

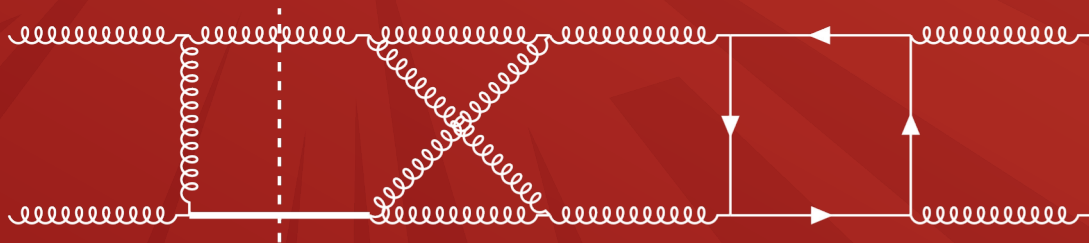
Taking the First Steps Toward Higgs Production at N⁴LO in QCD

Adi Suresh

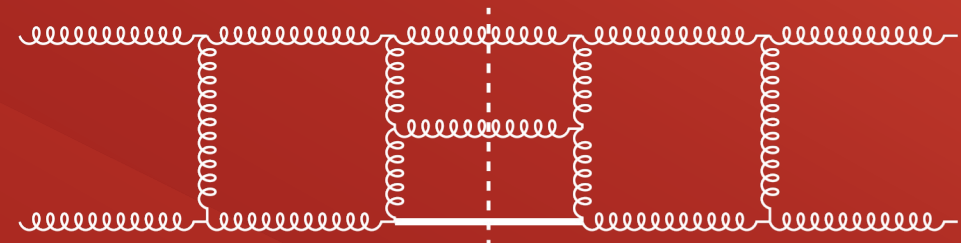
In collaboration with Bernhard Mistlberger, Xin Guan, Michael Ruf, and Sumit Banik

QCD@LHC, University of Oxford

October 1st, 2026

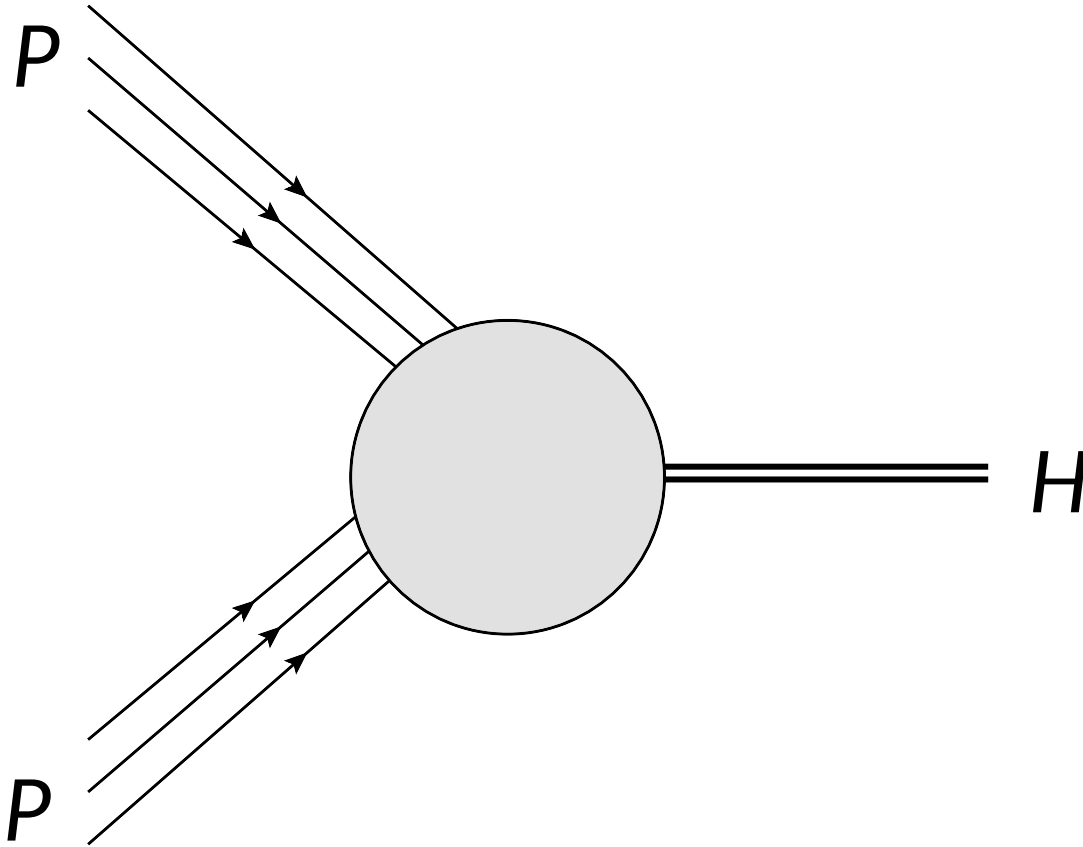


arXiv:26xx.xxxxx



arXiv:2606.xxxxx

How many Higgs bosons are produced at the LHC?



And how many did we expect...?

- i.e. what is the production rate?
- Signal strength:

$$\mu = \sigma_{\text{obs}} / \sigma_{\text{SM}}$$

Observed cross
section

Standard Model
calculation

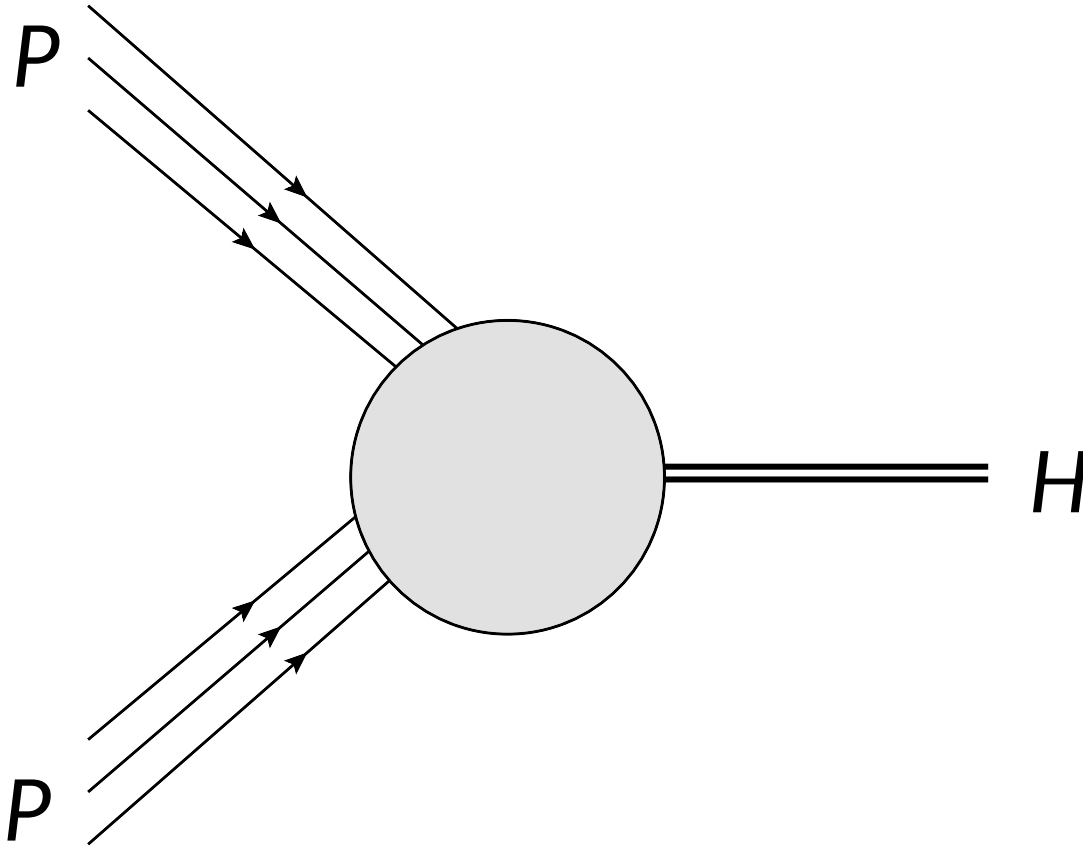
- ATLAS Run 2 (from *Nature* **607**, 52–59 (2022)):

$$\mu = 1.05 \pm 0.06$$

$$= 1.05 \pm 0.03(\text{stat.}) \pm 0.03(\text{exp.})$$

$$\pm 0.04(\text{sig. th.}) \pm 0.02(\text{bkg. th.})$$

How many Higgs bosons are produced at the LHC?



And how many did we expect...?

- i.e. what is the production rate?
- Signal strength:

$$\mu = \sigma_{\text{obs}} / \sigma_{\text{SM}}$$

Observed cross section

Standard Model calculation

- **ATLAS Run 2 Update**

(from ATLAS-CONF-2025-006):

$$\mu = 1.023^{+0.056}_{-0.053}$$

$$= 1.023 \pm 0.028(\text{stat.}) \pm 0.026^{+0.026}_{-0.025}(\text{exp.})$$

$$\pm 0.039^{+0.039}_{-0.036}(\text{sig. th.}) \pm 0.012(\text{bkg. th.})$$

In the foreseeable future:

LHC Era	(Projected) Integrated Luminosity (fb ⁻¹)
Run 2	140
Run 3 (2026)	330
HL-LHC (2040)	3000+

We will be collecting a lot more data!

- Along with improvements in experimental uncertainty, **much better statistics!**
- The LHC will deliver measurements at **the percent level!**
- We will need to get more precise on the theory side...

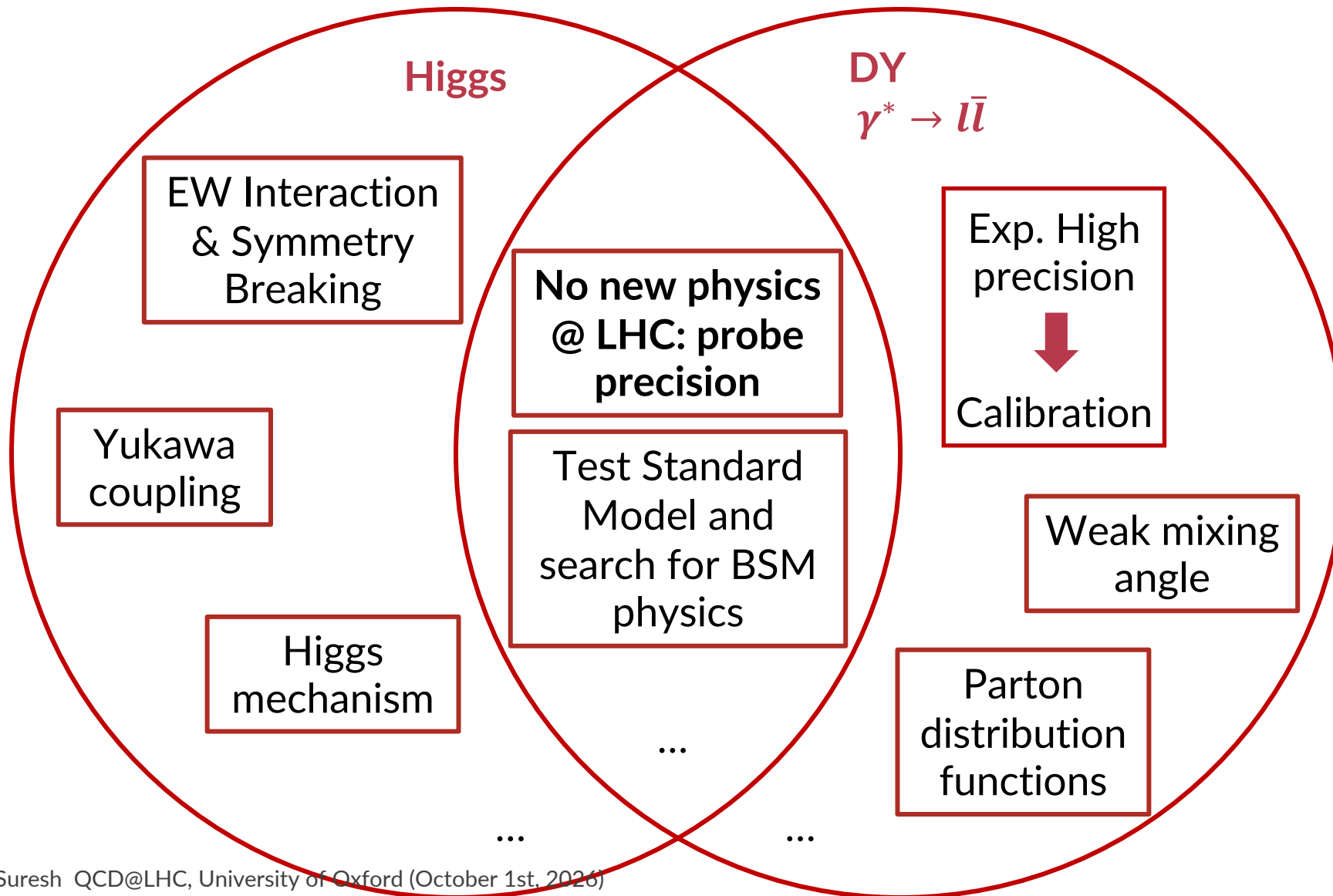
Outline

1. Higgs boson production at hadron colliders.
 - How and why do we count them?
 - How are they produced?
2. The journey to the precision frontier.
3. The single-real ingredients.
4. Toward double-real ingredients and beyond.

Outline

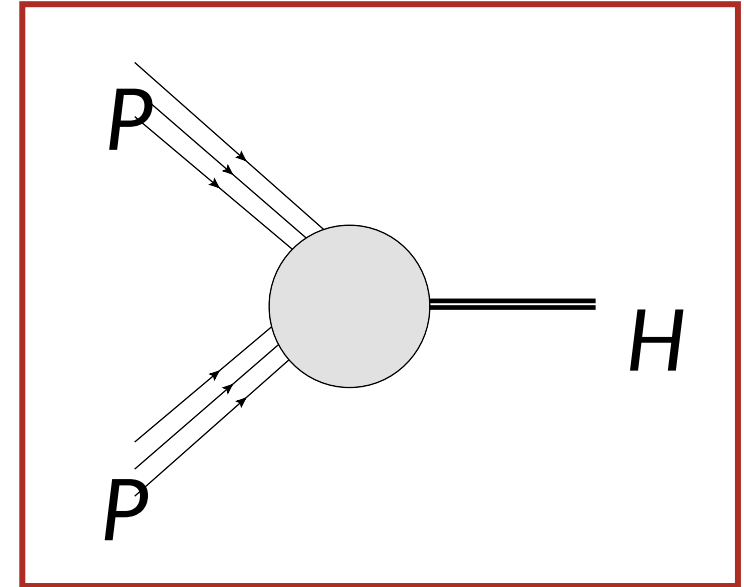
1. Higgs boson production at hadron colliders.
 - How and why do we count them?
 - How are they produced?
2. The journey to the precision frontier.
3. The single-real ingredients.
4. Toward double-real ingredients and beyond.

Why probe Higgs and Drell-Yan Production?



How are Higgs bosons produced at the LHC?

What we want: $\sigma_{PP \rightarrow H+X}$

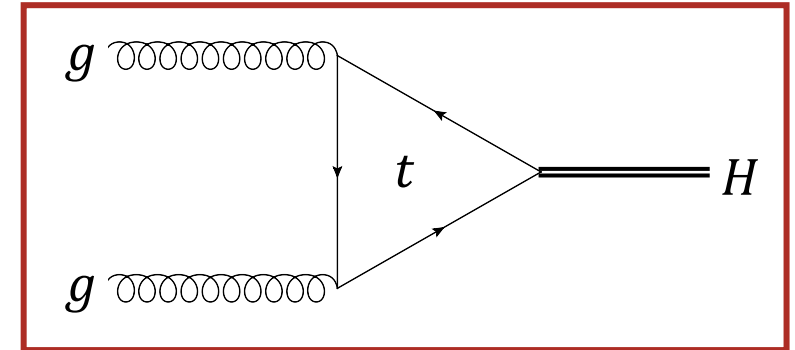
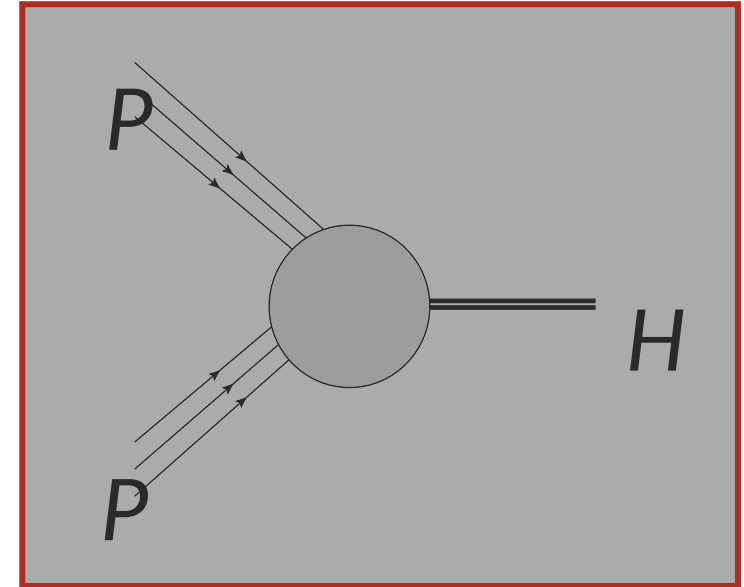


How are Higgs bosons produced at the LHC?

What we **want**: $\sigma_{PP \rightarrow H+X}$

vs.

What we can **compute**: $\hat{\sigma}_{ij \rightarrow H+X}$



How are Higgs bosons produced at the LHC?

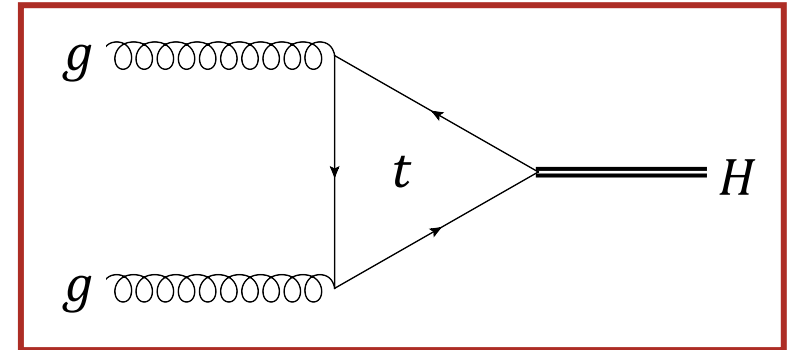
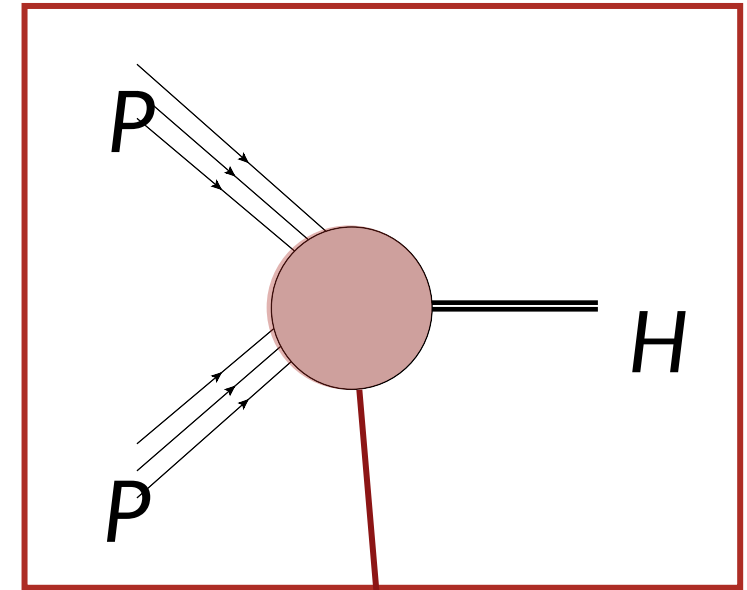
Factorization model:

$$\sigma = \sum_{i,j} f_i \otimes f_j \otimes \hat{\sigma}_{ij}$$

Hadronic process
(long-range,
nonperturbative)

PDFs (proton
substructure)

Partonic process
(short-range,
perturbative!)

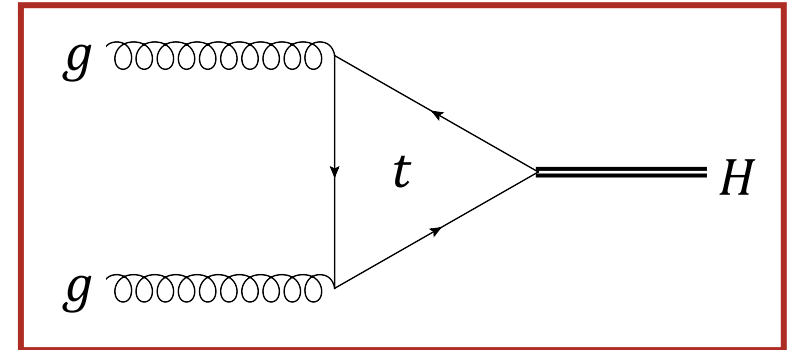
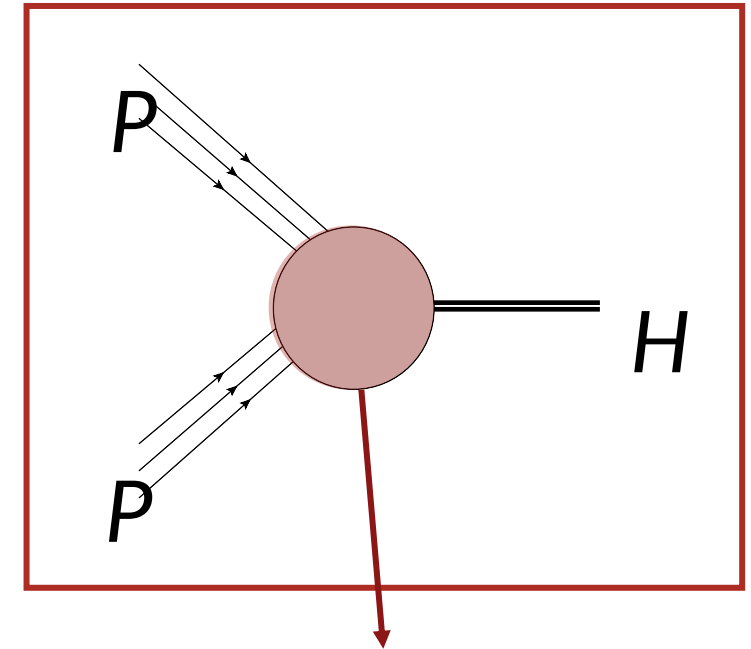


How are Higgs bosons produced at the LHC?

Factorization model theorem:

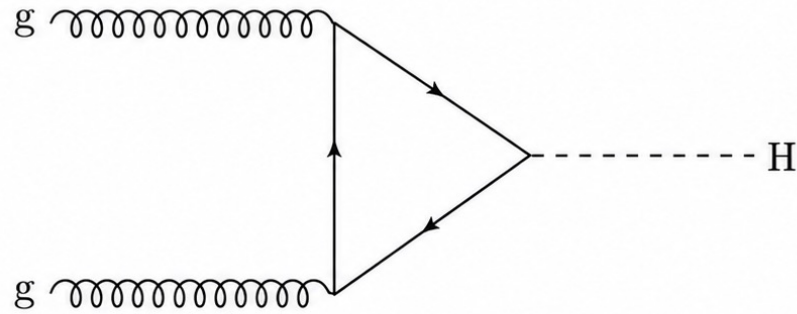
$$\sigma = \sum_{i,j} f_i \otimes f_j \otimes \hat{\sigma}_{ij} + \mathcal{O}(\Lambda_{QCD}^2/Q^2)$$

- Rigorously shown to be valid for inclusive color singlet production*

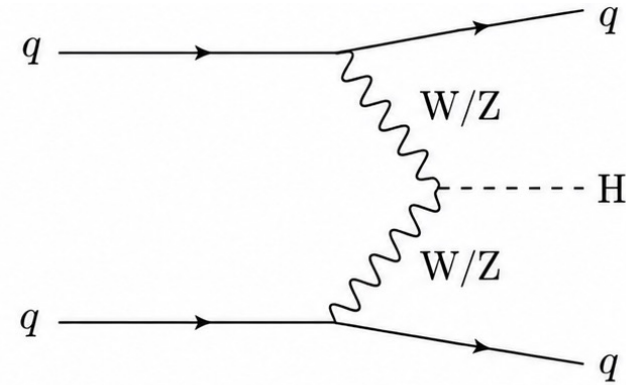


*[J. C. Collins, D. E. Soper, G. Sterman, Nucl. Phys. B261 (1985) 104; Nucl. Phys. B308 (1988) 833.]

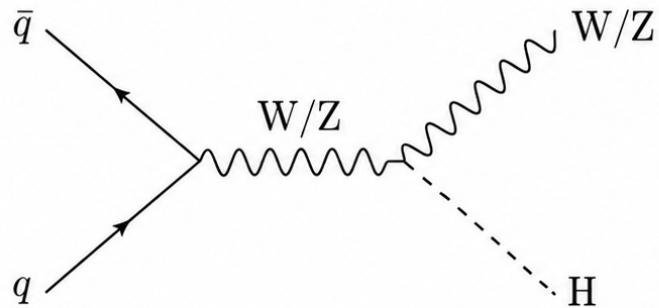
How are Higgs bosons produced at the LHC partonically?*



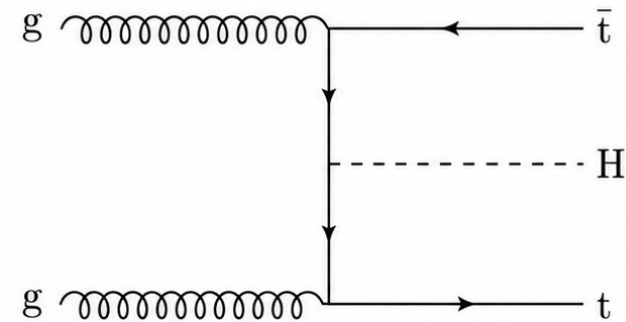
Gluon-fusion



Vector-boson fusion



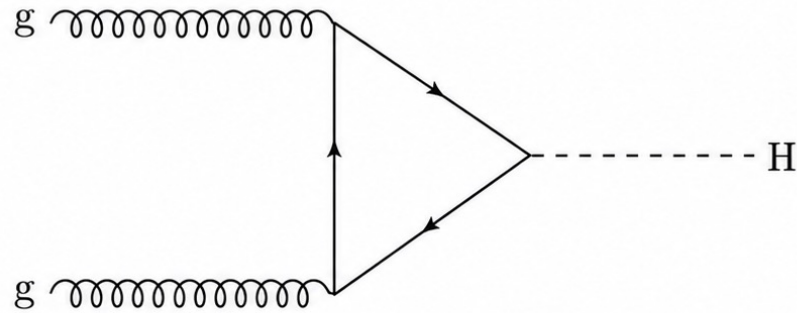
***W/Z* Associated**



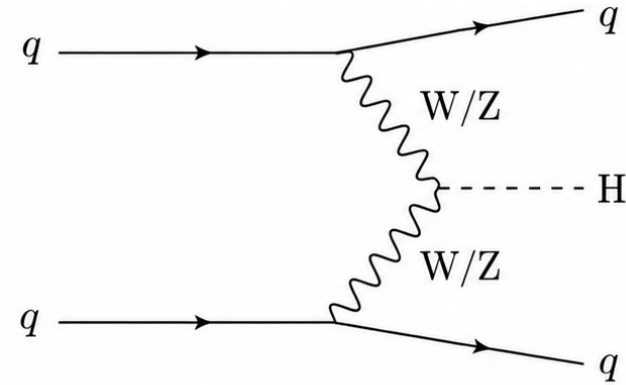
***t\bar{t}* associated**

*[Higgs Cross Section
Working Group;
arXiv:1610.07922]

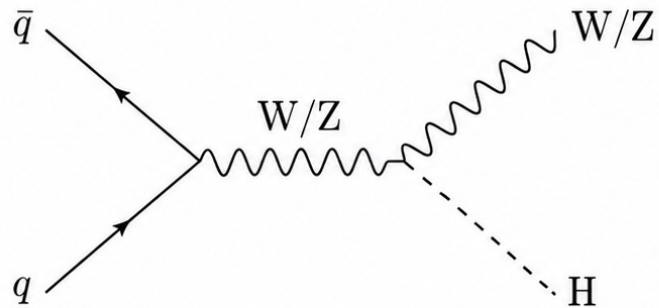
How are Higgs bosons produced at the LHC partonically?*



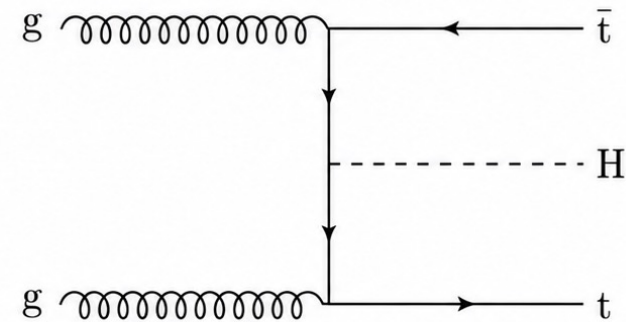
Gluon-fusion 87%



Vector-boson fusion 7%



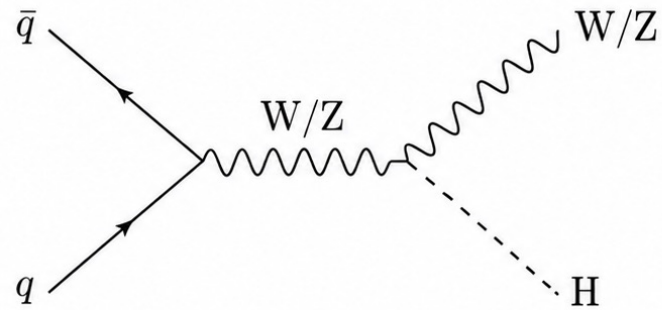
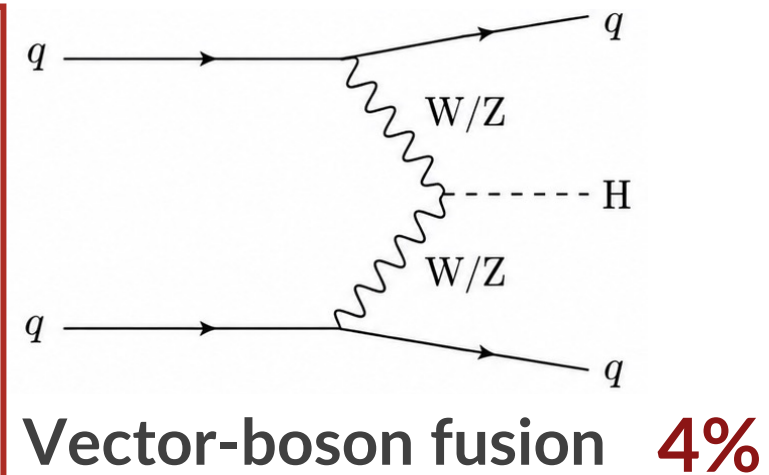
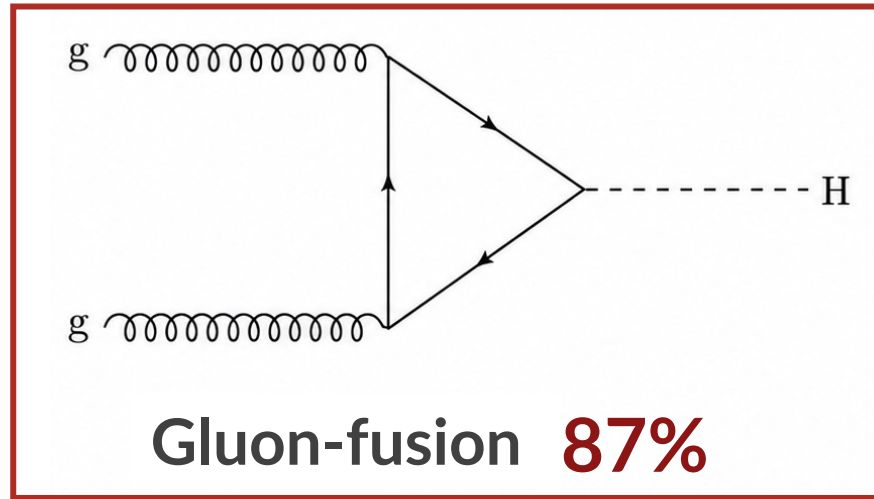
W/Z Associated 4%



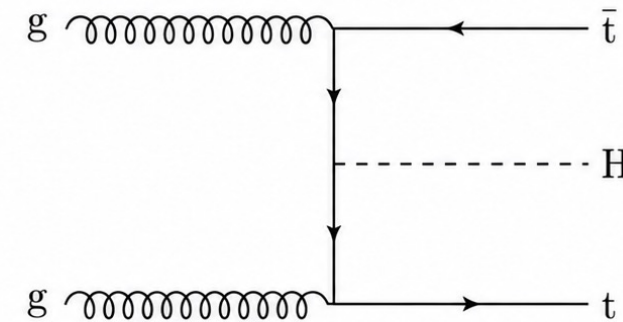
$t\bar{t}$ associated 1%

*[Higgs Cross Section Working Group; arXiv:1610.07922]

How are Higgs bosons produced at the LHC partonically?*



W/Z Associated 7%

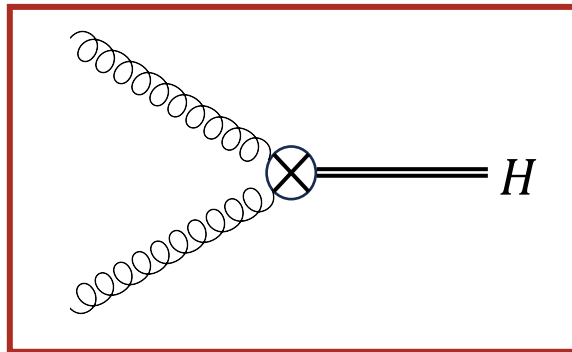
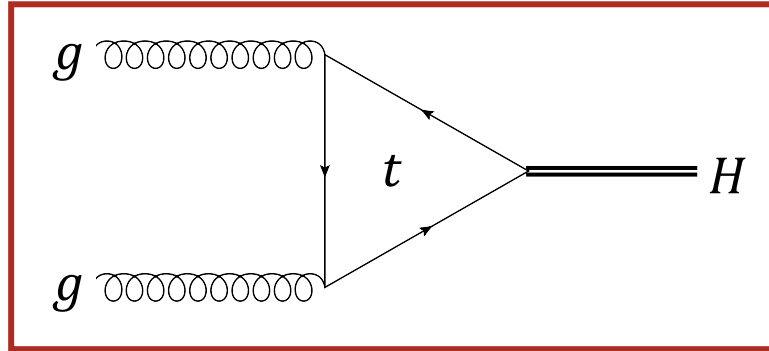


$t\bar{t}$ associated 1%

*[Higgs Cross Section Working Group; arXiv:1610.07922]

How are Higgs bosons produced at the LHC partonically?

About 90% are produced via gluon fusion!



$$\mathcal{L} \supset -\frac{1}{4} C^0 H G_{\mu\nu}^a G_a^{\mu\nu}$$

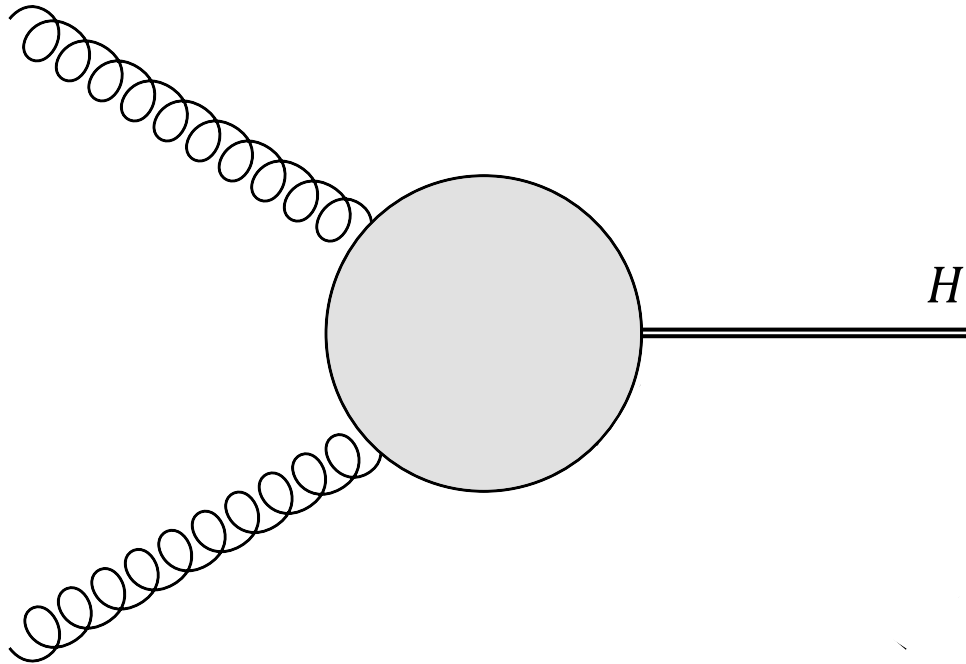
- $gg \rightarrow H$ via top quark loop
- We can simplify:
take “heavy-top” limit $m_{\text{top}} \rightarrow \infty$
 - i.e. leading term in m_H/m_{top} expansion
 - Higgs looks like it directly couples to gluons

Outline

1. Higgs boson production at hadron colliders.
 - How and why do we count them?
 - How are they produced?
2. The journey to the precision frontier.
3. The single-real ingredients.
4. Toward double-real ingredients and beyond.

How do we compute how many Higgs bosons are produced?

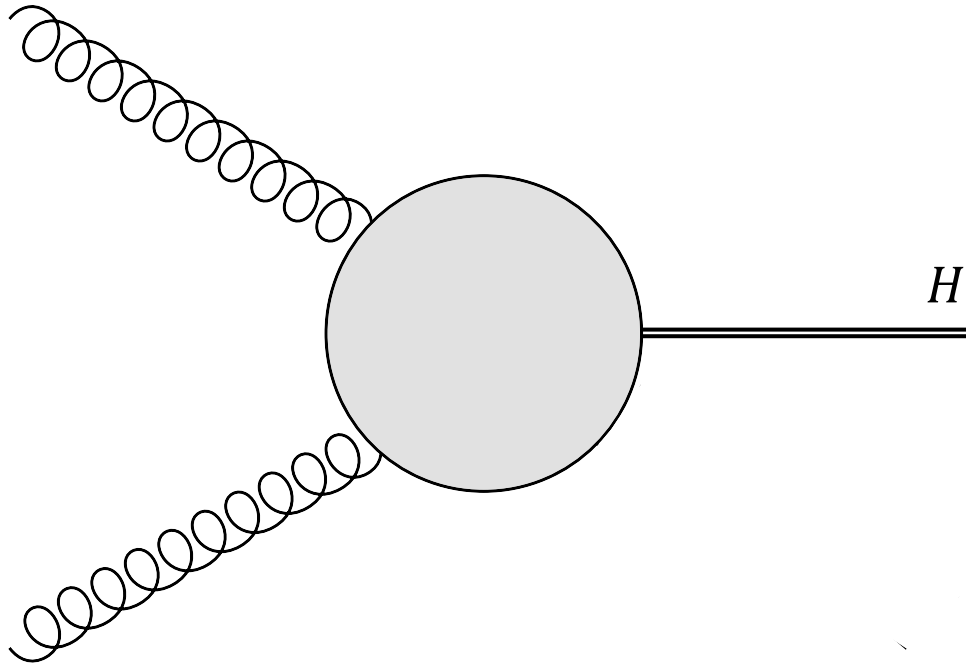
To correctly compute an **inclusive** cross section:



- i.e. probability of Higgs produced

How do we compute how many Higgs bosons are produced?

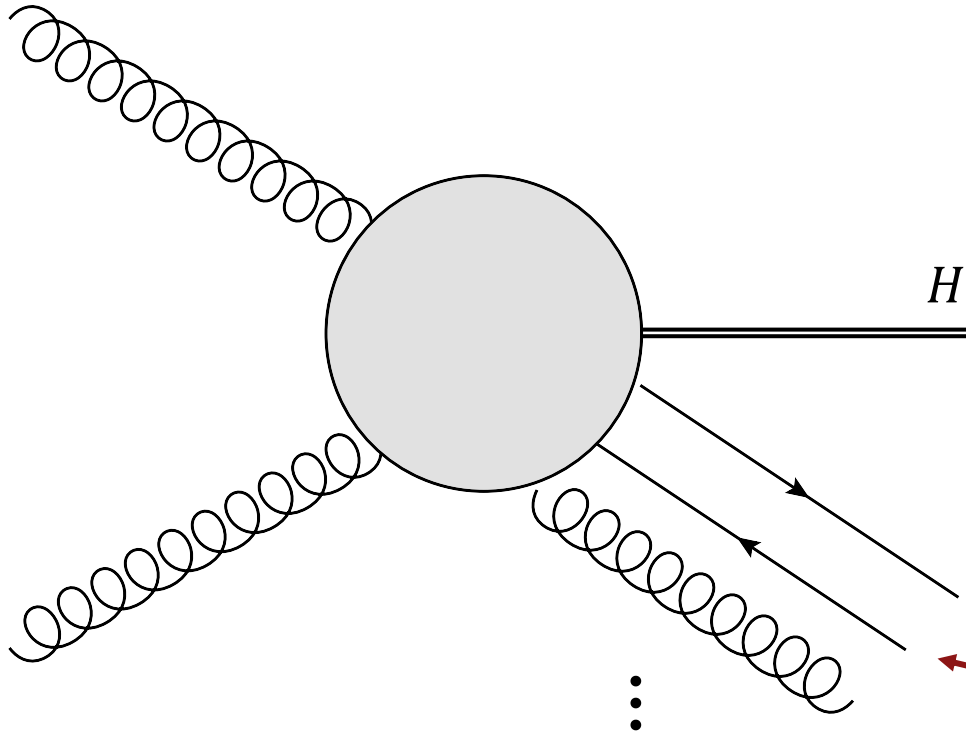
To correctly compute an **inclusive** cross section:



- i.e. probability of Higgs produced
- KLN theorem:
Perturbative corrections must include:
 - loops

How do we compute how many Higgs bosons are produced?

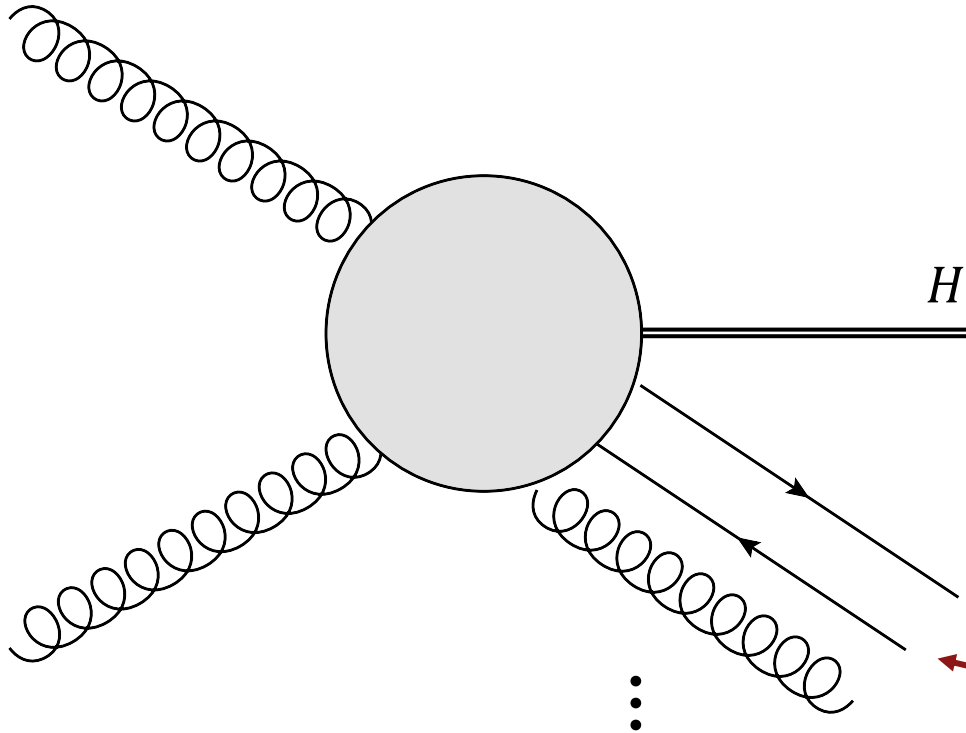
To correctly compute an **inclusive** cross section:



- i.e. probability of Higgs produced
- KLN theorem:
Perturbative corrections must include:
 - loops &
 - **Radiation**

How do we compute how many Higgs bosons are produced?

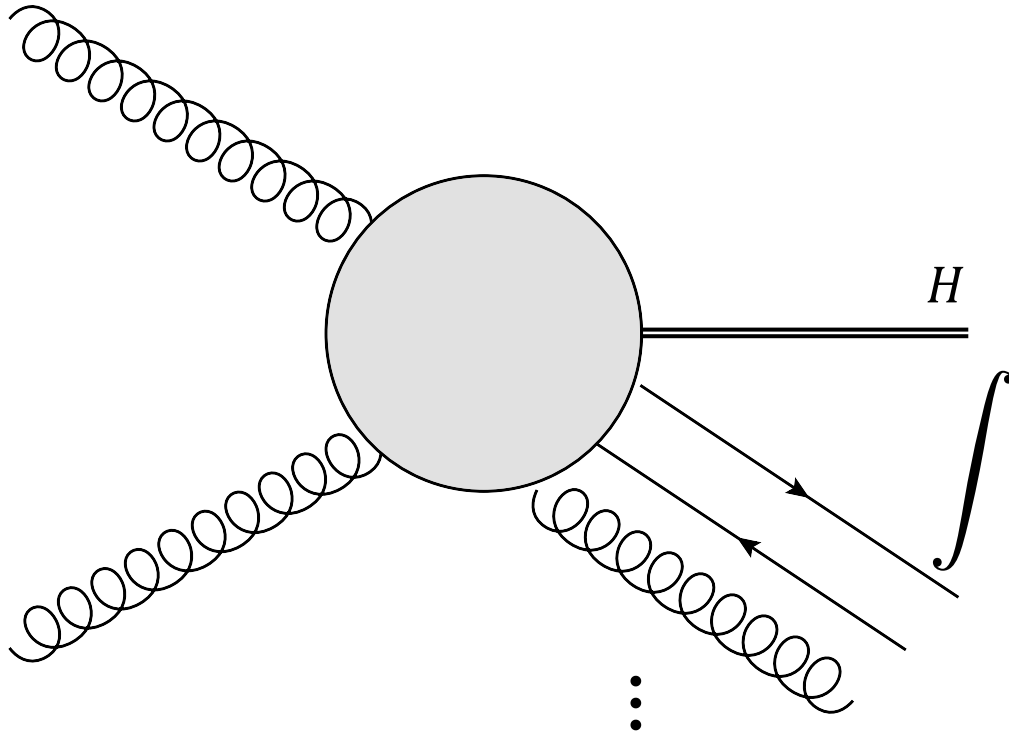
To correctly compute an **inclusive** cross section:



- i.e. probability of *Higgs + X* produced
- KLN theorem:
Perturbative corrections must include:
 - loops &
 - **Radiation**

Building blocks at $N^k\text{LO}$:

To correctly compute an **inclusive** cross section:



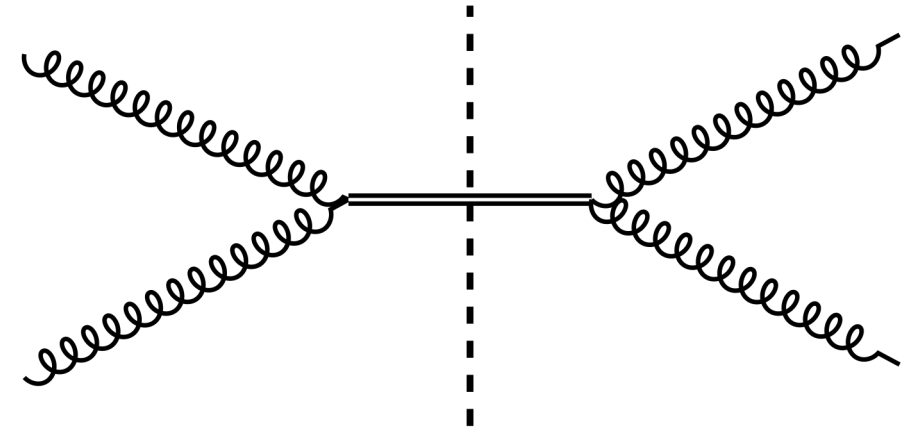
- Interfere up to k loops and k additional partons: $L + M = k$
- Integrate over $M + 1$ final state phase space
- Each piece is well-defined, gauge invariant
- Natural way to partition the computation

Theoretical Uncertainty for Gluon-fusion Higgs

Contribution* of the QCD perturbative expansion at each order (at 13 TeV):

*[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger; arXiv:1602.00695]

$$\begin{aligned}\hat{\sigma} = & \hat{\sigma}_{\text{LO}} \\ & + \hat{\sigma}_{\text{NLO}} \\ & + \hat{\sigma}_{\text{NNLO}} \\ & + \hat{\sigma}_{\text{N3LO}} \\ & + \dots\end{aligned}$$



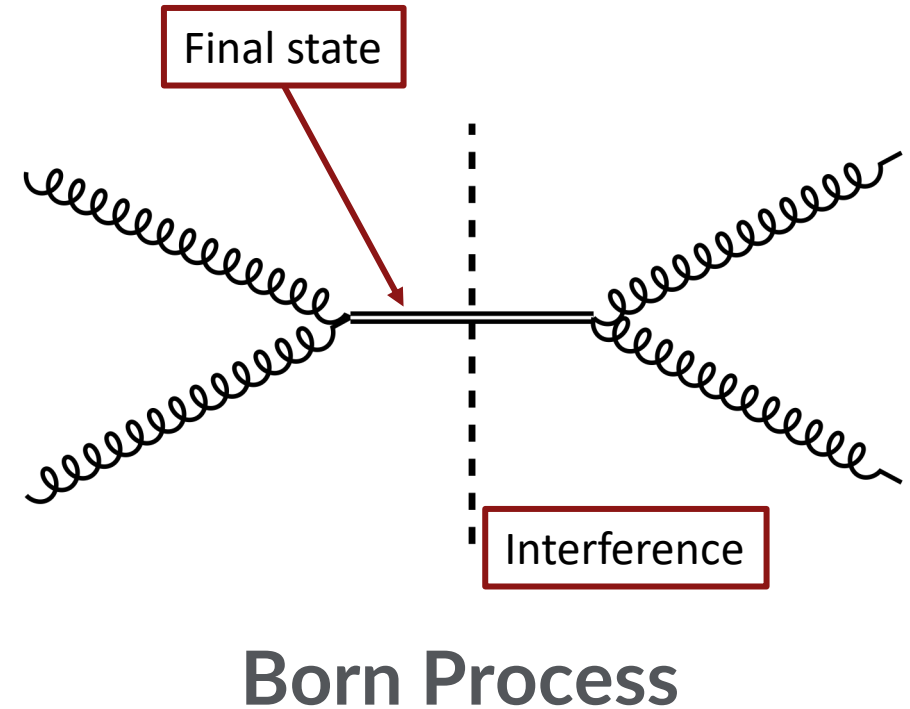
Born Process

Theoretical Uncertainty for Gluon-fusion Higgs

Contribution* of the QCD perturbative expansion at each order (at 13 TeV):

*[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger; arXiv:1602.00695]

$$\begin{aligned}\hat{\sigma} = & \hat{\sigma}_{\text{LO}} \\ & + \hat{\sigma}_{\text{NLO}} \\ & + \hat{\sigma}_{\text{NNLO}} \\ & + \hat{\sigma}_{\text{N3LO}} \\ & + \dots\end{aligned}$$



Theoretical Uncertainty for Gluon-fusion Higgs

Correction* of the QCD perturbative expansion at each order (at 13 TeV):

*[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger; arXiv:1602.00695]

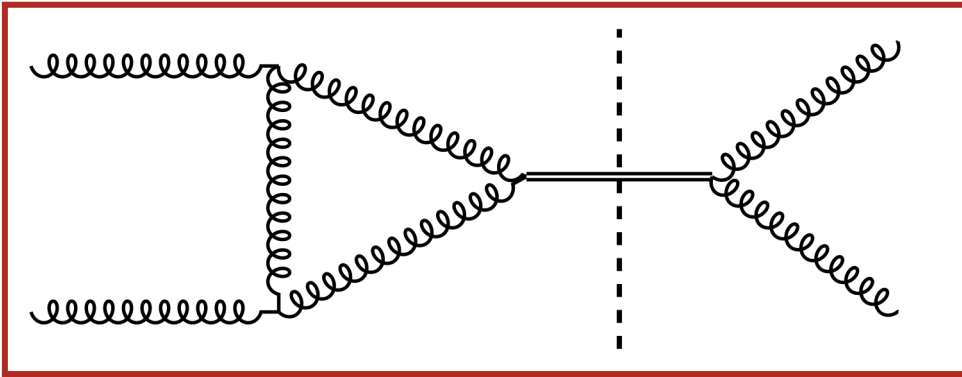
$$\begin{aligned}\hat{\sigma} = & \hat{\sigma}_{\text{LO}} \\ & + \hat{\sigma}_{\text{NLO}}^{**} \\ & + \hat{\sigma}_{\text{NNLO}} \\ & + \hat{\sigma}_{\text{N3LO}} \\ & + \dots\end{aligned}$$

$O(100\%)$

**[Dawson; Nucl.Phys.B 359 (1991), 283-300]

The Gluon-fusion Higgs at NLO*

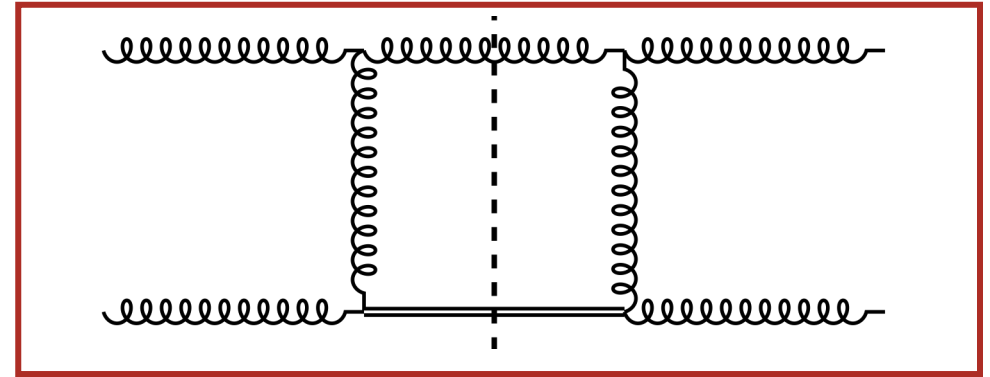
Ingredients:



V

Virtual Correction

+



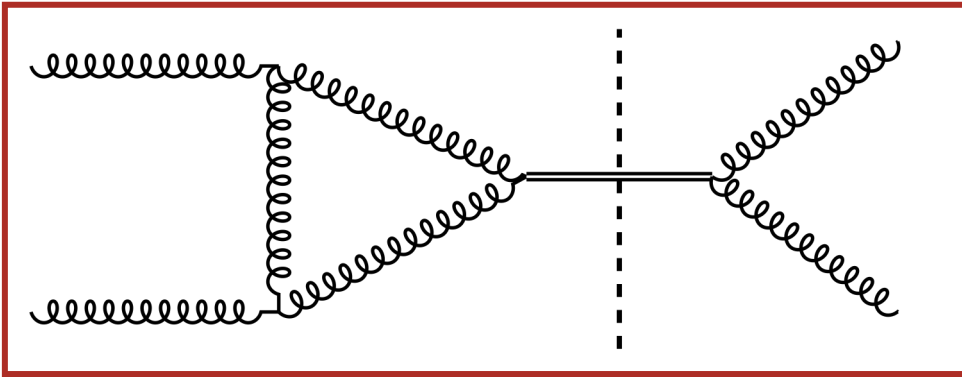
R

Real Correction

*[Dawson; Nucl.Phys.B 359 (1991), 283-300]

The Gluon-fusion Higgs at NLO*

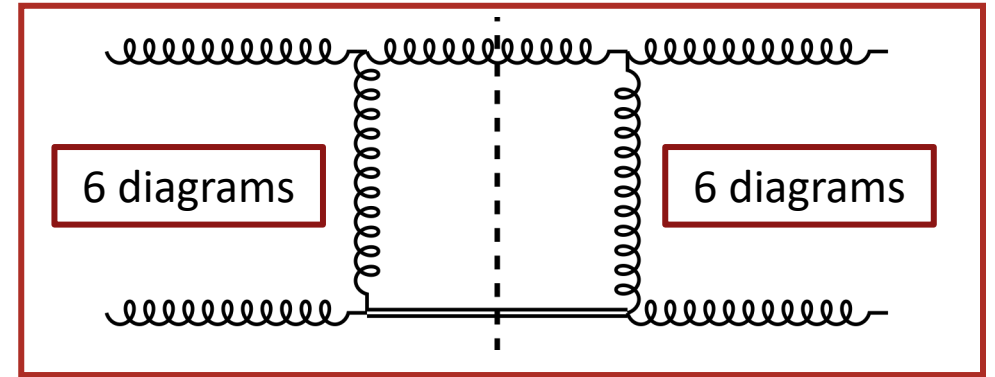
Ingredients:



V

Virtual Correction

+



R

Real Correction

*[Dawson; Nucl.Phys.B 359 (1991), 283-300]

Theoretical Uncertainty for Gluon-fusion Higgs

Correction* of the QCD perturbative expansion at each order (at 13 TeV):

*[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger; arXiv:1602.00695]

$$\begin{aligned}\hat{\sigma} = & \hat{\sigma}_{\text{LO}} \\ & + \hat{\sigma}_{\text{NLO}}^{**} \\ & + \hat{\sigma}_{\text{NNLO}}^{***} \\ & + \hat{\sigma}_{\text{N3LO}} \\ & + \dots\end{aligned}$$

$O(100\%)$

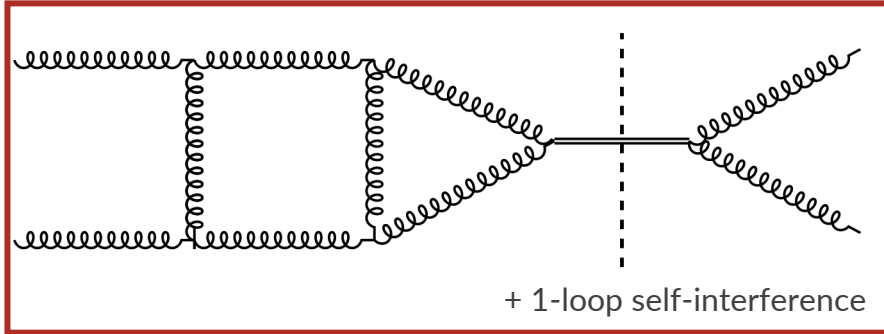
$O(25\%)$

**[Dawson; Nucl.Phys.B 359 (1991), 283-300]

***[Harlander, Kilgore; arXiv:hep-ph/0201206], [Anastasiou, Melnikov; arXiv:hep-ph/0207004], [Ravindran, Smith, Neerven; arXiv: hep-

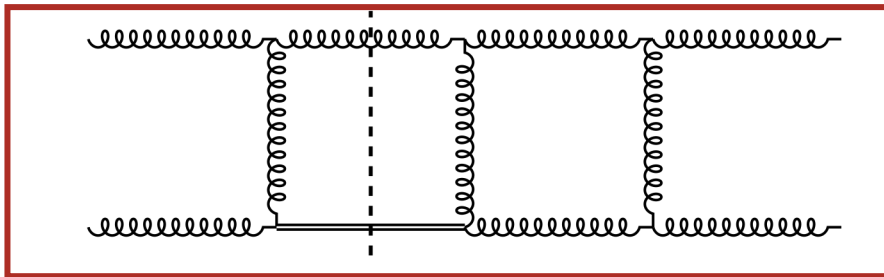
The Gluon-fusion Higgs at NNLO*

Ingredients:



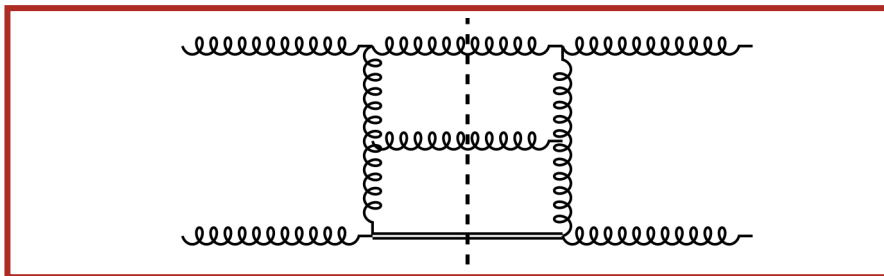
VV

Double-Virtual Correction



RV

Real-Virtual Correction

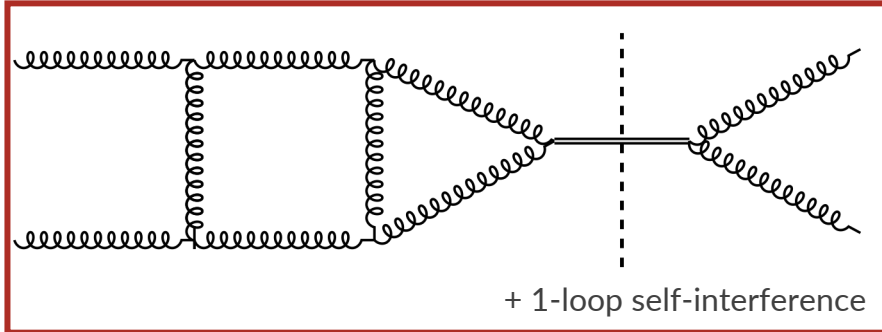


RR

Double-Real Correction

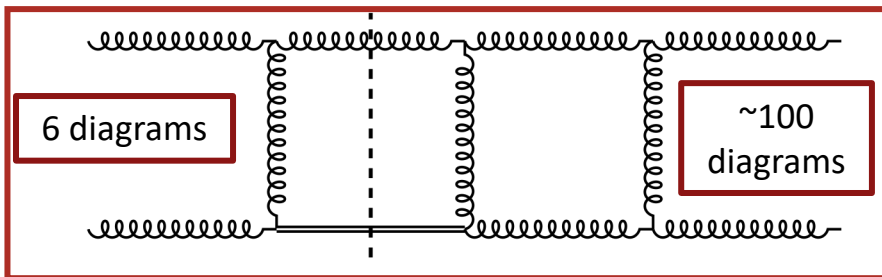
The Gluon-fusion Higgs at NNLO*

Ingredients:



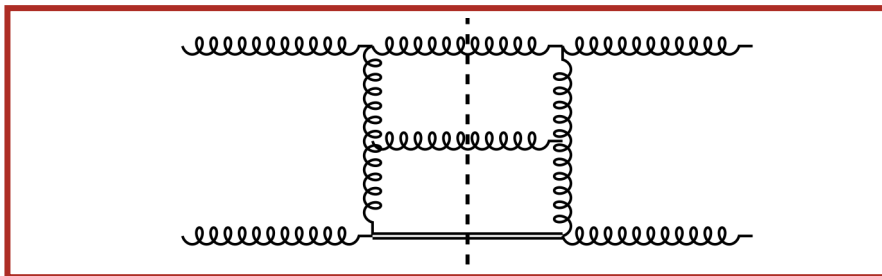
VV

Double-Virtual Correction



RV

Real-Virtual Correction

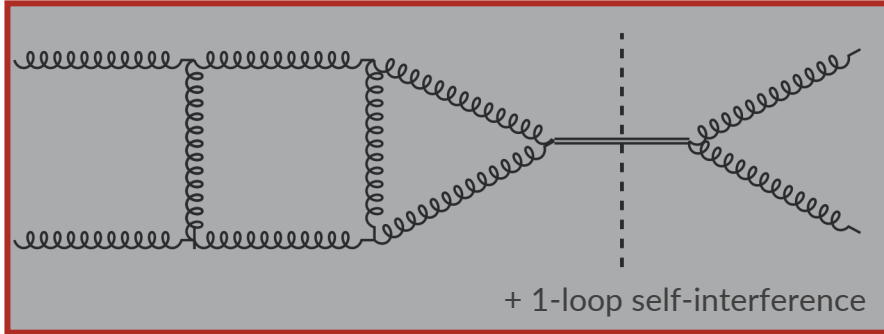


RR

Double-Real Correction

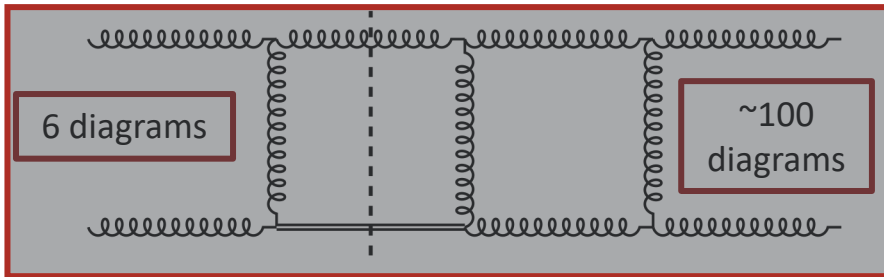
The Gluon-fusion Higgs at NNLO*

Ingredients:



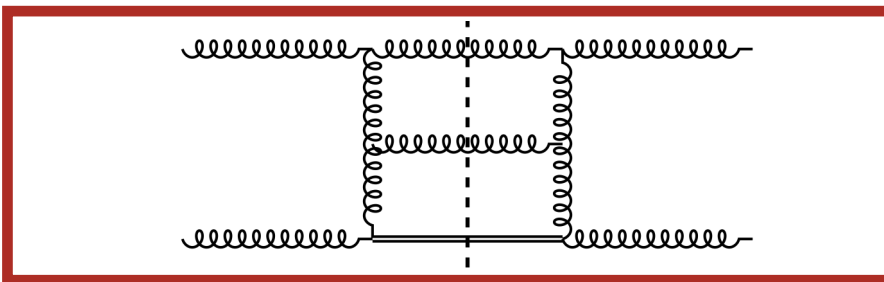
VV

Double-Virtual Correction



RV

Real-Virtual Correction

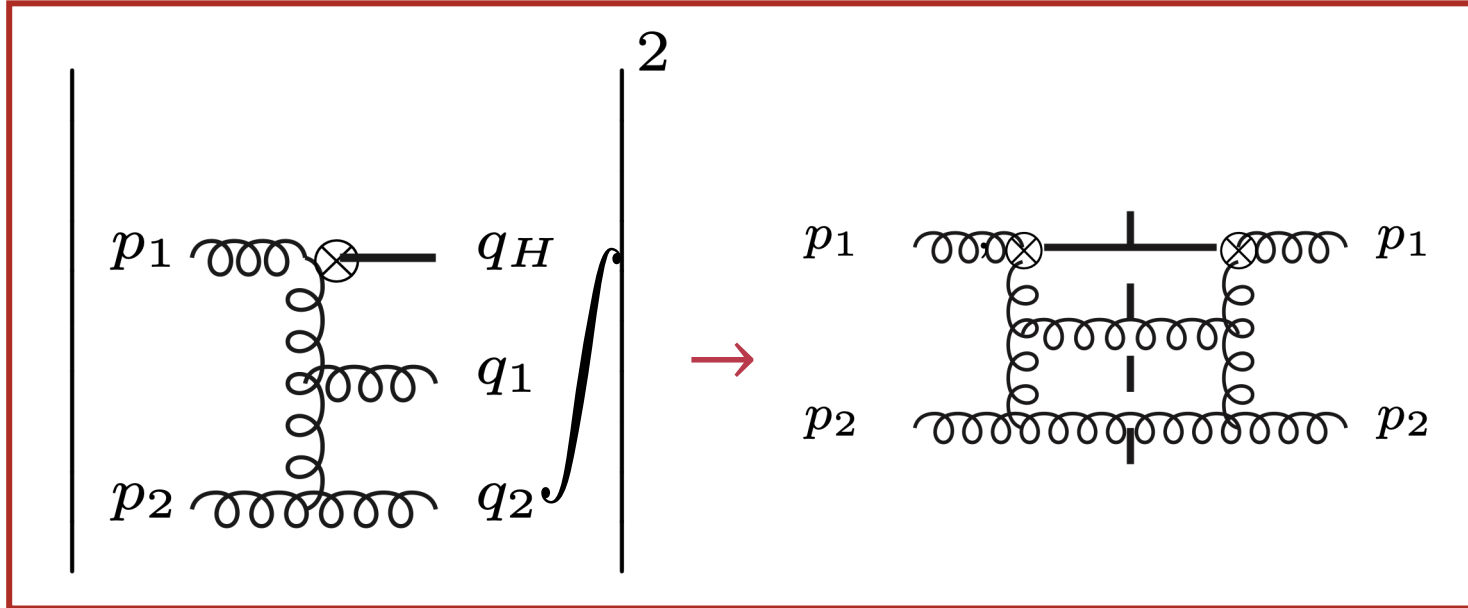


RR

Double-Real Correction

The Gluon-fusion Higgs at NNLO*

The reverse unitarity approach



$$(2\pi) \delta(q^2 - m^2) \rightarrow \frac{i}{q^2 - m^2 + i0^+} - \text{c. c.}$$

- Phase space integration becomes loop integration...
- Can use loop integration technology!

Theoretical Uncertainty for Gluon-fusion Higgs

Correction* of the QCD perturbative expansion at each order (at 13 TeV):

*[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger; arXiv:1602.00695]

$$\begin{aligned}\hat{\sigma} = & \hat{\sigma}_{\text{LO}} \\ & + \hat{\sigma}_{\text{NLO}}^{**} \\ & + \hat{\sigma}_{\text{NNLO}}^{***} \\ & + \hat{\sigma}_{\text{N3LO}}^* \\ & + \dots\end{aligned}$$

$O(100\%)$

$O(25\%)$

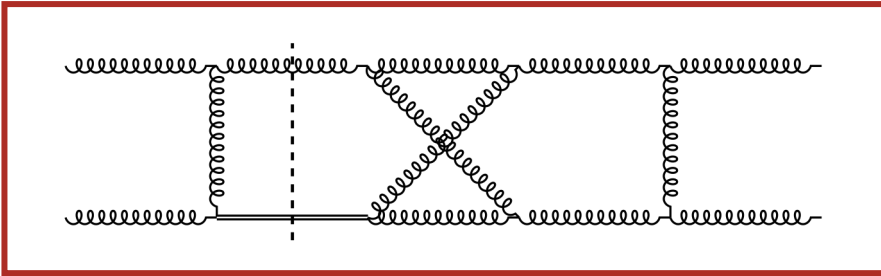
$O(5\%)$

**[Dawson; Nucl.Phys.B 359 (1991), 283-300]

***[Harlander, Kilgore; arXiv:hep-ph/0201206], [Anastasiou, Melnikov; arXiv:hep-ph/0207004]

The Gluon-fusion Higgs at N^3LO^*

Ingredients:



VVV

Triple-Virtual Corrections

RV²

Single-real Corrections

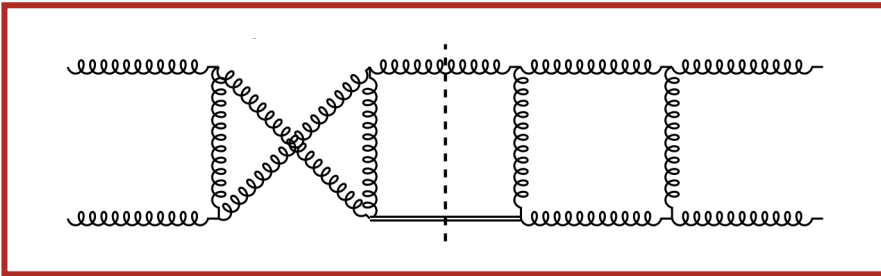
RVV

RRV

Double-real Correction

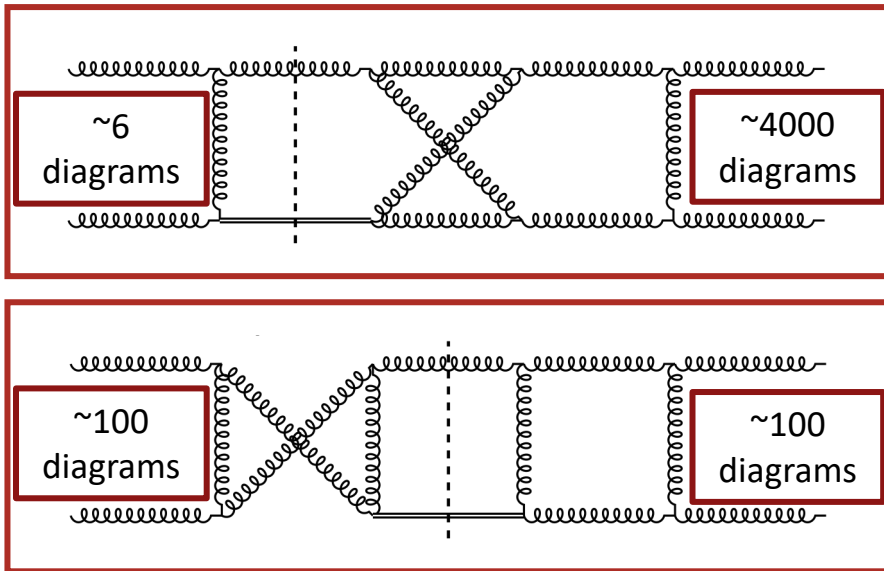
RRR

Triple-Real Correction



The Gluon-fusion Higgs at N^3LO^*

Ingredients:



VVV

Triple-Virtual Corrections

RV²

Single-real Corrections

RVV

RRV

Double-real Correction

RRR

Triple-Real Correction

Current Theoretical Uncertainty for Gluon-fusion Higgs

Uncertainty* due to solely truncating the QCD perturbative expansion (at 13 TeV):

*[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger; arXiv:1602.00695]

$$\begin{aligned}\hat{\sigma} = & \hat{\sigma}_{\text{LO}} \\ & + \hat{\sigma}_{\text{NLO}} \\ & + \hat{\sigma}_{\text{NNLO}} \\ & + \hat{\sigma}_{\text{N3LO}} \\ & + \times \dots\end{aligned}$$

- State of the art sufficient for current experimental needs
- But **N4LO** will take time to compute – **start now!**

$\sim 2.5\%$

Current Theoretical Uncertainty for Gluon-fusion Higgs

Uncertainty due to solely truncating the QCD perturbative expansion?

$$\begin{aligned}\hat{\sigma} = & \hat{\sigma}_{\text{LO}} \\ & + \hat{\sigma}_{\text{NLO}} \\ & + \hat{\sigma}_{\text{NNLO}} \\ & + \hat{\sigma}_{\text{N3LO}} \\ & + \hat{\sigma}_{\text{N4LO}}\end{aligned}$$

- State of the art sufficient for current experimental needs
- But **N4LO** will take **years** to compute – **start now!**

Aim for $\sim 1\%$

Outline

1. Higgs boson production at hadron colliders.
 - How and why do we count them?
 - How are they produced?
2. The journey to the precision frontier.
3. **The single-real ingredients.**
4. Toward double-real ingredients and beyond.

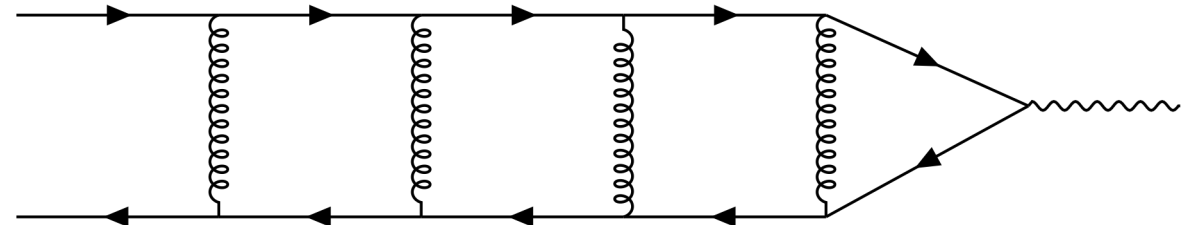
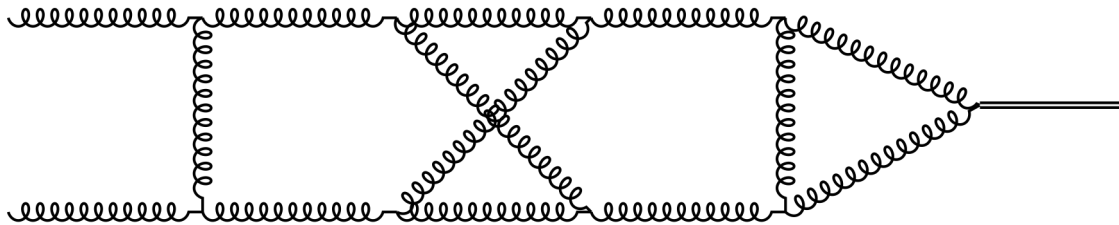
Fully virtual correction at N⁴LO

Four loop amplitudes, “VVVV” contribution:

- Higgs and Drell-Yan first known approximately (2020) *
- Full computation (2022) **

*[Das, Moch, Vogt; arXiv: 2004.00563]

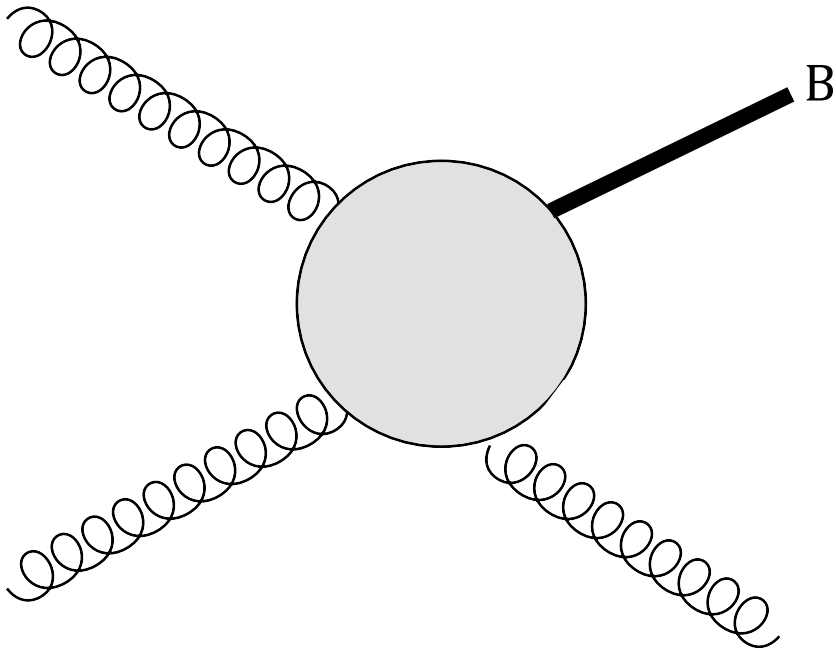
**[Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser; arXiv: 2202.04660]



Single-real Contributions

The single-real ingredient as a probe

Only one additional emission is “easy”:

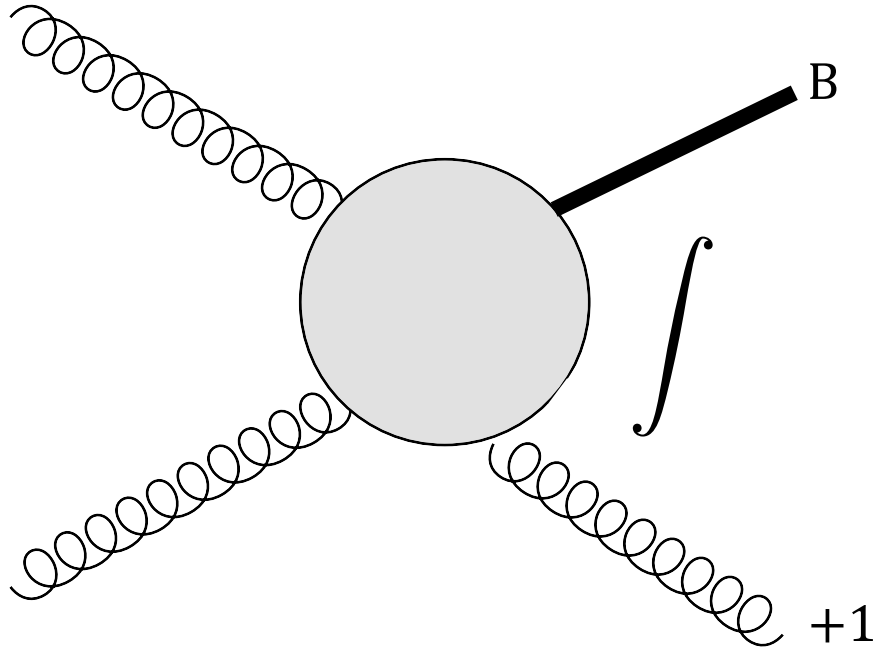


- Simplest (non-trivial) phase space
 - Can do **direct** phase space integration

Phase Space Integration

We only care about Higgs / DY production → Integrate over d.o.f. of final state

$$\hat{\sigma} \sim \int d\Phi_{B+1} A^{(1)} A^{*(2)}$$



Momentum conservation & rotational symmetry



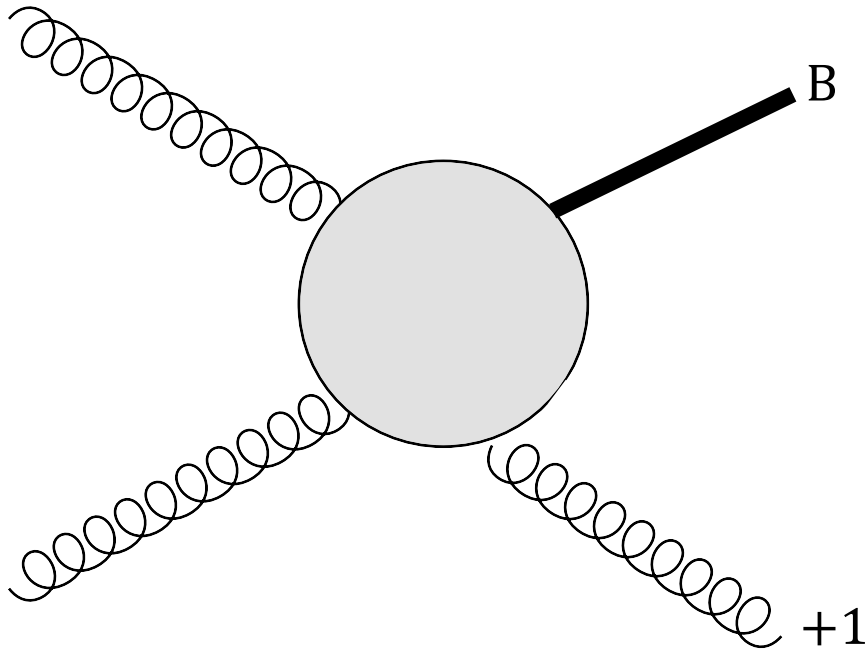
Only **one** actual d.o.f.:

Angle w.r.t. beamline:

$$\lambda \in [0, 1]$$

The single-real ingredient as a probe

Only one additional emission is “easy”:



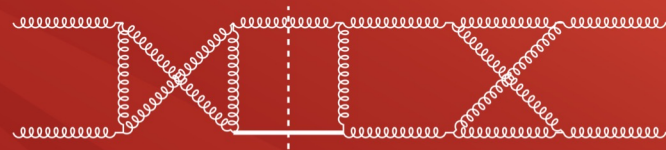
- Simplest (non-trivial) phase space
 - Can do **direct** phase space integration
 - Higher multiplicity involves the reverse unitarity approach (**loops + PS together**)
- Probe complexity at $N^4\text{LO}$:
 - Function space
 - Color
 - Computational tools

Last year's QCD@LHC

RVVxV: Interference contributions to inclusive Higgs boson and Drell-Yan production at N⁴LO in QCD

Adi Suresh & Bernhard Mistlberger

SLAC National Accelerator Laboratory



SLAC NATIONAL ACCELERATOR LABORATORY

arXiv:2504.10574

Stanford University U.S. DEPARTMENT of ENERGY

RVVxV Contribution*

- 2 loop x 1 loop interference
- Development:
 - integration technology
 - Computational pipeline
- Warm up for...

[Submitted on 14 Apr 2025]

RVVxV: Interference contributions to inclusive Higgs boson and Drell-Yan production at N⁴LO in QCD

Bernhard Mistlberger, Adi Suresh

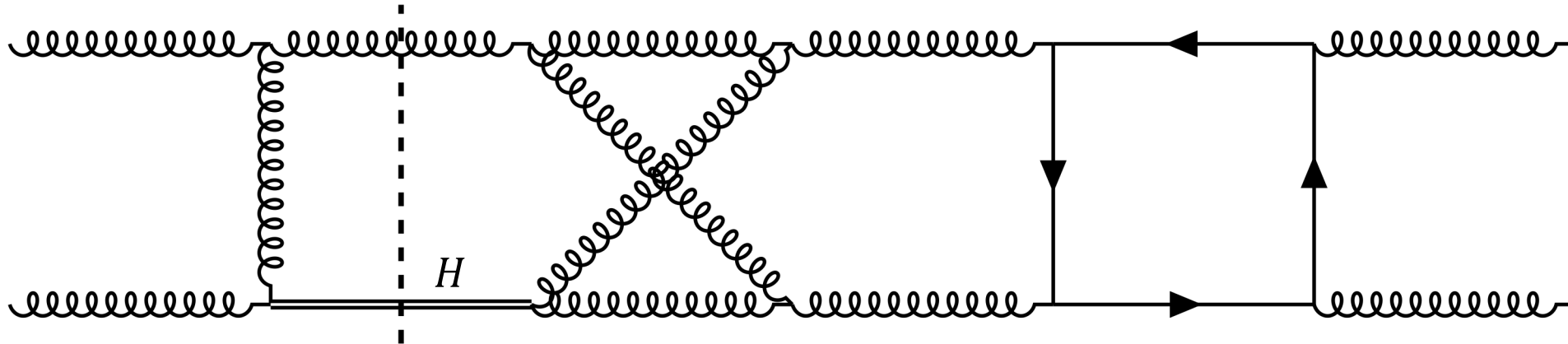
*[Mistlberger, Suresh; arXiv: 2504.10574]

RVVV Contribution

The other single-real ingredient at N⁴LO:

RVVV: "single-real, triple-virtual".

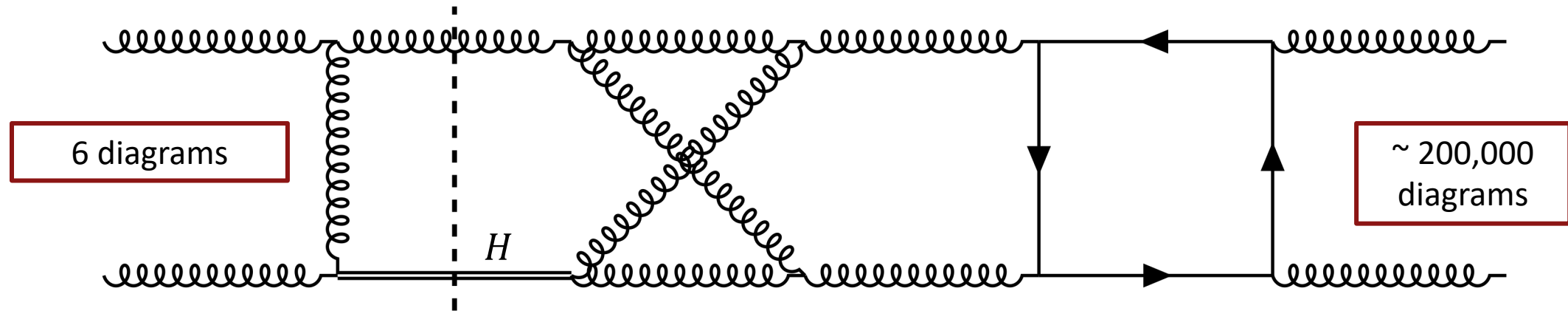
RVVV



The other single-real ingredient at N⁴LO:

RVVV: "single-real, triple-virtual".

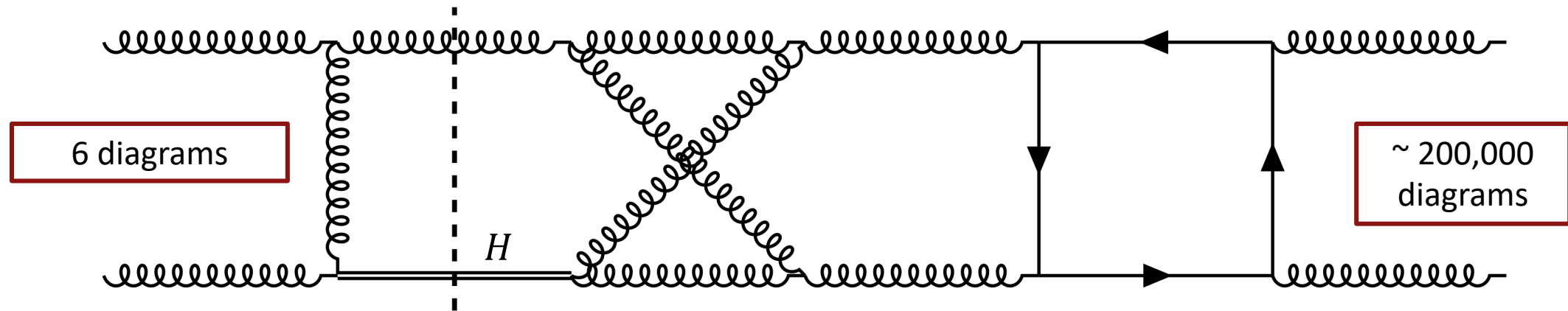
RVVV



The other single-real ingredient at N⁴LO:

RVVV: "single-real, triple-virtual".

RVVV

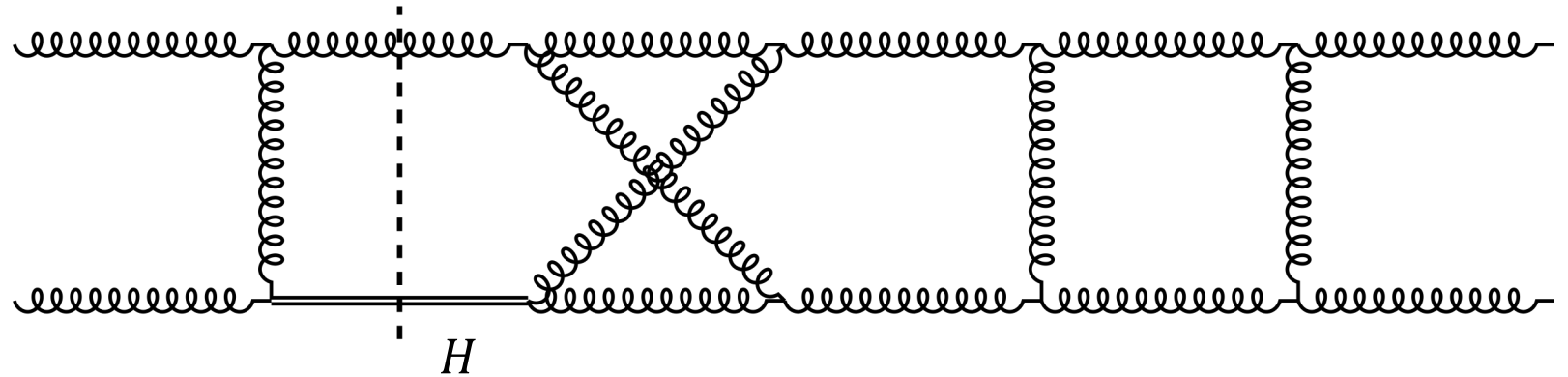


Complexity grows fast!

Production Channels

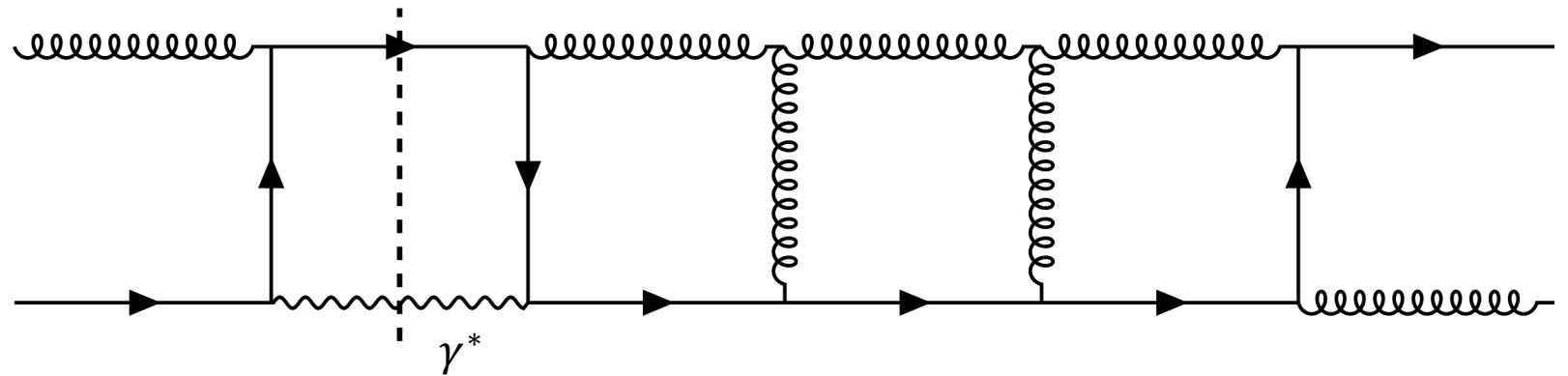
Higgs Production

- $g + g \rightarrow H + g$
- $q + \bar{q} \rightarrow H + g$
- $q + g \rightarrow H + q$



Drell-Yan Production

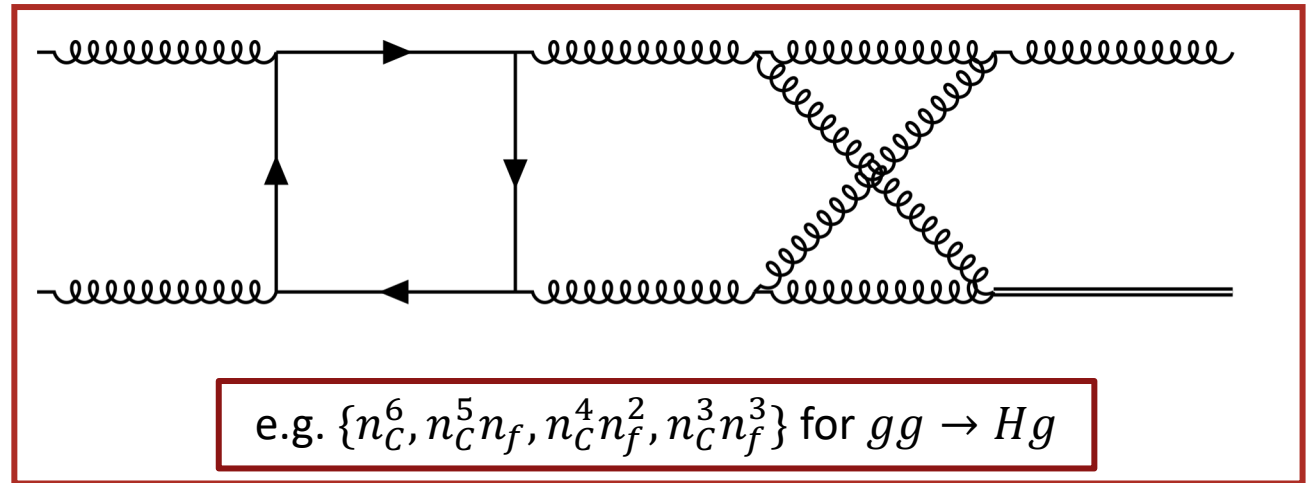
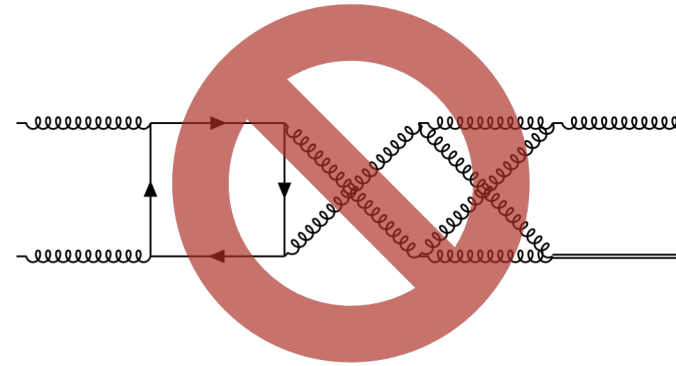
- ~~$g + g \rightarrow \gamma^* + g$~~
 - Starts at 1-loop order
 - Only $RVV \times V$
- $q + \bar{q} \rightarrow \gamma^* + g$
- $q + g \rightarrow \gamma^* + q$



Three-loop amplitudes

State-of-the-art: Leading color amplitudes?

- Only leading power in $n_C^\alpha n_f^\beta$
- Drell-Yan known^{*}
- Higgs recently computed^{**}



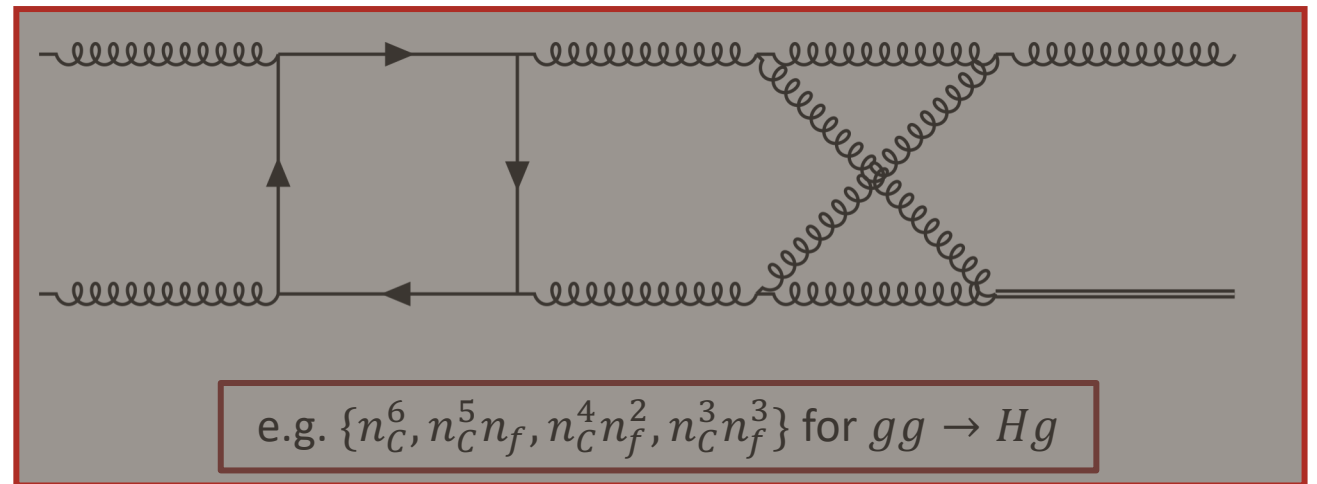
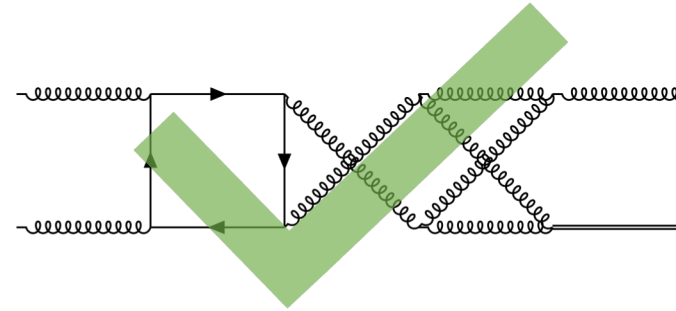
^{*}[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi; arXiv:2307.15405]

^{**}[Chen, Guan, Mistlberger; arXiv: 2504.06490]

Three-loop amplitudes

State-of-the-art: Leading color amplitudes?

- Only leading power in $n_C^\alpha n_f^\beta$
- Drell-Yan known^{*}
- Higgs recently computed^{**}
 - Full color now available!^{***}



^{*}[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi; arXiv:2307.15405]

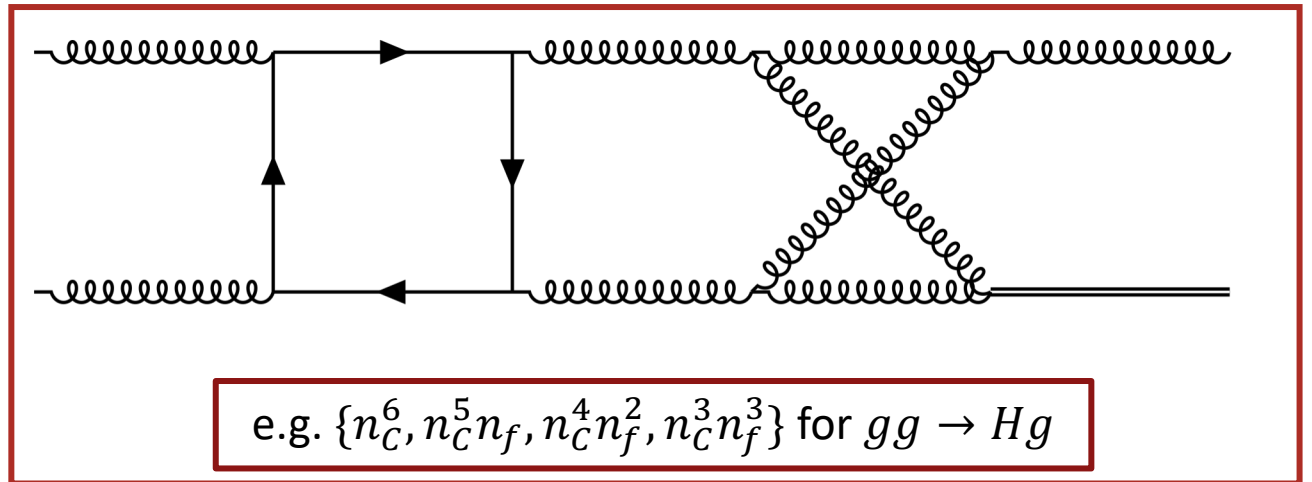
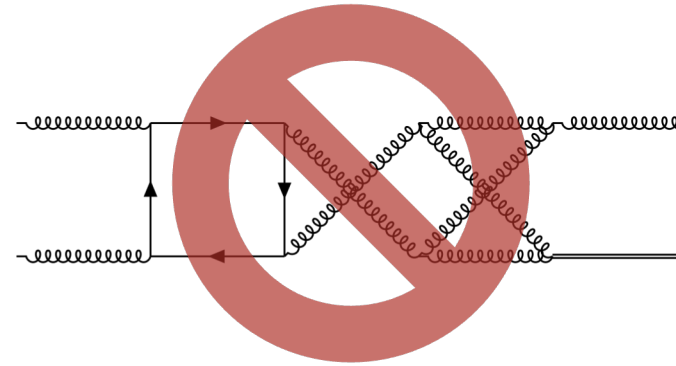
^{**}[Chen, Guan, Mistlberger; arXiv: 2504.06490]

^{***}[Guan, Mistlberger, Ruf; arXiv: 2607.12033]

Three-loop amplitudes

State-of-the-art: Leading color amplitudes?

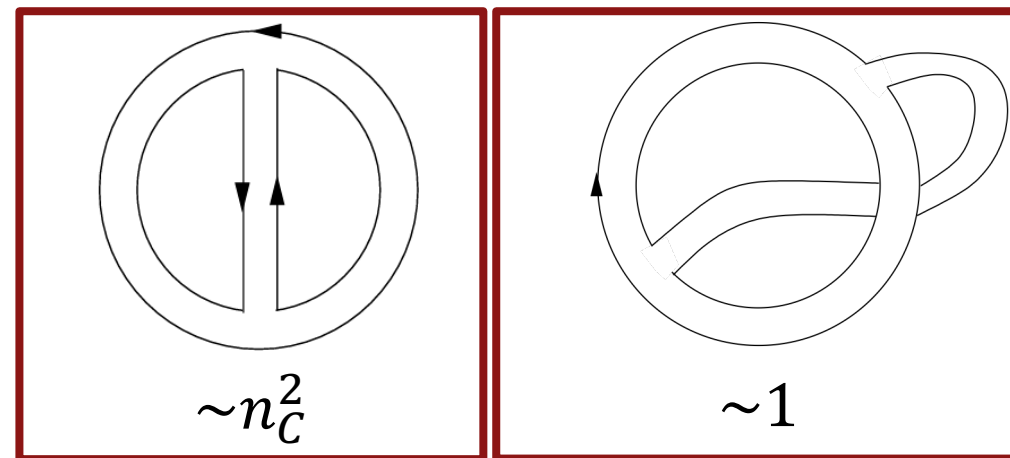
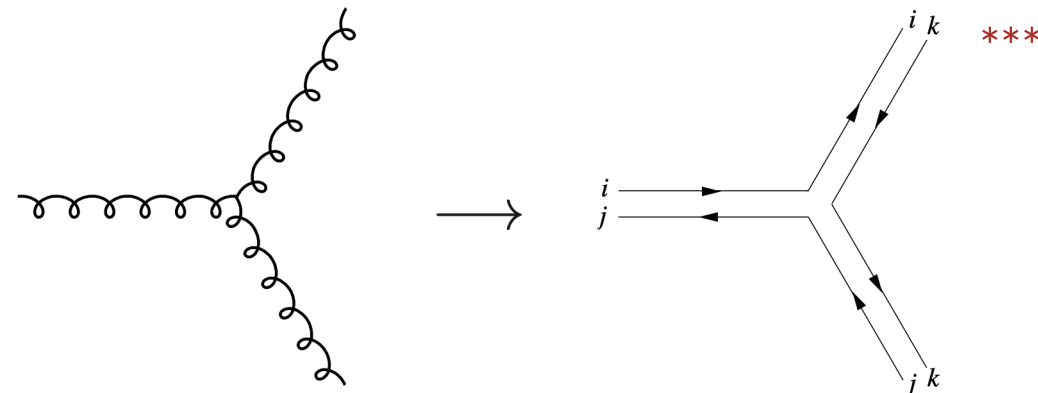
- We use the leading color amplitudes for our computation
- Full-color function space is a bit trickier to integrate



Generalized leading-color limit

Take $n_f, n_C \rightarrow \infty$

- Only planar color diagrams contribute
- Much easier to compute
 - E.g. Γ_{cusp} known to all orders in planar $N=4$ sYM^{*} vs. 4-loop for non-planar^{**}
- Subleading color suppressed by $\frac{1}{n_C^2} \sim \frac{1}{10}$.
 - For XS @ N⁴LO, $\frac{1}{10}$ of 1%?



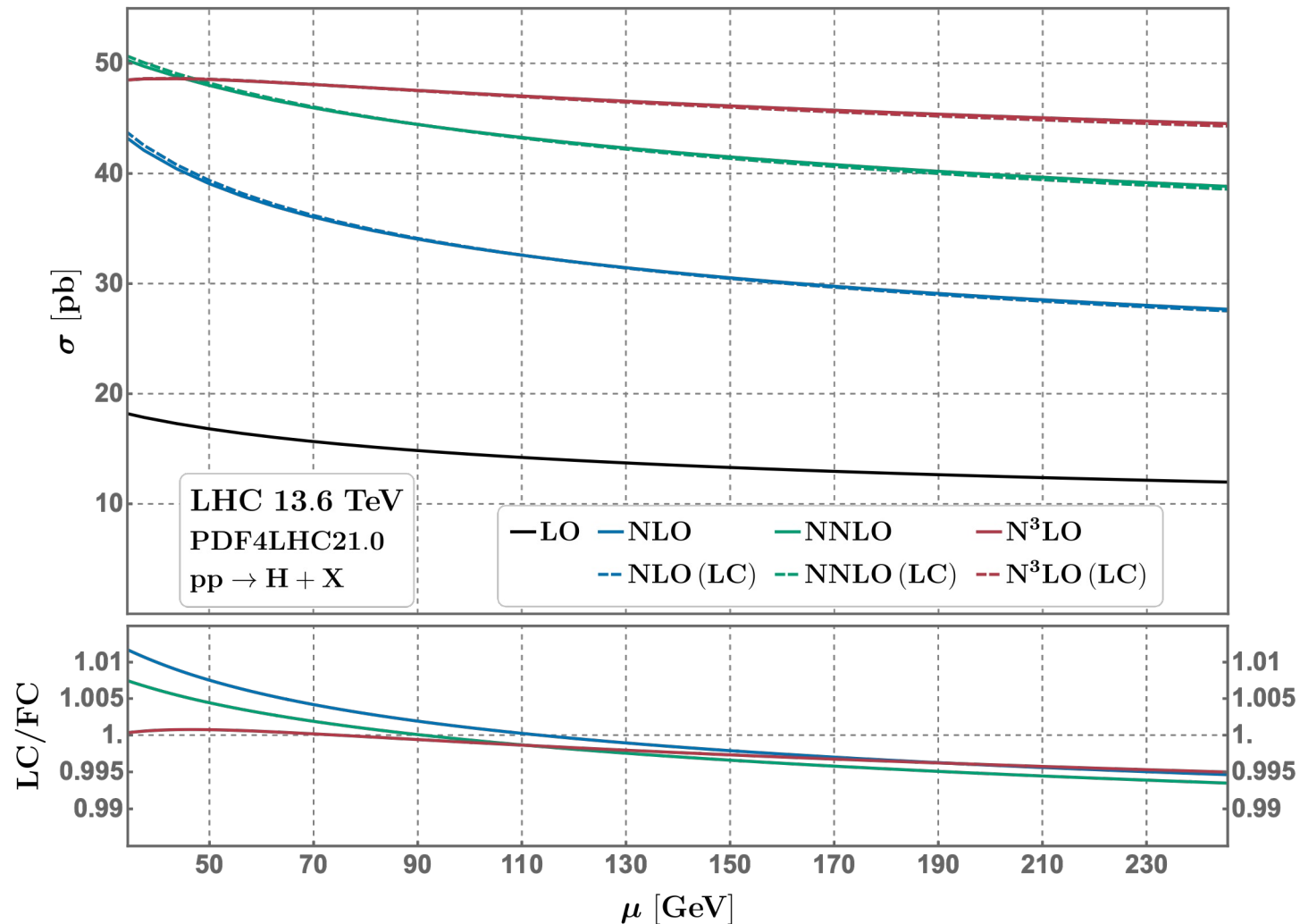
^{*}[Beisert, Eden, Staudacher; arXiv:hep-th/0610251]

^{**}[Boels, Huber, Yang; arXiv:1705.03444]

^{***}[davidtong.org/teaching/gauge-theory/]

Can we do **phenomenology** with only leading color?

Hadronic Cross Section up to N³LO: Leading vs. Full Color



Leading color appears to capture the cross section at the <0.5% level!

Encouraging for N⁴LO.

Results

RVVV (LC): After Performing Phase Space Integration

Results

- With $\bar{z} = 1 - \frac{m_B^2}{s}$, contains:
 - Everything in RVVxV (i.e. polylogs) and
 - More general class of iterated integrals:
 - New (algebraic) letters:
$$\frac{1}{\bar{z}\sqrt{1-\bar{z}}}, \frac{1}{\sqrt{3+\bar{z}}\sqrt{1-\bar{z}}}$$
 - Entered at RRV and RRR for $N^3\text{LO}$ ***

Checks

- Regulated IR singularities with known results:
 - Three-loop single-emission soft current*
 - Three-loop splitting amplitudes**
- Computed threshold expansion near $\bar{z} = 0$ before and after phase-space integration

*[Herzog, Ma, Mistlberger, Suresh; arXiv:2309.07884]

**[Guan, Herzog, Ma, Mistlberger, Suresh; arXiv:2408.03019]

***[Mistlberger; arXiv:1802.00833]

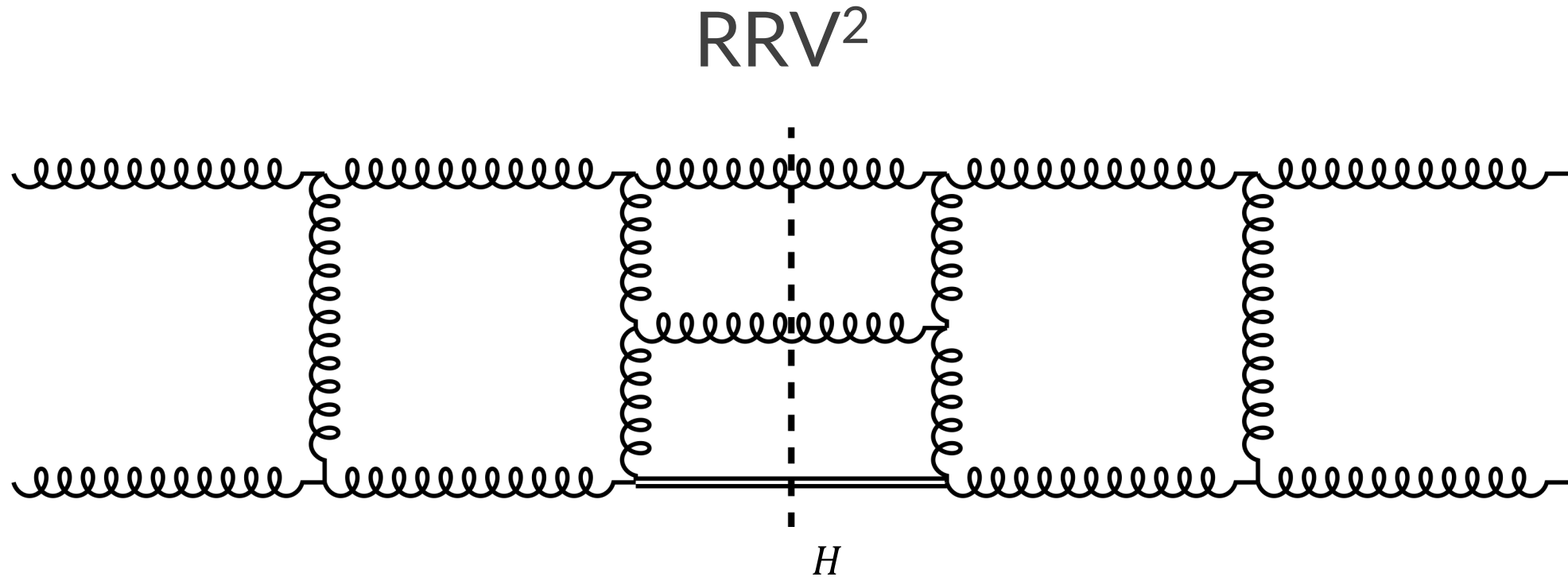
Outline

1. Higgs boson production at hadron colliders.
 - How and why do we count them?
 - How are they produced?
2. The journey to the precision frontier.
3. The single-real ingredients.
4. **Toward double-real ingredients and beyond.**

Double-real Contributions

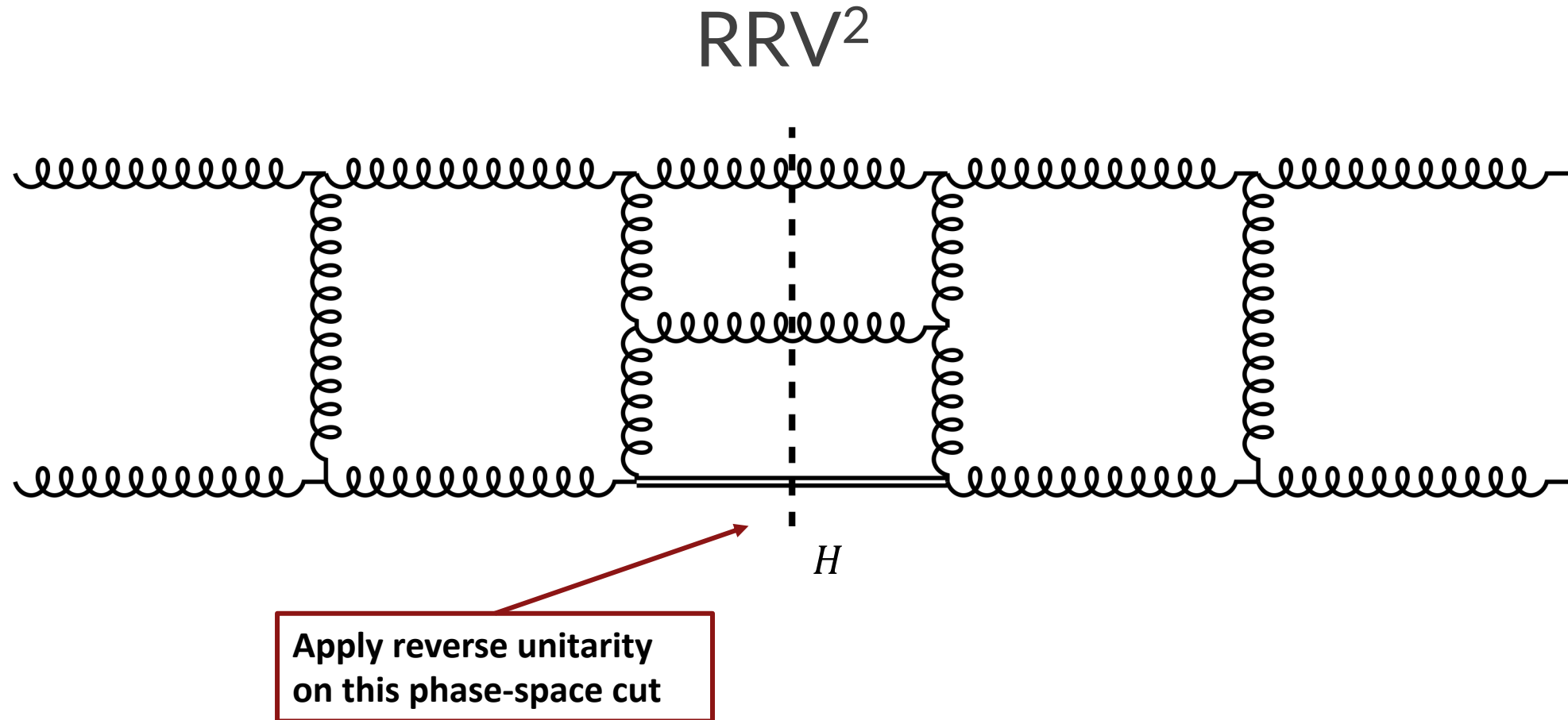
First double-real ingredient at N⁴LO:

RRV²: "double-real, single-virtual-squared".



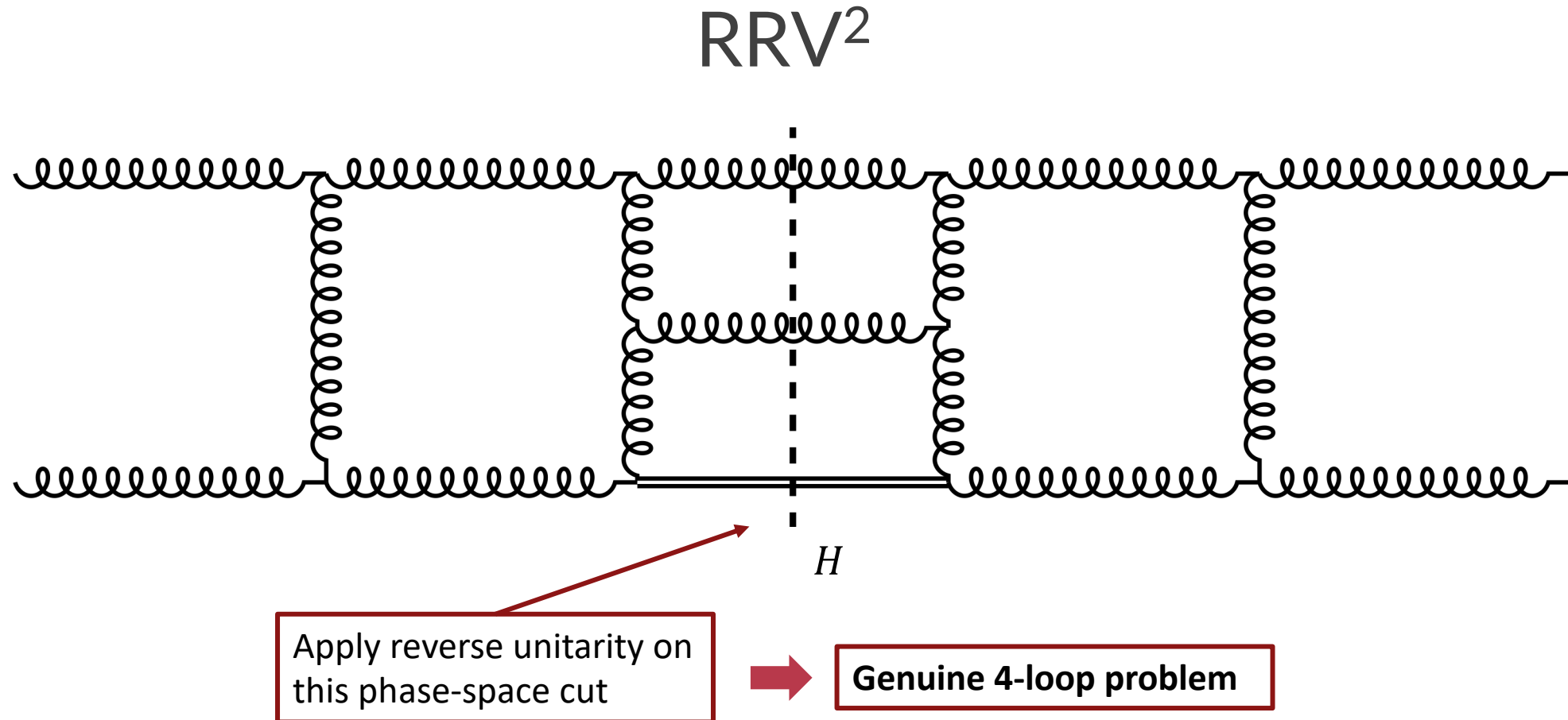
First double-real ingredient at N⁴LO:

RRV²: "double-real, single-virtual-squared".



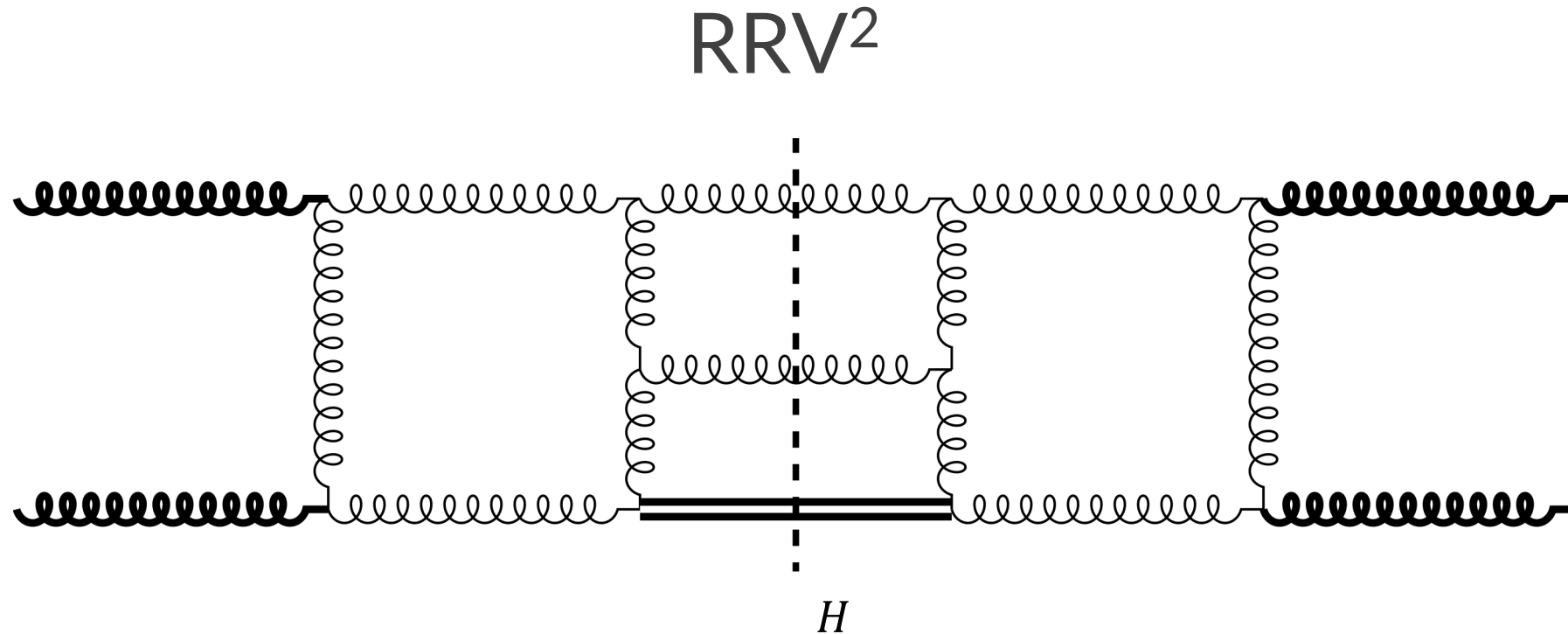
First double-real ingredient at N⁴LO:

RRV²: "double-real, single-virtual-squared".



First double-real ingredient at N⁴LO:

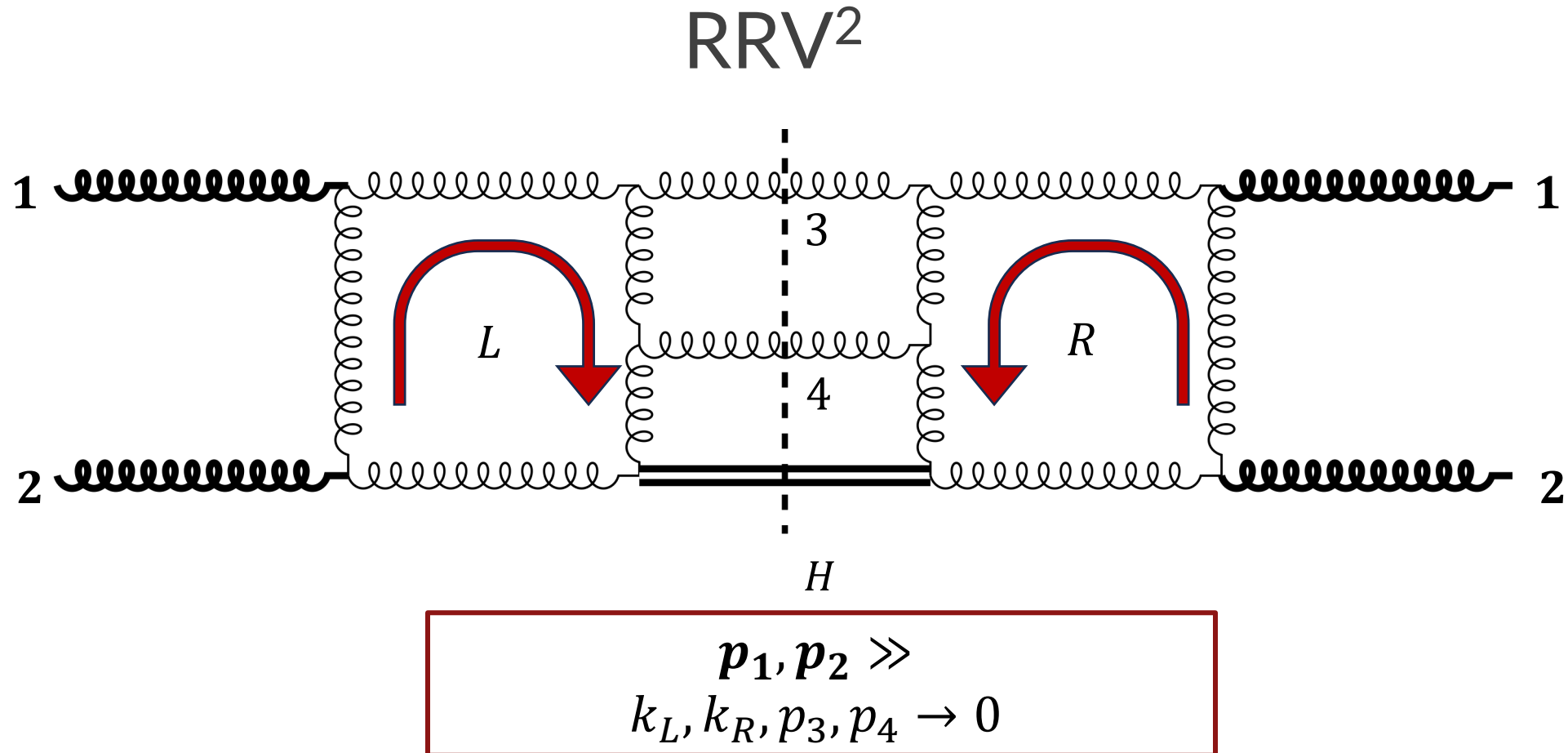
RRV²: "double-real, single-virtual-squared".



Start with maximally soft region

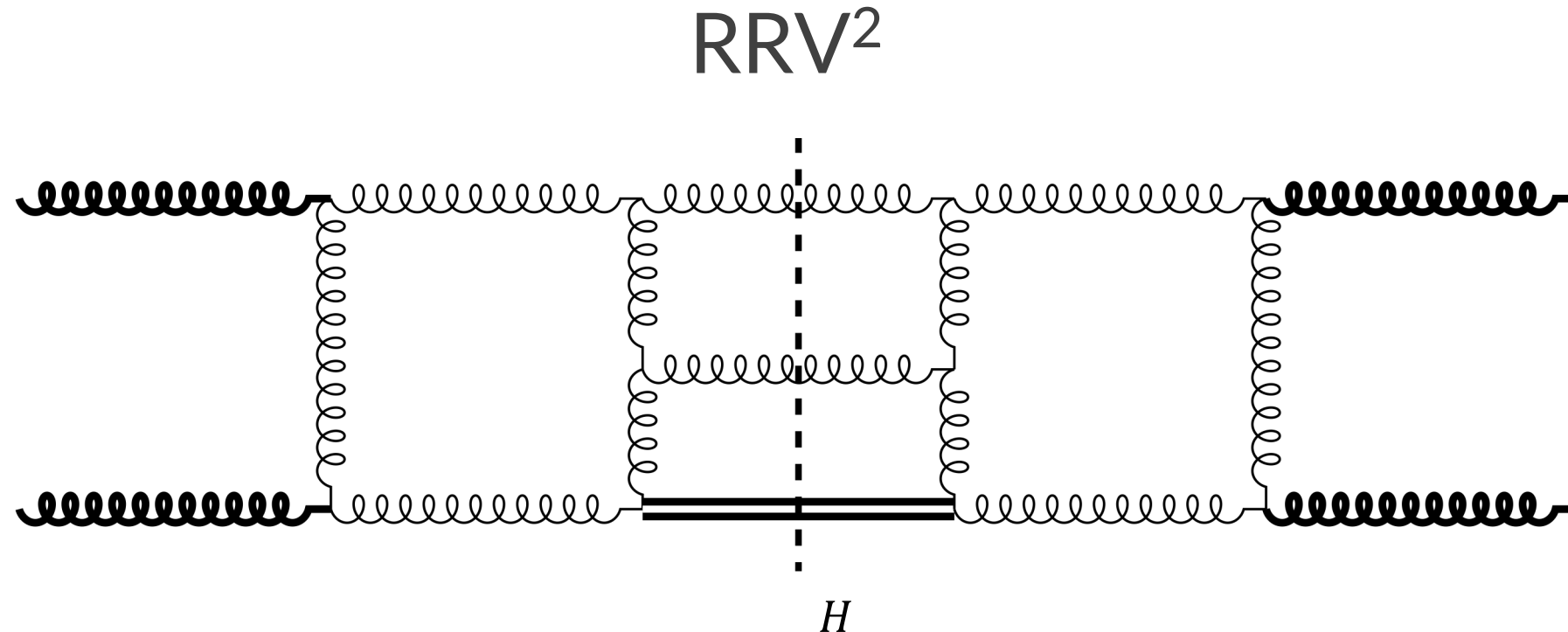
First double-real ingredient at N⁴LO:

RRV²: "double-real, single-virtual-squared".



First double-real ingredient at N⁴LO:

RRV²: "double-real, single-virtual-squared".



Start with maximally soft region



**Contributes to LP in the
threshold expansion**

Constructing the RRV² Contribution

Generate all interfered
Feynman diagrams



Do color, spin, and Lorentz algebra

All particles live in $d = 4 - 2\epsilon$ dimensions
(conventional dimensional regularization)



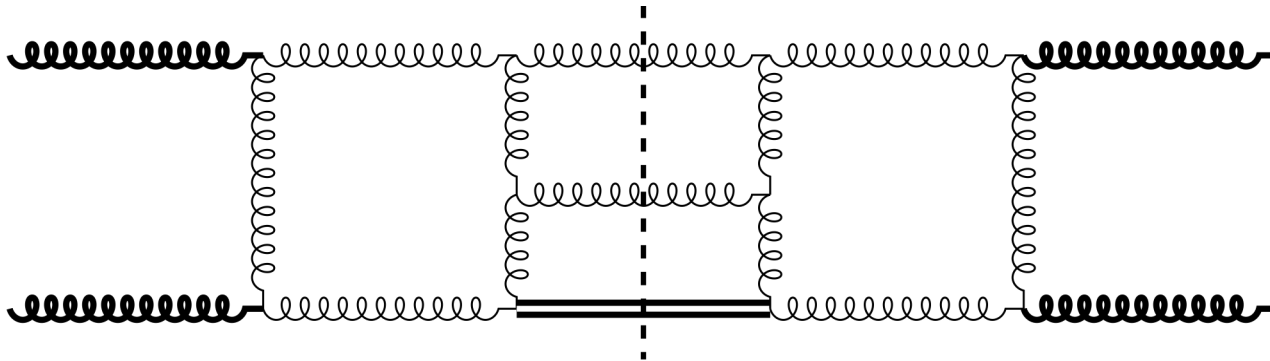
Extract max-soft region

i.e. uniformly scale
 $k_L, k_R, p_3, p_4 \sim \bar{z} \rightarrow 0$



Refine Loop integrals

Use symmetry & IBPs to obtain basis of
“master integrals”



Constructing the RRV² Contribution

Generate all interfered
Feynman diagrams



Do **color**, spin, and Lorentz algebra

All particles live in $d = 4 - 2\epsilon$ dimensions
(conventional dimensional regularization)



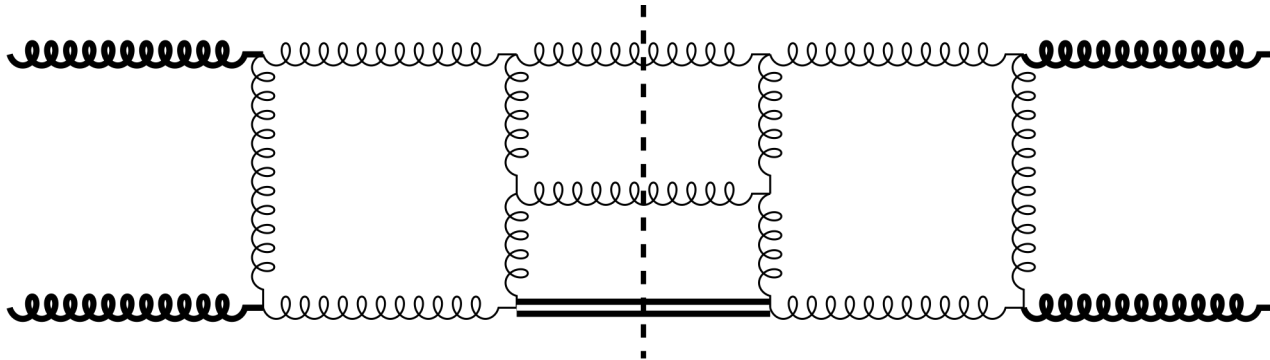
Extract max-soft region

i.e. uniformly scale
 $k_L, k_R, p_3, p_4 \sim \bar{z} \rightarrow 0$



Refine Loop integrals

Use symmetry & IBPs to obtain basis of $O(50)$
“master integrals”



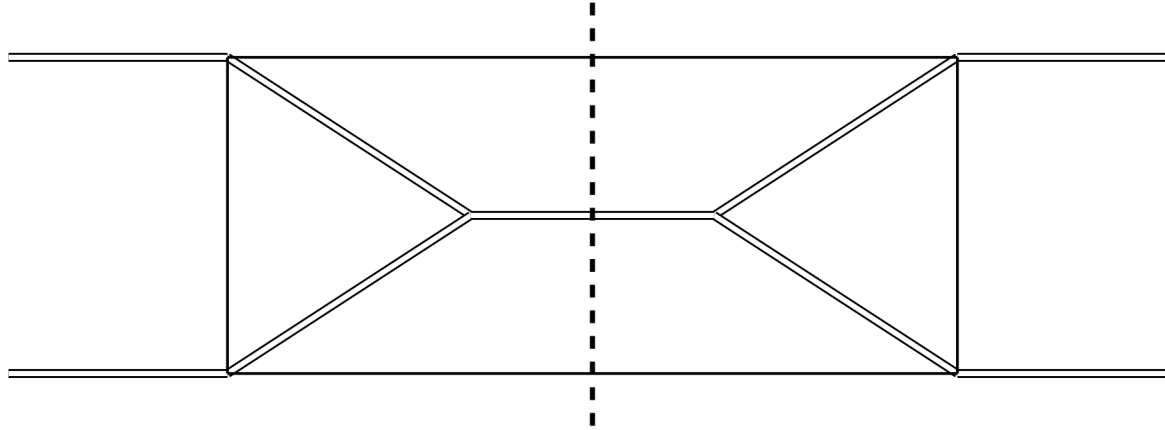
*<https://github.com/adisurtya/CIFAR>

**<https://gitee.com/multiloop-pku/blade>

Community tools:
CIFAR* and BLADE**

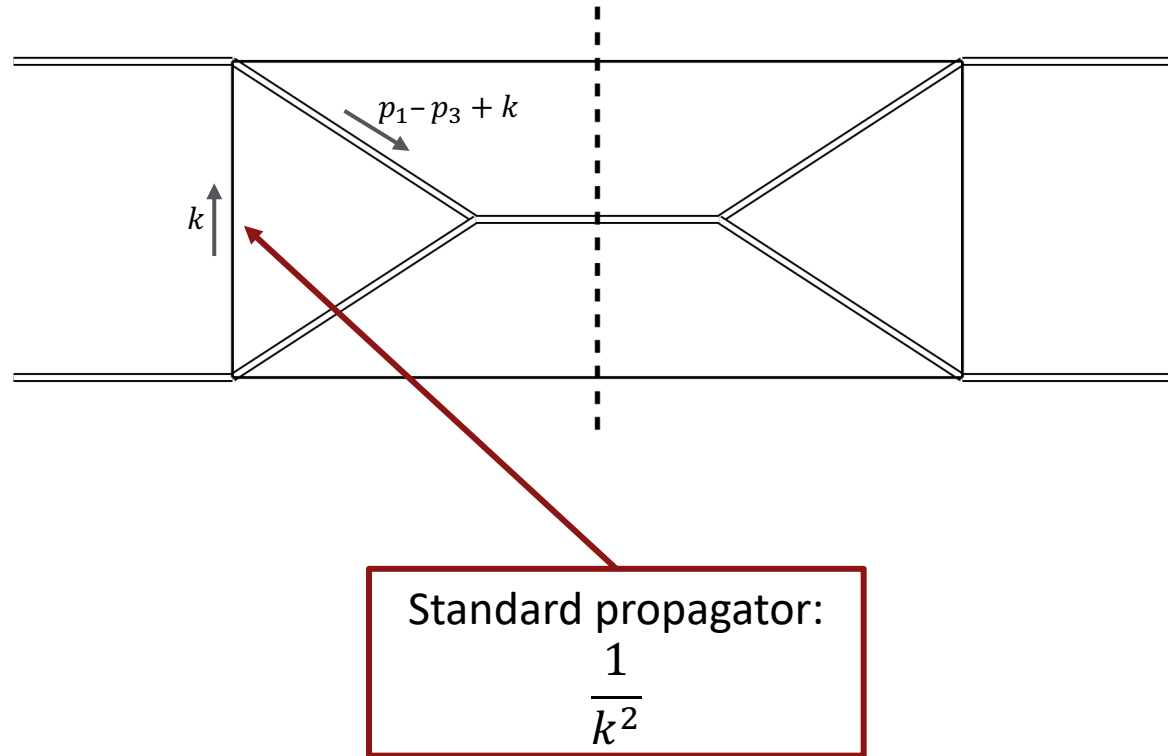
Now we are left with soft master integrals to compute

One of our masters:



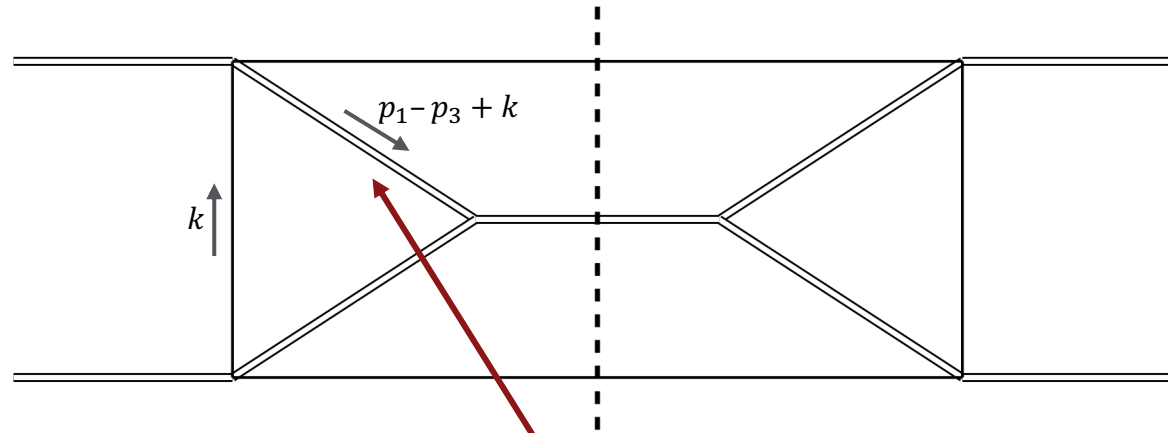
Now we are left with soft master integrals to compute

One of our masters:



Now we are left with soft master integrals to compute

One of our masters:

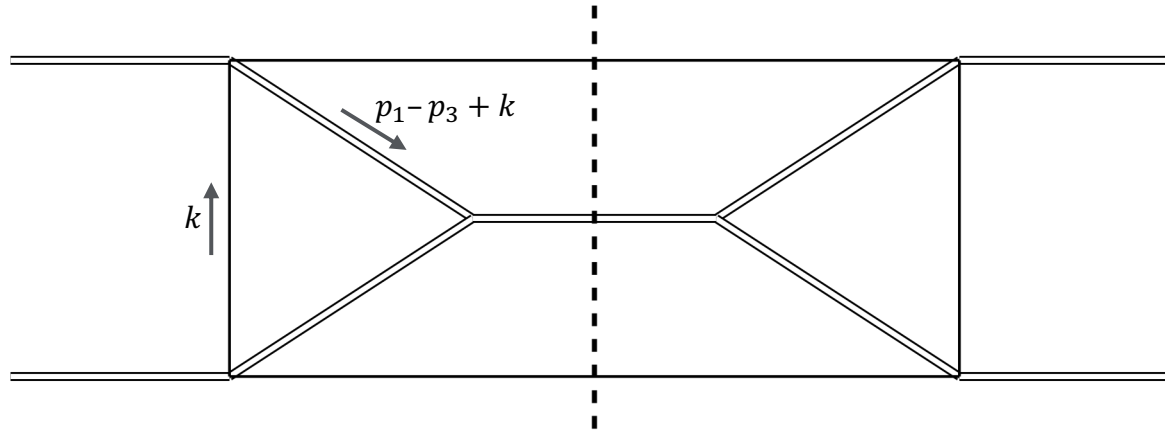


Eikonal propagator:

$$\frac{1}{2p_1 \cdot (k - p_3)}$$

Now we are left with soft master integrals to compute

One of our masters:



(max-)Soft integrals:

$$I_{\text{soft}} = \bar{z}^{-1-8\epsilon} C(\epsilon)$$

Standard propagator:

$$\frac{1}{k^2}$$

Eikonal propagator:

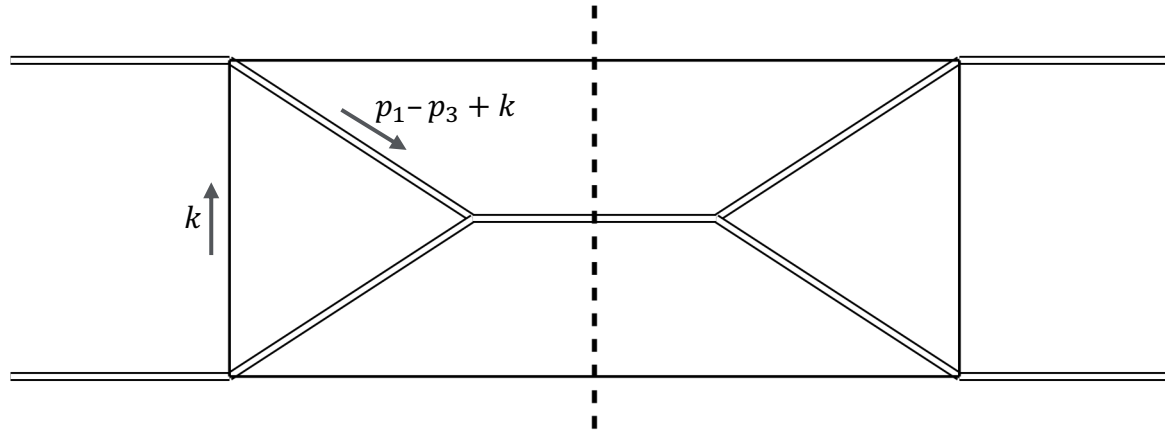
$$\frac{1}{2p_1 \cdot (k - p_3)}$$

Soft integral:

$$\int d\Phi_S d^d k \cdots \frac{1}{k^2} \frac{1}{2p_1 \cdot (k - p_3)} \cdots$$

Now we are left with soft master integrals to compute

One of our masters:



Standard propagator:

$$\frac{1}{k^2}$$

Eikonal propagator:

$$\frac{1}{2p_1 \cdot (k - p_3)}$$

Soft integral:

$$\int d\Phi_S d^d k \dots \frac{1}{k^2} \frac{1}{2p_1 \cdot (k - p_3)} \dots$$

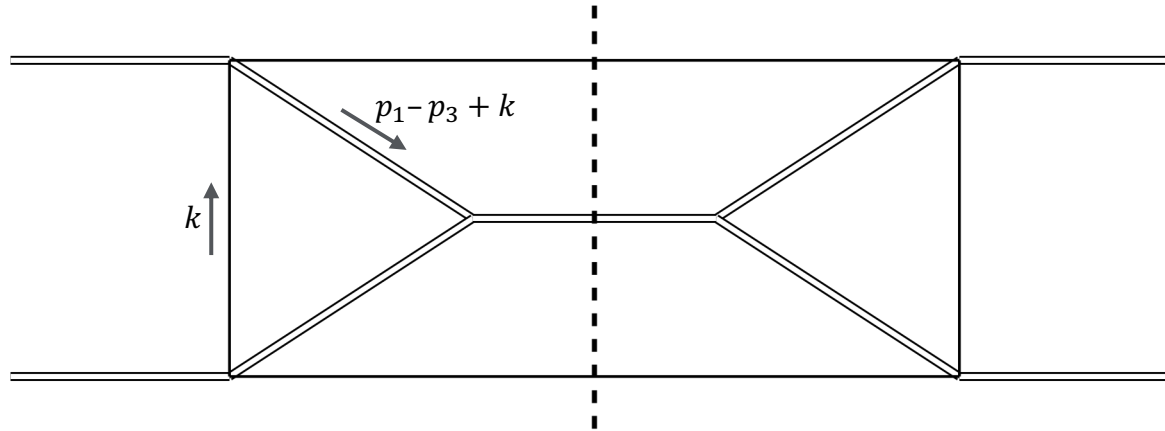
(max-)Soft integrals:

$$I_{\text{soft}} = \bar{Z}^{-1-8\epsilon} C(\epsilon)$$

Depend trivially on
scales/kinematics

Now we are left with soft master integrals to compute

One of our masters:



Standard propagator:

$$\frac{1}{k^2}$$

Eikonal propagator:

$$\frac{1}{2p_1 \cdot (k - p_3)}$$

Soft integral:

$$\int d\Phi_S d^d k \dots \frac{1}{k^2} \frac{1}{2p_1 \cdot (k - p_3)} \dots$$

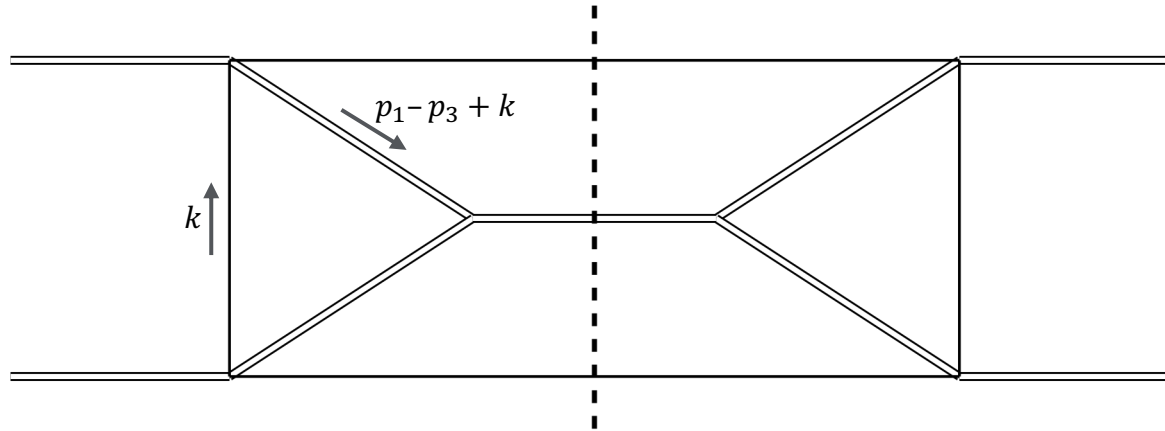
(max-)Soft integrals:

$$I_{\text{soft}} = \bar{z}^{-1-8\epsilon} \boxed{C(\epsilon)}$$

So.. how to compute this?

Now we are left with soft master integrals to compute

One of our masters:



Standard propagator:

$$\frac{1}{k^2}$$

Eikonal propagator:

$$\frac{1}{2p_1 \cdot (k - p_3)}$$

Soft integral:

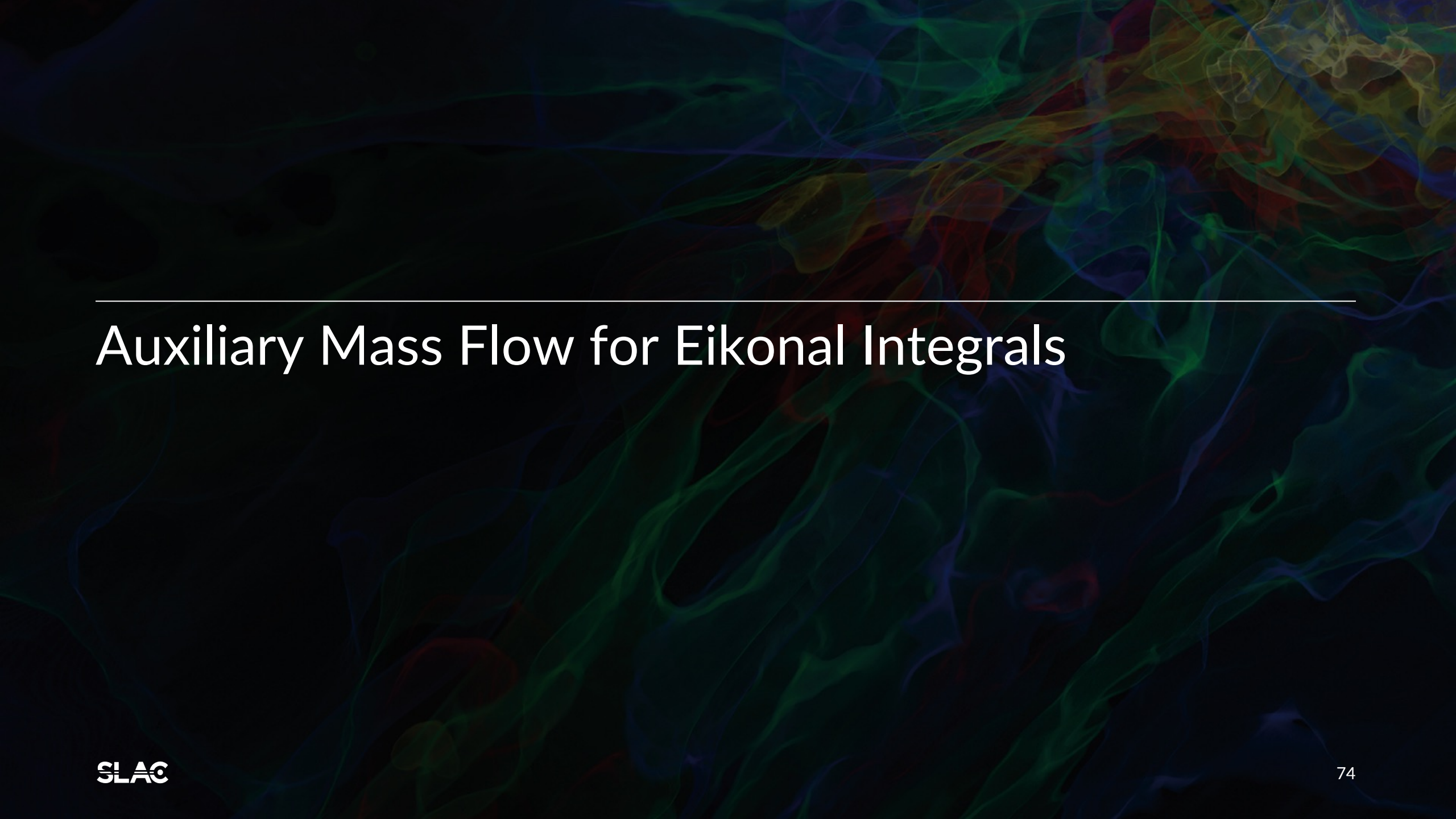
$$\int d\Phi_S d^d k \dots \frac{1}{k^2} \frac{1}{2p_1 \cdot (k - p_3)} \dots$$

(max-)Soft integrals:

$$I_{\text{soft}} = \bar{z}^{-1-8\epsilon} \boxed{C(\epsilon)}$$

So.. how to compute this?

Parametrization/Direct integration, Mellin-Barnes, etc ...
but no standard / universal approach



Auxiliary Mass Flow for Eikonal Integrals

Auxiliary Mass Flow for Eikonal Integrals

Auxiliary mass flow^{*}

- Introduce an auxiliary mass in selected propagators:
 - $D \mapsto D - \eta$
- IBP reduction gives a differential system:
 - $dJ/d\eta = A(\eta, \varepsilon)J$
- Fix the boundary at $\eta \rightarrow \infty$.
- Flow to the physical point $\eta \rightarrow 0$.

^{*}[X. Liu, Ma, Wang; arXiv:1711.09572]
[X. Liu, Ma, Tao, Zhang; arXiv:2009.07987]
[X. Liu, Ma; arXiv:2107.01864, arXiv:2201.11669]
[Z.-F. Liu, Ma; arXiv:2201.11636, arXiv:2201.11637]
[Huang, X. Liu, Ma; arXiv:2607.08477]

Auxiliary Mass Flow for Eikonal Integrals

Auxiliary mass flow*

- Introduce an auxiliary mass in selected propagators:
 - $D \mapsto D - \eta$
- IBP reduction gives a differential system:
 - $dJ/d\eta = A(\eta, \varepsilon)J$
- Fix the boundary at $\eta \rightarrow \infty$.
- Flow to the physical point $\eta \rightarrow 0$.

Standard treatment of eikonal propagators

- For $1/[2p_1 \cdot (k - p_3)]$, add xk^2 , then an auxiliary mass:
 - $xk^2 + 2p_1 \cdot (k - p_3) - x\eta$
 $= x(k + p_1/x)^2 - 2p_1 \cdot p_3 - x\eta$
- At fixed x_0 , flow $\eta: \infty \rightarrow 0$.
- Then flow $x: x_0 \rightarrow 0$.

*[X. Liu, Ma, Wang; arXiv:1711.09572]
[X. Liu, Ma, Tao, Zhang; arXiv:2009.07987]
[X. Liu, Ma; arXiv:2107.01864, arXiv:2201.11669]
[Z.-F. Liu, Ma; arXiv:2201.11636, arXiv:2201.11637]
[Huang, X. Liu, Ma; arXiv:2607.08477]

Auxiliary Mass Flow for Eikonal Integrals

Auxiliary mass flow*

- Introduce an auxiliary mass in selected propagators:
 - $D \mapsto D - \eta$
- IBP reduction gives a differential system:
 - $dJ/d\eta = A(\eta, \varepsilon)J$
- Fix the boundary at $\eta \rightarrow \infty$.
- Flow to the physical point $\eta \rightarrow 0$.

*[X. Liu, Ma, Wang; arXiv:1711.09572]
[X. Liu, Ma, Tao, Zhang; arXiv:2009.07987]
[X. Liu, Ma; arXiv:2107.01864, arXiv:2201.11669]
[Z.-F. Liu, Ma; arXiv:2201.11636, arXiv:2201.11637]
[Huang, X. Liu, Ma; arXiv:2607.08477]

Standard treatment of eikonal propagators

- For $1/[2p_1 \cdot (k - p_3)]$, add xk^2 , then an auxiliary mass:
 - $xk^2 + 2p_1 \cdot (k - p_3) - x\eta$
 $= x(k + p_1/x)^2 - 2p_1 \cdot p_3 - x\eta$
- At fixed x_0 , flow $\eta: \infty \rightarrow 0$.
- Then flow $x: x_0 \rightarrow 0$.

Quadratic propagators largely restore the original standard four-loop complexity

Auxiliary Mass Flow for Eikonal Integrals

Introducing eikonal mode!

- Keep propagators linear. Only shift those:
 - $1/[2p_1 \cdot (k - p_3)]$
 $\mapsto 1/[2p_1 \cdot (k - p_3) - \eta]$
- Keep the cuts unchanged.
- For RRV^2 , deform one virtual side at a time.

Auxiliary Mass Flow for Eikonal Integrals

Introducing eikonal mode!

- Keep propagators linear. Only shift those:
 - $1/[2p_1 \cdot (k - p_3)]$
 $\mapsto 1/[2p_1 \cdot (k - p_3) - \eta]$
- Keep the cuts unchanged.
- For RRV^2 , deform one virtual side at a time.

Boundaries adapted to the phase space

- Analyze the deformed virtual loop in its large- η regions.
- Iterate the AMFlow procedure
- Reduce to simpler (factorized) boundary integrals
 - E.g. Soft PS Volume $\times$ eikonal triangle $\times$ tadpole

Auxiliary Mass Flow for Eikonal Integrals

Introducing eikonal mode!

- Keep propagators linear. Only shift those:
 - $1/[2p_1 \cdot (k - p_3)]$
 $\mapsto 1/[2p_1 \cdot (k - p_3) - \eta]$
- Keep the cuts unchanged.
- For RRV^2 , deform one virtual side at a time.

Boundaries adapted to the phase space

- Analyze the deformed virtual loop in its large- η regions.
- Iterate the AMFlow procedure
- Reduce to simpler (factorized) boundary integrals
 - E.g. Soft PS Volume $\times$ eikonal triangle $\times$ tadpole

Retain the soft-integral simplification throughout the auxiliary flow

Auxiliary Mass Flow for Eikonal Integrals

In practice

- Tell AMFlow to fit the Laurent series coefs. in ϵ to 400 digits precision
- Use PSLQ* to do a numeric fit to transcendental constants
 - Only see Zeta values!

$$\text{Out}[..]=13.48 \dots 20`400$$



$$\frac{12}{7}\pi^2 + \frac{5}{8}\pi^4 - \frac{107}{2}\zeta_3$$

*[Ferguson, Bailey, Arno; Math. Comp. 68 (1999) 351–369, DOI: 10.1090/S0025-5718-99-00995-3]

Auxiliary Mass Flow for Eikonal Integrals

Need high precision to fit up to transcendental weight 8!

Out[...]= 25.53 ... 13`400



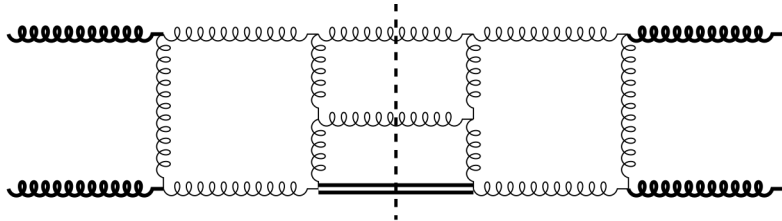
- Fit on first 350 digits
- Validate on last 50 digits

$$\begin{aligned} & \frac{17}{5} \zeta_{5,3} + \frac{2568}{35} \zeta_6 + \frac{38917}{360} \zeta_8 - \frac{256}{315} \zeta_2 \zeta_3 \\ & - \frac{856}{21} \zeta_4 \zeta_3 + \frac{32}{5} \zeta_3^2 + \frac{4}{3} \zeta_2 \zeta_3^2 - \frac{992}{105} \zeta_5 \\ & - \frac{856}{105} \zeta_2 \zeta_5 + 31 \zeta_3 \zeta_5 - \frac{13589}{105} \zeta_7 \end{aligned}$$

RRV²

Work in progress! We have:

- All integrals computed via eikonal mode.
- Independent Mellin–Barnes calculations cross-check various AMFlow results.
- Reproduced known results with eikonal mode (e.g. soft pentagon^{*})
- Numerical reconstruction to only zeta values also gives a strong check.



Very

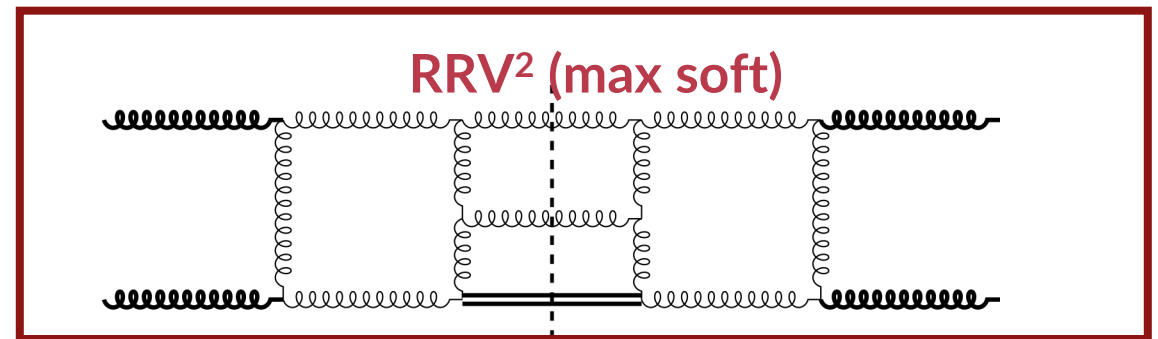
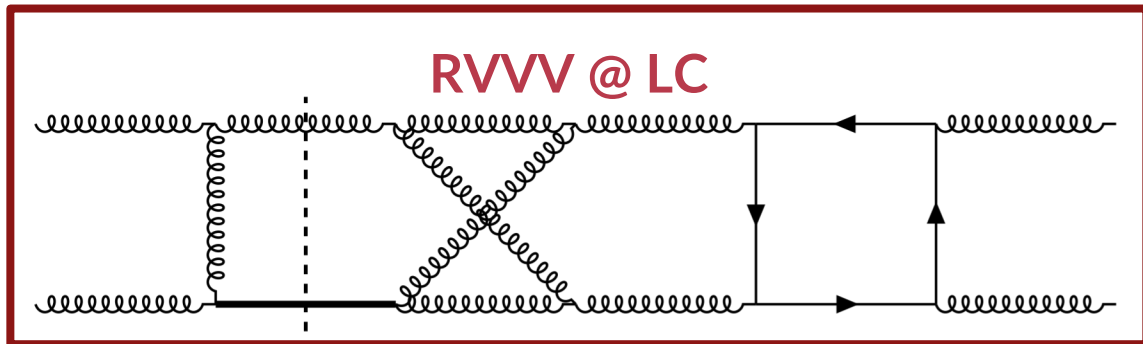
ABSTRACT: We compute some part of the Higgs cross-section at N4LO

^{*}[Anastasiou, Duhr, Dulat, Furlan, Herzog, Mistlberger; arXiv:1505.04110]

Conclusion and Next Steps

Unlock Full Phenomenological Potential of LHC & Higgs Physics: Need N⁴LO Precision

- Two more key steps in computing the production cross section at N⁴LO:



- Understanding complexity and building up technology (e.g. AMFlow “eikonal mode”) to tackle further N⁴LO calculations
- Compute other (soft) ingredients: other double-real RRVV. And beyond: RRRV, RRRR
- Full-color computation of RVVV

Backup slides

Thank you!

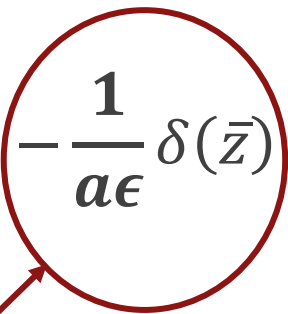
Regulation of Soft Singularity

Expand soft singularity:

$$\left(\frac{1}{\bar{z}}\right)^{1+a\epsilon} = -\frac{1}{a\epsilon} \delta(\bar{z}) + \sum_{k=0}^{\infty} \frac{(-a\epsilon)^k}{k!} \left[\frac{\log^k \bar{z}}{\bar{z}} \right]_+,$$

Regulation of Soft Singularity

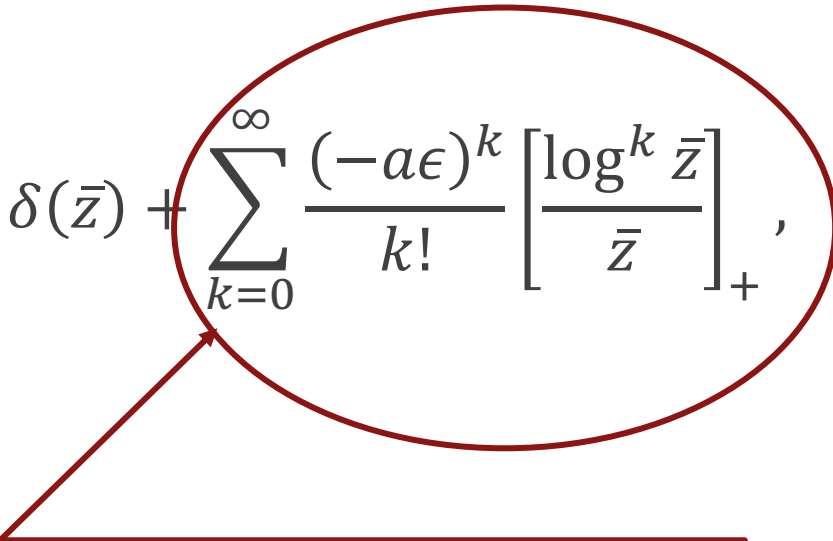
Expand soft singularity:

$$\left(\frac{1}{\bar{z}}\right)^{1+a\epsilon} = -\frac{1}{a\epsilon} \delta(\bar{z}) + \sum_{k=0}^{\infty} \frac{(-a\epsilon)^k}{k!} \left[\frac{\log^k \bar{z}}{\bar{z}} \right]_+,$$


Additional ϵ pole at soft limit $\bar{z} \rightarrow 0$

Regulation of Soft Singularity

Expand soft singularity:

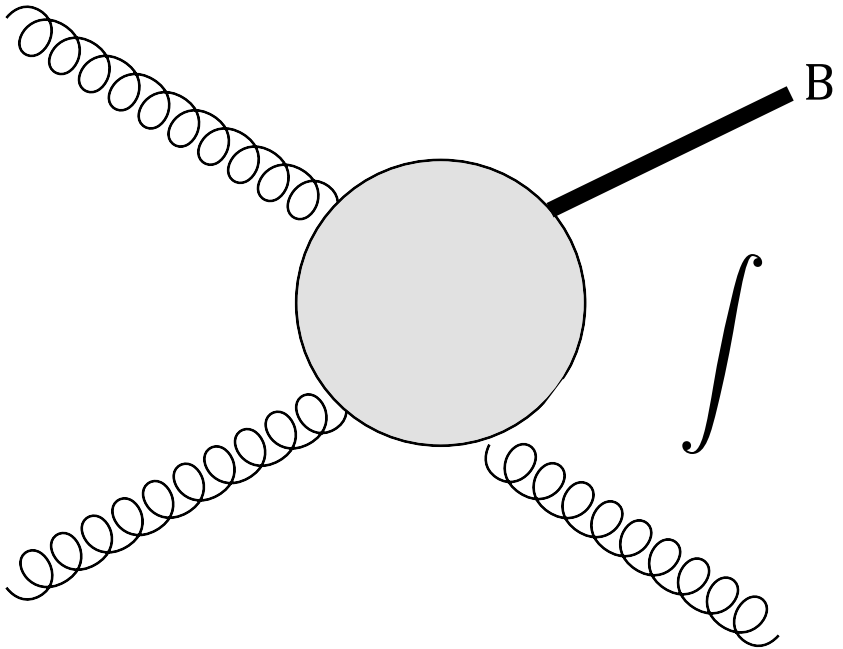
$$\left(\frac{1}{\bar{z}}\right)^{1+a\epsilon} = -\frac{1}{a\epsilon} \delta(\bar{z}) + \sum_{k=0}^{\infty} \frac{(-a\epsilon)^k}{k!} \left[\frac{\log^k \bar{z}}{\bar{z}} \right]_+,$$


Plus distribution:

$$\int_0^1 d\bar{z} \left[\frac{\log^k \bar{z}}{\bar{z}} \right]_+ f(\bar{z}) \equiv \int_0^1 d\bar{z} \frac{\log^k \bar{z}}{\bar{z}} (f(\bar{z}) - f(0)).$$

Phase Space Integration

What do these integrals typically look like?

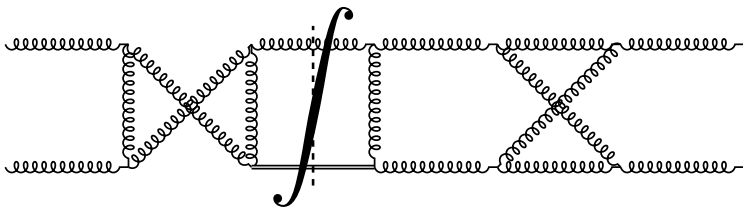
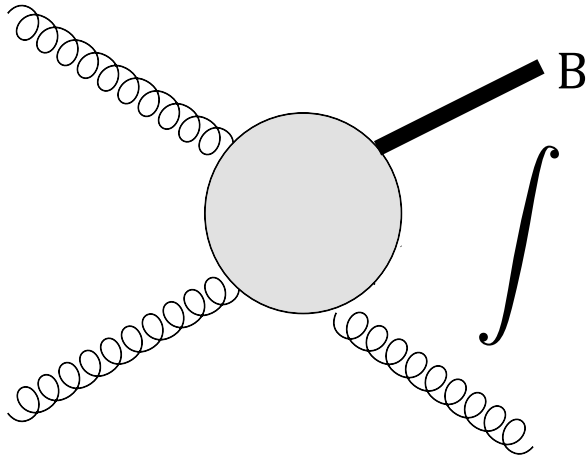


$$\hat{\sigma} \sim \int d\lambda \frac{\log \lambda}{\lambda - 1/\bar{z}} = \log \lambda \log (1 - \bar{z}\lambda) + \text{Li}_2(\bar{z}\lambda)$$

Parametrizes energy of scattering

Phase Space Integration

What do these integrals typically look like in practice?



$gg \rightarrow Hg$ integrand:

$$\begin{aligned}
 & -\frac{1521745ca^4}{20736} + \frac{18895ca^3nf}{5184} + \frac{59743ca^2cfnf}{2304} + \frac{97ca^2nf^2}{144} - \frac{1}{36}cacfnf^2 - \frac{5cannf^3}{216} - \\
 & \frac{\dots 1 \dots}{20736} + \frac{\dots 732496 \dots}{64512} + \frac{\dots 1 \dots}{64512} + \frac{\dots 1 \dots}{64512} + \frac{288761ca^4\text{del}[1-\lambda]\times\text{PlusD}[zb,0]\text{Zeta}[7]}{10752} + \\
 & \frac{288761ca^4\text{del}[\lambda]\times\text{PlusD}[zb,0]\text{Zeta}[7]}{10752} + \frac{288761ca^4\text{del}[zb]\times\text{PlusD}[1-\lambda,0]\text{Zeta}[7]}{21504} + \\
 & \frac{288761ca^4\text{del}[zb]\times\text{PlusD}[\lambda,0]\text{Zeta}[7]}{21504} + \frac{1}{\epsilon} \left(-\frac{56339ca^4}{3456} + \frac{5705ca^3nf}{1728} + \frac{741}{128}ca^2cfnf + \frac{\dots 172439 \dots}{258048} + \right. \\
 & \left. \frac{1923ca^4\text{del}[zb]\times\text{PlusD}[\lambda,1]\text{Zeta}[5]}{1280} - \frac{3613931ca^4\text{del}[zb]\times\text{del}[1-\lambda]\text{Zeta}[7]}{258048} - \frac{3613931ca^4\text{del}[zb]\times\text{del}[\lambda]\text{Zeta}[7]}{258048} \right)
 \end{aligned}$$

Size in memory: 456.6 MB

+ Show more

Show all

Iconize ▼

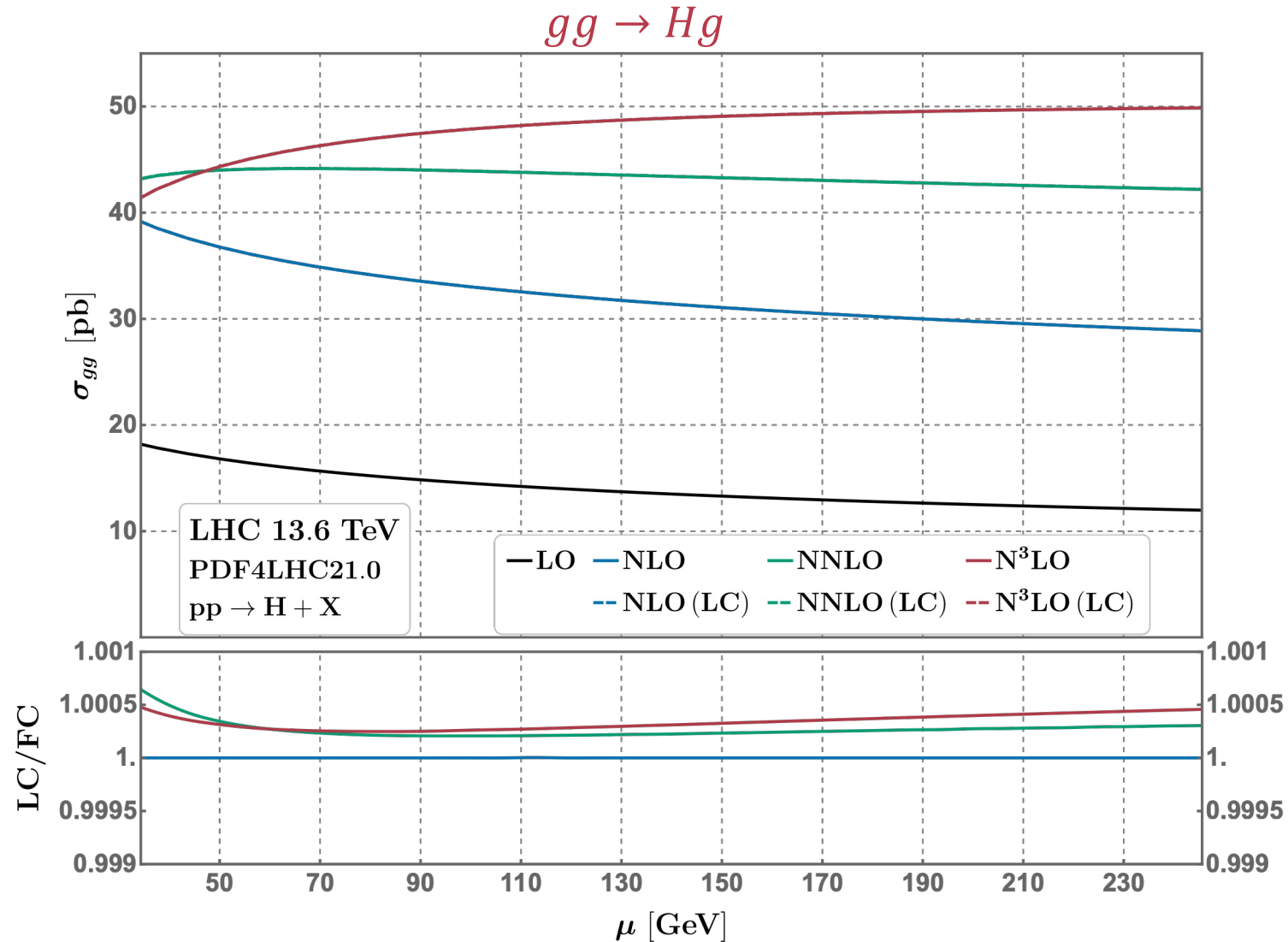
Store full expression in notebook



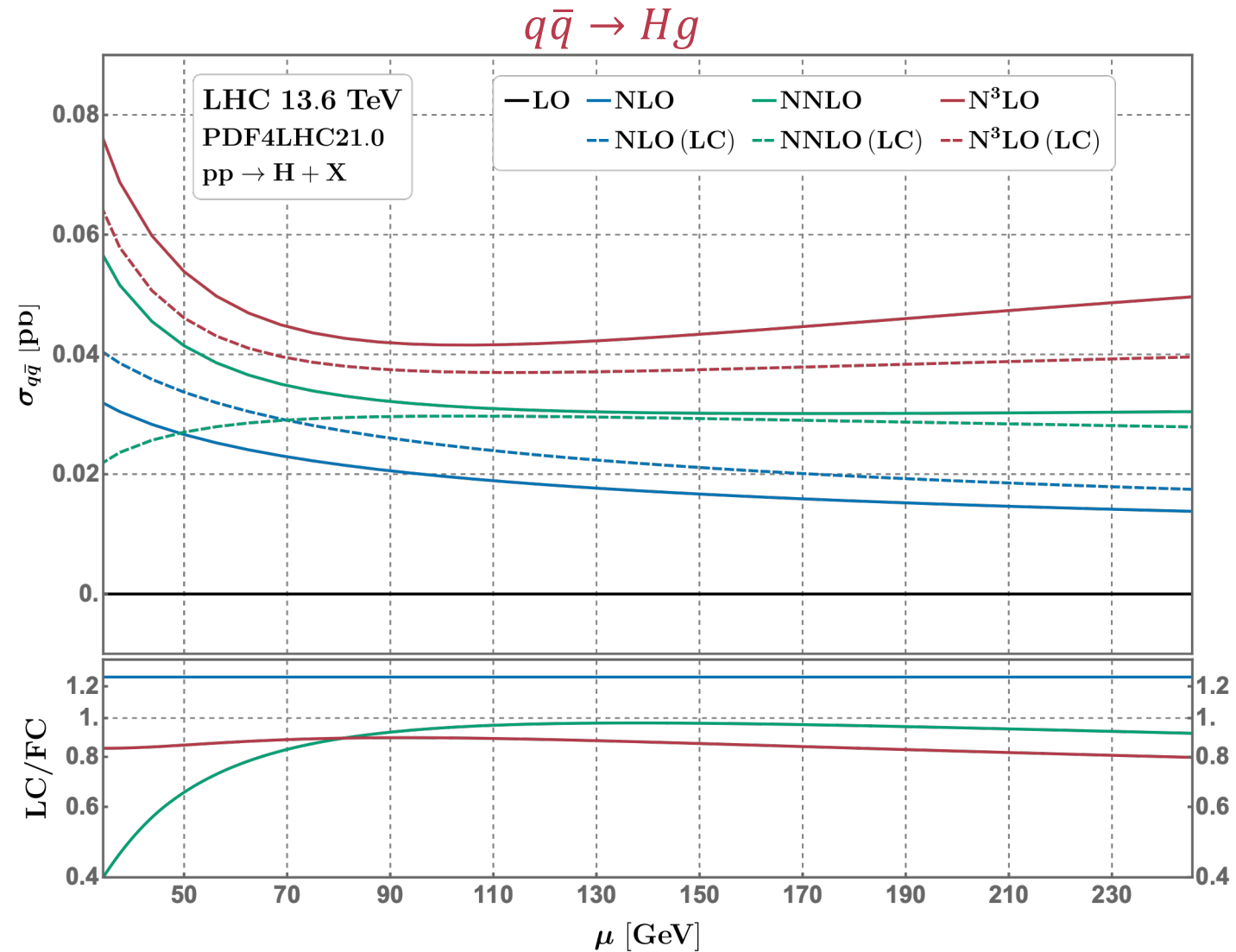
Huge and complicated! Contains various polylogs, distributions, renormalization factors...

Integration software based on PolyLogTools*

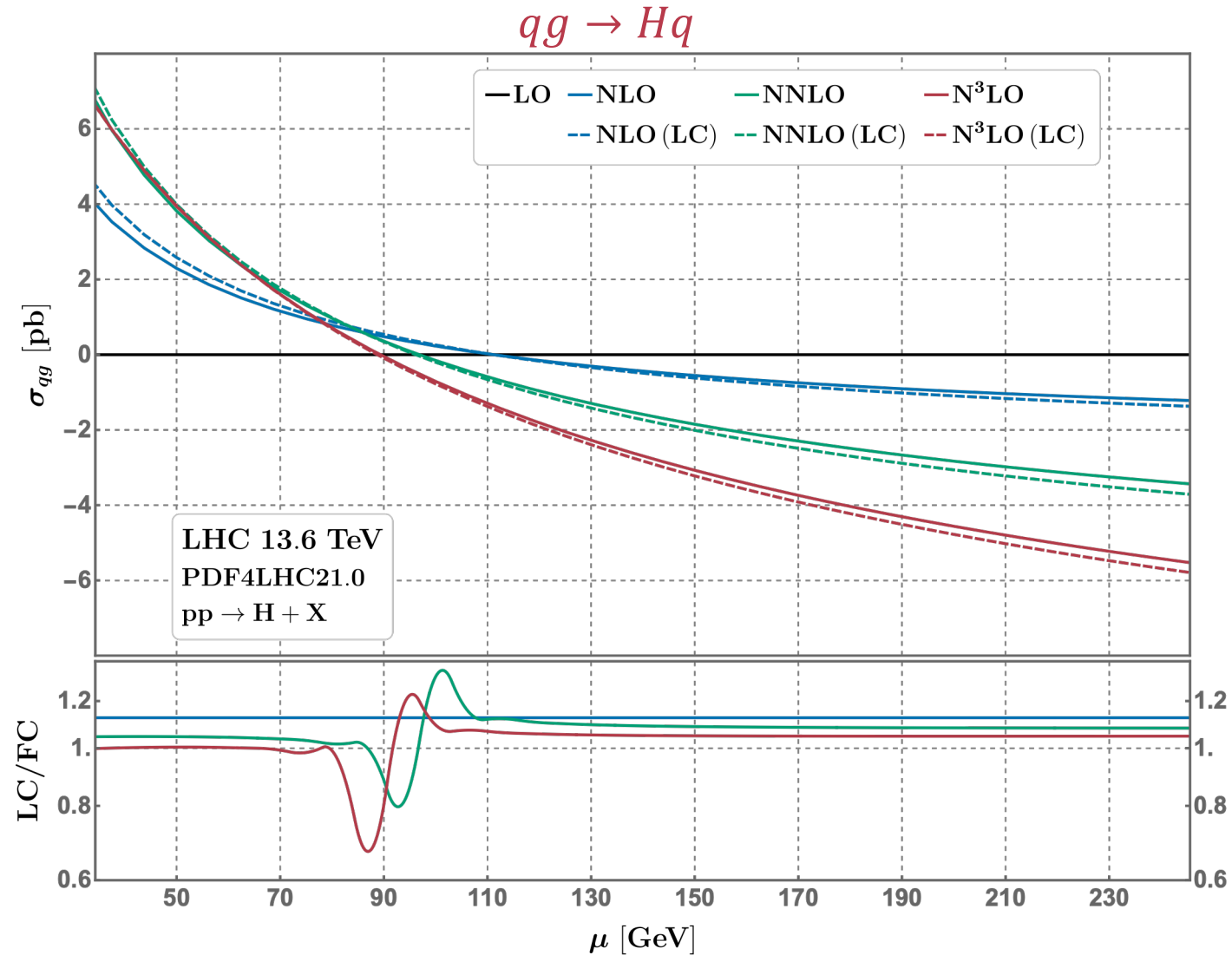
More Leading Color Phenomenology



More Leading Color Phenomenology

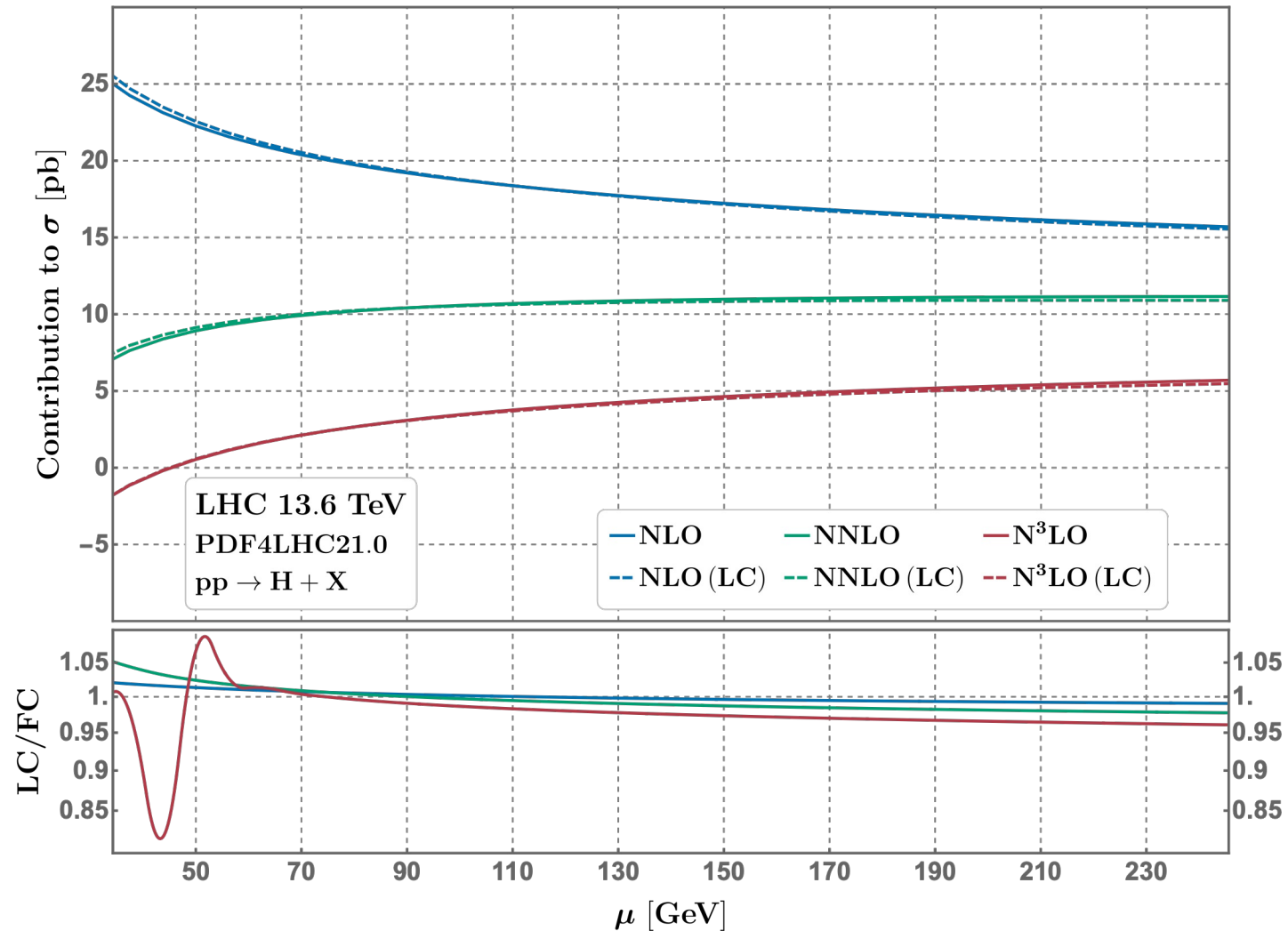


More Leading Color Phenomenology



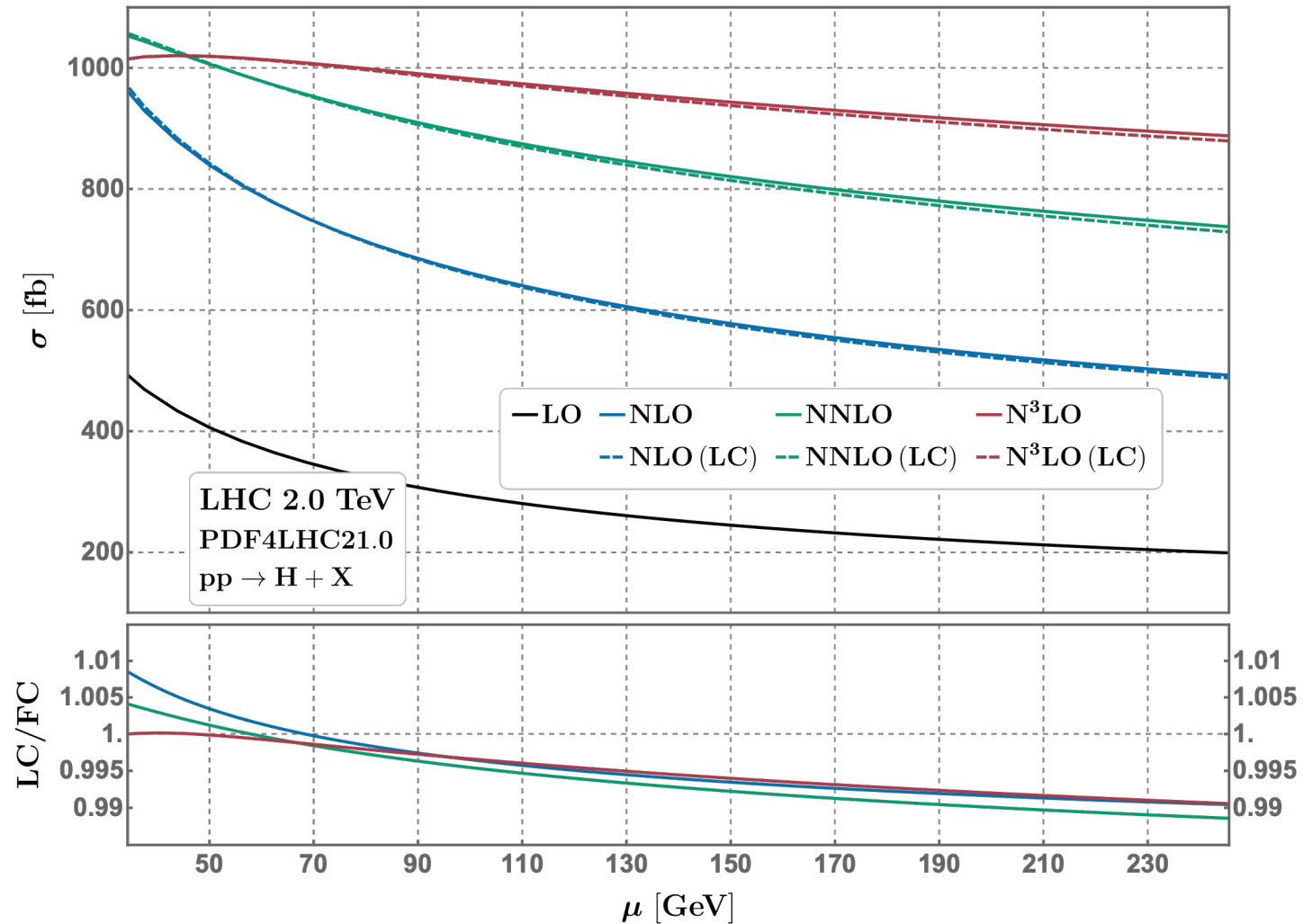
More Leading Color Phenomenology

Contribution to cross section at each order:



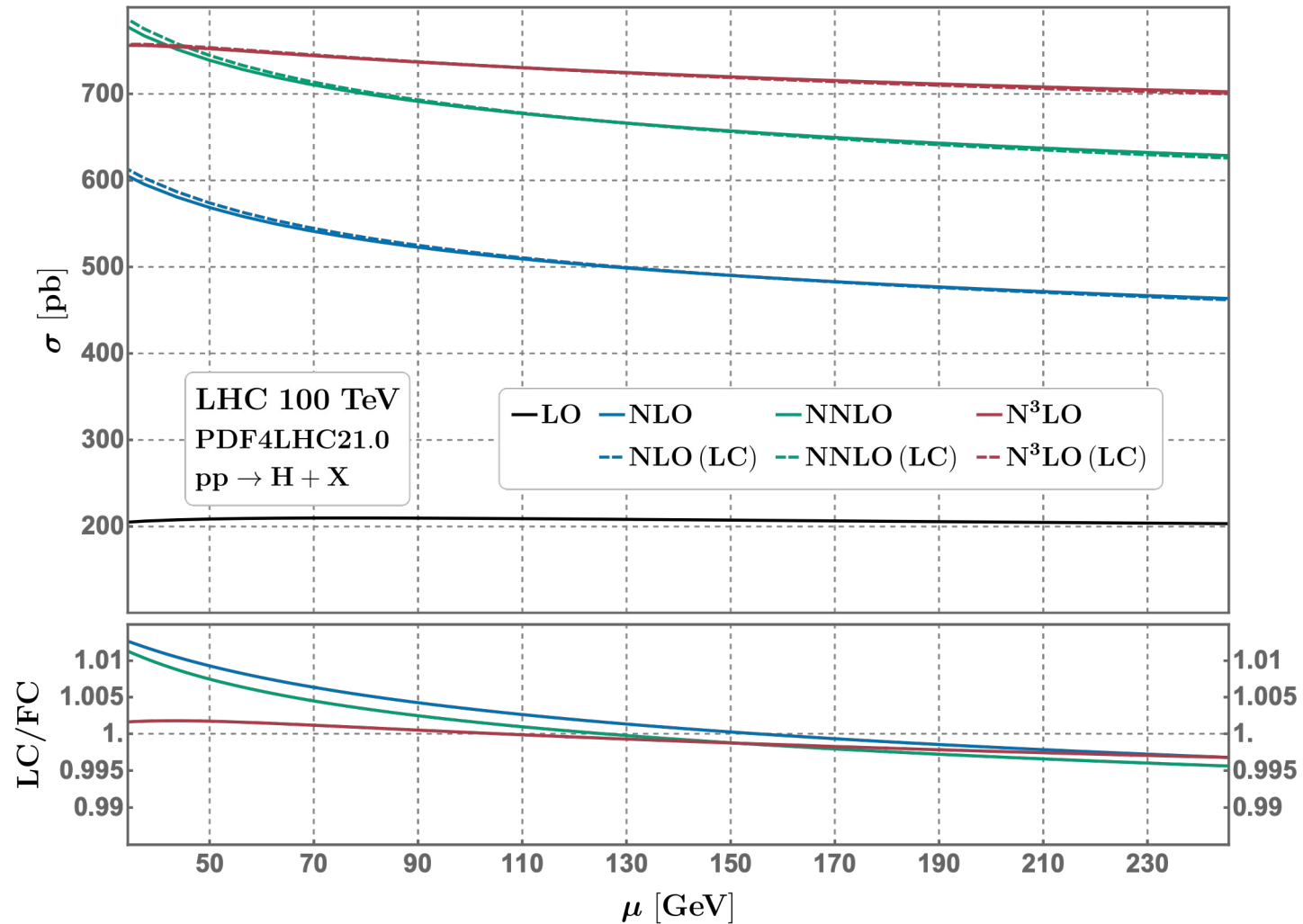
More Leading Color Phenomenology

Cross section at $\sqrt{s} = 2$ TeV

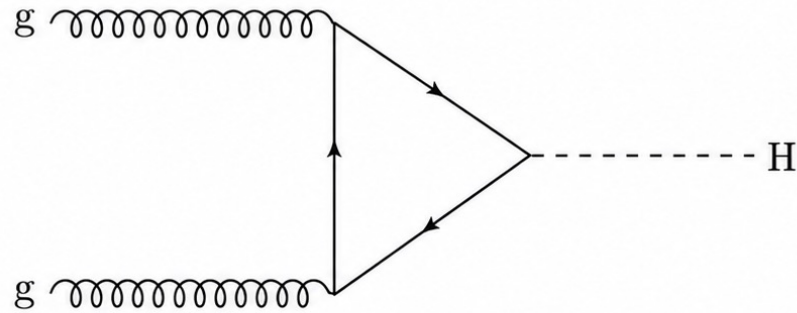


More Leading Color Phenomenology

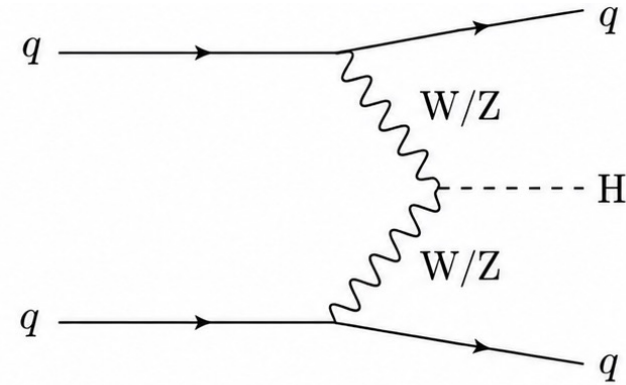
Cross section at $\sqrt{s} = 100$ TeV



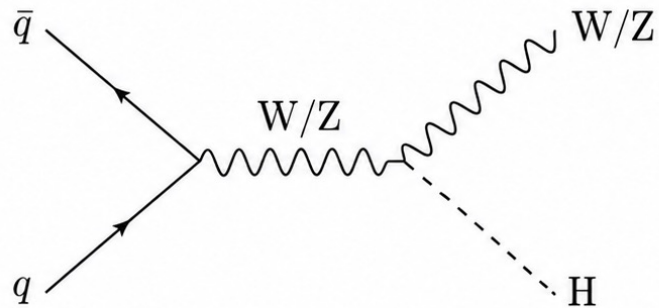
How are Higgs bosons produced at the LHC partonically?*



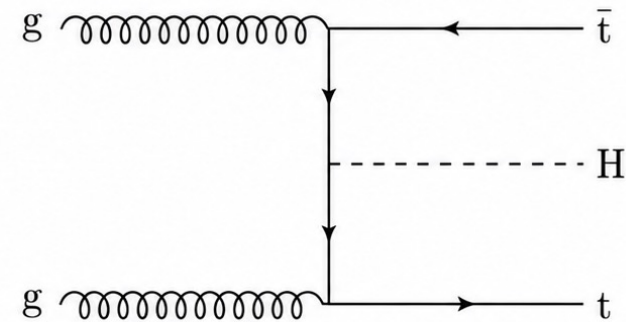
Gluon-fusion 88%



Vector-boson fusion 4%



W/Z Associated 7%



$t\bar{t}$ associated 1%

*[Higgs Cross Section Working Group; arXiv:1610.07922]

The Inclusive Cross Section

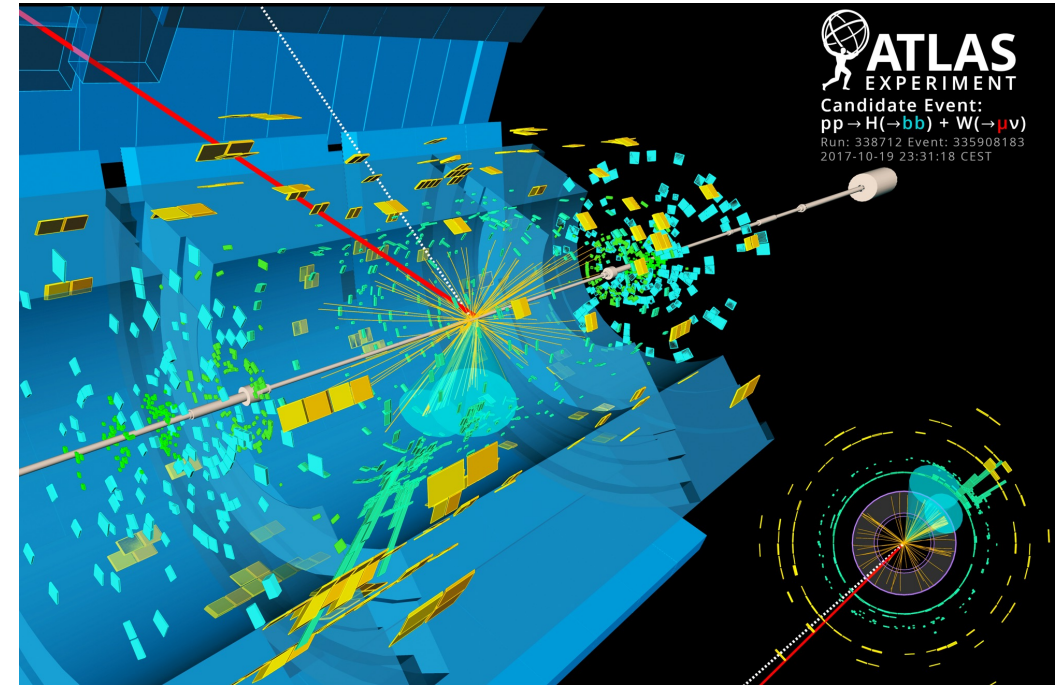
- The inclusive cross section serves as a fundamental benchmark for all other measurements
- Enters into global fits
 - Higgs couplings
 - PDF constraints
- Precise differential predictions require precise inclusive benchmarks.



From Counts to Cross Sections

How do we determine the cross section from what we see in the detector?

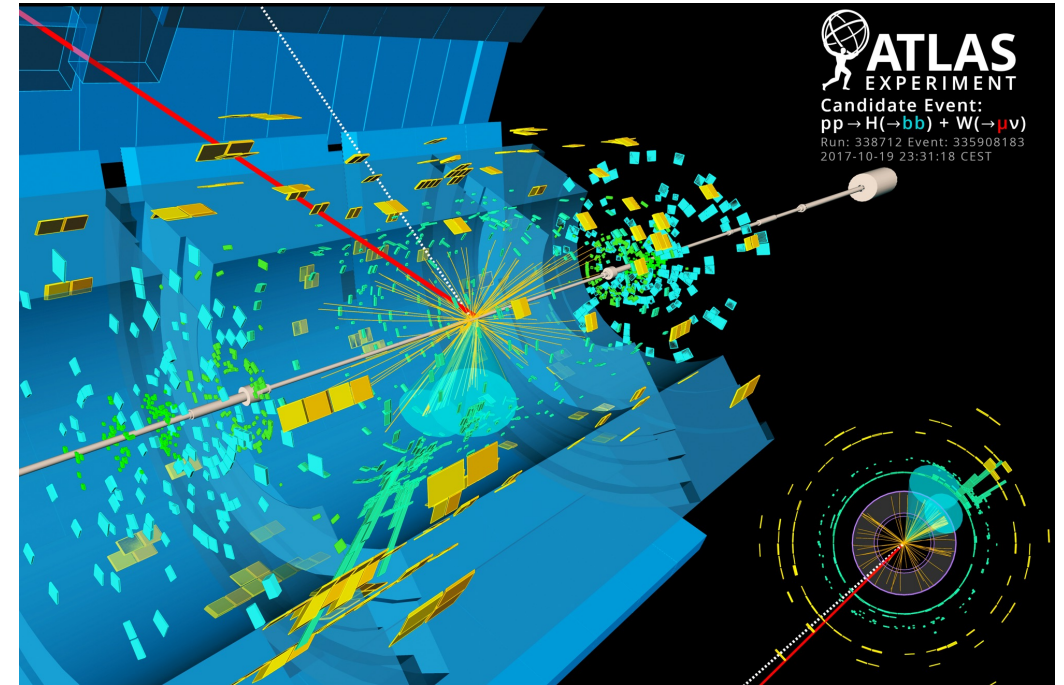
- $N_{PP \rightarrow H} = \mathcal{L} \sigma_{PP \rightarrow H+X} \text{BR}_{H \rightarrow Y} A \epsilon$
- We need to understand:
 - Luminosity
 - Branching Ratio
 - Acceptance and efficiency
 - Background N_{bkg} and interference N_{int}



From Counts to Cross Sections

How do we determine the cross section from what we see in the detector?

- $N_{PP \rightarrow H} = \mathcal{L} \sigma_{PP \rightarrow H+X} \text{BR}_{H \rightarrow Y} A \epsilon$
- We need to understand:
 - Luminosity
 - Branching Ratio
 - Acceptance and efficiency
 - Background N_{bkg} and interference N_{int}



The Gluon-fusion Higgs at NLO*

Result

$$\hat{\sigma}_{\text{NLO,reg}} \sim \frac{11n_c \bar{z}^3}{6(\bar{z}-1)} + \frac{2n_c (\bar{z}^2 - \bar{z} + 1)^2 \log(1 - \bar{z})}{(\bar{z}-1) \bar{z}} - \frac{4n_c (\bar{z}^3 - 2\bar{z}^2 + 3\bar{z} - 1) \log(\bar{z})}{\bar{z}-1}$$

- $\bar{z} = 1 - \frac{m_B^2}{s}$
- Simple rational functions and logs!

*[Dawson; Nucl.Phys.B 359 (1991), 283-300]

The Gluon-fusion Higgs at NNLO*

Result

$$\hat{\sigma}_{\text{NNLO,reg}} \sim$$

$$\begin{aligned} & \frac{4 G(2, 1, 0, z) (z^2 - 3z + 3)^2 n_C^2}{(z-2)(z-1)} + \frac{31 G(0, 1, 0, z) (z^2 - z + 1)^2 n_C^2}{(z-1)z} + \frac{3 G(2, 2, 1, z) (3z^2 - 8z + 6) n_C^2}{(z-2)(z-1)} - \frac{48 G(0, 0, 0, z) (z^3 - 2z^2 + 3z - 1) n_C^2}{z-1} + \frac{G(2, 1, z) (11z^3 - 54z^2 + 63z - 6) n_C^2}{6(z-1)} - \\ & \frac{G(1, 2, 1, z) (3z^4 - 16z^3 + 36z^2 - 40z + 19) n_C^2}{(z-2)(z-1)} - \frac{G(2, 1, 1, z) (4z^4 - 24z^3 + 75z^2 - 112z + 66) n_C^2}{2(z-2)(z-1)} + G(1, 1, 1, z) \left(\frac{(9z^5 - 28z^4 + 15z^3 + 15z^2 - z - 14) n_C^2}{(z-2)(z-1)z} - \frac{5}{4} (z-2) n_f n_C + \frac{5(z-2) n_f}{4 n_C} \right) + \\ & G(1, 0, 1, z) \left(\frac{(-37z^5 + 98z^4 - 10z^3 - 100z^2 + 11z + 78) n_C^2}{2(z-2)(z-1)z} + \frac{3}{2} (z-2) n_f n_C - \frac{3(z-2) n_f}{2 n_C} \right) + G(1, 1, 0, z) \left(-\frac{2(9z^5 - 26z^4 + 11z^3 + 14z^2 + z - 16) n_C^2}{(z-2)(z-1)z} + 2(z-2) n_f n_C - \frac{2(z-2) n_f}{n_C} \right) + \\ & \frac{G(0, 0, 1, z) (2(59z^4 - 120z^3 + 183z^2 - 124z + 61) n_C^3 - z(z^2 + 1) n_f)}{4(z-1)z n_C} + \frac{G(0, 1, 1, z) (z(z^2 + 1) n_f - 8(6z^4 - 13z^3 + 21z^2 - 15z + 7) n_C^3)}{4(z-1)z n_C} + \frac{G(1, 0, 0, z) ((31z^4 - 30z^3 - 3z^2 + 2z + 31) n_C^3 - 2z(z^2 - 3z + 2) n_f n_C^2 + 2z(z^2 - 3z + 2) n_f)}{(z-1)z n_C} + \\ & \frac{G(0, 0, z) ((220z^3 - 428z^2 + 450z - 22) n_C^3 + (-8z^3 + 23z^2 - 27z + 4) n_f n_C^2 + z(4z^2 - 15z + 15) n_f)}{3(z-1) n_C} + \frac{G(0, z) ((-2295z^3 + 4991z^2 - 5256z + 60\pi^2(z^3 - 2z^2 + 3z - 1) + 265) n_C^3 + (182z^3 - 533z^2 + 534z - 37) n_f n_C^2 - 4z(17z^2 - 84z + 75) n_f)}{36(z-1) n_C} + \\ & \frac{G(1, z) ((4054z^4 - 9789z^3 + 10698z^2 - 3162z - 12\pi^2(9z^4 - 2z^3 - 21z^2 + 14z + 9) + 268) n_C^3 + (-280z^4 + 3(261 + 4\pi^2)z^3 - 18(37 + 2\pi^2)z^2 + 3(19 + 8\pi^2)z - 40) n_f n_C^2 - 4z(-34z^3 + 138z^2 - 120z + 3\pi^2(z^2 - 3z + 2) + 8) n_f)}{72(z-1)z n_C} + \\ & \frac{G(1, 1, z) (2(374z^4 - 669z^3 + 462z^2 - 24z + 11) n_C^3 + (-32z^4 + 45z^3 + 33z^2 - 58z - 4) n_f n_C^2 + z(24z^3 - 51z^2 - 9z + 44) n_f)}{24(z-1)z n_C} + \\ & \frac{G(1, 0, z) (-2(187z^4 - 383z^3 + 339z^2 - 66z + 11) n_C^3 + (12z^4 - 19z^3 - 6z^2 + 17z + 4) n_f n_C^2 + z(-8z^3 + 15z^2 + 6z - 17) n_f)}{6(z-1)z n_C} + \\ & \frac{G(0, 1, z) ((-605z^4 + 1240z^3 - 1317z^2 + 96z - 55) n_C^3 + (24z^4 - 67z^3 + 75z^2 - 18z + 10) n_f n_C^2 - 3z(4z^3 - 16z^2 + 15z + 1) n_f)}{12(z-1)z n_C} + \\ & \frac{1}{432(z-1) n_C} ((21165 - 8424\zeta(3))z^3 + 16(-2734 + 1053\zeta(3))z^2 - 36(-1181 + 702\zeta(3))z - 6\pi^2(363z^3 - 856z^2 + 900z - 44) + 8424\zeta(3) - 1616) n_C^3 + \\ & (-2030z^3 + 5941z^2 - 5778z + 12\pi^2(8z^3 - 23z^2 + 27z - 4) + 224) n_f n_C^2 - 4z(-124z^2 + 987z + 3\pi^2(4z^2 - 15z + 15) - 825) n_f) \end{aligned}$$

- *Not just* simple rational functions and logs!
- MPLs $G(\dots, \bar{z})$ and transcendental constants enter

Multiple polylogarithms (MPLs)*

Recursively defined iterated integrals:

$$G(a_1, a_2, \dots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \dots, a_n, t)$$

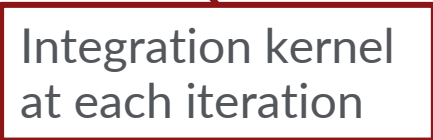
$$\text{And } G(\vec{0}_n, x) = \frac{\log^n x}{n!}$$

- Ubiquitous in multiloop amplitudes and generally well-understood

*A. B. Goncharov, Math. Res. Lett. 5 (1998) 497; arXiv:1105.2076.

The Symbol of MPLs

Decompose an MPL into “letters” s_j

$$\mathcal{S}[G(a_1, a_2, \dots, a_n, x)] = \sum_i c_i s_{i_1} \otimes \dots \otimes s_{i_n}$$


- For MPLs, “ $d\log$ ” letters
 - i.e. $s_j = d\log[x - a] = \frac{1}{x-a}$
- Non-“ $d\log$ ” letters describe a more general class of functions

*A. B. Goncharov, Math. Res. Lett. 5 (1998) 497; arXiv:1105.2076.

The Gluon-fusion Higgs at N³LO*

Result

$\hat{\sigma}_{\text{N3LO,reg}} \sim$

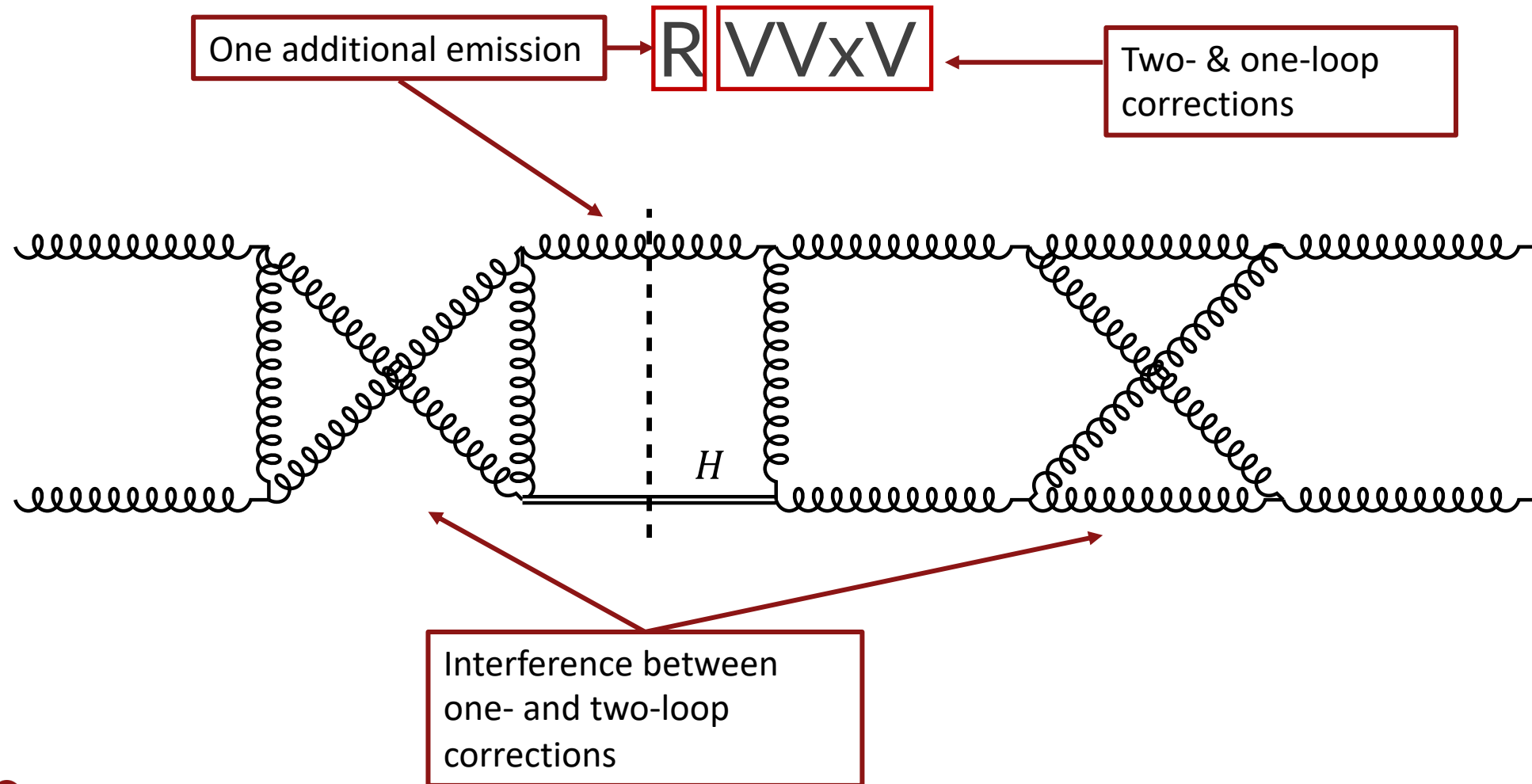
$$\begin{aligned} & \frac{11 \text{nc}^3 \pi^2 \text{zb}^7}{216 (\text{zb}-2) (\text{zb}-1)^2} - \frac{47 \text{nc}^2 \text{nf} \pi^2 \text{zb}^7}{864 (\text{zb}-2) (\text{zb}-1)^2} - \frac{\text{nf} \pi^2 \text{zb}^7}{216 (\text{zb}-2) (\text{zb}-1)^2} - \frac{\text{nc}^3 \text{zb}^6}{24 (\text{zb}-2) (\text{zb}-1)^2} + \frac{77 \text{nc}^2 \text{nf} \text{zb}^6}{1440 (\text{zb}-2) (\text{zb}-1)^2} + \frac{7 \text{nf} \text{zb}^6}{1440 (\text{zb}-2) (\text{zb}-1)^2} - \frac{3085 \text{nc}^3 \pi^2 \text{zb}^6}{3456 (\text{zb}-2) (\text{zb}-1)^2} + \\ & \frac{2131 \text{nc}^2 \text{nf} \pi^2 \text{zb}^6}{3456 (\text{zb}-2) (\text{zb}-1)^2} + \frac{5 \text{nf} \pi^2 \text{zb}^6}{2304 \text{nc}^2 (\text{zb}-2) (\text{zb}-1)^2} - \frac{239 \text{nf} \pi^2 \text{zb}^6}{6912 (\text{zb}-2) (\text{zb}-1)^2} + \dots 2478 \dots + \frac{\text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{1-\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{1-\text{zb}}, \text{zb}\right)}{2 \text{zb}} - \\ & \frac{3 \text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{\text{zb}}, \frac{1}{1-\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \text{zb}\right)}{2 \text{zb}} + \frac{27 \text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{\text{zb}}, \frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \text{zb}\right)}{2 (\text{zb}-1) \text{zb}} + \frac{9 \text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \text{zb}\right)}{(\text{zb}-1) \text{zb}} - \\ & \frac{3 \text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{1-\text{zb}}, \text{zb}\right)}{(\text{zb}-1) \text{zb}} + \frac{9 \text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \text{zb}\right)}{2 (\text{zb}-1) \text{zb}} - \frac{3 \text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{1-\text{zb}}, \text{zb}\right)}{2 (\text{zb}-1) \text{zb}} + \\ & \frac{3 \text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{1-\text{zb}}, \frac{1}{1-\text{zb}}, \text{zb}\right)}{(\text{zb}-1) \text{zb}} + \frac{21 \text{nc}^3 J\left(\frac{1}{\text{zb}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\sqrt{1-\text{zb}} \sqrt{\text{zb}+3}}, \frac{1}{\text{zb}}, \frac{1}{1-\text{zb}}, \text{zb}\right)}{4 (\text{zb}-1) \text{zb}} \end{aligned}$$

Size in memory: 4 MB + Show more Show all Iconize Store full expression in notebook

- Now square root (from RRV) and elliptic letters (from RRR) enter!

What are the single-real ingredients? At N^4LO :

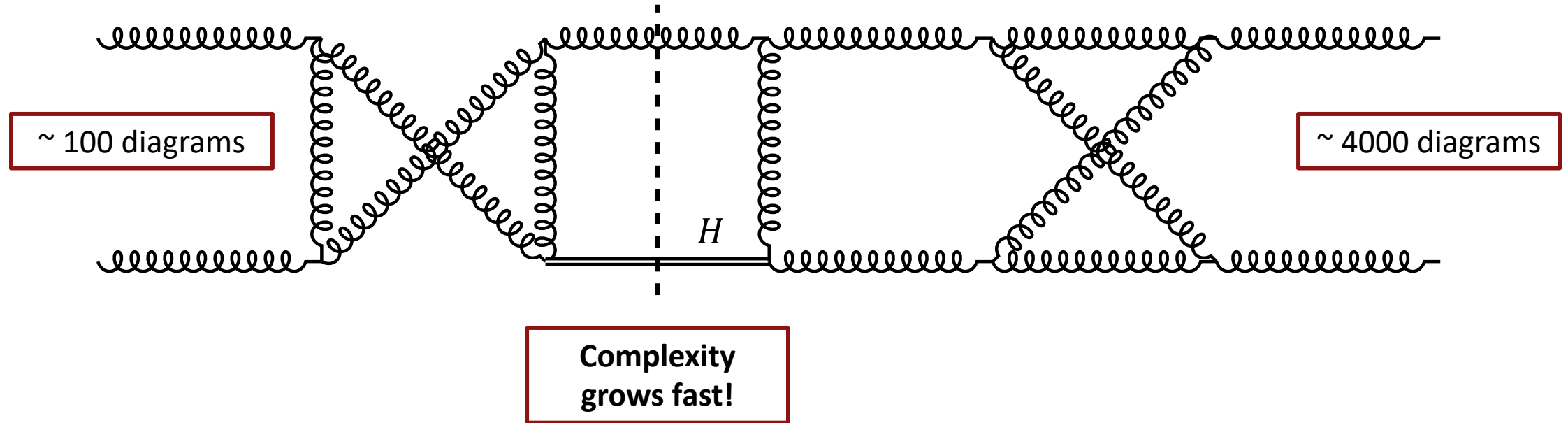
Now $RVVxV$: "single-real, double-virtual cross single-virtual".



What are the single-real ingredients? At N⁴LO:

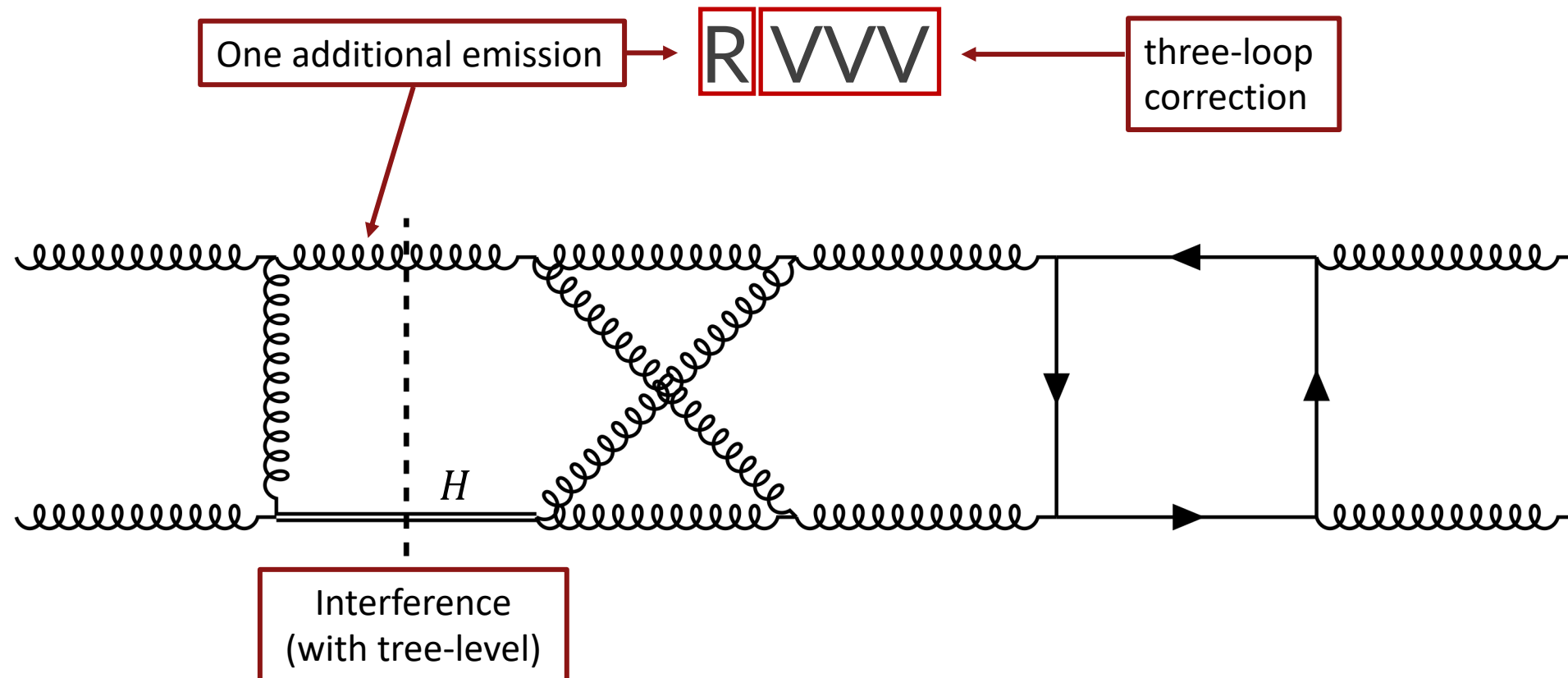
Now RVVxV: "single-real, double-virtual cross single-virtual".

RVVxV



What are the single-real ingredients? At N⁴LO:

Now RVVV: "single-real, triple-virtual".



RVVxV Contribution

(Re)computing one- and two^{**}-loop amplitudes

Generate all Feynman diagrams



Project onto gauge-invariant Lorentz basis

All particles live in $d = 4 - 2\epsilon$ dimensions
(conventional dimensional regularization)



Project onto basis of color tensors



Compute Loop integrals

- Use IBPs to obtain basis of “master integrals”
- Evaluate with method of differential equations
- In terms of generalized polylogarithms $G(\dots)$ up to transcendental weight eight

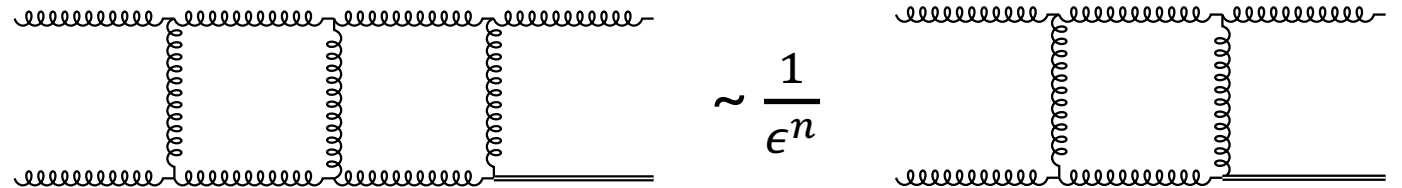
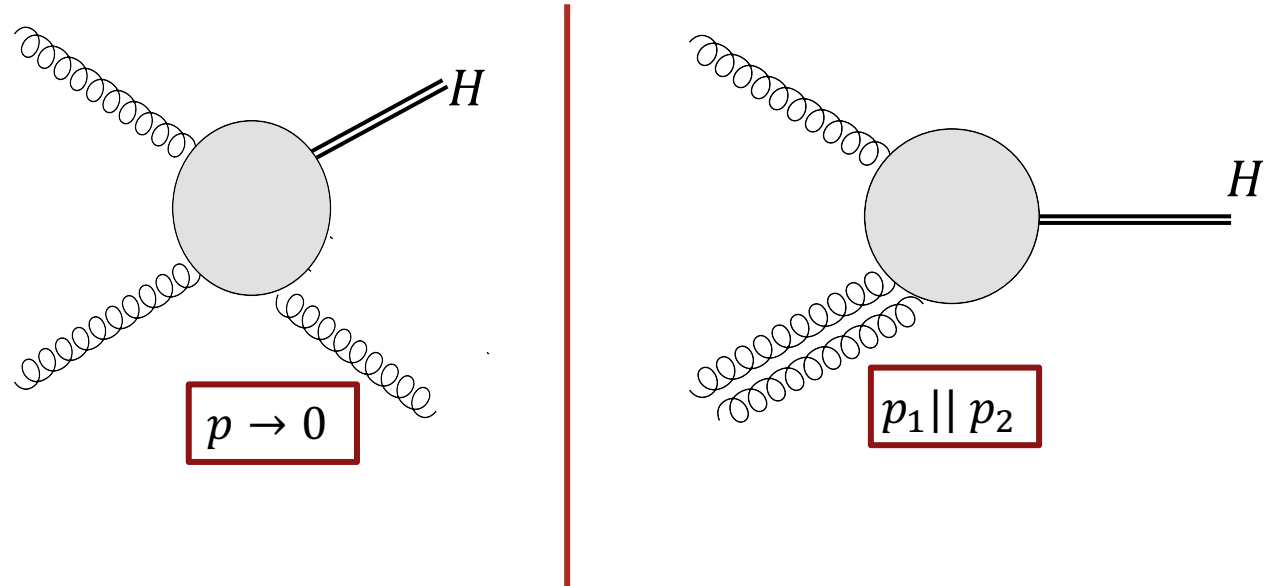
^{**}[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi; arXiv:2301.10849]

[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi; arXiv:2306.10170]

Checks on Amplitudes

Amplitudes factorize in:

- Soft ($p \rightarrow 0$) limits
- Collinear ($p_1 || p_2$) limits
- Poles in $\epsilon \sim$ lower loop amplitudes



(Re)computing one- and two^{**}-loop amplitudes

Generate all Feynman diagrams



Project onto gauge-invariant Lorentz basis

All particles live in $d = 4 - 2\epsilon$ dimensions (conventional dimensional regularization)



Project onto basis of color tensors

Debut Mathematica color algebra package: CIFAR



Compute Loop integrals

- Use IBPs to obtain basis of “master integrals”
- Evaluate with method of differential equations
- In terms of generalized polylogarithms up to transcendental weight eight

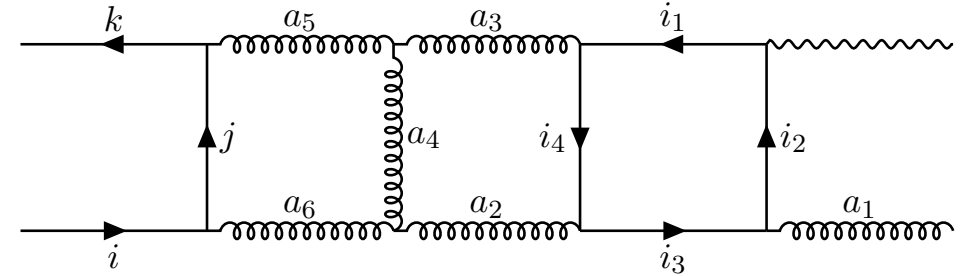
^{**}[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi; arXiv:2301.10849]

[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi; arXiv:2306.10170]

Computing Color Factors in Gauge Theories

QCD gauge group $\rightarrow$ “color” factors in amplitude

- Gauge group is $SU(n_c)$, with n_c number of colors
 - Factors reduce to polynomials in n_c
- Can instead consider **generalized gauge group**
 - Factors reduce to color group (“Casimir”) invariants $\mathcal{C}$



$\text{Proj. } t_{ik}^{a_1}$

$$\delta_{i_1 i_2} t_{i_2 i_3}^{a_1} t_{i_3 i_4}^{a_2} t_{i_4 i_1}^{a_3} t_{ik}^{a_1} t_{kj}^{a_5} t_{ji}^{a_6} f^{a_3 a_4 a_5} f^{a_2 a_6 a_4}$$

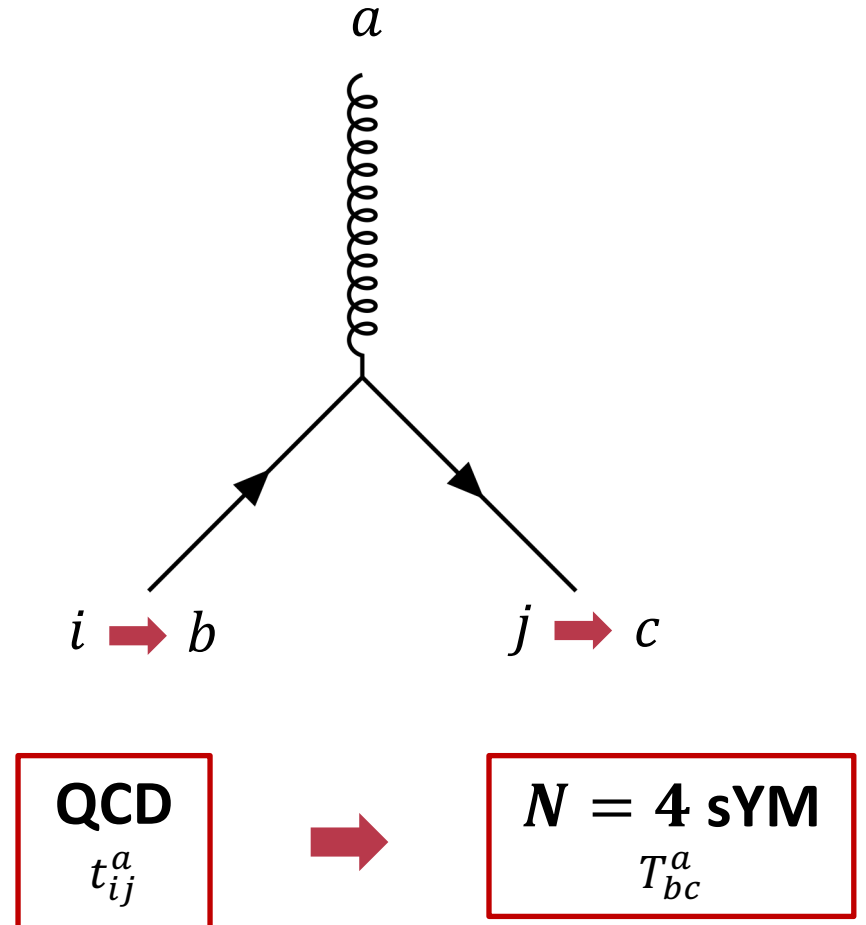
$$-\frac{1}{2} C_A C_3^{FF} - \frac{1}{8} T_F^2 D_A C_A^2$$

$$-\frac{1}{8} + \frac{3}{16} n_c^2 - \frac{1}{16} n_c^4$$

Computing Color Factors in Gauge Theories

QCD gauge group $\rightarrow$ “color” factors in amplitude

- Ubiquitous step in amplitude calculations
- Generalized gauge group $\rightarrow$ allows for comparison across theories
 - e.g. Set $C_F = C_A$ for $N = 4$ sYM (“Principle of maximal transcendentality”)
- Need an efficient and reliable computational pipeline



CIFAR: A Color Algebra Package

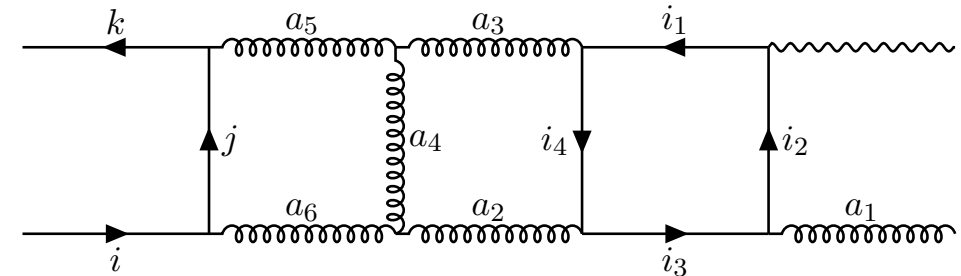
QCD gauge group → “color” factors in amplitude

- CIFAR can do this computation for us!

```
In[1]:= D1 = deltaF[i1,i2]*
        TT[{a1},i2,i3]*TT[{a2},i3,i4]*TT[{a3},i4,i1]*
        TT[{a1},i,k]*TT[{a5},k,j]*TT[{a6},j,i]*
        ff[a3,a4,a5]*ff[a2,a6,a4];
In[2]:= D1CIFAR = CIFARReduce[ D1 ]

Out[2]:= -1/2*(C3FF*CA) - (CA^2*DA*TF^2)/8
```

- Quartic Casimirs: Ready for N⁴LO
- Available at:
github.com/adisurtya/CIFAR



Proj. $t_{ik}^{a_1}$

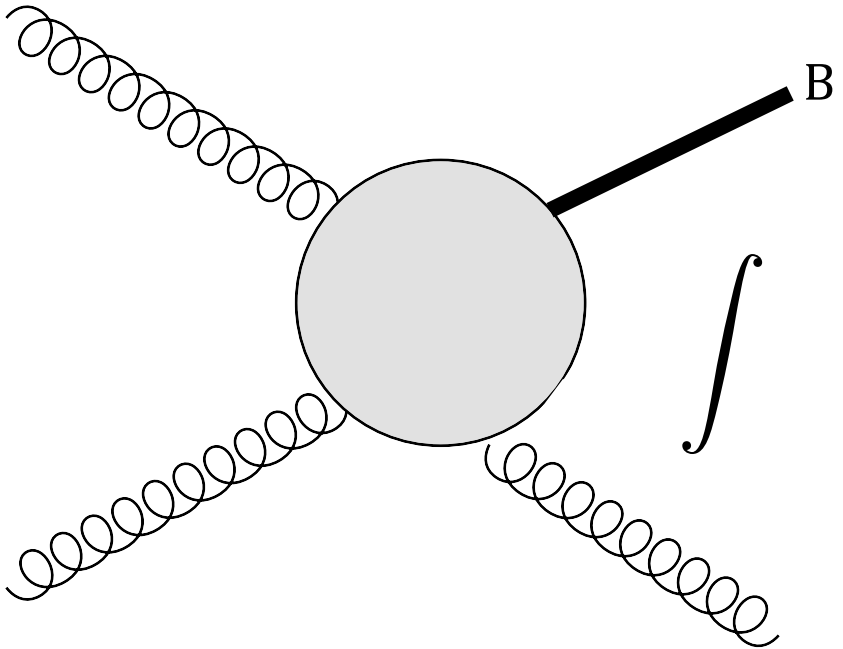
$$\delta_{i_1 i_2} t_{i_2 i_3}^{a_1} t_{i_3 i_4}^{a_2} t_{i_4 i_1}^{a_3} t_{i k}^{a_1} t_{k j}^{a_5} t_{j i}^{a_6} f^{a_3 a_4 a_5} f^{a_2 a_6 a_4}$$

$$-\frac{1}{2} C_A C_3^{FF} - \frac{1}{8} T_F^2 D_A C_A^2$$

$$-\frac{1}{8} + \frac{3}{16} n_c^2 - \frac{1}{16} n_c^4$$

Phase Space Integration

What do these integrals typically look like?

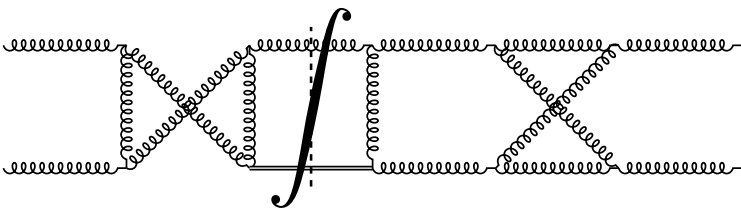
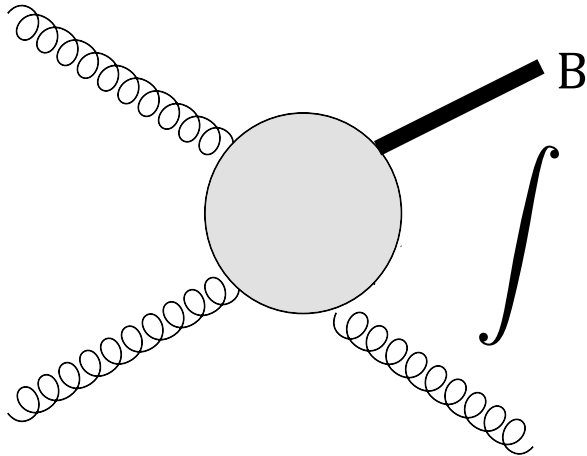


$$\hat{\sigma} \sim \int d\lambda \frac{\log \lambda}{\lambda - 1/\bar{z}} = \log \lambda \log (1 - \bar{z}\lambda) + \text{Li}_2(\bar{z}\lambda)$$

Parametrizes energy of scattering

Phase Space Integration

What do these integrals typically look like in practice?



$gg \rightarrow Hg$ integrand:

$$\begin{aligned}
 & -\frac{1521745ca^4}{20736} + \frac{18895ca^3nf}{5184} + \frac{59743ca^2cfnf}{2304} + \frac{97ca^2nf^2}{144} - \frac{1}{36}cacfnf^2 - \frac{5cannf^3}{216} - \\
 & \frac{\dots 1 \dots}{20736} + \frac{\dots 732496 \dots}{64512} + \frac{\dots 1 \dots}{64512} + \frac{\dots 1 \dots}{64512} + \frac{288761ca^4\text{del}[1-\lambda]\times\text{PlusD}[zb,0]\text{Zeta}[7]}{10752} + \\
 & \frac{288761ca^4\text{del}[\lambda]\times\text{PlusD}[zb,0]\text{Zeta}[7]}{10752} + \frac{288761ca^4\text{del}[zb]\times\text{PlusD}[1-\lambda,0]\text{Zeta}[7]}{21504} + \\
 & \frac{288761ca^4\text{del}[zb]\times\text{PlusD}[\lambda,0]\text{Zeta}[7]}{21504} + \frac{1}{\epsilon} \left(-\frac{56339ca^4}{3456} + \frac{5705ca^3nf}{1728} + \frac{741}{128}ca^2cfnf + \dots 172439 \dots + \right. \\
 & \left. \frac{1923ca^4\text{del}[zb]\times\text{PlusD}[\lambda,1]\text{Zeta}[5]}{1280} - \frac{3613931ca^4\text{del}[zb]\times\text{del}[1-\lambda]\text{Zeta}[7]}{258048} - \frac{3613931ca^4\text{del}[zb]\times\text{del}[\lambda]\text{Zeta}[7]}{258048} \right)
 \end{aligned}$$

Size in memory: 456.6 MB

+ Show more

Show all

Iconize

Store full expression in notebook



Huge and complicated! Contains various polylogs, distributions, renormalization factors...

Integration software based on PolyLogTools*

RVVxV: After Performing Phase Space Integration

Results

- Laurent expansion in dimensional regulator, ϵ
- With $\bar{z} = 1 - \frac{m_B^2}{s}$, contains:
 - Rational functions
 - Multiple Polylogs: $G(\dots, \bar{z})$
 - Distributions: $\delta(\bar{z}), \left[\frac{\log^k \bar{z}}{\bar{z}} \right]_+$
 - (soft singularities)

Checks

- Recomputed all lower-order single-real (R...) interference contributions
- Soft-singular parts for:
 - $g + g \rightarrow H + g$
 - $q + \bar{q} \rightarrow \gamma^* + g$
 - Applied single-emission soft current to $g + g \rightarrow H$ and $q + \bar{q} \rightarrow \gamma^*$ to find agreement

RVVV Computation: Integration strategy for Higgs Channels

Problem: $\int_0^1 r(\lambda) G(a_1, \dots, a_n, \lambda) d\lambda$

hard to evaluate when $a_i = \sqrt{\alpha + \beta \bar{z} + \gamma \bar{z}^2}$

RVVV Computation: Integration strategy for Higgs Channels

Problem: $\int_0^1 r(\lambda) G(a_1, \dots, a_n, \lambda) d\lambda$

hard to evaluate when $a_i = \sqrt{\alpha + \beta \bar{z} + \gamma \bar{z}^2}$



**Canonical
Master Integral**

Partial Integration Idea: Evaluate $\int_0^1 r(\lambda) M_i(\lambda) d\lambda$!

RVVV Computation: Integration strategy for Higgs Channels

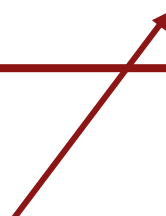
Problem: $\int_0^1 r(\lambda) G(a_1, \dots, a_n, \lambda) d\lambda$

hard to evaluate when $a_i = \sqrt{\alpha + \beta \bar{z} + \gamma \bar{z}^2}$



Partial Integration Idea: Evaluate $\int_0^1 r(\lambda) M_i(\lambda) d\lambda$!

$$= R M \Big|_0^1 - \int_0^1 R \partial_\lambda M_i d\lambda$$


$$R(\lambda) = \int r(\lambda) d\lambda$$

RVVV Computation: Integration strategy for Higgs Channels

Problem: $\int_0^1 r(\lambda) G(a_1, \dots, a_n, \lambda) d\lambda$

hard to evaluate when $a_i = \sqrt{\alpha + \beta \bar{z} + \gamma \bar{z}^2}$



Partial Integration Idea: Evaluate $\int_0^1 r(\lambda) M_i(\lambda) d\lambda$!

$$= R M \Big|_0^1 - \int_0^1 R \partial_\lambda M_i d\lambda$$

Know from differential equations!

$$\partial_\lambda M_i = \epsilon A_{ij} M_j$$

RVVV Computation: Integration strategy for Higgs Channels

Problem: $\int_0^1 r(\lambda) G(a_1, \dots, a_n, \lambda) d\lambda$

hard to evaluate when $a_i = \sqrt{\alpha + \beta \bar{z} + \gamma \bar{z}^2}$



Partial Integration Idea: Evaluate $\int_0^1 r(\lambda) M_i(\lambda) d\lambda!$

$$= R M \Big|_0^1 - \int_0^1 R \partial_\lambda M_i d\lambda$$



Next Iteration: $\epsilon \int_0^1 A_{ij} R(\lambda) M_j(\lambda) d\lambda$