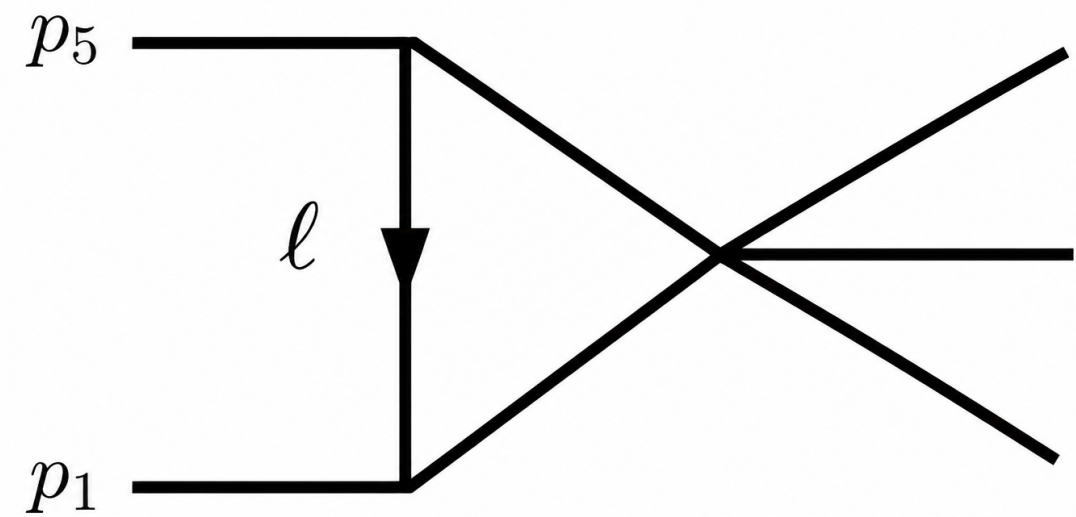
$$0 = a_0 B + \sum_{i \in \text{ISPs}(\Gamma)} a_i \partial_i B + \sum_{e \in \text{Props}(\Gamma)} \tilde{a}_e z_e B + \sum_{e \in \text{Props}(\Gamma)} \bar{a}_e z_e \partial_e B$$

■ a_0
■ $a_i, \tilde{a}_e, \bar{a}_e$

Syzygies from Landau

- All the computations can be done using Singular [\[Decker, Greuel, Pfister, Schönemann\]](#)
- Computation with numerical evaluation of the external kinematics over finite fields $\mathbb{F}_p$ is easy
- Analytic building blocks are obtained by expressing the generators of J_{syz}^Γ in terms of pre-localized generators
- The syzygies are obtained from the maximal minors of the matrices F_i

Example : One-loop triangle



$$p_i^2 = 0, \quad \sum_{i=1}^5 p_i = 0, \quad q_j = \sum_{j=1}^i p_j, \quad D_i = (\bar{l} - q_i)^2$$

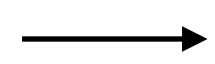
$$\text{Props}(\Gamma) : \quad z_1 = (\bar{l} - q_1)^2, \quad z_4 = (\bar{l} - q_4)^2, \quad z_5 = \bar{l}^2$$

$$\text{ISPs}(\Gamma) : \quad z_2 = (\bar{l} - q_2)^2, \quad z_3 = (\bar{l} - q_3)^2$$

$$R \simeq \mathbb{C} [l^0, l^1, l^2, l^3, \mu_{11}]$$

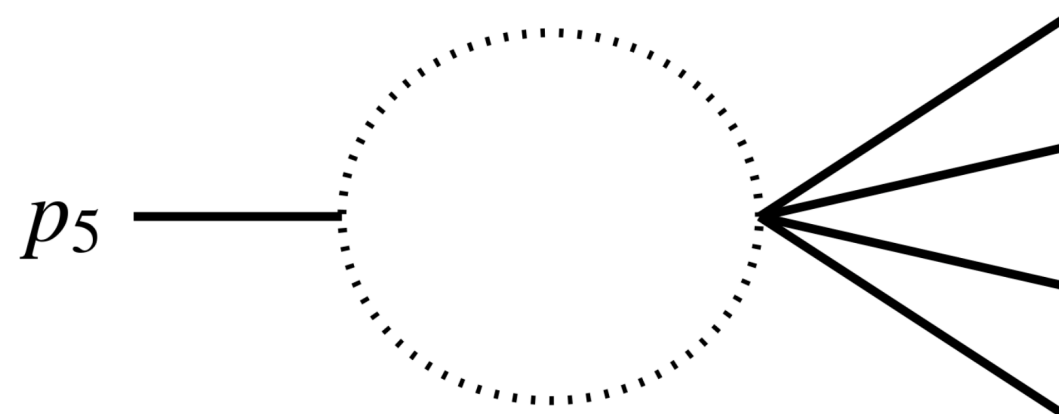
■ 4-d
■ (-2ϵ) -d

$$\bar{l} = \boxed{l} + \boxed{\hat{l}}$$

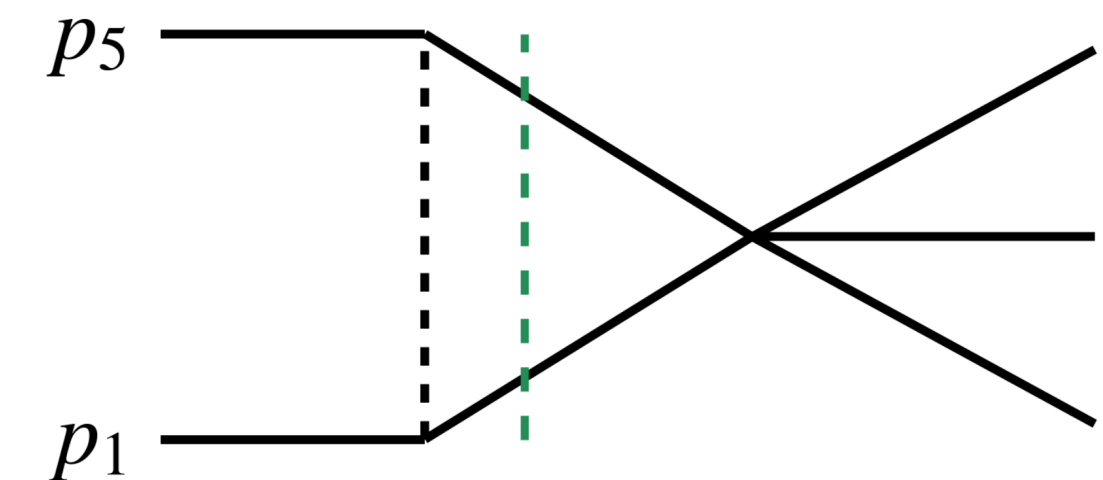
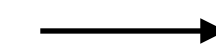
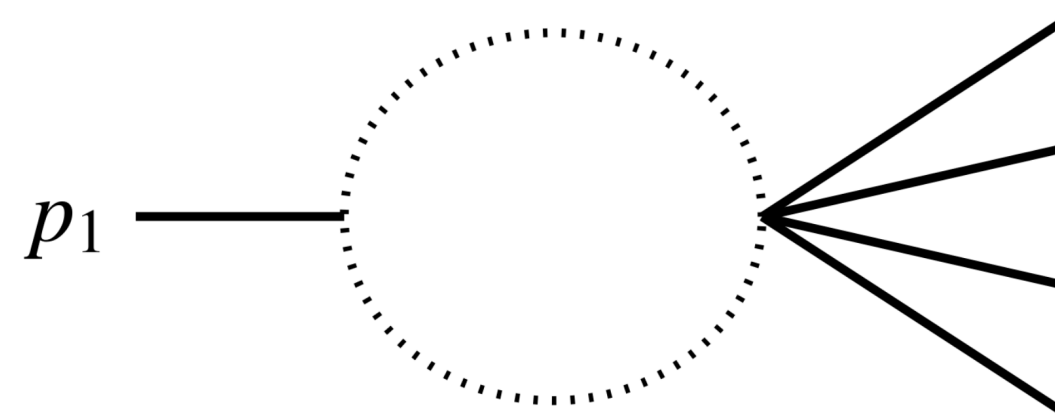


$$\begin{aligned} \blacksquare \quad & \bar{l} \cdot p_j = l \cdot p_j \\ \blacksquare \quad & \bar{l} \cdot \bar{l} = l \cdot l + \hat{l} \cdot \hat{l} = l \cdot l + \mu_{11} \end{aligned}$$

$$\text{Baikov polynomial :} \quad B = \mu_{11} = \frac{G(\bar{l}_1 p_1 p_2 p_3 p_4)}{G(p_1 p_2 p_3 p_4)}$$



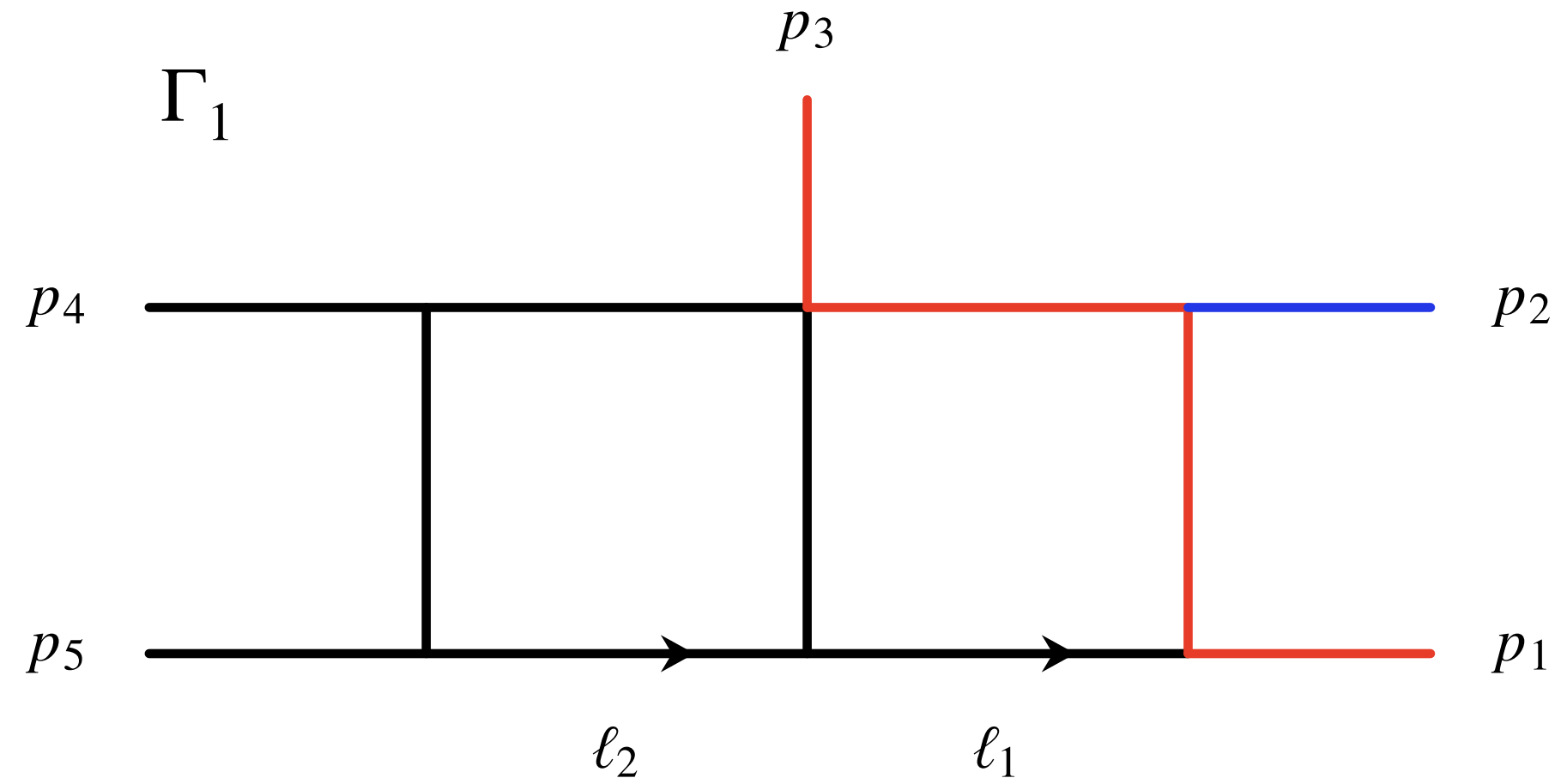
U



$V(J_{\text{syz}})$ decomposes into two irreducible components, corresponding to collinear configurations

The maximal cut Landau locus is soft

$$pp \rightarrow t\bar{t}H$$



$$p_1^2 = p_3^2 = m_t^2, \quad p_2^2 = m_H^2, \quad p_4^2 = p_5^2 = 0,$$

$$q_k = \sum_{j=1}^k p_j$$

$$z_{i,j} = (\bar{l}_i - q_j)^2 - m_{i,j}^2,$$

$$\tilde{z} = (\bar{l}_1 - \bar{l}_2)^2,$$

$$B = \mu_{11}\mu_{22} - \mu_{12}^2$$

$$U_{Landau}^{\Gamma_1} = U_0^{\Gamma_1} \cup U_1^{\Gamma_1} \cup U_2^{\Gamma_1}$$

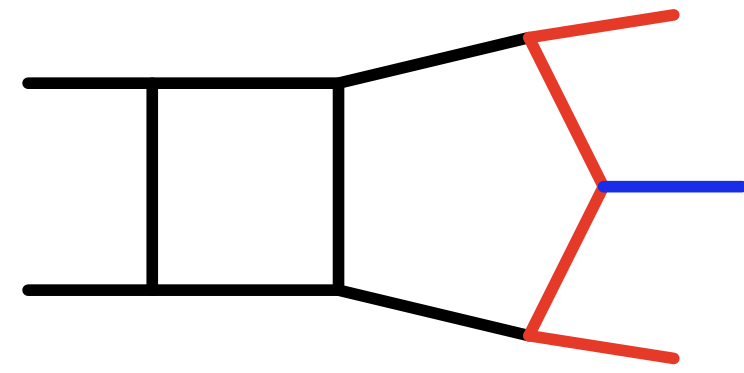
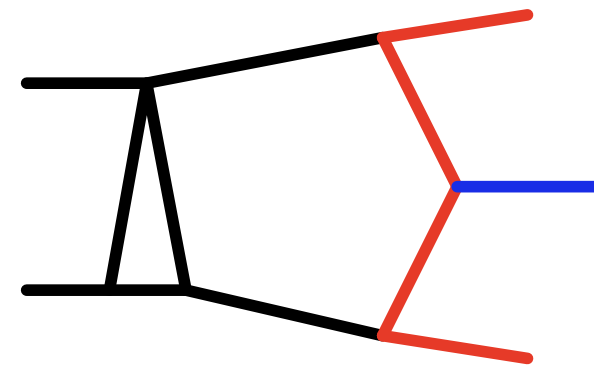
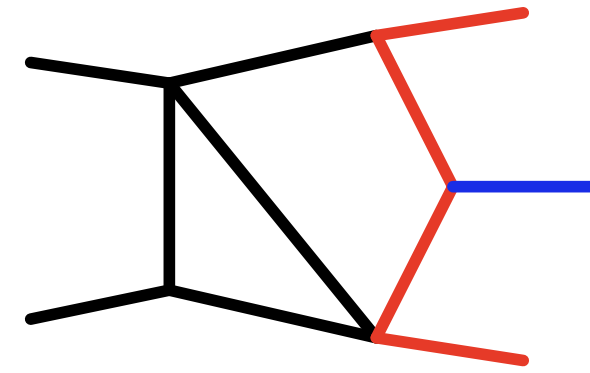
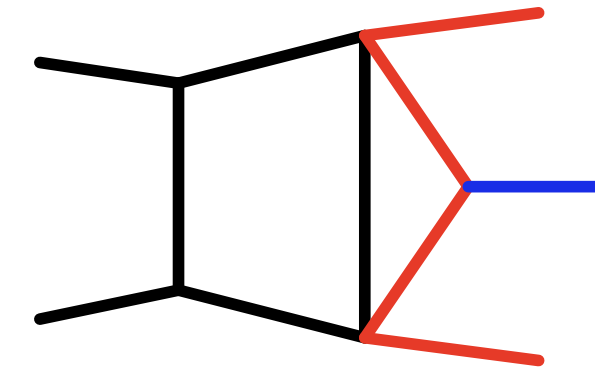
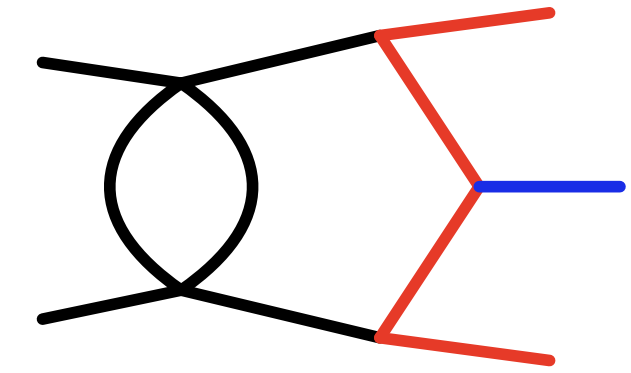
- U_0^{Γ} is associated with the 4d locus on the maximal cut
- On $U_1^{\Gamma_1}$: $\bar{l}_2 - q_4$ is soft
- On $U_2^{\Gamma_1}$: $\bar{l}_1$ is collinear to $\bar{l}_2$

$$J_{\text{syz}}^{\Gamma_1, [U_1^{\Gamma_1}]} = J_{4d} \cap J_{\text{e-col}}^{l_2, p_5} \cap J_{\text{e-col}}^{l_2 - q_4, p_4} = \langle \mu_{12}, \mu_{22}, \mu_{11} l_2^\perp, z_{2,0} z_{2,3} \mu_{11}, z_{2,4} \mu_{11} \rangle$$

$$J_{\text{syz}}^{\Gamma_1, [U_2^{\Gamma_1}]} = J_{4d} \cap J_{\text{soft}}^{l_1 - l_2} \cap J_{\text{i-col}}^{l_1, l_2} = \langle l_1^\nu \mu_{22} - l_2^\nu \mu_{12}, z_{1,0}(\mu_{22} - \mu_{12}), z_{1,0}(\mu_{11} - \mu_{22}), B \rangle + (1 \leftrightarrow 2)$$

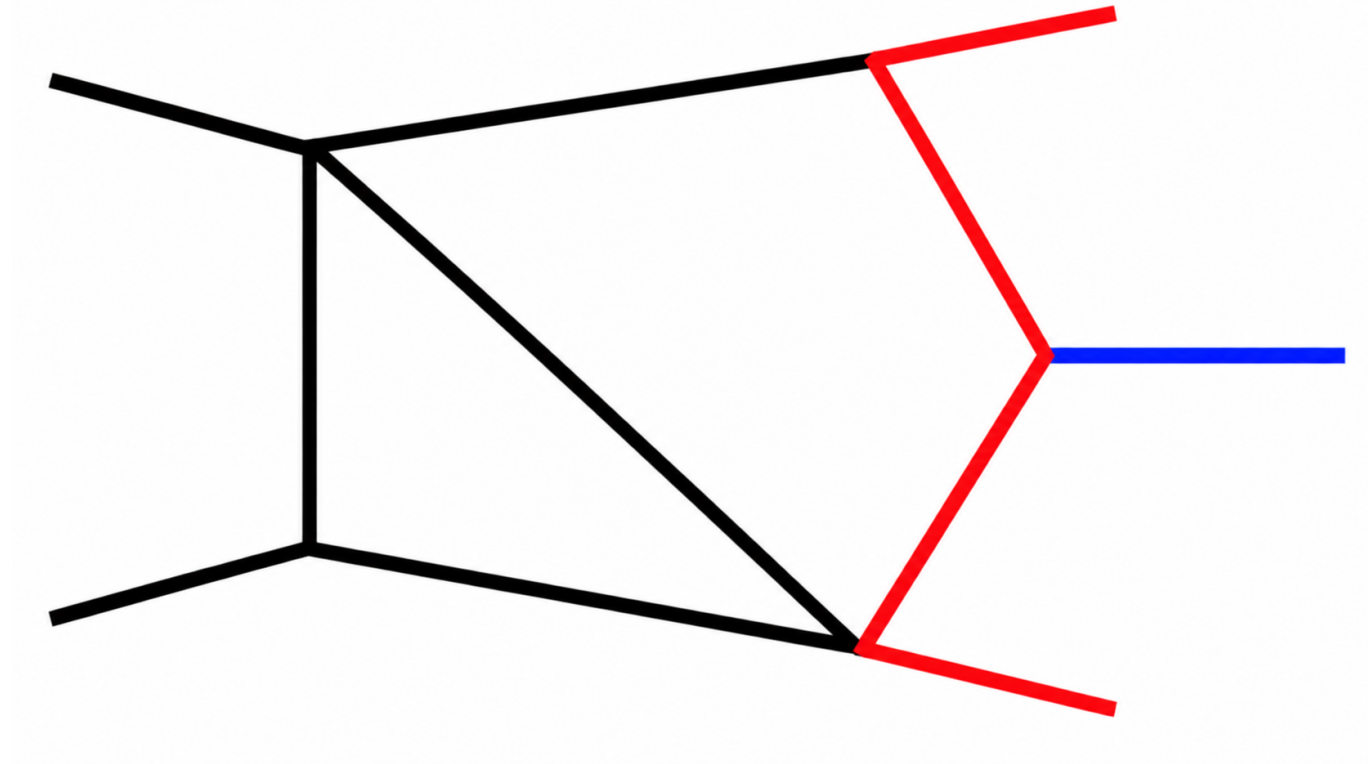
Expressing the generators of J_{syz} in terms of those explicit representations, we obtain the matrices C_1, C_2

$$pp \rightarrow t\bar{t}H$$


$$\Gamma_2$$

$$\Gamma_3$$

$$\Gamma_4$$

$$\Gamma_5$$

$$\Gamma_6$$

- Universality of IR singularities is reflected in our construction
- It is possible to reuse blocks of the Fitting construction
- For these six topologies, only a small set of new distinct building blocks need to be computed

$$pp \rightarrow t\bar{t}H$$



$$U_{Landau}^{\Gamma_4} = U_0^{\Gamma_4} \cup U_1^{\Gamma_4} \cup U_2^{\Gamma_4}$$

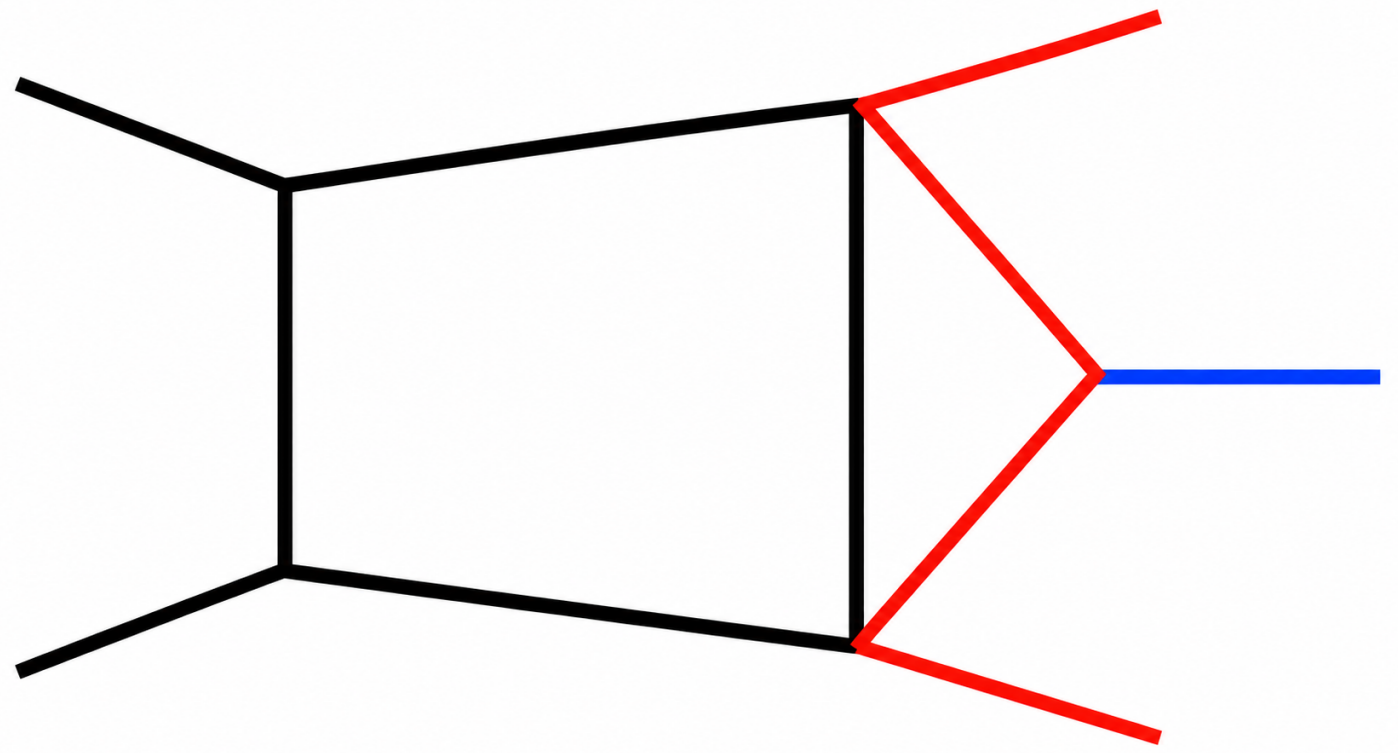
- $U_0^{\Gamma_4}$ is associated with the 4d locus on the maximal cut
- On $U_1^{\Gamma_4}$: $\bar{l}_2$ is collinear to p_5
- On $U_2^{\Gamma_4}$: $\bar{l}_1 - \bar{l}_2$ is soft

$$J_{sy\bar{z}}^{\Gamma_4, [U_1^{\Gamma_4}]} = \langle \mu_{11}, \mu_{22}, \mu_{11} l_2^{\perp \nu}, z_{2,0} \mu_{11}, z_{2,4} \mu_{11} \rangle$$

$$J_{sy\bar{z}}^{\Gamma_4, [U_2^{\Gamma_4}]} = \langle \mu_{11} - \mu_{12}, \mu_{11} - \mu_{22}, \mu_{11} (l_1 - l_2)^\nu \rangle$$

Here, the pre-localization computation is novel and we need to compute the matrices C_1, C_2 from the new generators

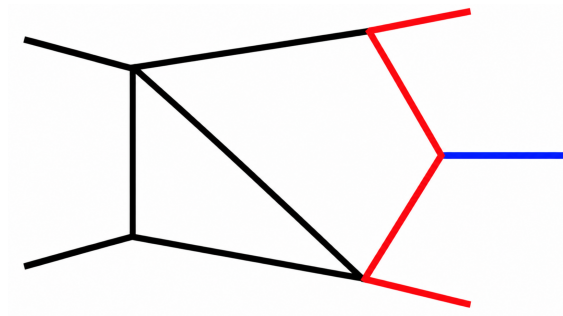
$pp \rightarrow t\bar{t}H$



$$U_{Landau}^{\Gamma_5} = U_0^{\Gamma_5} \cup U_1^{\Gamma_5} \cup U_2^{\Gamma_5}$$

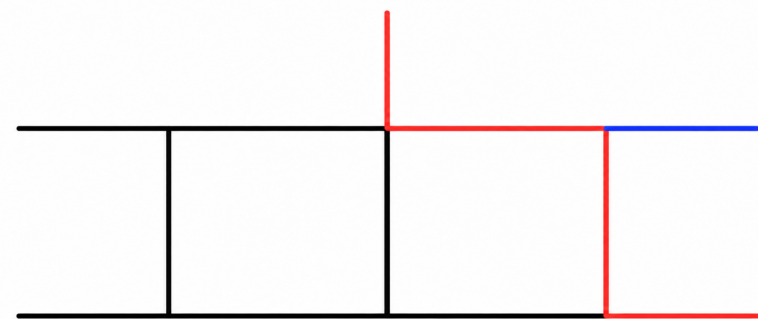
- $U_0^{\Gamma_5}$ is associated with the 4d locus on the maximal cut
- On $U_1^{\Gamma_5}$: $\bar{l}_1 - \bar{l}_2$ is soft
- On $U_2^{\Gamma_5}$: $\bar{l}_2 - q_4$ is soft

$$J_{syz}^{\Gamma_5, [U_1^{\Gamma_5}]}$$



$$J_{syz}^{\Gamma_4, [U_2^{\Gamma_4}]} = \langle \mu_{11} - \mu_{12}, \mu_{11} - \mu_{22}, \mu_{11}(l_1 - l_2)^\nu \rangle$$

$$J_{syz}^{\Gamma_5, [U_2^{\Gamma_5}]}$$



$$J_{syz}^{\Gamma_1, [U_1^{\Gamma_1}]} = \langle \mu_{12}, \mu_{22}, \mu_{11} l_2^\perp, z_{2,0} z_{2,3} \mu_{11}, z_{2,4} \mu_{11} \rangle$$

In both cases, no further computation is required

Conclusions

- Under certain hypotheses, we only need a subset of critical syzygies Often satisfied in practice
- Integral relations are controlled by IR singularities
- Universality of IR singularities is reflected in the construction of critical syzygies
- Determinantal construction of analytic syzygies

Thank you for your attention