

On the N3LO evolution of the PDFs from moments

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Ingredients for precision

$$d\sigma = \int dx_1 dx_2 \, f_i(x_1, \mu) \cdot f_j(x_2, \mu) \cdot d\hat{\sigma}_{ij}(s, x_1 x_2, \mu)$$

Partonic Cross Section

N3LO required!

$$d\hat{\sigma}_{ij} = d\hat{\sigma}^{(0)} + a d\hat{\sigma}^{(1)} + a^2 d\hat{\sigma}^{(2)} + a^3 d\hat{\sigma}^{(3)} + \dots \quad a = \frac{\alpha_s}{4\pi}$$

Parton Distribution Functions

$$f_{\text{ns},ij}^{\pm} = (f_{q_i} \pm f_{\bar{q}_i}) - (f_{q_j} \pm f_{\bar{q}_j}) \quad f_{\text{ns}}^{\text{v}} = \sum_{i=1}^{n_f} (f_{q_i} - f_{\bar{q}_i})$$

Flavour Non-Singlet

$$f_{\text{s}} = \sum_{i=1}^{n_f} (f_{q_i} + f_{\bar{q}_i})$$

$$f_g = g(x, \mu^2)$$

Flavour singlet

Scale evolution

The scale dependence controlled by the DGLAP equation

$$\mu^2 \frac{d}{d\mu^2} \begin{pmatrix} f_{\text{ns}}^{\pm, \nu} \\ f_s \\ f_g \end{pmatrix} = \begin{pmatrix} P_{\text{ns}}^{\pm, \nu} & 0 & 0 \\ 0 & P_{qq} & P_{qg} \\ 0 & P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} f_{\text{ns}}^{\pm, \nu} \\ f_s \\ f_g \end{pmatrix}$$

$\otimes$ is the convolution in x -space

$$P \otimes f[x] \equiv \int_x^1 \frac{dz}{z} P(z) \cdot f\left(\frac{x}{z}\right)$$

Perturbative information: the **splitting functions**

$$P_{ij}(x) = a P_{ij}^{(0)}(x) + a^2 P_{ij}^{(1)}(x) + a^3 P_{ij}^{(2)}(x) + \underbrace{a^4 P_{ij}^{(3)}(x)}_{\text{N3LO}} + \dots$$

Moments

The evolution in moment space is controlled by

$$\gamma_{ij}^{(k)}(N) = - \int_0^1 dx x^{N-1} P_{ij}^{(k)}(x)$$

γ_{ij} are the **anomalous dimensions** of gauge invariant operators [Curci,Furmanski,Petronzio 1980;Collins,Soper 1981] that arise the OPE [Gross,Wilczek 1974;Georgi,Politzer 1974]

$$O_{\text{ns}}^{\mu_1 \dots \mu_N; a}(N) = \frac{i^{N-1}}{2} \mathcal{S} \left[\bar{\psi}_i \gamma^{\mu_1} \lambda_{ij}^a D^{\mu_2} \dots D^{\mu_N} \psi_j \right] - \text{traces}$$

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$$O_g^{\mu_1 \dots \mu_N}(N) = \frac{i^{N-2}}{2} \mathcal{S} \left[F_{\mu_1 \rho}^{a_1} D_{\mu_2}^{a_1 a_2} \dots D_{\mu_{N-1}}^{a_{N-2} a_{N-1}} F_{\mu_N \rho}^{a_{N-1}} \right] - \text{traces}$$

Overview of the calculation

$$O_g(N) + \dots = -\frac{a}{\epsilon} \left(\gamma_3 + \gamma_{gg}^{(0)}(N) \right) O_g(N) - \frac{a}{\epsilon} \eta^{(0)} O_A^I$$

The **anomalous dimensions** are **UV poles** of propagator-like diagrams

- Four-Loop Master Integrals are known [Baikov,Chetyrkin 2010;Lee,A.Smirnov,V.Smirnov 2012]
- Reductions are automated for **fixed N** with Forcer [Ruijl,Ueda,Vermaseren 2016]
- Care required: **unphysical operators** in the **singlet**! [Dixon,Taylor;Joglekar,Lee 1976;Hamberg,van Neerven 1992] [GF,Herzog 2022;Gehrmann,von Manteuffel,Yang 2023]

Development of the Fixed-moment calculation

Non-singlet moments

[Moch,Ruijl,Ueda,Vermaseren,Vogt 2017]: Direct application of Forcer

- Eight even moments $\gamma_{\text{ns}}^{+(3)}(N)$, $N = 2, 4, \dots, 16$
- Eight odd moments $\gamma_{\text{ns}}^{-(3)}(N)$, $N = 1, 3, \dots, 15$
- Nine odd moments of the sea contribution $\gamma_{\text{ns}}^{s(3)} = \gamma_{\text{ns}}^{v(3)} - \gamma_{\text{ns}}^{-(3)}$

[Moch,Ruijl,Ueda,Vermaseren,Vogt 2024 unpub]:

- Extension to **eleven** moments $\gamma_{\text{ns}}^{\pm(3)}(N \leq 22)$

Singlet anomalous dimensions

[GF,Herzog,Moch,Pelloni,Vogt 2023-2024]: **Ten** even moments $N \leq 20$

$$\left(\begin{array}{cc} \gamma_{qq}^{(3)} = \gamma_{\text{ns}}^{+(3)} + \gamma_{\text{ps}}^{(3)} & \gamma_{qg}^{(3)} \\ \gamma_{gq}^{(3)} & \gamma_{gg}^{(3)} \end{array} \right)$$

How to use a few moments

k **moments**, $m_1 \dots m_k$, $\Rightarrow$ k -parameter expressions in x -space, e.g.

$$P(x) = c_1 (1-x) \frac{\log^2(x)}{x} + c_2 \frac{\log(x)}{x} + \frac{c_3}{1-x}_+ + \dots c_k$$

such that

$$m_i = \int_0^1 dx x^{i-1} P(x), \quad \forall i \in 1 \dots k.$$

The space of functions is **large** [Moch,Vermaseren,Vogt 2004;
Gehrmann,von Manteuffel,Sotnikov,Yang 2026]

- Distributions $\delta(1-x)$, $\frac{1}{1-x}_+$
- denominators x , $1 \pm x$, positive powers of x
- HPLs of weight $2(L-1)$, ζ of weight $2L-1$, L loop order

Singularities at $x \rightarrow 0$ and $x \rightarrow 1$

Endpoint logs: $L_0 = \log x$, $L_1 = \log(1 - x)$. Consider e.g. $P_{\text{ps}}^{(3)}$

$$P_{\text{ps}}^{(3)}(x \rightarrow 0) \simeq \underbrace{E_1 \frac{L_0^2}{x}}_{[1]} + \mathbf{E}_2 \frac{L_0}{x} + \frac{\mathbf{E}_3}{x} + \underbrace{F_1 L_0^6 + F_2 L_0^5 + F_3 L_0^4}_{[2]} + \cdots + \mathbf{F}_6 L_0$$

$$P_{\text{ps}}^{(3)}(x \rightarrow 1) \simeq (1 - x) \left[\underbrace{G_1 L_1^4 + G_2 L_1^3}_{[3]} + \mathbf{G}_3 L_1^2 + \mathbf{G}_4 L_1 \right]$$

[1] = [Catani,Hautmann 1994], [2] = [Davies,Kom,Moch,Vogt 2022],

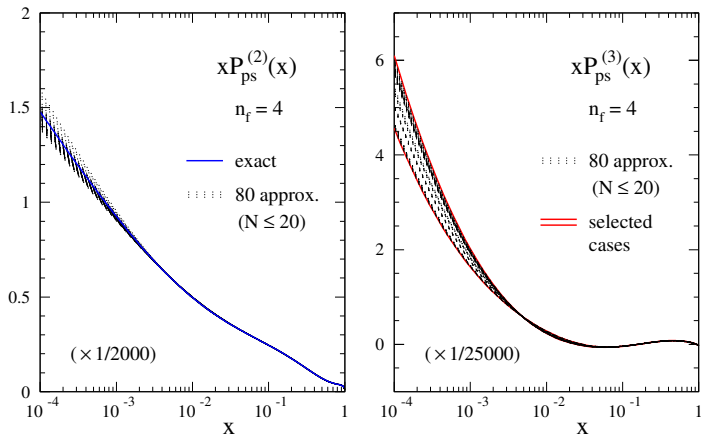
[3] = [Soar,Moch,Vermaseren,Vogt 2009]

Free parameters: $E_2, E_3, F_4, F_5, F_6, G_3, G_4 + 3 \rightarrow$ fit 10 moments

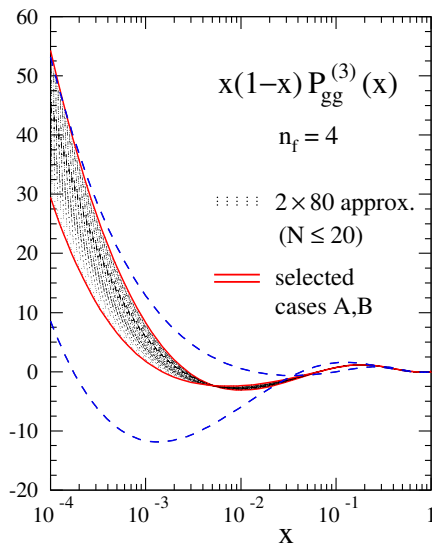
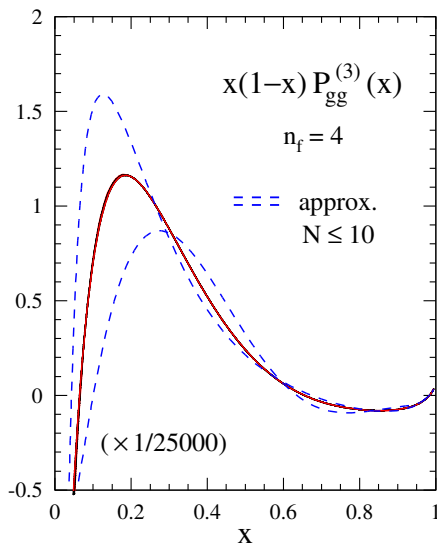
$$P_{\text{ps,param}}^{(3),i} = P_{\text{ps}}^{(3)}(x \rightarrow 0) + P_{\text{ps}}^{(3)}(x \rightarrow 1) + f_i(K_1, K_2, K_3)$$

Approximate DGLAP splitting function

Free choice of $f_i(K_1, K_2, K_3) \rightarrow$ construct $\mathcal{O}(100)$ parameterisations to estimate uncertainties. Below $\Delta P_{\text{ps}}^{(3)} / P_{\text{ps}}^{(3)} \simeq 15\%$ at $x = 10^{-4}$.



General features of the approximations

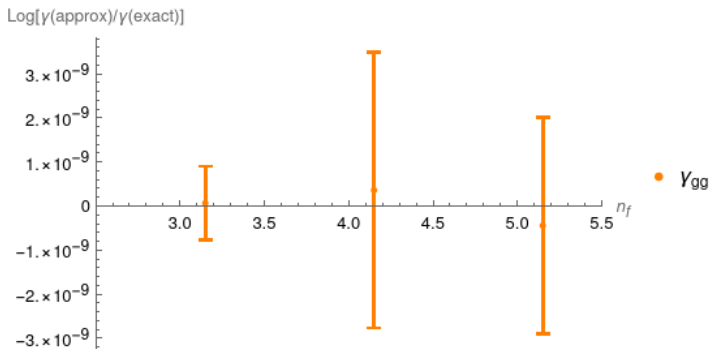


Validation: moment $N + 2$ in the singlet sector

- 1 Compute the next moment using the parameterisations

$$\gamma_{\text{up/low}}(N = 22) = \int_0^1 P_{\text{param}}^{(3),\text{up/low}}(x) x^{21} dx$$

- 2 Then compute it **analytically** [GF,Herzog,Moch,Pelloni,Vogt 2025]

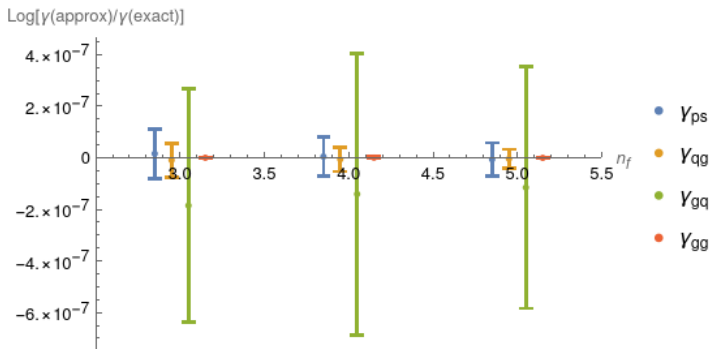


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Validation: a few comments

- All the predictions are consistent with the exact result.
- $\gamma_{gq}^{(3)}(22)$ has the **largest uncertainties** and is slightly underestimated. The pitfalls come from small- x physics
 - ▶ $P_{gq}^{(3)}(x) \simeq E_{LL} \frac{\log^3(x)}{x}$ [Jaroszewicz 82; Catani, Fiorani, Marchesini 91]
 - ▶ Assume Casimir scaling at NLL and use $P_{gg}^{(3)}$ results from [Fadin, Lipatov 1998; Ciafaloni, Camici 1998] [Ciafaloni, Colferai, Salam, Stasto 2006]
 - ▶ **Estimate** NNLL from the size of NLL.

Updates [GF, Herzog, Moch, Pelloni, Vogt 2025]

- Increased NNLL and error from $\gamma_{gq}^{(3)}(22)$ and NNLL/NLL ratio in $P_{gg}^{(3)}$: **new** parameterisation for $P_{gq}^{(3)}(x)$.
- Parameterisations of all the splitting functions at $n_f = 6$.

Validation: Non-singlet sector

The non-singlet parameterisations [Moch,Ruijl,Ueda,Vermaseren,Vogt 2017] included fewer theoretical inputs

- **Eight** computed moments.
- Numerical error on the **cusp anomalous dimension**

$$P_{\text{ns}}^{(3)}(x \rightarrow 1) \simeq \frac{A_q}{1-x_+} + B_q \delta(1-x) + C_q \log(1-x) + D_q - A_q$$

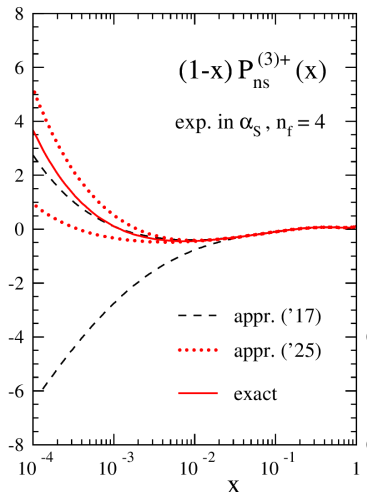
Results available at the end of 2025:

- 3 more moments [Moch,Ruijl,Ueda,Vermaseren,Vogt 2024 unpub]
- A_q [Henn,Korchensky,Mistlberger; von Manteuffel,Panzer,Schabinger 20]
- **Exact** n_f dependent piece [Kniehl,Moch,Velizhanin,Vogt 2025]

The non-singlet parameterisations are affected.

Comparison with the exact results

The **exact** non-singlet DGLAP splitting functions are now computed
[Gehrmann,von Manteuffel,Sotnikov,Yang 2026]



[A. Vogt, S. Moch LL2026]

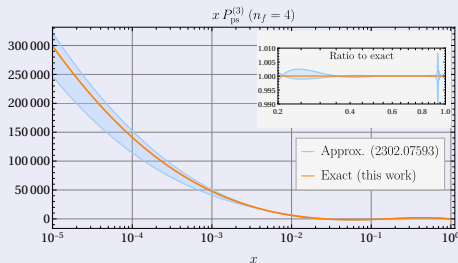
Impact of 3 more moments and theoretical input:

- Small shift in the parameterised curves
- The error bands shrink
- The approximate $P_{ns}^{+(3)}$ becomes **compatible** with the exact results.

New comparisons!

The **pure singlet** has just been published
[Gehrmann,von Manteuffel,Sotnikov,Yang 2026]

Accuracy of the parameterisation



[From 2609.37563v1 Gehrmann et al.]

- Confirmed moments $N = 2 \dots 22$.
- Parameterisation consistent through $x = 10^{-5}$.
- New term at small- x
 - ▶ NNLL and beyond the leading colour.

Approximate scale evolution

$P_{ij}^{(3)}(x)$ enter the scale evolution equations via **convolution**

$$\mu^2 \frac{d}{d\mu^2} f_i(x, \mu^2) = \int_x^1 \frac{dz}{z} P_{ij}(z) \cdot f_j\left(\frac{x}{z}\right)$$

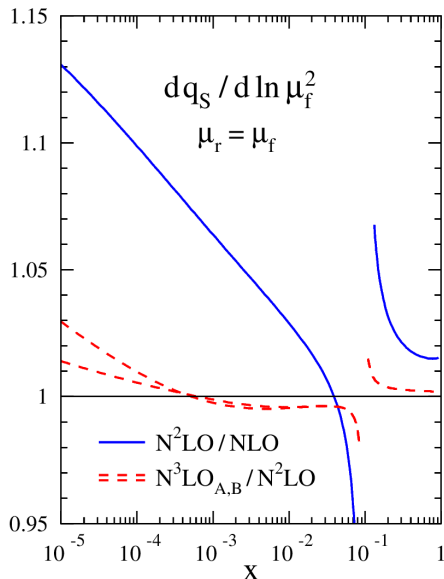
Approx

- $P_{ij}^{(3)}(z)$ have **large uncertainties** for $z \simeq x \ll 1$
- PDFs $f_j\left(\frac{x}{z}\right)$ **suppress** the small- z region: using model PDFs
[Moch,Vermaseren,Vogt 2004]

$$\begin{aligned} x g(x) &= 1.6 x^{-0.3} (1-x)^{4.5} (1-0.6x^{0.3}), \\ x q_s(x) &= 0.6 x^{-0.3} (1-x)^{3.5} (1+5.0x^{0.8}) \end{aligned}$$

$$f_j\left(\frac{x}{z}\right) \xrightarrow{z \rightarrow x} f_j(1) \sim 0$$

Scale evolution of the singlet quark PDF

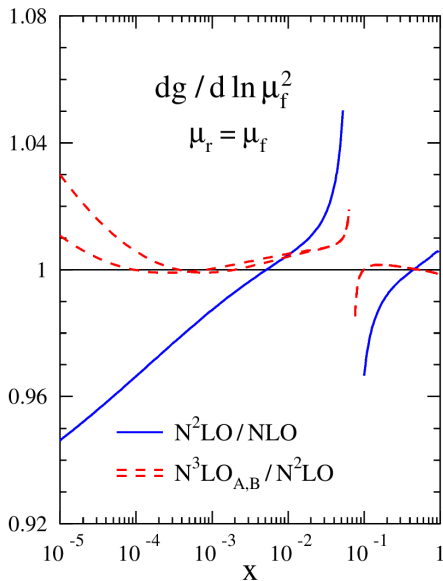


Evolution of the quark (S) PDF

$$\mu^2 \frac{d}{d\mu^2} f_q = P_{qq} \otimes f_q + P_{qg} \otimes f_g$$

- N³LO effect is **small** compared to NNLO
 - ▶ **subpercent** for $x > 10^{-4}$
 - ▶ $\left. \frac{\dot{q}_s^{\text{N}^3\text{LO}}}{\dot{q}_s^{\text{NNLO}}} \right|_{x=10^{-5}} \sim (2 \pm 1)\%$
- Uncertainties
 - ▶ **Negligible** for $x \gtrsim 10^{-4}$
 - ▶ $\delta \dot{q}_s = \pm 1\%$ at $x = 10^{-5}$

Evolution of the gluon PDF



Evolution of the gluon PDF

$$\mu^2 \frac{d}{d\mu^2} f_g = P_{gq} \otimes f_q + P_{gg} \otimes f_g$$

- NNLO effects already small.
 Same picture at $N^3\text{LO}$
 - ▶ **subpercent** for $x > 10^{-4}$
 - ▶ $\left. \frac{\dot{g}^{N^3\text{LO}}}{\dot{g}^{\text{NNLO}}} \right|_{x=10^{-5}} \sim (2 \pm 1)\%$
- Uncertainties
 - ▶ **Negligible** for $x \gtrsim 10^{-4}$
 - ▶ $\delta \dot{g} = \pm 1\%$ at $x = 10^{-5}$

Conclusion

- The evolution of the singlet PDFs is parameterised in x -space, using physical constraints at $x \rightarrow 0$ and $x \rightarrow 1$ and the fixed **moments** up to $N = 20$.
- The approximation has been verified by computing one new moment, $N = 22$, analytically
 - ▶ All the approximations are **consistent** with the new result.
 - ▶ $P_{gq}^{(3)}(x)$ has been slightly updated.
- The impact of the N3LO corrections on the evolution of the gluon and the quark singlet is **small**
 - ▶ Subpercent for $x \lesssim 10^{-4}$ and about 2% at $x = 10^{-5}$.
 - ▶ Subpercent errors up to $x = 10^{-5}$.

Conclusion

- The precision of the approximations is suitable for phenomenology.
- The bulk of the uncertainty originates from the **small-x** behaviour: interesting area for theory improvement!
- Getting $P_{ij}^{(3)}$ exactly requires $10^3 - 10^4$, e.g. with generating functions techniques

[Ablinger,Blümlein,Hasselhuhn,Klein,Schneider,Wissbrock 2012]

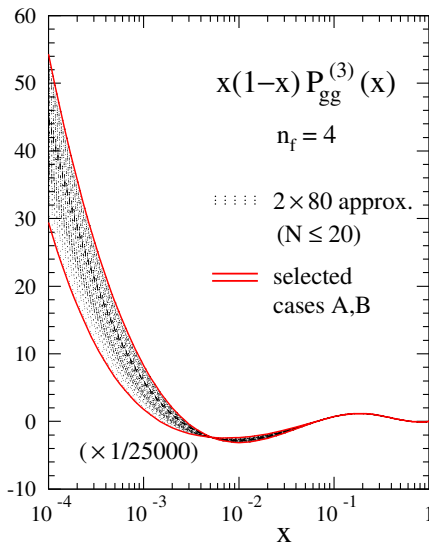
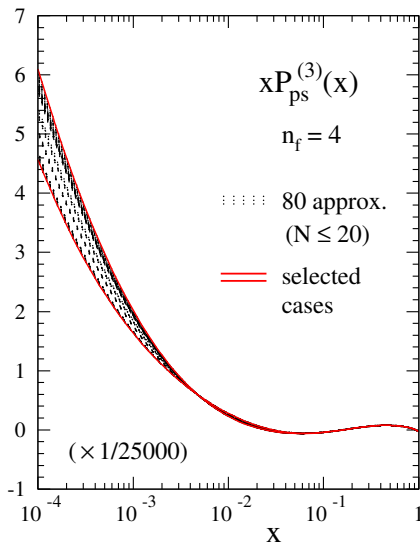
[Ablinger,Behring,Blümlein,De Freitas,von Manteuffel,Schneider 2014]

[Gehrmann,von Manteuffel,Sotnikov,Yang 2026].

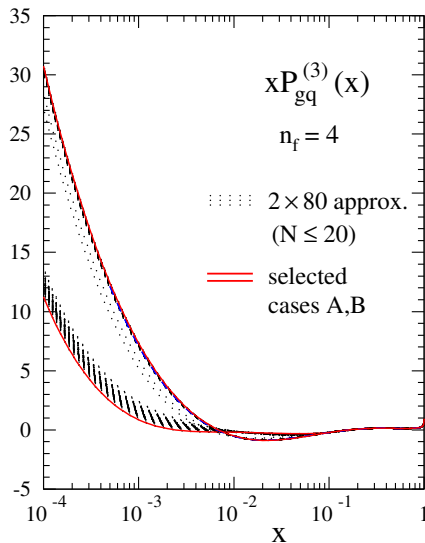
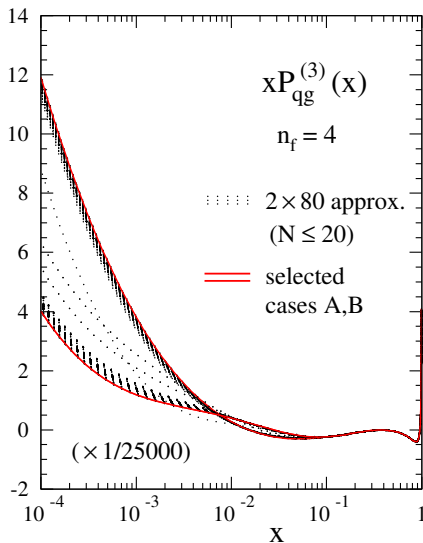
Thank you!

Backup

Approximate $P_{ps}^{(3)}$ and $P_{gg}^{(3)}$



Approximate $P_{qg}^{(3)}$ and $P_{gq}^{(3)}$



Renormalisation scale stability

