



The variable flavor number scheme to three-loop order

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DESY AND TU DORTMUND

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[JHEP 01 (2026) 111, JHEP 03 (2026) 248, Phys. Lett. B (2026) 140411, and in preparation]

What is the use of variable flavor number schemes ?



- Account for universal heavy flavor contributions in massless and massive parton distribution functions.
- Matching conditions at (or around) flavor thresholds.
- Full account of the heavy flavor contributions in deep-inelastic structure functions in the region $Q^2 \gg m_Q^2$ by using massless Wilson coefficients.
- Single-mass and two-mass decoupling, since m_c^2/m_b^2 is not very small compared to one.
- The treatment is completely based on the **renormalization group**.

[Buza, Matiounine, Smith, van Neerven 1996]

■ single-mass case:

- one loop: $A_{Qg}, A_{gg,Q}$
- two loops: $A_{qq,Q}^{NS}, A_{Qq}^{PS}, A_{Qg}, A_{gq,Q}, A_{gg,Q}$
- three loops: $A_{qq,Q}^{NS}, A_{Qq}^{PS}, A_{qq,Q}^{PS}, A_{Qg}, A_{gq,Q}, A_{gq,Q}, A_{gg,Q}$

■ two-mass case:

- one loop: no contribution
- two loops: $A_{Qg}^{(1)} \cdot A_{gg,Q}^{(1)}$ factorizing
- three loops: $A_{qq,Q}^{NS,two}, A_{Qq}^{PS,two}, A_{Qg}^{two}, A_{gq,Q}^{two}, A_{gg,Q}^{two}$

The variable flavor number scheme



■ Matching conditions for parton distribution functions: 2-mass case

$$f_k(N_F + 2) + \bar{f}_k(N_F + 2) = A_{qq,Q}^{NS} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot [f_k(N_F) + \bar{f}_k(N_F)] + \frac{1}{N_F} A_{qq,Q}^{PS} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma^+(N_F) \\ + \frac{1}{N_F} A_{qg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F),$$

$$f_k(N_F + 2) - \bar{f}_k(N_F + 2) = A_{qq,Q}^{NS} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot [f_k(N_F) - \bar{f}_k(N_F)]$$

$$f_Q(N_F + 2) + \bar{f}_Q(N_F + 2) = A_{Qq}^{PS} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma^+(N_F) + A_{Qg} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F),$$

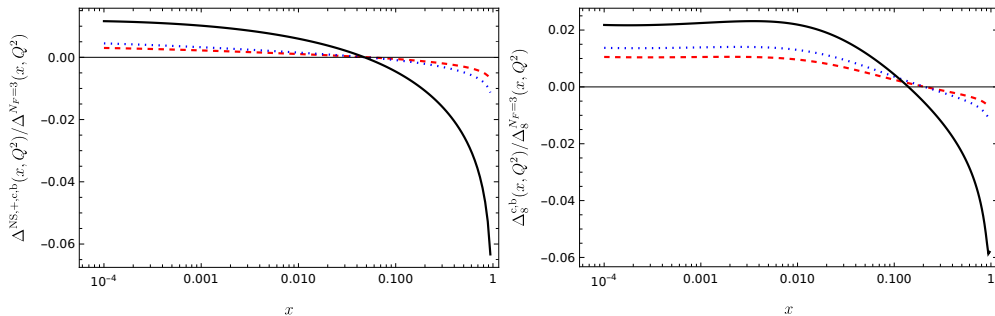
$$f_c(N_F + 1) - \bar{f}_c(N_F + 1) = A_{Qq}^{PS,s} \left(\frac{m_c^2}{\mu^2} \right) \cdot \Sigma^-(N_F), \quad f_b(N_F + 1) - \bar{f}_b(N_F + 1) = A_{Qq}^{PS,s} \left(\frac{m_b^2}{\mu^2} \right) \cdot \Sigma^-(N_F)$$

$$\Sigma^+(N_F + 2) = \left[A_{qq,Q}^{NS} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{qq,Q}^{PS} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{Qq}^{PS} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot \Sigma^+(N_F) \\ + \left[A_{qg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{Qg} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot G(N_F),$$

$$G(N_F + 2) = A_{qg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma^+(N_F) + A_{gg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F).$$

$$\Sigma^\pm = [u \pm \bar{u}] + [d \pm \bar{d}] + [s \pm \bar{s}]$$

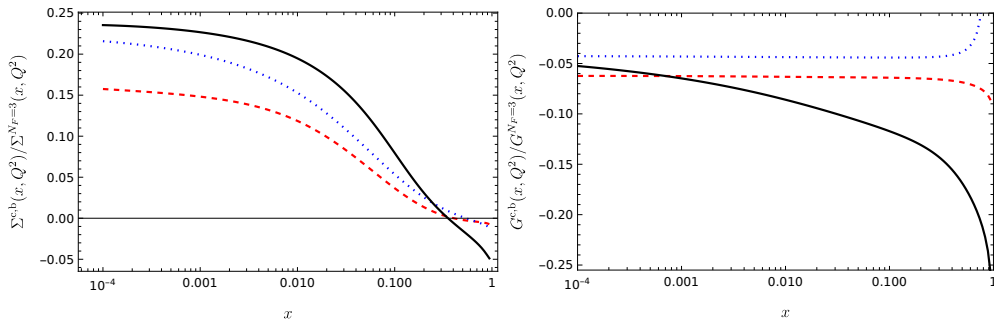
Non-singlet distributions: unpolarized & polarized



Left panel: The ratio $\Delta^{c,b}(x, Q^2)/\Delta^{N_F=3}(x, Q^2)$ as a function of x , referring to the matching scales $Q^2 = 30 \text{ GeV}^2$ (dotted line), $Q^2 = 100 \text{ GeV}^2$ (dashed line), $Q^2 = 10000 \text{ GeV}^2$ (full line) in the unpolarized case. Right panel: polarized distribution

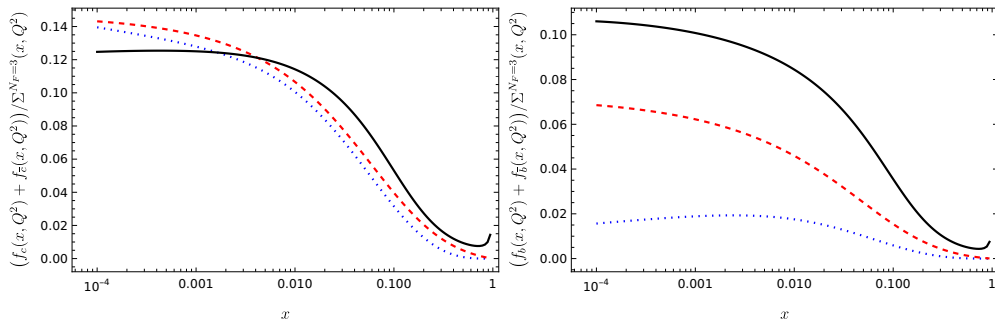
$$D^{c,b} = D^c + D^b - 2D^{N_F=3}$$

Unpolarized singlet and gluon distributions



Left panel: The ratio $\Sigma^{c,b}(x, Q^2) / \Sigma^{N_F=3}(x, Q^2)$ as a function of x , referring to the matching scales $Q^2 = 30 \text{ GeV}^2$ (dotted line), $Q^2 = 100 \text{ GeV}^2$ (dashed line), $Q^2 = 10000 \text{ GeV}^2$ (full line) in the unpolarized case. Right panel: the same for $G^{c,b}(x, Q^2) / G^{N_F=3}(x, Q^2)$.

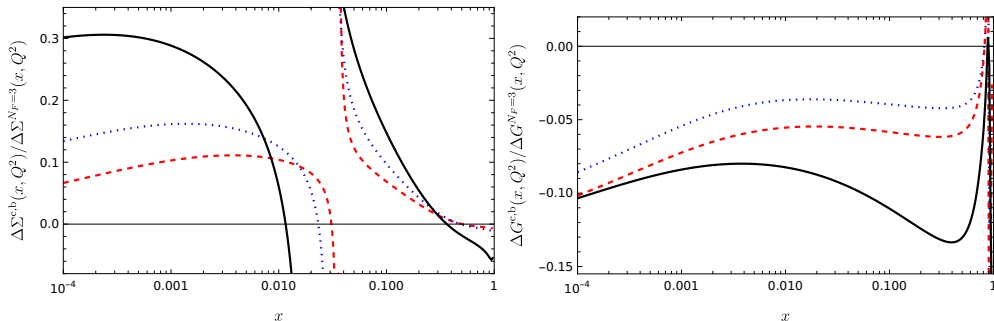
Unpolarized charm and bottom distributions



Left panel: the ratio $(f_c(x, Q^2) + f_{\bar{c}}(x, Q^2)) / \Sigma^{N_F=3}(x, Q^2)$ as a function of x referring, to the matching scales $Q^2 = 30 \text{ GeV}^2$ (dotted line), $Q^2 = 100 \text{ GeV}^2$ (dashed line), $Q^2 = 10000 \text{ GeV}^2$ (full line) in the unpolarized case.

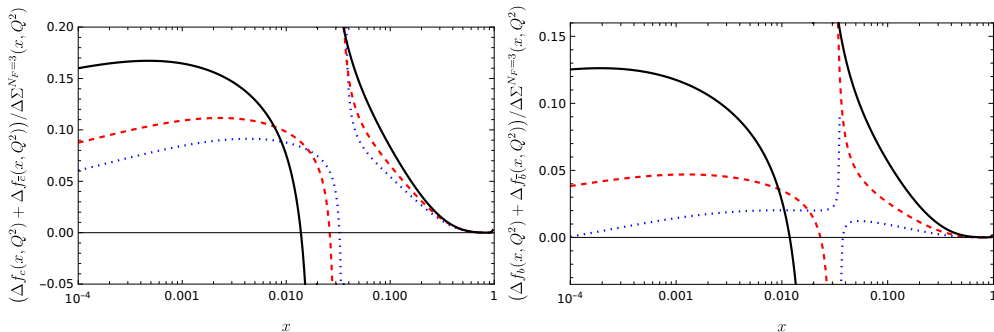
Right panel: the same for the ratio $(f_b(x, Q^2) + f_{\bar{b}}(x, Q^2)) / \Sigma^{N_F=3}(x, Q^2)$.

Polarized singlet and gluon distributions



Left panel: the ratio $\Delta\Sigma^{c,b}(x, Q^2)/\Delta\Sigma^{N_F=3}(x, Q^2)$ as a function of x , at the matching scales $Q^2 = 30 \text{ GeV}^2$ (dotted line), $Q^2 = 100 \text{ GeV}^2$ (dashed line), $Q^2 = 10000 \text{ GeV}^2$ (full line) in the polarized case. Right panel: the same for the ratio $\Delta G^{c,b}(x, Q^2)/\Delta G^{N_F=3}(x, Q^2)$

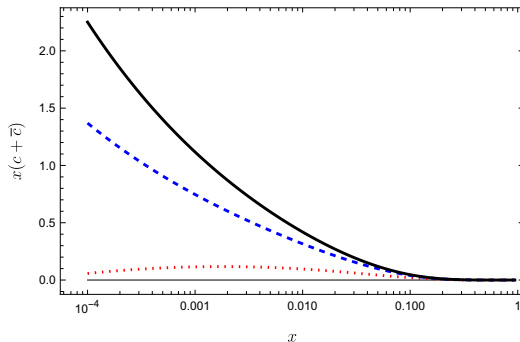
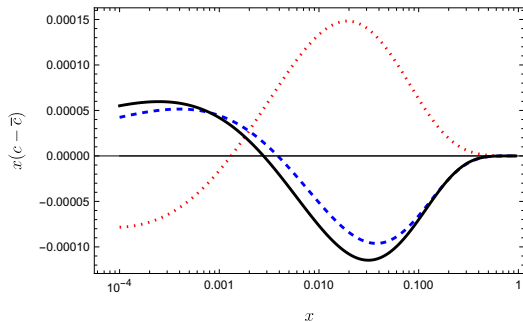
Polarized charm and bottom distributions



Left panel: the ratio $(\Delta f_c(x, Q^2) + \Delta f_{\bar{c}}(x, Q^2)) / \Delta \Sigma^{N_F=3}(x, Q^2)$ as a function of x referring, to the matching scales $Q^2 = 30 \text{ GeV}^2$ (dotted line), $Q^2 = 100 \text{ GeV}^2$ (dashed line), $Q^2 = 10000 \text{ GeV}^2$ (full line) in the unpolarized case.

Right panel: the same for the ratio $(\Delta f_b(x, Q^2) + \Delta f_{\bar{b}}(x, Q^2)) / \Delta \Sigma^{N_F=3}(x, Q^2)$.

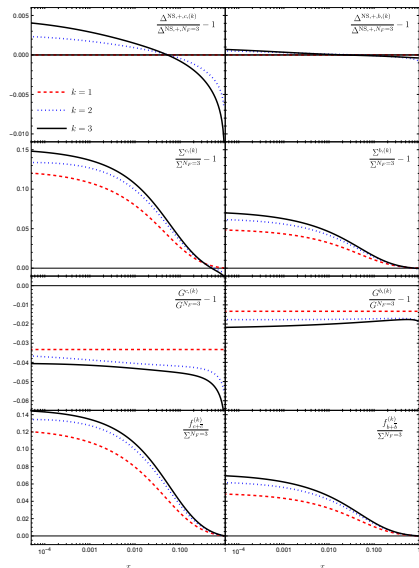
The difference and the sum of heavy flavor pdfs in the VFNS



The unpolarized distributions $x[c(x, Q^2) - \bar{c}(x, Q^2)]$ (left panel) and $x[c(x, Q^2) + \bar{c}(x, Q^2)]$ (right panel). Dotted lines: $Q^2 = 4 \text{ GeV}^2$. Dashed lines: $Q^2 = 30 \text{ GeV}^2$. Full lines: $Q^2 = 100 \text{ GeV}^2$. [A. Behring, JB, A. De Freitas, A. von Manteuffel, C. Schneider, K. Schönwald, Phys. Lett. B (2026) 140411: **1st correct calculation**

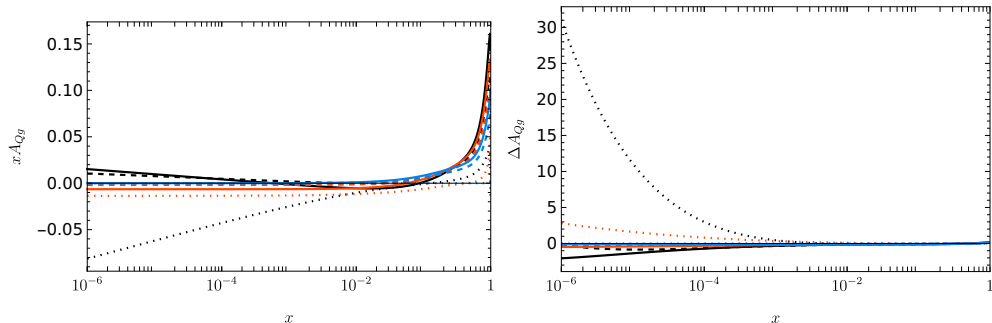
$$\Delta\gamma_{qq}^{\text{NS},s,(2)} = 4 \frac{d_{abc} d^{abc}}{N_c} N_F \frac{1}{2} [1 + (-1)^{N_l}] \left\{ \frac{S_1 Q_4}{N^4 (1+N)^4} + \left[-\frac{2(1+N+N^2)(2+N+N^2)}{N^3 (1+N)^3} - \frac{4(N-1)(2+N)}{N^2 (1+N)^2} S_1 \right] S_{-2} - \frac{(2+N+N^2)}{N^2 (1+N)^2} [S_3 - 2S_{-3} + 4S_{-2,1}] \right\}.$$

The HQ ratios from $O(a_s)$ to $O(a_s^3)$



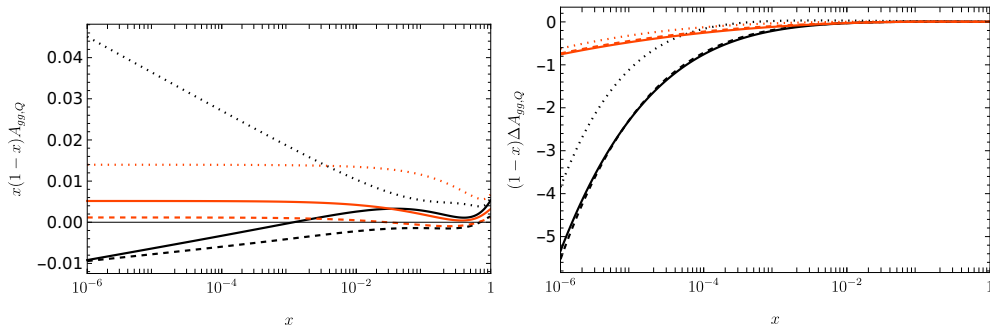
Left panel: relative heavy-quark contributions due to charm to $\Delta^{\text{NS},+}$, Σ , G and $f_{c+\bar{c}}$. Right panel: relative heavy-quark contributions due to bottom to $\Delta^{\text{NS},+}$, Σ , G and $f_{b+\bar{b}}$, both at $Q^2 = 100 \text{ GeV}^2$. The massive OMEs have been truncated to the perturbative order $O(a_s^k)$. The denominators of the ratios are $\Sigma^{N_F=3}$ or $G^{N_F=3}$, respectively. Dashed lines: $O(a_s)$. Dotted lines: contributions to $O(a_s^2)$. Full lines: contributions to $O(a_s^3)$.

The single-mass OMEs A_{Qg} and ΔA_{Qg}



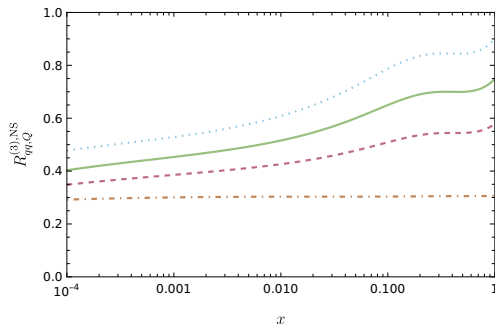
Left panel: $xA_{Qg}(x, Q^2)$. Right panel $\Delta A_{Qg}(x, Q^2)$. Dotted lines: $Q^2 = m_c^2$. Dashed lines: $Q^2 = 30 \text{ GeV}^2$. Full lines: $Q^2 = 100 \text{ GeV}^2$. Blue lines: $O(a_s)$. Red lines: contributions up to $O(a_s^2)$. Black lines: contributions up to $O(a_s^3)$.

The OMEs $A_{gg,Q}$ and $\Delta A_{gg,Q}$



Left panel: $x(1-x)A_{gg,Q}(x, Q^2)$. Right panel $(1-x)\Delta A_{gg,Q}(x, Q^2)$. The notion of lines is the same as in the previous figure.

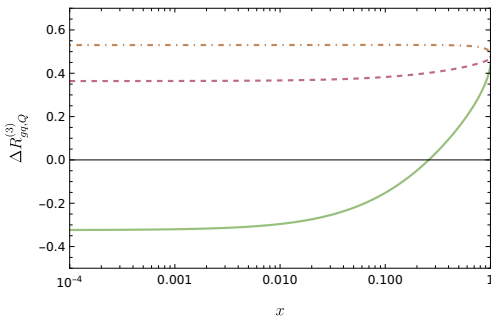
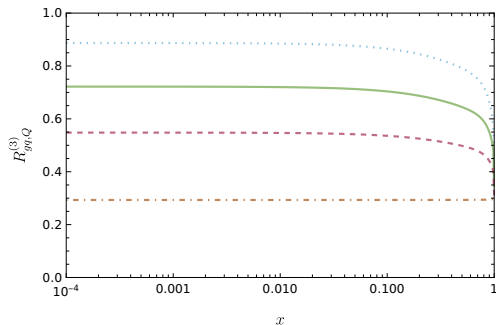
Two-mass Corrections: $(\Delta)A_{qq,Q}^{NS,2m}$



[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[Ablinger, JB, De Freitas, Hasselhuhn, Schneider, Wißbrock '17]

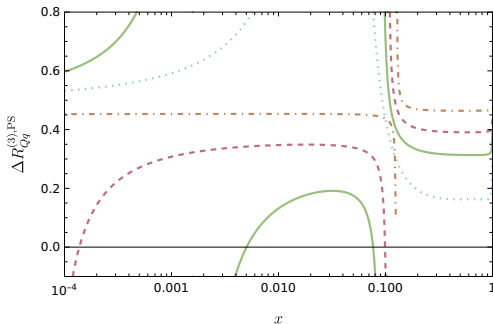
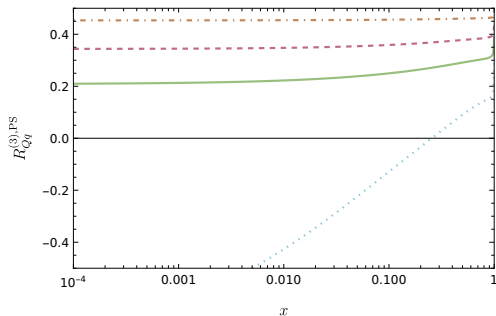
$$R_{ij} = \frac{A_{ij}^{2m, \text{irred}}}{A_{ij, \text{tot}}^{T_F^2}}$$



[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[Abinger, JB, De Freitas, Hasselhuhn, Schneider, Wißbrock '17]

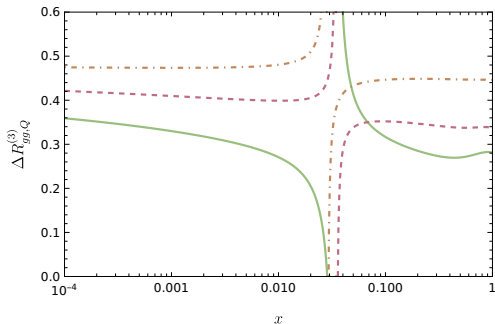
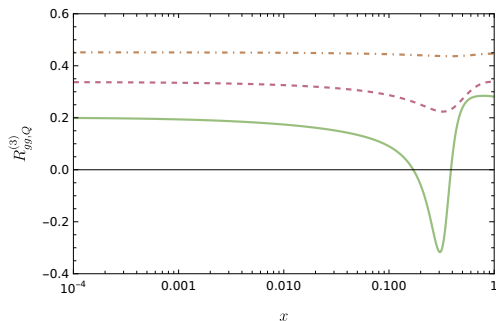
[Behring, JB, De Freitas, von Manteuffel, Schönwald, Schneider '21]



[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[Ablinger, JB, De Freitas, Schneider, Schönwald '17]

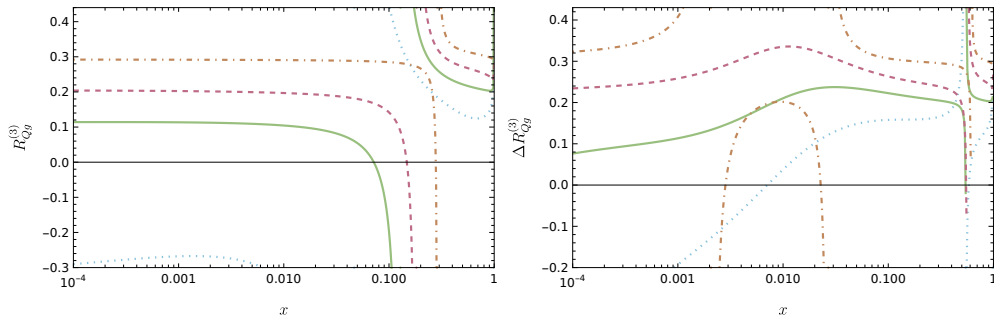
[Ablinger, JB, De Freitas, Saragnese, Schneider, Schönwald '19]



[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[Abinger, JB, De Freitas, Schneider, Schönwald '18]

[Abinger, JB, Goedicke, De Freitas, Saragnese, Schneider, Schönwald '20]

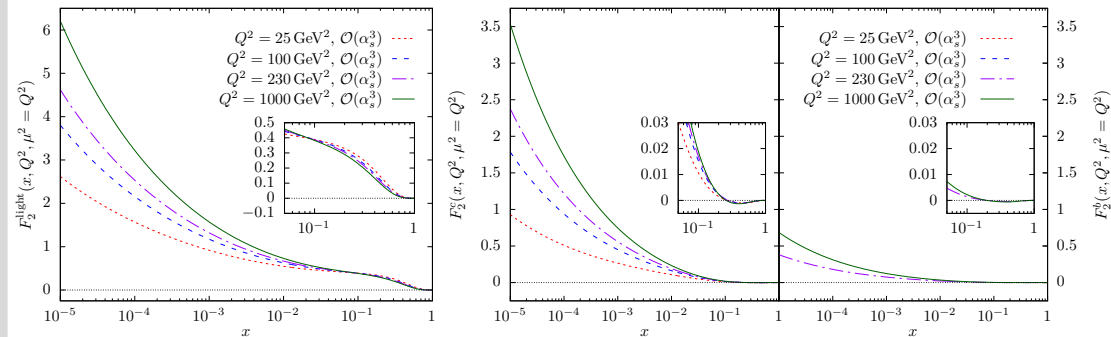


[Dotted line: $Q^2 = 30 \text{ GeV}^2$, full line: $Q^2 = 50 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, dash-dotted line: $Q^2 = 1000 \text{ GeV}^2$]

[J.Ablinger et al. 2025]

In summary: the 3-loop 2-mass OMEs form $\sim 50 \%$ of the $T_F^2 C_i$ contributions in all cases.

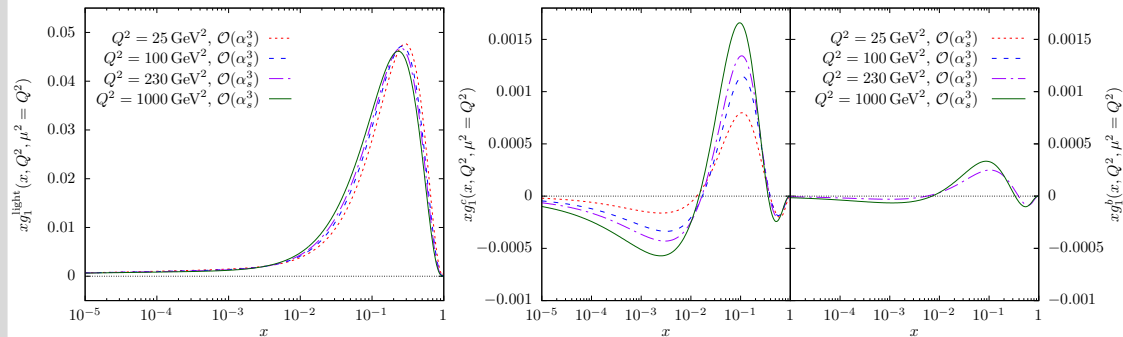
Massless and massive contributions to $F_2(x, Q^2)$



Left panel: The massless contributions to the structure function $F_2(x, Q^2)$ at NNLO. Right panel: The heavy-flavor contributions due to charm and bottom quarks to the structure function $F_2(x, Q^2)$ at NNLO in the region $Q^2/m_Q^2 > 10$. Left panel: charm contributions. Right panel: bottom contributions.

[J.Ablinger et al. 2025]

Massless and massive contributions to $g_1(x, Q^2)$



Codes

- We have designed numerical x -space codes for all unpolarized and polarized single-mass OMEs to three loop order. [C++, Fortran: [arXiv:2510.02175](#)].
- As well we have released x -space codes for the massless and asymptotic massive Wilson coefficients to three loop order: [arXiv:2509.16124](#), [arXiv:2602.10334](#).
- The codes are very fast and precise and consist of analytic small- and large- x expansions and cubic splines over fine-grided data arrays for the remainder part, leaving a_s , $\ln(Q^2/\mu^2)$, $\ln(m^2/\mu^2)$, N_F and charge weights as free parameters.
- They can be efficiently used in QCD-fitting codes.

Conclusions

- The single and two-mass VFNS has been promoted from two to three loop order both in the unpolarized and polarized case.
- The mass-modified parton distribution functions are **uniquely** implied by the massive renormalization group equation.
- These schemes describe the deep-inelastic structure functions in the region $Q^2 \gg m_Q^2$.
- Fast and precise numerical implementations were released as public codes for experimental use in the massless and single mass case. Soon corresponding codes for the two-mass OMEs will be available.