

LOCAL INFRARED FACTORISATION

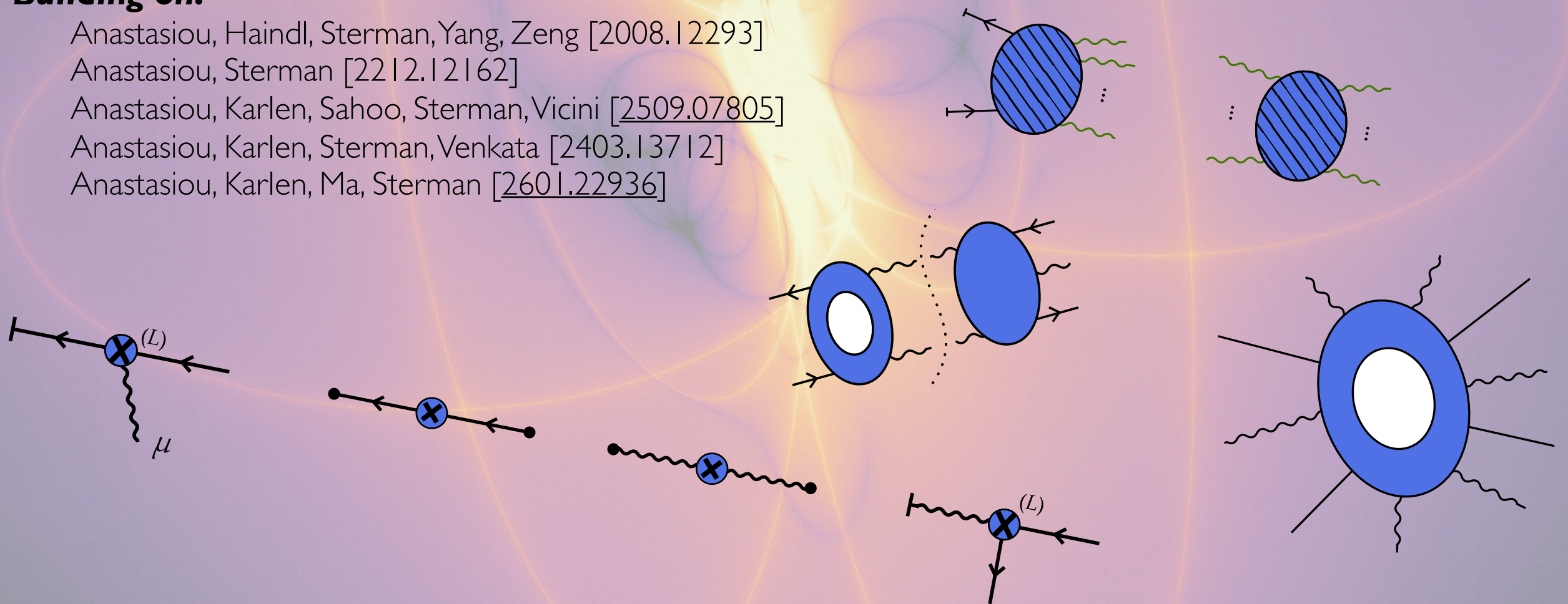
A ROAD TO NUMERICAL SCATTERING AMPLITUDES

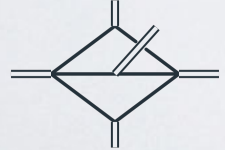
To appear in: Anastasiou, Biello, Favorito, **GG**, Sahoo, Sterman [2650.01151]

.....

Building on:

Anastasiou, Haindl, Sterman, Yang, Zeng [2008.12293]
Anastasiou, Sterman [2212.12162]
Anastasiou, Karlen, Sahoo, Sterman, Vicini [2509.07805]
Anastasiou, Karlen, Sterman, Venkata [2403.13712]
Anastasiou, Karlen, Ma, Sterman [2601.22936]



		dimensionless variables	geometry
	double box 4 massless legs	1	HPLs
	double box 1 off-shell leg (V, H + jet)	2	2dHPLs
	penta-box 5 massless legs (3 jets)	4	pentagon functions
	penta-box 1 off-shell leg (W + 2 jets)	5	one-mass pentagon functions
	sunrise 3 equal masses	1	elliptic: eMPLs, modular forms
	double box closed top loop ($gg \rightarrow t\bar{t}$)	2	elliptic
	crossed box massive lines ($t\bar{t}$, Møller)	2	hyperelliptic, genus 2
	tardigrade 5 off-shell legs	9	K3 surface

AMPLITUDE IR FACTORISATION

$$A = J_1 J_2 \dots J_n \mathcal{S} \cdot \mathcal{H}$$

Collins, Soper, Sterman ['88];

Catani: [9802439];

Kidonakis, Oderda, Sterman [9803241];

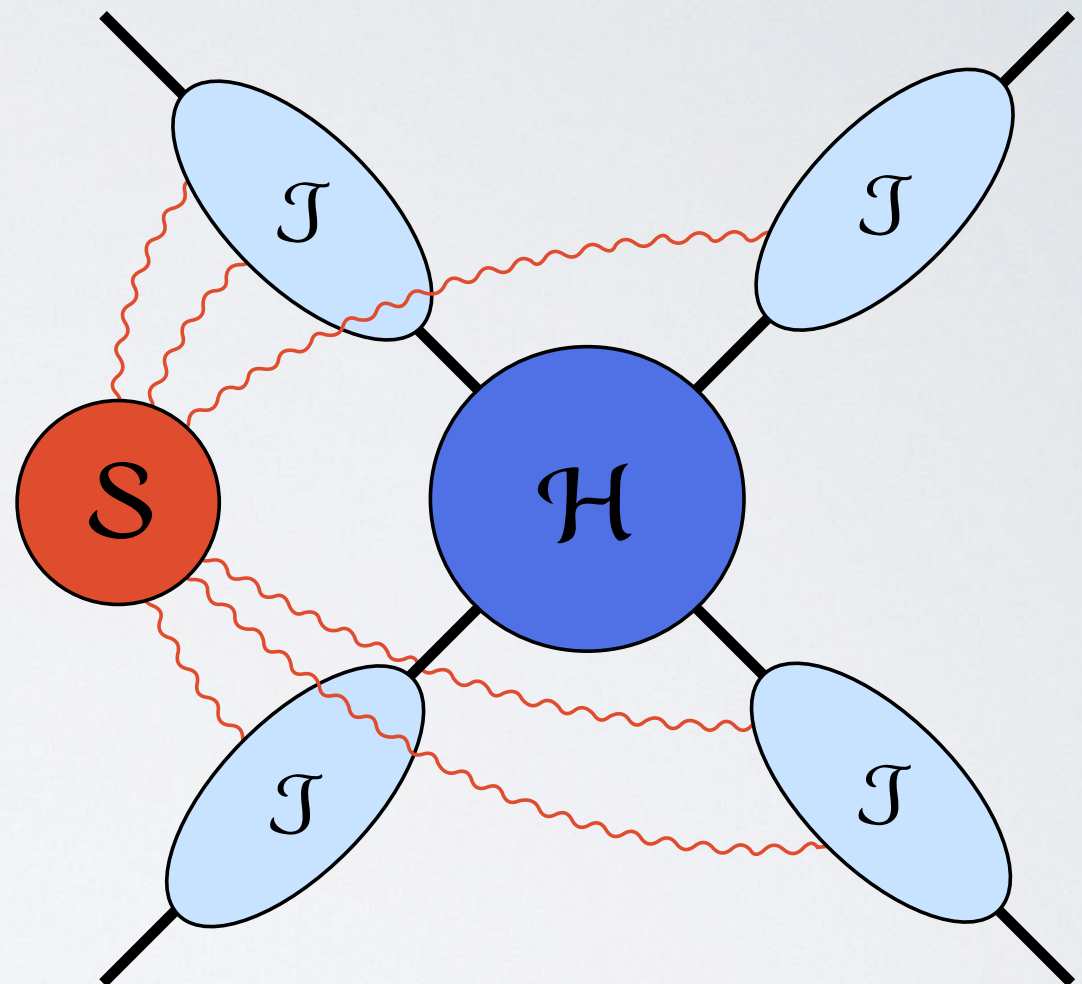
Sterman, Tejeda-Yeomans [0210130];

Gardi, Magnea: [0901.1091],[0908.3273];

Becher, Neubert: [0903.1126];

Feige, Schwartz: [1403.6472];

+ many more!!



AMPLITUDE IR FACTORISATION

$$A = J_1 J_2 \dots J_n \mathcal{S} \cdot \mathcal{H}$$

Collins, Soper, Sterman ['88];

Catani: [9802439];

Kidonakis, Oderda, Sterman [9803241];

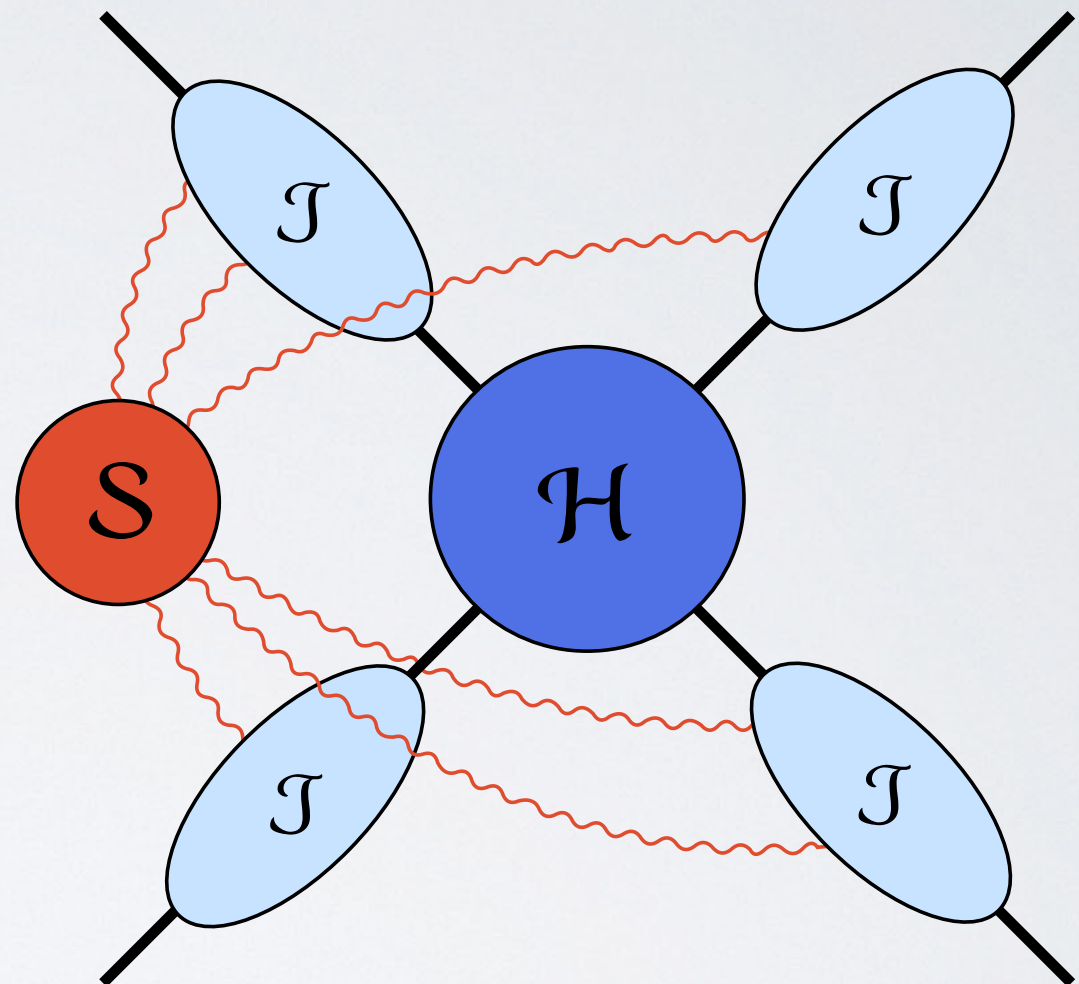
Sterman, Tejeda-Yeomans [0210130];

Gardi, Magnea: [0901.1091], [0908.3273];

Becher, Neubert: [0903.1126];

Feige, Schwartz: [1403.6472];

+ many more!!



perturbative analytic
finite remainders

$$\mathcal{H}^{(L)} = A^{(L)} - (I^{(1)} \cdot A^{(L-1)}) - (I^{(2)} \cdot A^{(L-2)}) - \dots$$

Can we find
AMPLITUDE REPRESENTATIONS
whose
IR FACTORISATION IS MANIFEST
LOCALLY?*

**before loop integration*

Can we find
AMPLITUDE REPRESENTATIONS
whose
IR FACTORISATION IS MANIFEST
LOCALLY?*

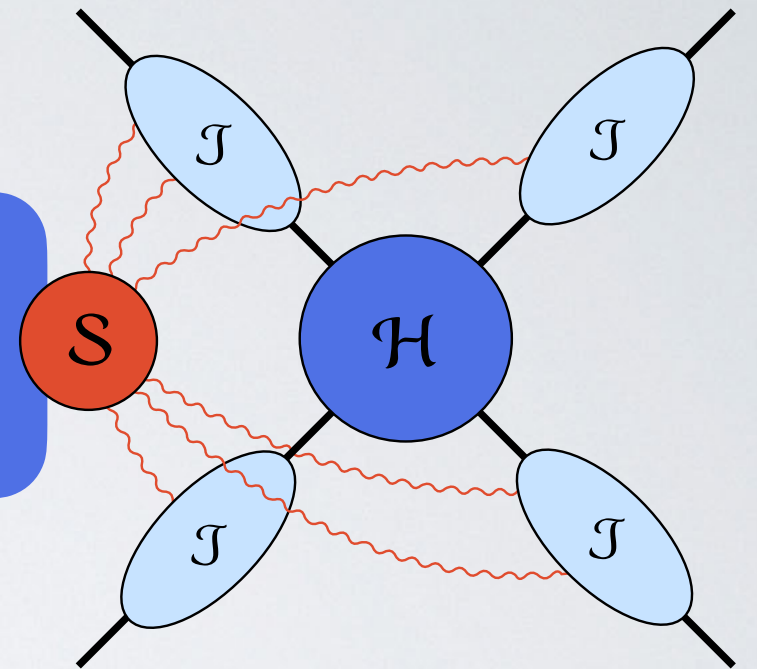
We use QED as a testing ground.

**before loop integration*

At two loops: YES,
for arbitrary amplitudes

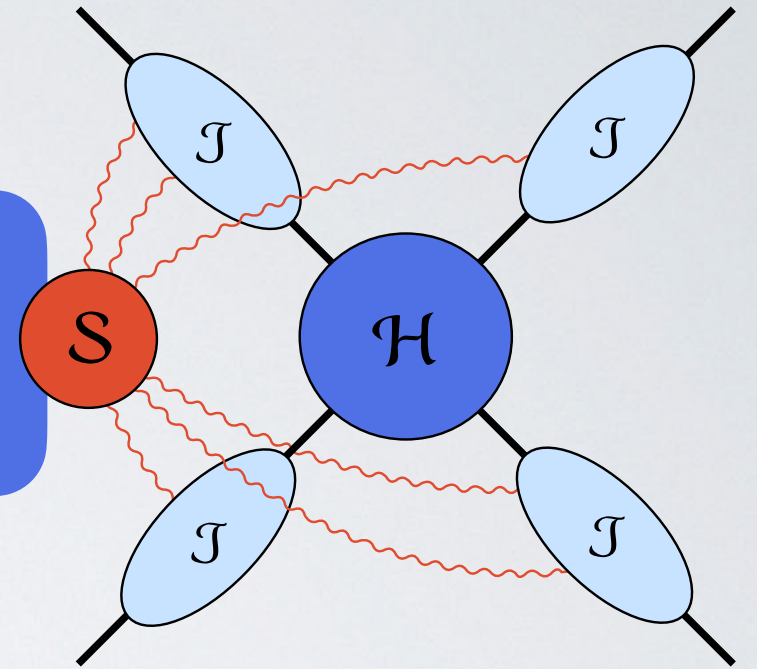
FOR EVERY LIMIT

$$\mathcal{A} \xrightarrow{IR} \kappa \times \mathcal{A}_{\text{lower order}}$$



FOR EVERY LIMIT

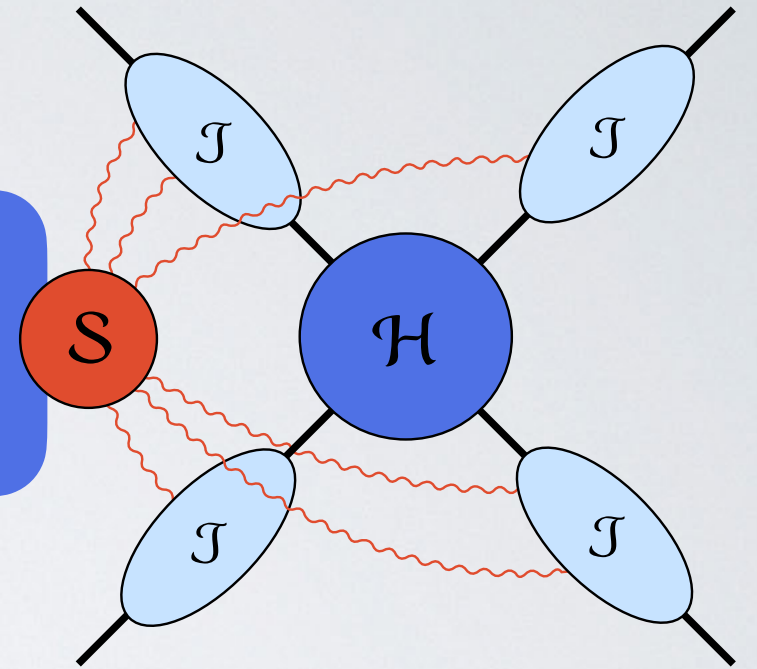
$$\mathcal{A} \xrightarrow{IR} \kappa \times \mathcal{A}_{lower\ order}$$



- I. within the full gauge theory, Feynman gauge

FOR EVERY LIMIT

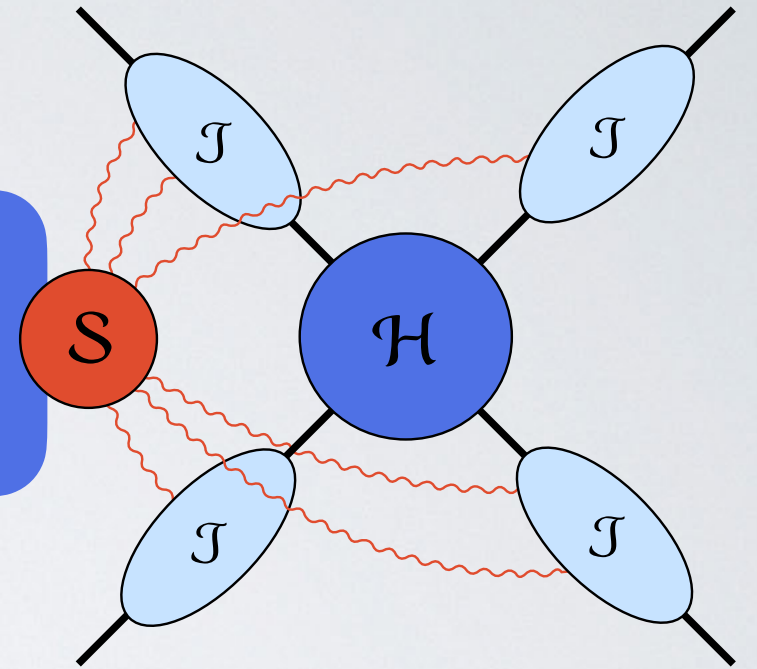
$$\mathcal{A} \xrightarrow{IR} \kappa \times \mathcal{A}_{lower\ order}$$



1. within the full gauge theory, Feynman gauge
2. loop-momentum integrand level

FOR EVERY LIMIT

$$\mathcal{A} \xrightarrow{IR} \kappa \times \mathcal{A}_{lower\ order}$$



1. within the full gauge theory, Feynman gauge
2. loop-momentum integrand level
3. scalar kernels are universal

THE ENGINE_(S) OF LOCAL IR FACTORISATION

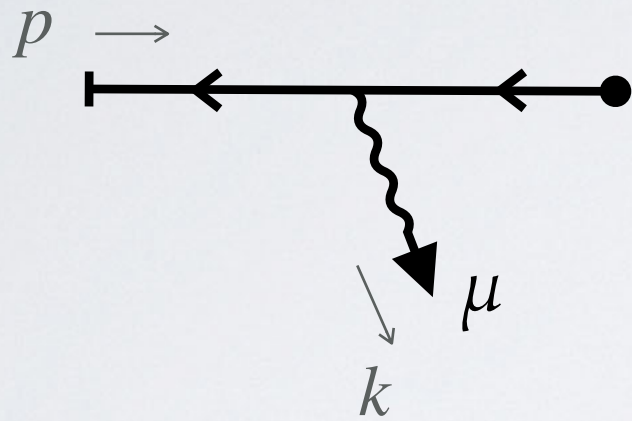
1. longitudinal polarisations

2. tree-level Ward identities

3. amplitude sewing

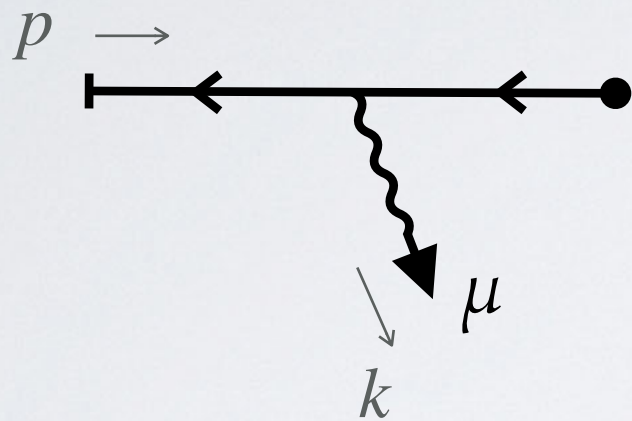
I. longitudinal polarisations

I. longitudinal polarisations



$$k^\mu \rightarrow \alpha p^\mu$$

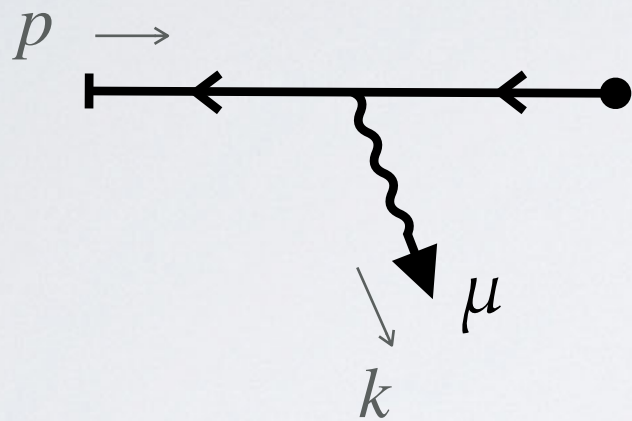
I. longitudinal polarisations



$$k^\mu \rightarrow \alpha p^\mu$$

$$\bar{u}(p)(-ie\gamma^\mu)\frac{-i(\not{p}-\not{k})}{(p-k)^2} \simeq e \left(\frac{2(p^\mu - k^\mu)}{(p-k)^2} \right) \bar{u}(p) \simeq e \left(\frac{2(1/\alpha - 1)k^\mu}{(p-k)^2} \right) \bar{u}(p)$$

I. longitudinal polarisations



$$k^\mu \rightarrow \alpha p^\mu$$

$$\bar{u}(p)(-ie\gamma^\mu)\frac{-i(\not{p}-\not{k})}{(p-k)^2} \simeq e \left(\frac{2(p^\mu - k^\mu)}{(p-k)^2} \right) \bar{u}(p) \simeq e \left(\frac{2(1/\alpha - 1)k^\mu}{(p-k)^2} \right) \bar{u}(p)$$

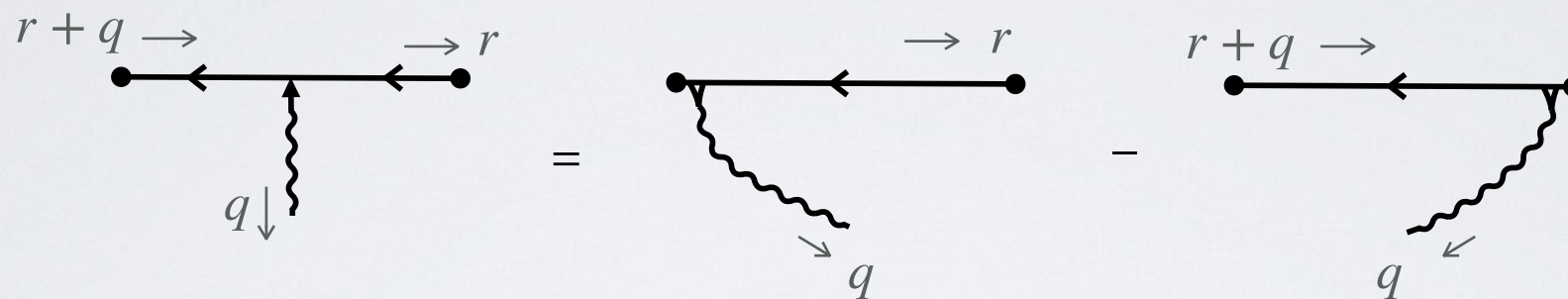
2. tree-level Ward identities

2. tree-level Ward identities

$$(r + q)^{-1} q r^{-1} = r^{-1} - (r + q)^{-1}$$

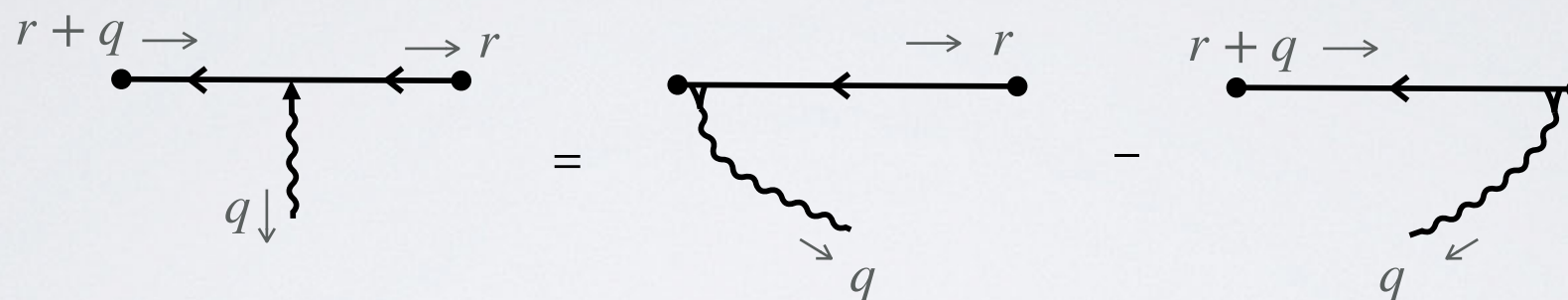
2. tree-level Ward identities

$$(r + q)^{-1} \not{q} r^{-1} = \not{r}^{-1} - (r + q)^{-1}$$

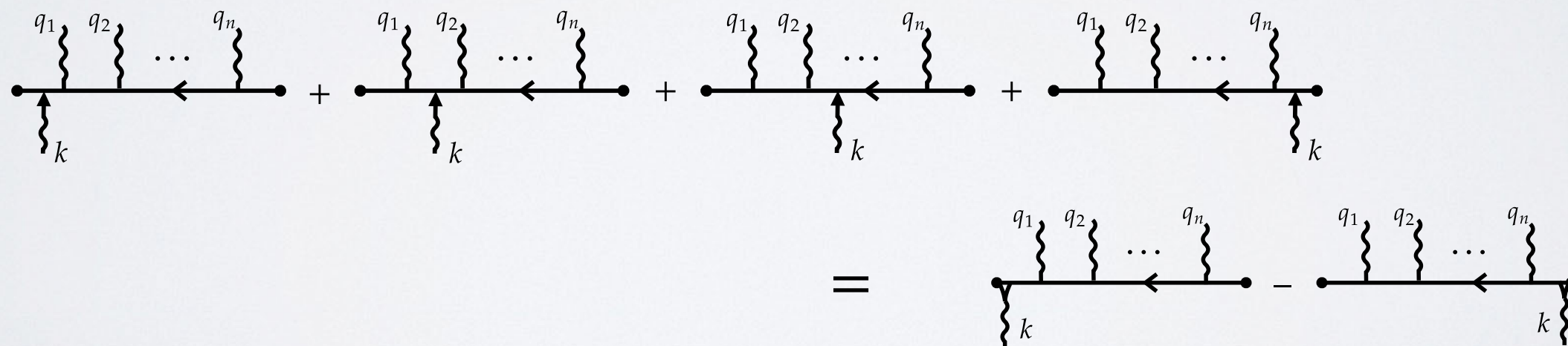


2. tree-level Ward identities

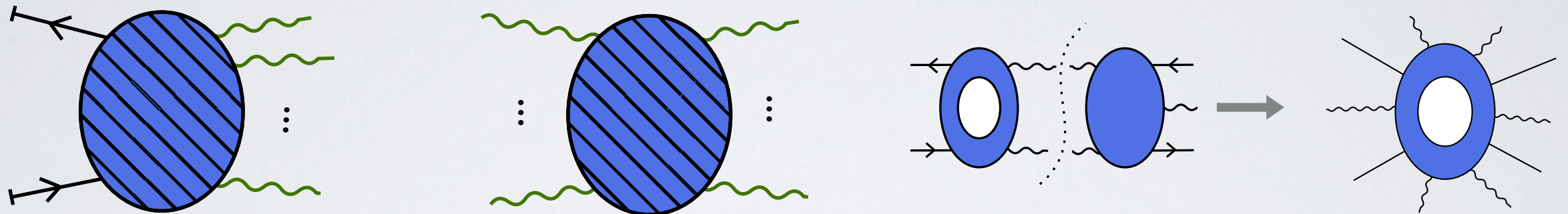
$$(r + q)^{-1} q r^{-1} = r^{-1} - (r + q)^{-1}$$



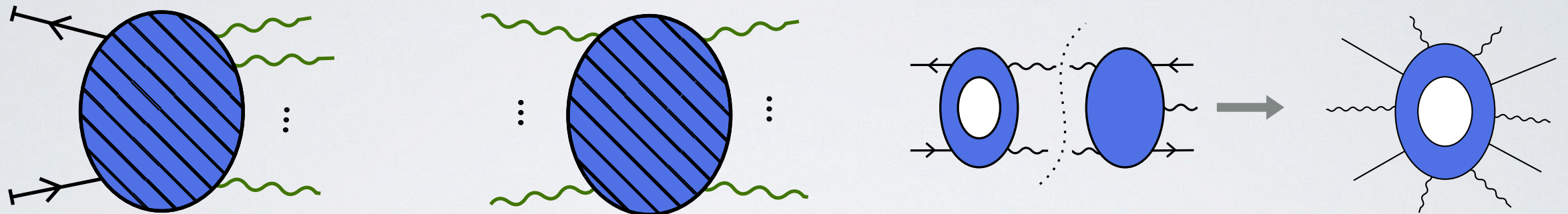
(local) Ward cancellation chains !



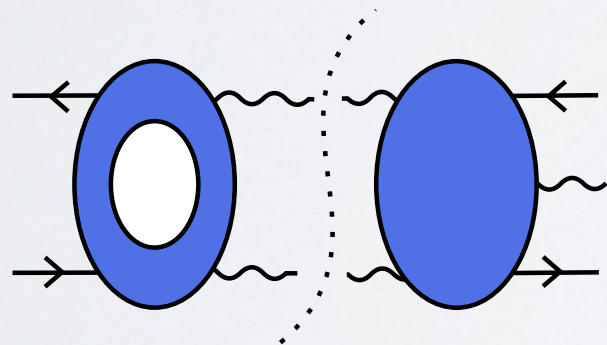
3. amplitude sewing



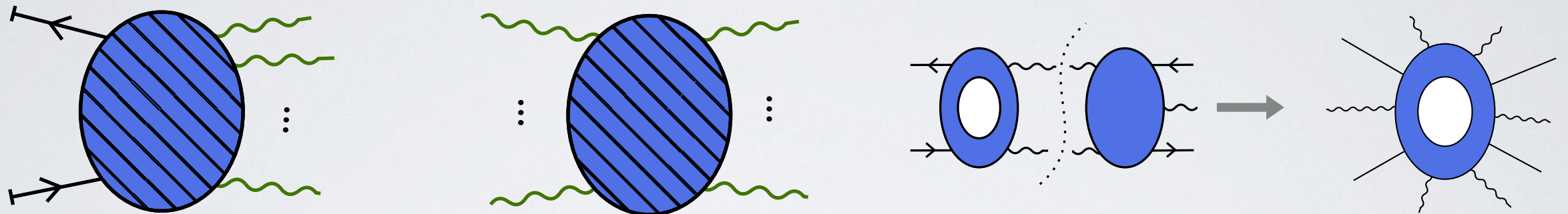
3. amplitude sewing



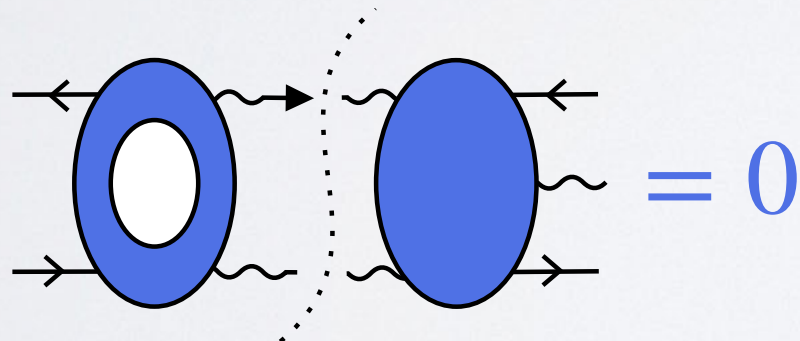
No new collinear
divergences in sewing:



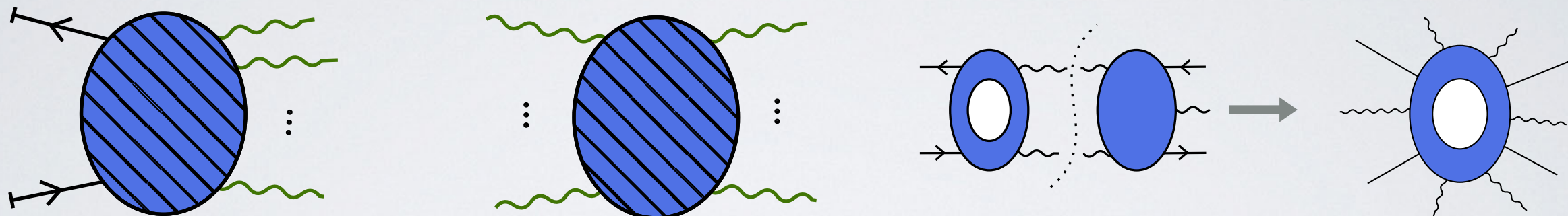
3. amplitude sewing



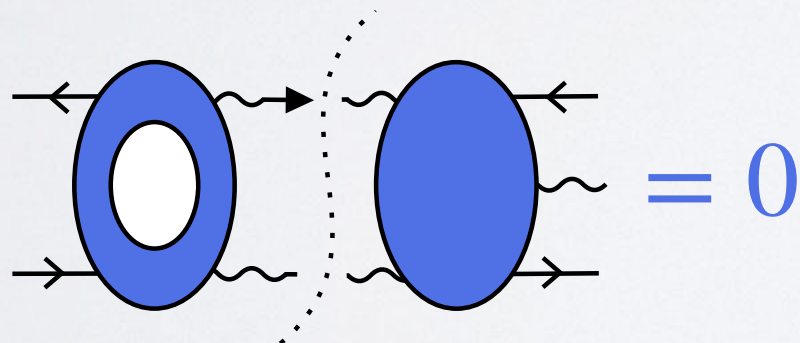
No new collinear
divergences in sewing:



3. amplitude sewing



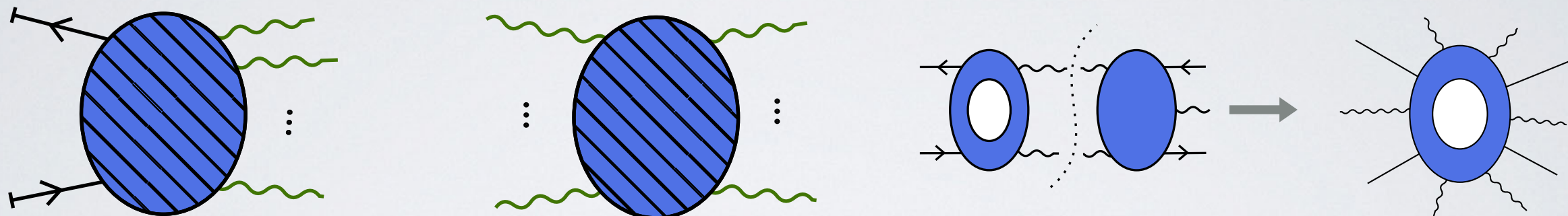
No new collinear
divergences in sewing:



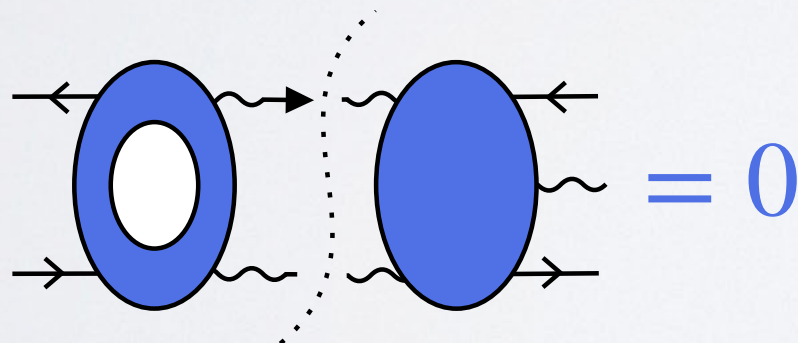
A gluing propagator:*

$$k \rightarrow \text{wavy line} \xrightarrow{\text{gluing}} \frac{-i}{k^2} \left(g_{\mu\nu} - 2 \frac{k_\mu \rho_\nu + k_\nu \rho_\mu}{(k + \rho)^2 - \rho^2} \right)$$

3. amplitude sewing



No new collinear divergences in sewing:



A gluing propagator:*

$$G_{\mu\nu} k^\nu \xrightarrow{k^2 \rightarrow 0} 0$$

$$k \rightarrow \text{wavy line} \xrightarrow{\text{gluing}} \frac{-i}{k^2} \left(g_{\mu\nu} - 2 \frac{k_\mu \rho_\nu + k_\nu \rho_\mu}{(k + \rho)^2 - \rho^2} \right)$$

WHAT BREAKS BEYOND 1-LOOP?

1. longitudinal polarisations

2. tree-level Ward identities

3. amplitude sewing

WHAT BREAKS BEYOND 1-LOOP?

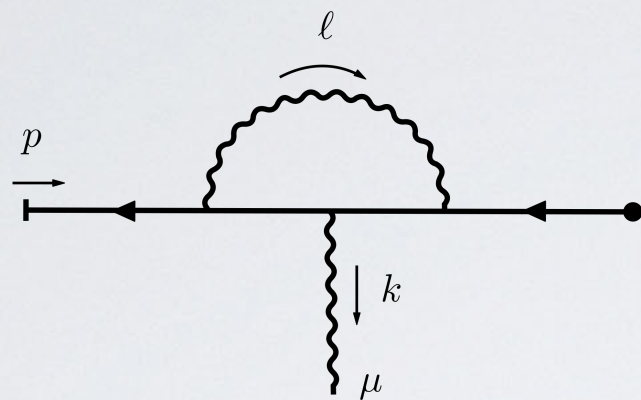


1. longitudinal polarisations

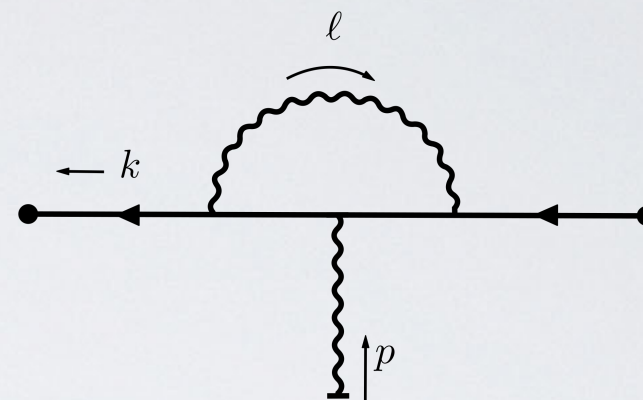


2. tree-level Ward identities

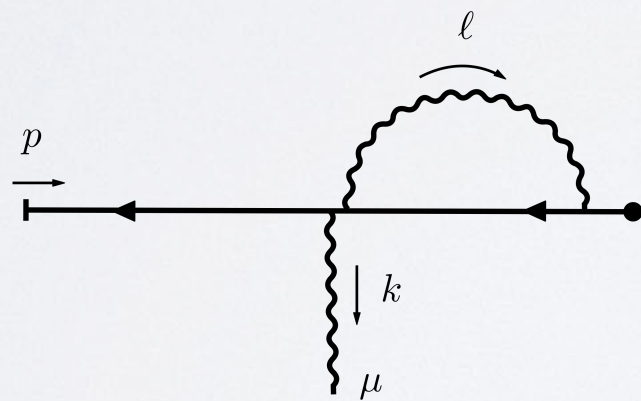
3. amplitude sewing



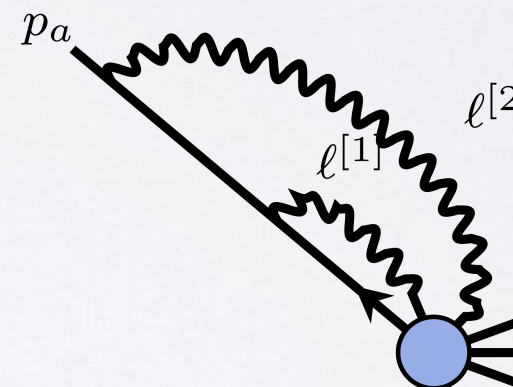
loop polarisations



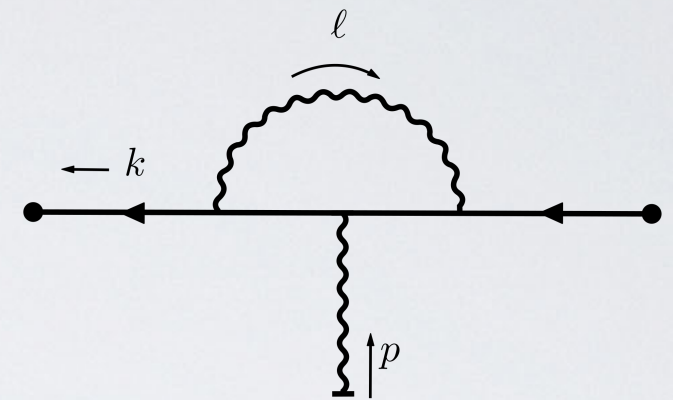
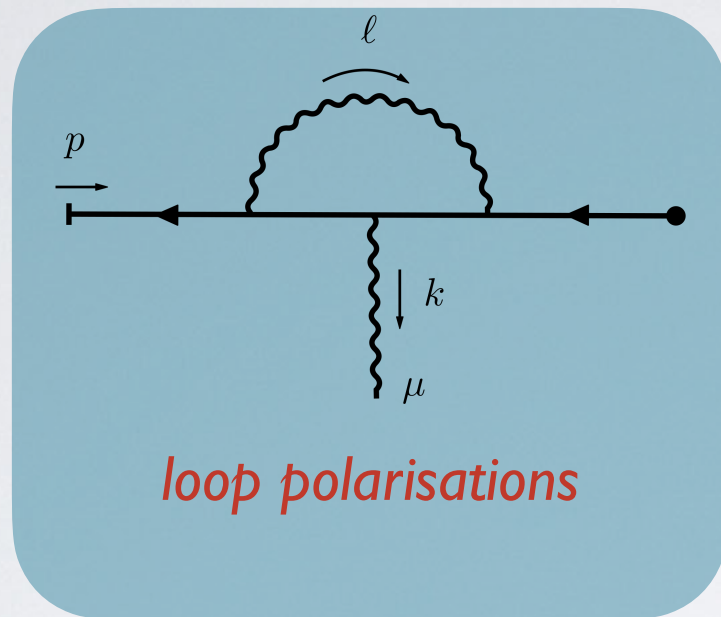
photon transient singularities



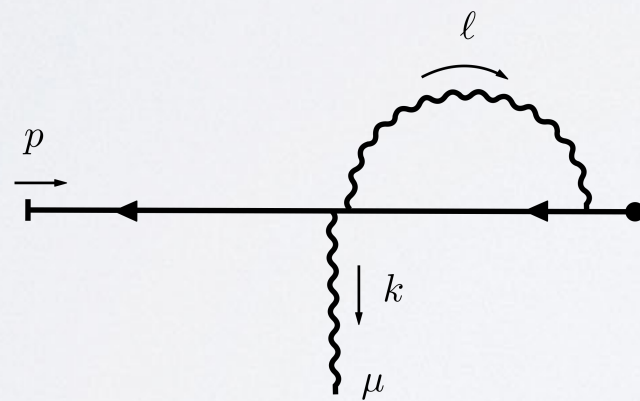
power-like singularities



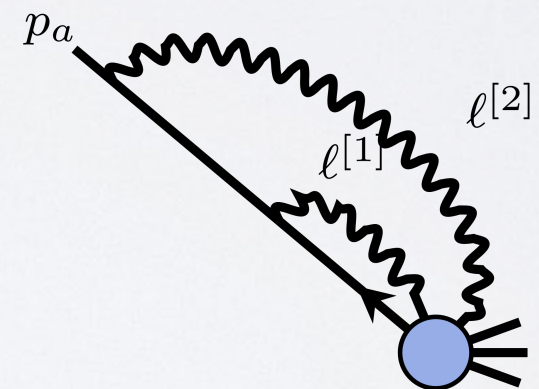
*triple-collinear
little group rotation*



photon transient singularities

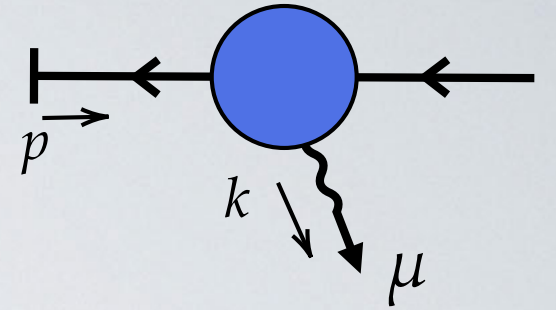


power-like singularities

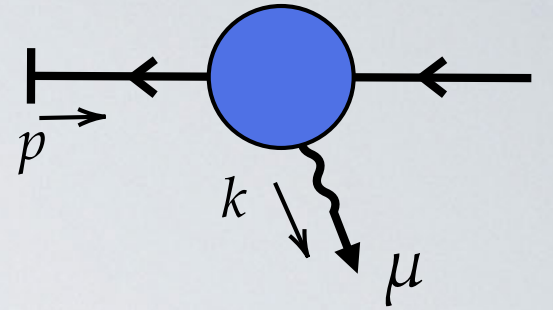
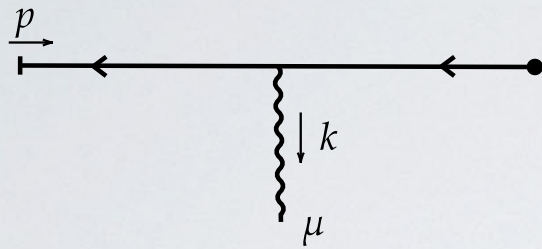


*triple-collinear
little group rotation*

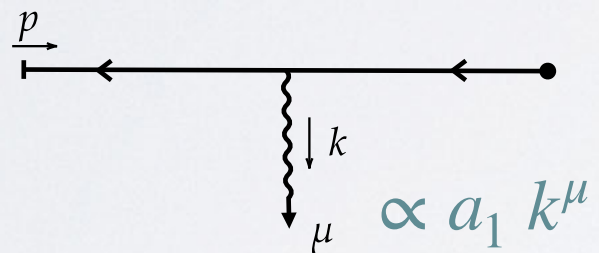
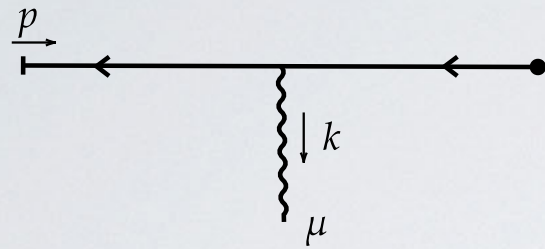
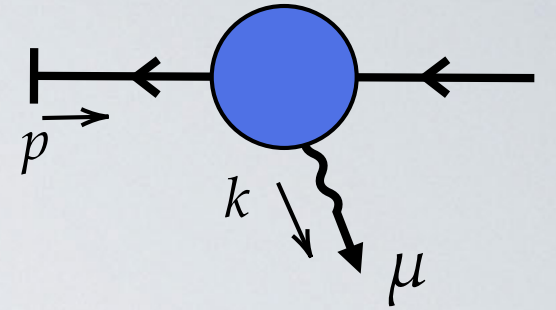
LOOP POLARISATIONS



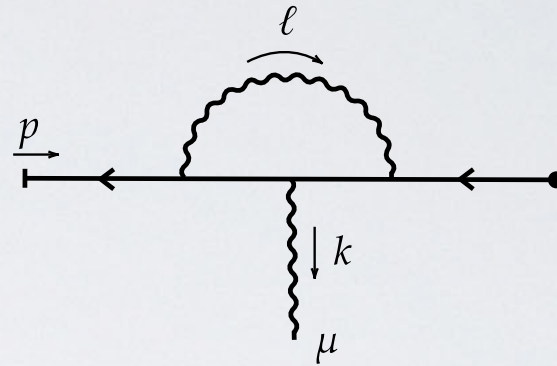
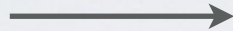
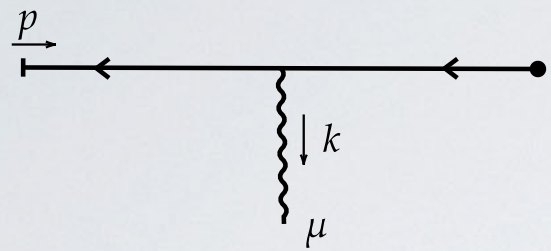
LOOP POLARISATIONS



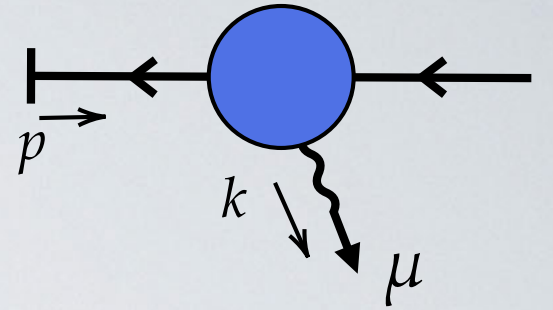
LOOP POLARISATIONS



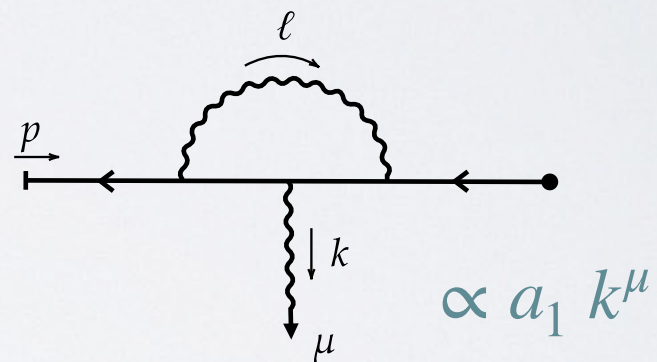
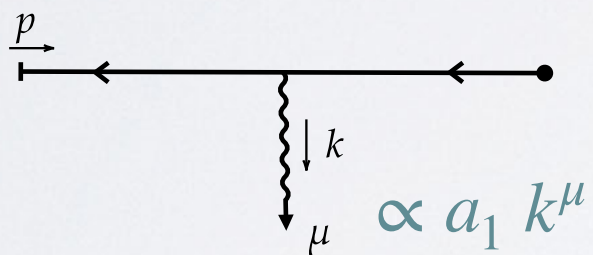
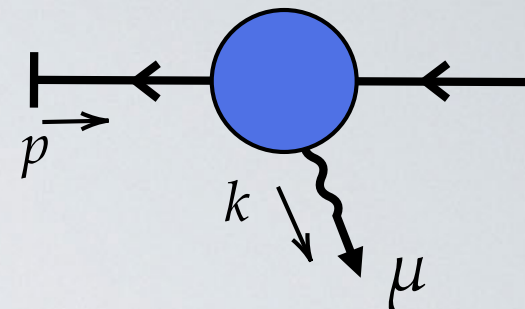
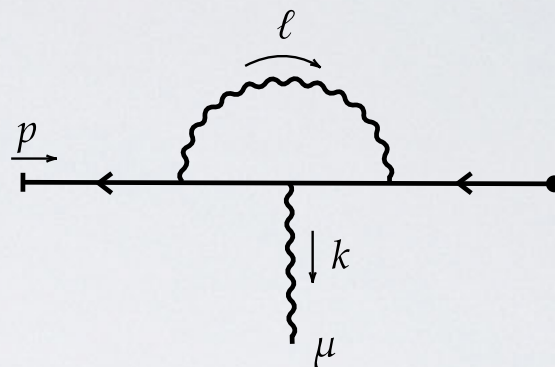
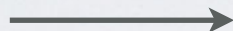
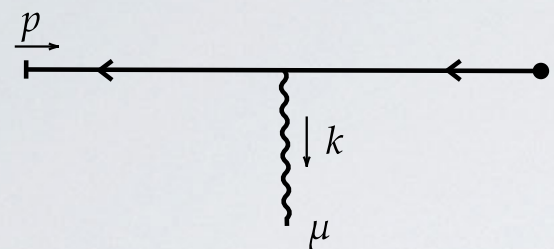
LOOP POLARISATIONS



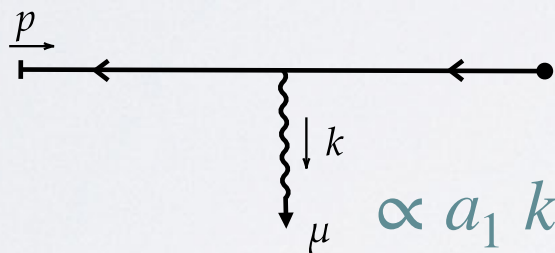
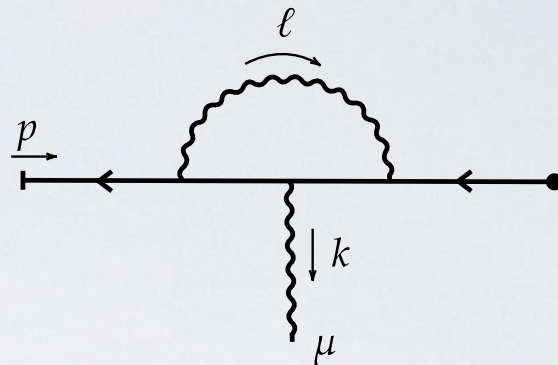
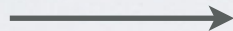
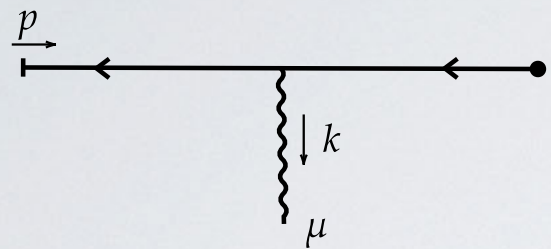
$$\propto a_1 k^\mu$$



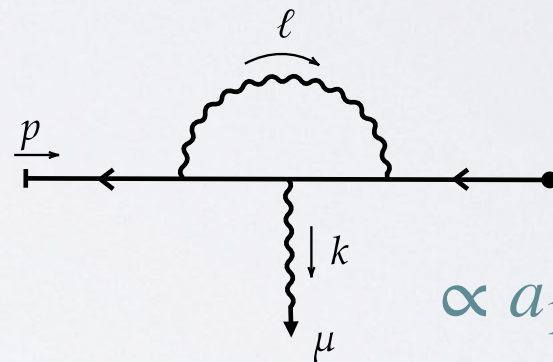
LOOP POLARISATIONS



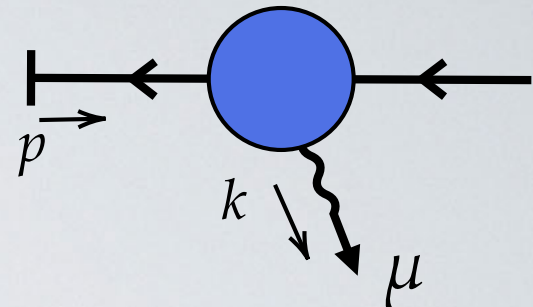
LOOP POLARISATIONS



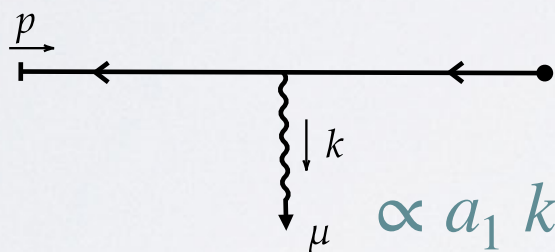
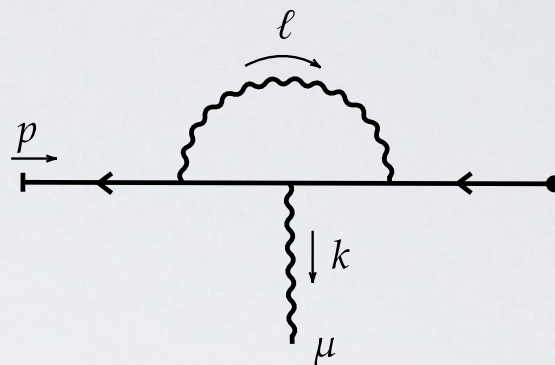
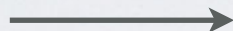
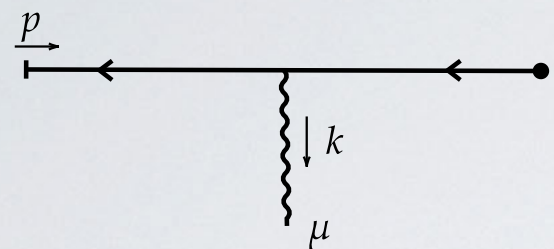
$$\propto a_1 k^\mu$$



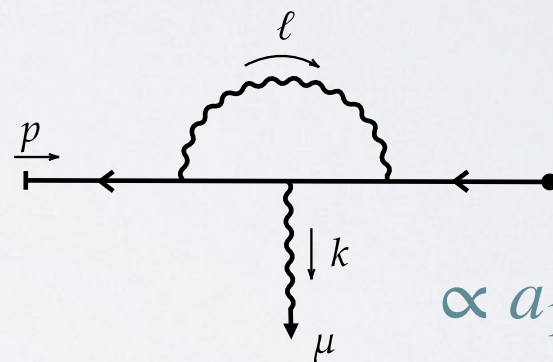
$$\propto a_1 k^\mu + a_2 \ell^\mu$$



LOOP POLARISATIONS

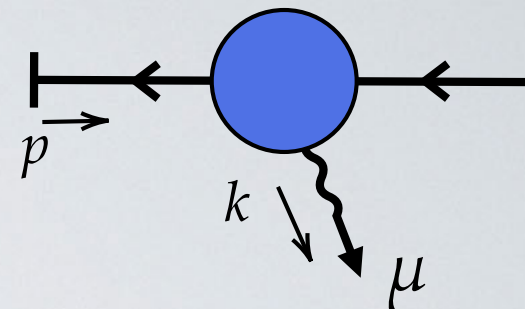


$$\propto a_1 k^\mu$$

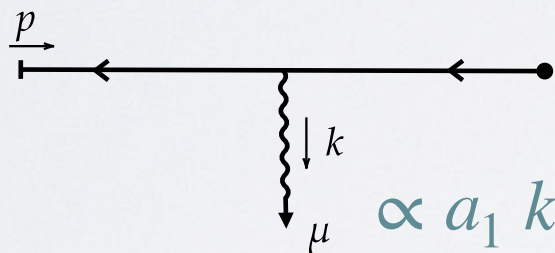
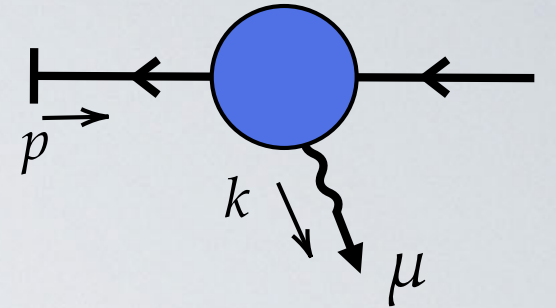
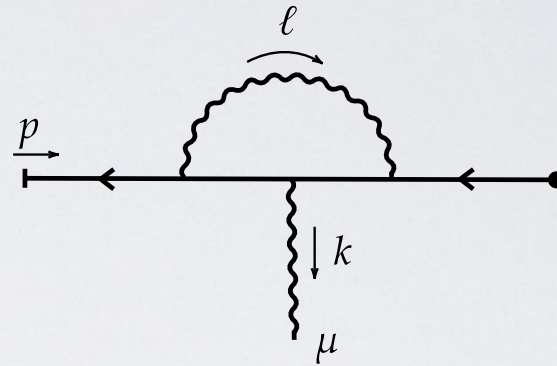
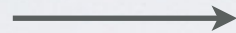
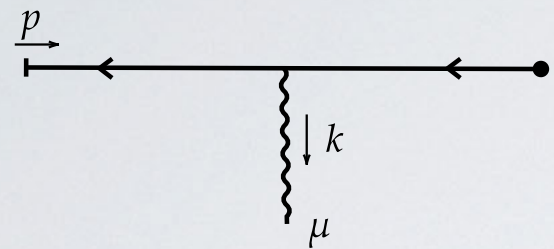


$$\propto a_1 k^\mu + a_2 \ell^\mu$$

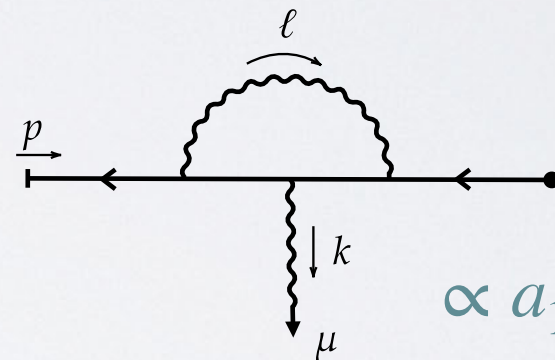
loop polarisation



LOOP POLARISATIONS

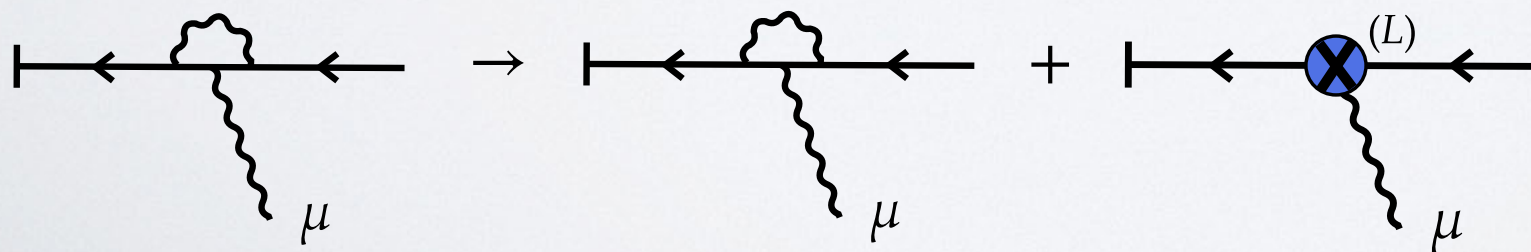


$$\propto a_1 k^\mu$$



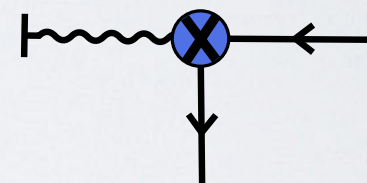
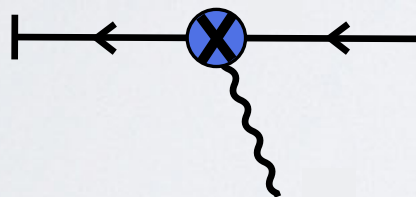
$$\propto a_1 k^\mu + a_2 \ell^\mu$$

loop polarisation

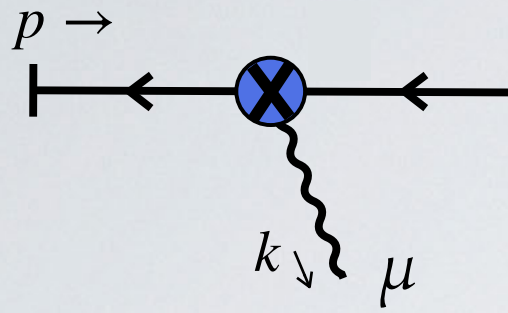


derived once
and for all

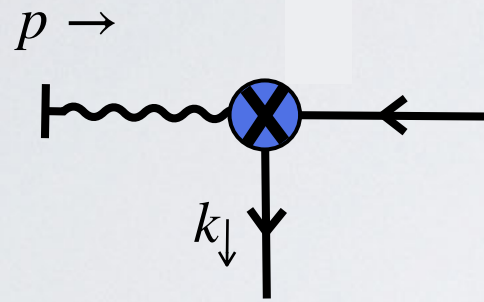
BOUNDARY COUNTERTERMS



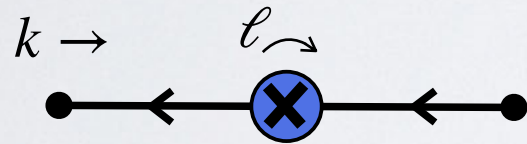
$$\eta^2 = 0$$



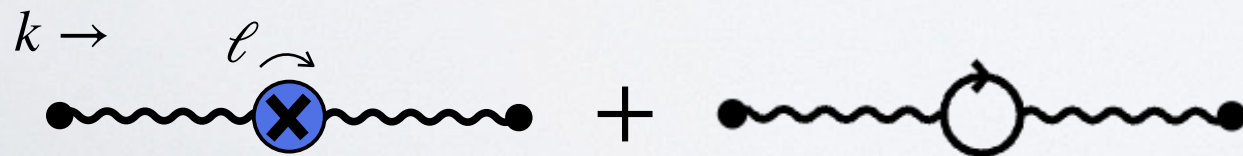
$$ie^2(2-d)\frac{\not{\eta}}{2p \cdot \eta} \left[\frac{(p-2\ell-k)^\mu}{\ell^2(p-\ell-k)^2} - \frac{(2p-2\ell-k)^\mu}{(p-\ell)^2(p-\ell-k)^2} \right]$$



$$-ie^2(2-d)\frac{\not{\eta}}{2p \cdot \eta} \left[\frac{(-k^\mu - 2\ell^\mu)}{\ell^2(p-k-\ell)^2} - \frac{(-k^\mu - 2\ell^\mu)}{(k+\ell)^2\ell^2} \right]$$



$$(ie^2)(2-d)\frac{\frac{k}{2} - \not{\ell}}{(k-\ell)^2\ell^2}$$



$$e^2 \frac{d-2}{d-1} \frac{1}{(k-\ell)^2\ell^2} \eta_{\mu\nu}$$

A local subtraction formula

$$\mathcal{A} \xrightarrow{IR} \kappa \times \mathcal{A}_{lower\ order}$$

LOCALLY FINITE REMAINDERS!

$$\mathcal{F} = \hat{\mathcal{S}}[e^{-\hat{\mathcal{K}}} \mathcal{A}_r]$$

LOCALLY FINITE REMAINDERS!

$$\mathcal{F} = \hat{\mathcal{S}}[e^{-\hat{\mathcal{K}}} \mathcal{A}_r]$$

$$\hat{\mathcal{K}} = \hat{\alpha} \hat{\mathcal{K}}^{(1)} + \hat{\alpha}^2 \hat{\mathcal{K}}^{(2)} + \mathcal{O}(\hat{\alpha}^3), \quad \hat{\mathcal{K}}^{(L)} = \hat{\mathcal{S}}^{(L)} + \hat{\mathcal{C}}^{(L)}$$

LOCALLY FINITE REMAINDERS!

$$\mathcal{F} = \hat{\mathcal{S}}[e^{-\hat{\mathcal{K}}} \mathcal{A}_r]$$

$$\hat{\mathcal{K}} = \hat{\alpha} \hat{\mathcal{K}}^{(1)} + \hat{\alpha}^2 \hat{\mathcal{K}}^{(2)} + \mathcal{O}(\hat{\alpha}^3), \quad \hat{\mathcal{K}}^{(L)} = \hat{\mathcal{S}}^{(L)} + \hat{\mathcal{C}}^{(L)}$$

every object is an integrand here

$$\mathcal{F}^{(1)} = \mathcal{A}_r^{(1)} - \hat{\mathcal{S}}\left[\hat{\mathcal{K}}_{[1]}^{(1)} \mathcal{A}^{(0)}\right]$$

LOCALLY FINITE REMAINDERS!

$$\mathcal{F} = \hat{\mathcal{S}}[e^{-\hat{\mathcal{K}}} \mathcal{A}_r]$$

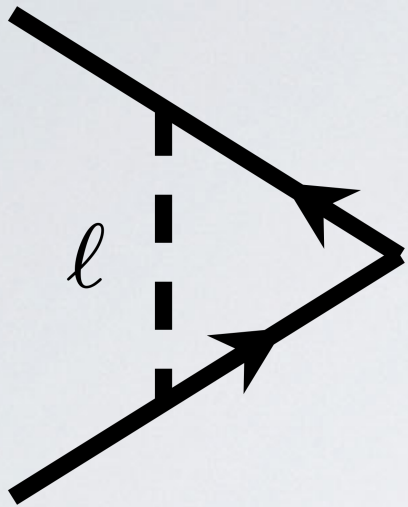
$$\hat{\mathcal{K}} = \hat{\alpha} \hat{\mathcal{K}}^{(1)} + \hat{\alpha}^2 \hat{\mathcal{K}}^{(2)} + \mathcal{O}(\hat{\alpha}^3), \quad \hat{\mathcal{K}}^{(L)} = \hat{\mathcal{S}}^{(L)} + \hat{\mathcal{C}}^{(L)}$$

every object is an integrand here

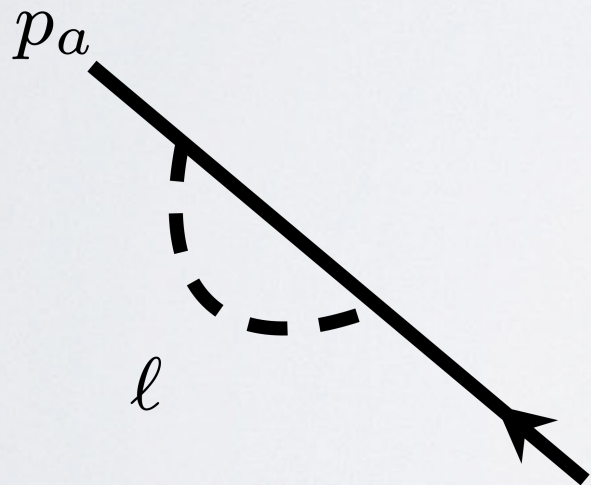
$$\mathcal{F}^{(1)} = \mathcal{A}_r^{(1)} - \hat{\mathcal{S}}\left[\hat{\mathcal{K}}_{[1]}^{(1)} \mathcal{A}^{(0)}\right]$$

$$\mathcal{F}^{(2)} = \mathcal{A}_r^{(2)} - \hat{\mathcal{S}}\left[\hat{\mathcal{K}}_{[1]}^{(1)} \mathcal{F}^{(1)}\right] - \frac{1}{2!} \hat{\mathcal{S}}\left[\hat{\mathcal{K}}_{[1]}^{(1)} \hat{\mathcal{K}}_{[2]}^{(1)} \mathcal{A}^{(0)}\right] - \hat{\mathcal{S}}\left[\hat{\mathcal{K}}_{[1,2]}^{(2)} \mathcal{A}^{(0)}\right]$$

ONE-LOOP LOCAL KERNELS

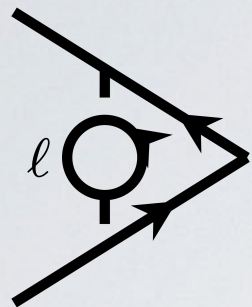


$$\mathcal{S}_{ab}^{(1)}(\ell) = e_a e_b \frac{(2\sigma_a p_a) \cdot (2\sigma_b p_b)}{\ell^2 (p_a - \ell)^2 (p_b + \ell)^2}$$

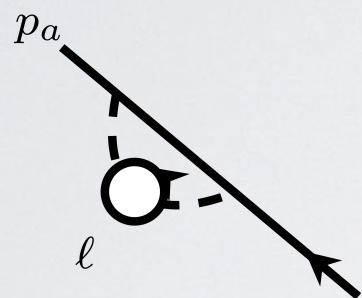


$$\mathcal{C}_a^{(1)}(\ell) = 2e_a^2 \left[\frac{1}{\ell^2 (p_a - \ell)^2} - \frac{1}{(\ell^2 - M^2)^2} \right]$$

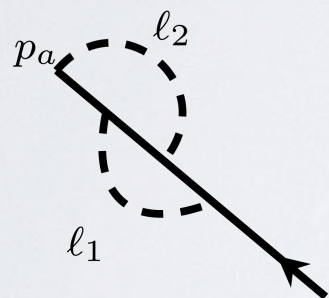
TWO-LOOP LOCAL KERNELS



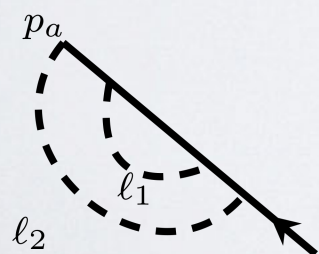
$$\mathcal{S}_{ab;f}^{(2)}(\ell) = -2e_f^2 \frac{d-2}{d-1} \left[\frac{1}{\ell_f^2 (\ell_f - \ell)^2} - \frac{1}{(\ell_f^2 - M^2)^2} \right] \mathcal{S}_{ab}^{(1)}(\ell)$$



$$\mathcal{C}_{a;f}^{(2)}(\ell) = -2e_f^2 \frac{d-2}{d-1} \left[\frac{1}{\ell_f^2 (\ell_f - \ell)^2} - \frac{1}{(\ell_f^2 - M^2)^2} \right] \mathcal{C}_a^{(1)}(\ell)$$



$$\mathcal{C}_a^{(2,sc)}(\ell_1, \ell_2) = -2e_a^4 (d-2) \left[\frac{(1-x_1) p_a \cdot \ell_2}{\ell_1^2 \ell_2^2 (p_a - \ell_1)^2 (p_a - \ell_1 - \ell_2)^2 (p_a - \ell_2)^2} + \frac{x_1 p_a \cdot \ell_2}{(\ell_1^2 - M^2)^2 (\ell_2^2 - M^2)^2 ((\ell_1 + \ell_2)^2 - M^2)} \right]$$



$$\delta \mathcal{C}_a^{(2,dc)} = -e_a^4 (2p_a \cdot \bar{p}_a)^2 \frac{[(d-2)(x_1^2 + x_2^2) - 4(x_1 + x_2) + 4] (\ell_{1\perp} \cdot \ell_{2\perp})}{\ell_1^2 \ell_2^2 (p_a - \ell_1)^2 (p_a - \ell_2)^2 (p_a - \ell_1 - \ell_2)^2 (\bar{p}_a + \ell_1)^2 (\bar{p}_a + \ell_2)^2}$$

$$\mathcal{F} = \hat{\mathcal{S}}[e^{-\hat{\mathcal{K}}} \mathcal{A}_r]$$

$$\mathcal{F} = \hat{\mathcal{S}}[e^{-\hat{\mathcal{K}}} \mathcal{A}_r]$$

$$\mathcal{A} = \mathbf{Z}_{\overline{\text{MS}}} \mathcal{F} \quad \Longleftrightarrow \quad \ln \mathbf{Z}_{\overline{\text{MS}}} \stackrel{\text{poles}}{=} \int \hat{\mathcal{K}} + \frac{1}{2} \sum_{\phi} \ln Z_{\phi}$$

Becher, Neubert [0901.0722], [0903.1126]

$$\mathcal{F} = \hat{\mathcal{S}} \left[e^{-\hat{\mathcal{K}}} \mathcal{A}_r \right]$$

$$\mathcal{A} = \mathbf{Z}_{\overline{\text{MS}}} \mathcal{F} \quad \Longleftrightarrow \quad \ln \mathbf{Z}_{\overline{\text{MS}}} \stackrel{\text{poles}}{=} \int \hat{\mathcal{K}} + \frac{1}{2} \sum_{\phi} \ln Z_{\phi}$$

Becher, Neubert [0901.0722], [0903.1126]

$$\ln \mathbf{Z}_{\overline{\text{MS}}} = \hat{\alpha} \left[\frac{\Gamma'_0}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon} \right] + \hat{\alpha}^2 \left[-\frac{3\beta_0\Gamma'_0}{16\epsilon^3} + \frac{\Gamma'_1 - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon} \right] + \mathcal{O}(\hat{\alpha}^3)$$

$$\mathcal{F} = \hat{\mathcal{S}}[e^{-\hat{\mathcal{K}}} \mathcal{A}_r]$$

$$\mathcal{A} = \mathbf{Z}_{\overline{\text{MS}}} \mathcal{F} \quad \Longleftrightarrow \quad \ln \mathbf{Z}_{\overline{\text{MS}}} \stackrel{\text{poles}}{=} \int \hat{\mathcal{K}} + \frac{1}{2} \sum_{\phi} \ln Z_{\phi}$$

Becher, Neubert [0901.0722], [0903.1126]

$$\ln \mathbf{Z}_{\overline{\text{MS}}} = \hat{\alpha} \left[\frac{\Gamma'_0}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon} \right] + \hat{\alpha}^2 \left[-\frac{3\beta_0\Gamma'_0}{16\epsilon^3} + \frac{\Gamma'_1 - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon} \right] + \mathcal{O}(\hat{\alpha}^3)$$

*We find full agreement!**

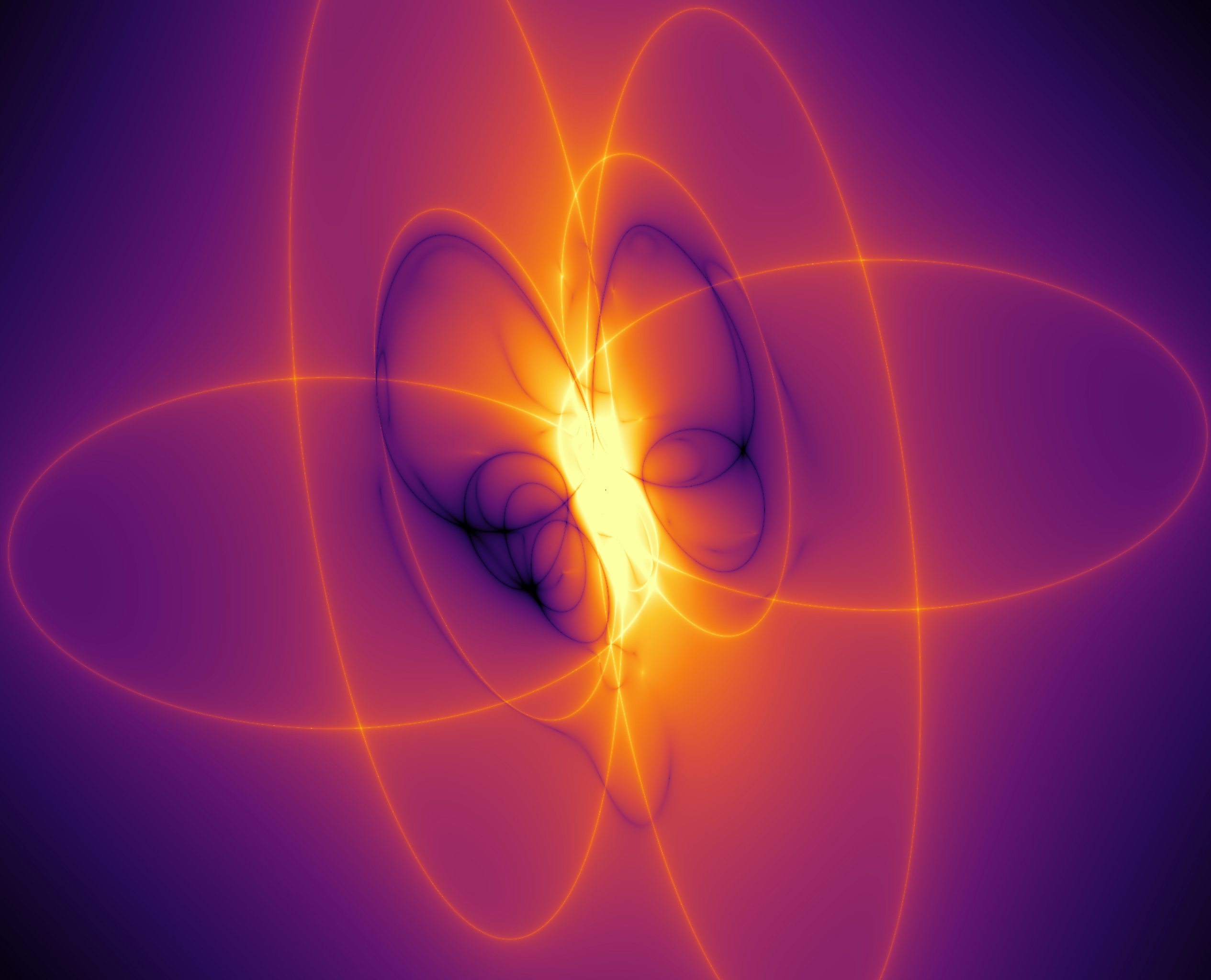
*after the change of renormalisation scheme, $\text{LM} \rightarrow \overline{\text{MS}}$

OUTLOOK

- Extension to 2-loop QCD
- A general study of numerical integration

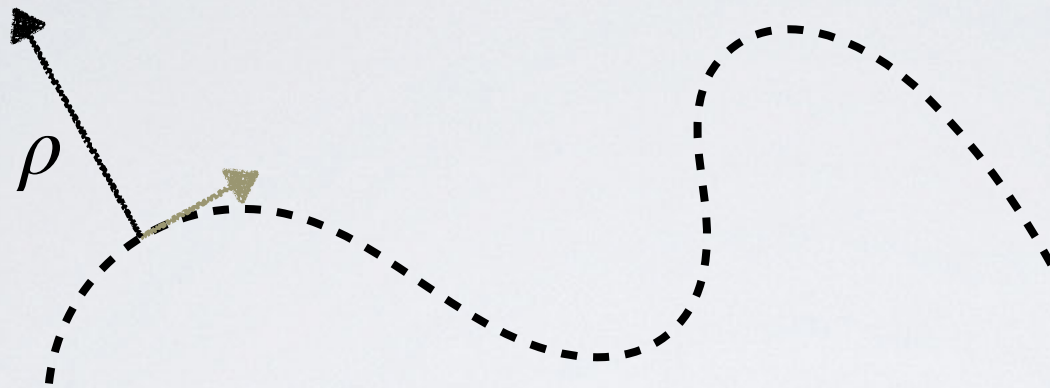
Already lots of results from

Capatti, Hirschi, Kermanschah, Ruijl, Vicini & others!



BACKUP

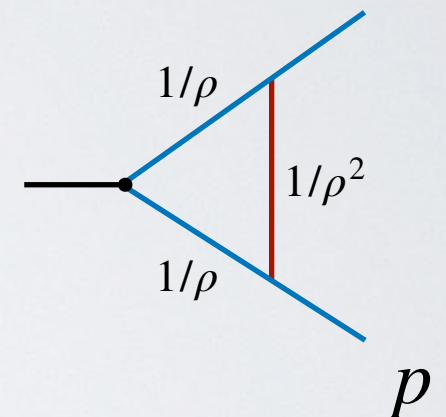
SOFT & COLLINEAR DIVERGENCES



$$\rho \rightarrow 0$$

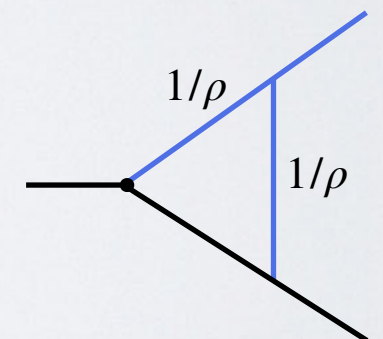
Soft:

$$\ell^\mu \xrightarrow{\text{soft}} \rho \hat{\ell}^\mu$$



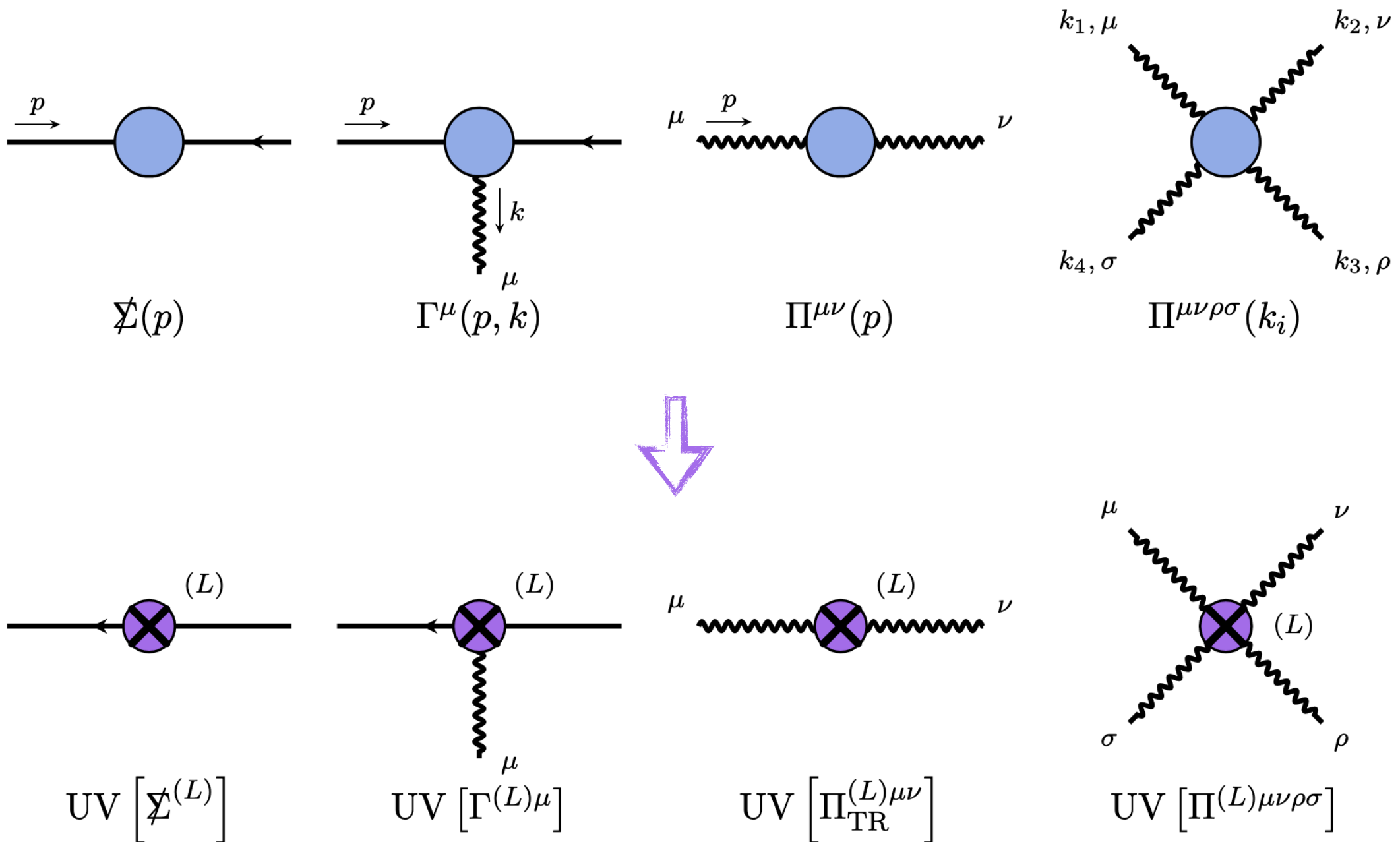
Collinear:

$$\ell^\mu \xrightarrow{\text{collinear to } p} \alpha p^\mu + \rho \beta \eta^\mu + \rho^{1/2} \hat{\ell}_\perp^\mu,$$



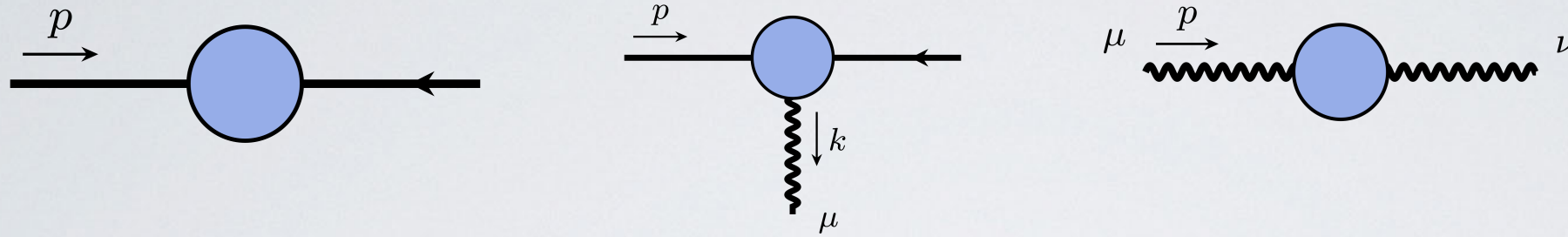
$$\eta^2 = \hat{\ell}_\perp \cdot p = \hat{\ell}_\perp \cdot \eta = 0$$

LOCAL RENORMALISATION



$$\int \text{UV} [\not{Z}^{(L)}] = \not{p} \delta_2^{(L)}, \quad \int \text{UV} [\Gamma^{(L)\mu}] = \gamma^\mu \delta_1^{(L)}, \quad \int \text{UV} [\Pi_{\text{TR}}^{(L)\mu\nu}] = g^{\mu\nu} p^2 \delta_3^{(L)}$$

LOCAL RENORMALISATION



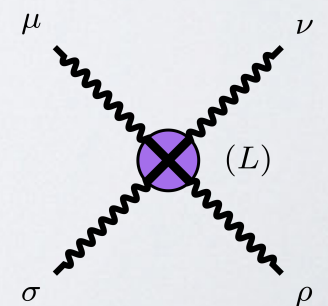
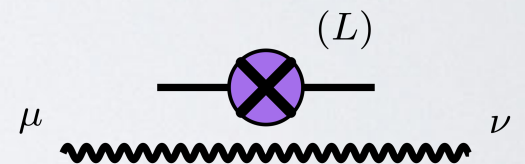
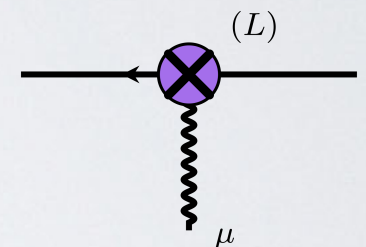
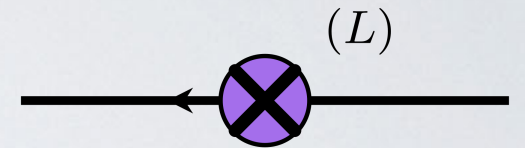
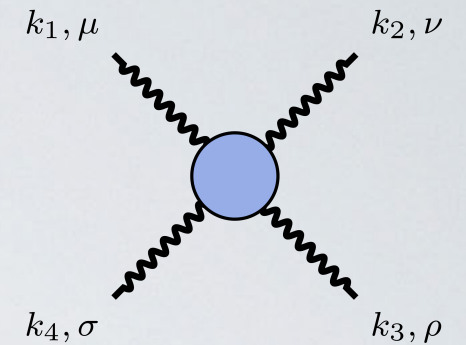
$$\text{UV} [\not{Z}^{(L)}(p)] = \mathcal{M} \left\{ \bar{R} \not{Z}^{(L)}(0) + p_\mu \left[\frac{\partial}{\partial p_\mu} \bar{R} \not{Z}^{(L)}(p) \right]_{p=0} \right\}$$

$$\text{UV} [\Gamma^{(L)\mu}] = \mathcal{M} \left[\frac{\partial}{\partial p_\mu} \bar{R} \not{Z}^{(L)}(p) \right]_{p=0}$$

$$\text{UV} [\Pi^{(L)\mu\nu}] = g^{\mu\nu} p^2 \mathcal{M} \bar{R} \Pi^{(L)}(0), \quad \Pi^{\mu\nu} \xrightarrow{\text{TR}} g^{\mu\nu} p^2 \Pi$$

$$\text{UV} [\Pi^{(L)\mu\nu\rho\sigma}] = e^2 \frac{\partial}{\partial \ell_{1\rho}} \frac{\partial}{\partial \ell_{1\sigma}} \mathcal{M} \left[\bar{R} \Pi^{(L)\mu\nu}(0) \right] + \text{perms}$$

$$M: q^2 \rightarrow q^2 - M^2$$



A RENORMALISATION SCHEME

$$\int^{\text{UV}} [\not{Z}^{(L)}] = \not{p} \delta_2^{(L)}, \quad \int^{\text{UV}} [\Gamma^{(L)\mu}] = \gamma^\mu \delta_1^{(L)}, \quad \int^{\text{UV}} [\Pi_{\text{TR}}^{(L)\mu\nu}] = g^{\mu\nu} p^2 \delta_3^{(L)}$$

the integrated counterterms are the standard Z's

$$Z_1 = Z_2 \quad \Longrightarrow \quad Z_e = \frac{Z_1}{Z_2 \sqrt{Z_3}} = Z_3^{-1/2}$$

nothing to add back: the local integrand is the renormalised amplitude in the M scheme

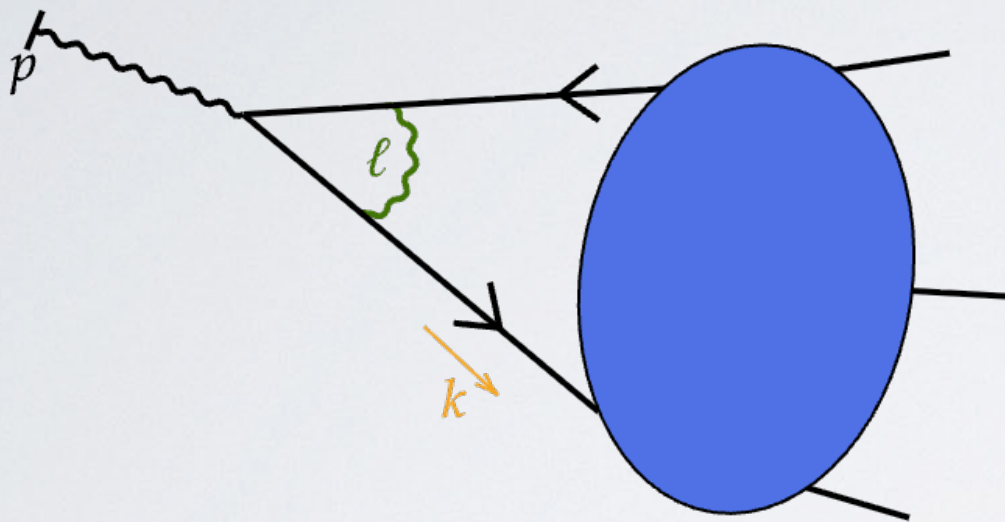
$$\delta_1^{(1)} = \delta_2^{(1)} = -\left[\frac{1}{\epsilon} + L_M - 1\right], \quad \delta_3^{(1)} = -\frac{4}{3}\left[\frac{1}{\epsilon} + L_M - \frac{1}{3}\right], \quad L_M = \ln \frac{\mu^2}{M^2}$$

$$\begin{aligned} \delta_1^{(2)} = \delta_2^{(2)} = & \frac{1}{\epsilon^2} \left(\frac{1}{2} - \frac{2}{3} n_f \right) + \frac{1}{\epsilon} \left(L_M - \frac{1}{4} + n_f \left(\frac{17}{9} - \frac{4}{3} L_M \right) \right) + L_M^2 - \frac{L_M}{2} - \frac{65}{72} - \frac{29\pi^2}{324} + \frac{7}{27} \psi' \left(\frac{1}{3} \right) \\ & + n_f \left(-\frac{4}{3} L_M^2 + \frac{34}{9} L_M - \frac{13}{54} + \frac{31\pi^2}{81} - \frac{20}{27} \psi' \left(\frac{1}{3} \right) \right), \quad \delta_3^{(2)} = -\frac{2}{\epsilon} - 4L_M - \frac{19}{3} - \frac{32\pi^2}{27} + \frac{16}{9} \psi' \left(\frac{1}{3} \right) \end{aligned}$$

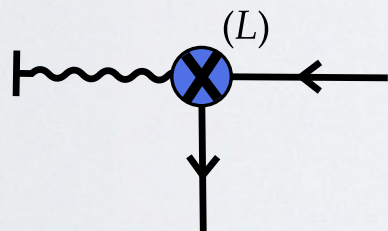
$$\alpha_0 = Z_3^{-1} \alpha \quad \Longrightarrow \quad \hat{\alpha}_M = \hat{\alpha}_{\overline{\text{MS}}} \left[1 + \hat{\alpha} n_f s(\epsilon) + \mathcal{O}(\hat{\alpha}^2) \right], \quad s(\epsilon) = \frac{4}{9} (1 - 3L_M) + \mathcal{O}(\epsilon)$$

→ MS-bar by a coupling shift

PHOTON TRANSIENT SINGULARITIES



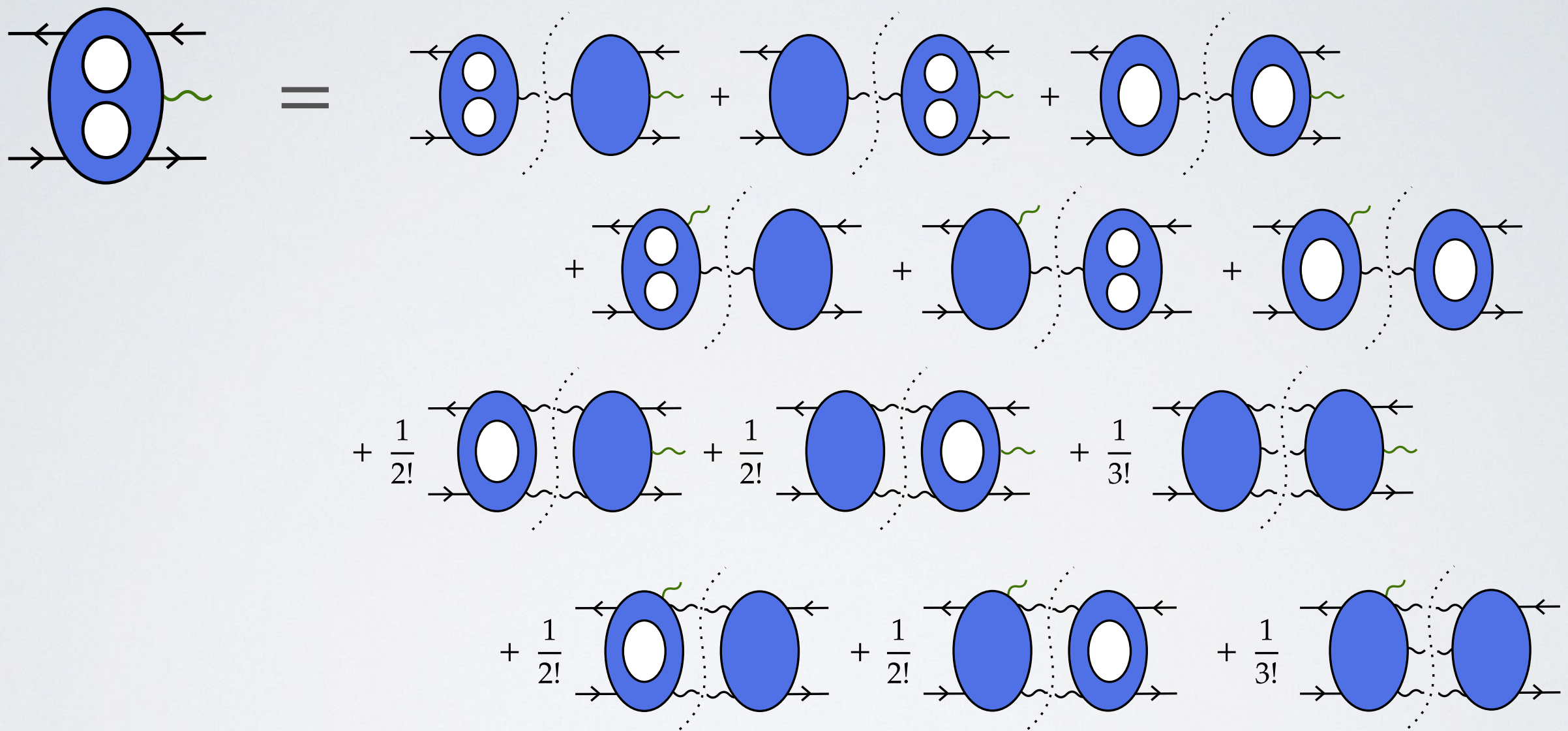
$$N \sim \not{q} \not{\epsilon}_p (\not{q} + \not{p}) \xrightarrow{q \parallel p} 0$$



$$N \sim (\not{k} - \not{p}) [\not{q} \not{\epsilon}_p (\not{q} + \not{p})] \not{k}$$

does not vanish (locally)

$$e^+ e^- \rightarrow \mu^+ \mu^- + \gamma \quad \text{at 2 loops}$$



$$\mathcal{A} \xrightarrow{IR\ 1} \boxed{S_1} \mathcal{A}_1$$

read off subtraction term

$$S_1 \mathcal{A}_1$$

$$\mathcal{A} - S_1 \mathcal{A}_1 \xrightarrow{IR\ 2} \boxed{S_2} \mathcal{A}_2$$

read off subtraction term

$$S_2 \mathcal{A}_2$$

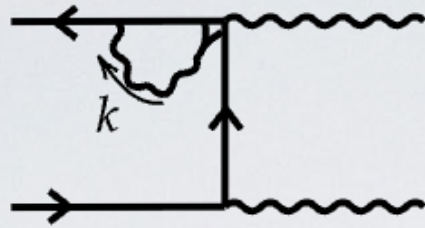
$$\mathcal{A} - S_1 \mathcal{A}_1 - S_2 \mathcal{A}_2$$

and so on ...

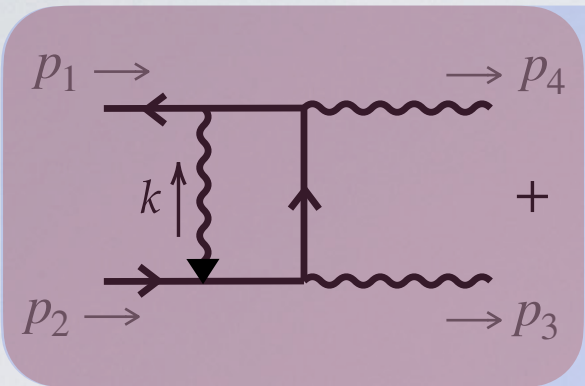
Order by: loop, volume size

1. Soft
2. Collinear
3. Double Soft
4. Soft-Collinear
5. Double Collinear

$$k^\mu \rightarrow \alpha p_1^\mu$$



$$= 2 \frac{1/\alpha - 1}{k^2(p_1 - k)^2} A^{(0)}$$

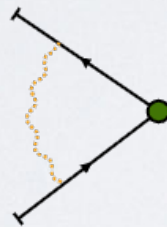


$$+ \text{[Diagram with momentum k flowing through the loop]} + \text{[Diagram with momentum k flowing through the loop]} + \text{[Diagram with momentum k flowing through the loop]} + (p_3 \leftrightarrow p_4)$$

$$- S(p_1, p_2; k) \times A_{ee \rightarrow \gamma\gamma}^{(0)}$$

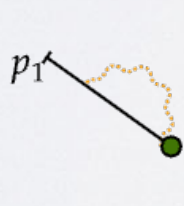
$$k^\mu \rightarrow 0$$

Soft limit



$$\left[\text{[Diagram with momentum k flowing through the loop]} + \text{[Diagram with momentum k flowing through the loop]} \right] = S(p_1, p_2; k) \times A_{ee \rightarrow \gamma\gamma}^{(0)}$$

Collinear limit



$$\left[\text{[Diagram with momentum k flowing through the loop]} + \text{[Diagram with momentum k flowing through the loop]} \right] = C(p_1; k) \times A_{ee \rightarrow \gamma\gamma}^{(0)}$$

$$A_{ee \rightarrow \gamma\gamma}^{(1)} - \left(C(p_1; k) + C(p_2; k) + S(p_1, p_2; k) \right) \times A_{ee \rightarrow \gamma\gamma}^{(0)} - \delta A_{ee \rightarrow \gamma\gamma}^{(1), UV}$$

$$A^{(1)} =$$

+

(

+

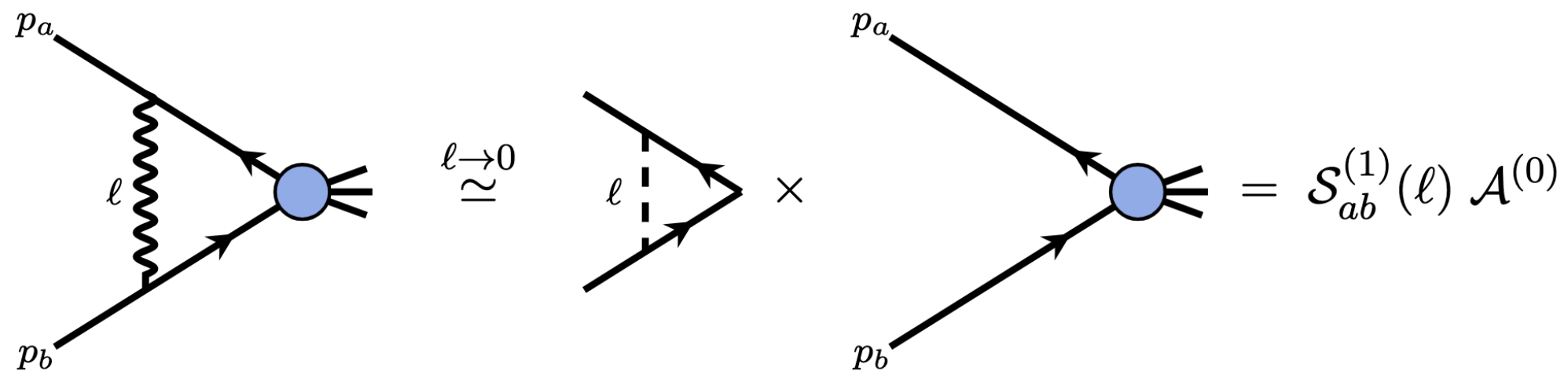
+

)

×

ONE LOOP

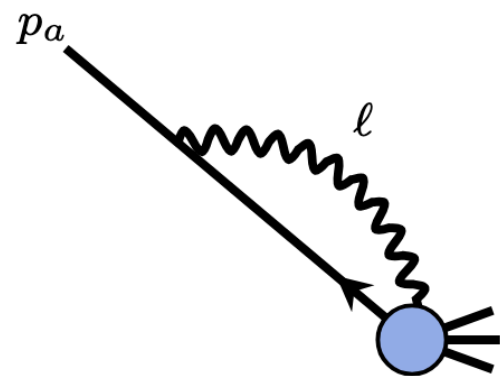
SOFT



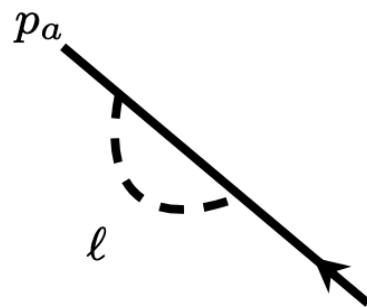
$$\mathcal{S}_{ab}^{(1)}(\ell) = e_a e_b \frac{(2\sigma_a p_a) \cdot (2\sigma_b p_b)}{\ell^2 (p_a - \ell)^2 (p_b + \ell)^2}$$

ONE LOOP

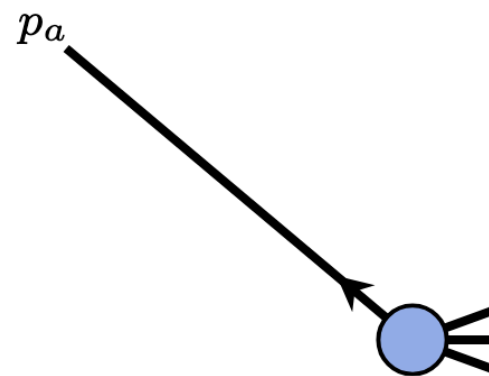
COLLINEAR



$\ell \parallel p_a$
 $\simeq$



$\times$

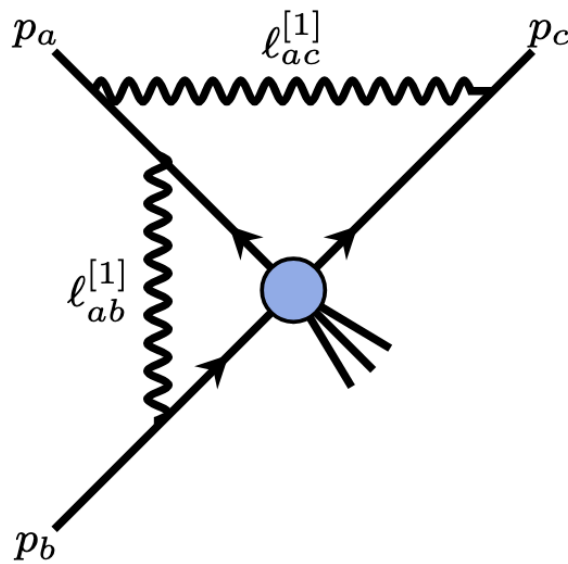


$$= C_a^{(1,b)}(\ell) \mathcal{A}^{(0)}$$

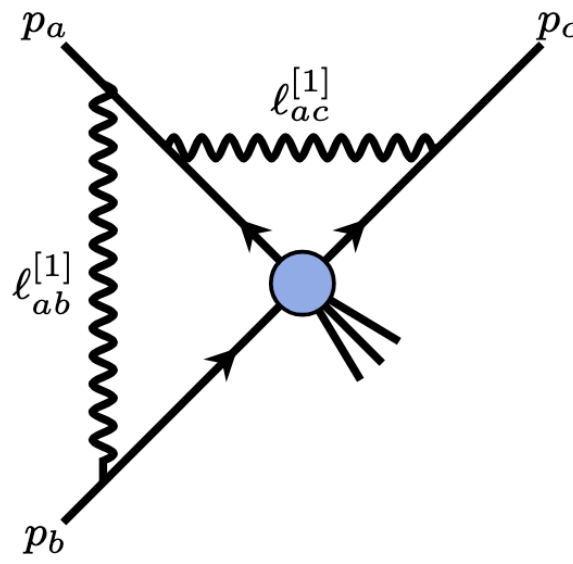
$$C_a^{(1,b)}(\ell) = 2 e_a^2 \frac{1 - 1/x_\ell}{\ell^2 (p_a - \ell)^2}$$

TWO LOOP

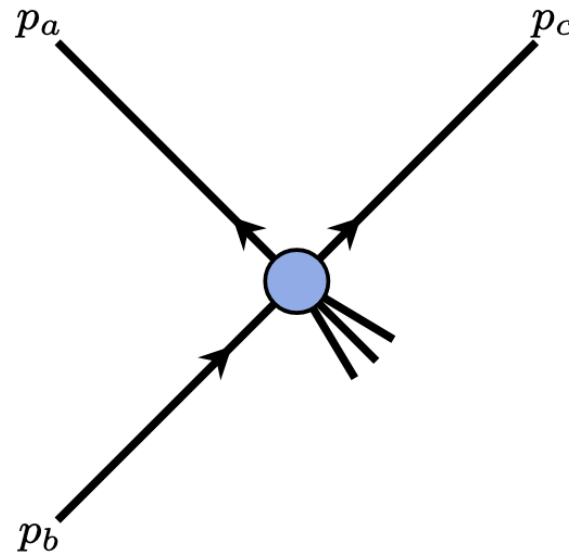
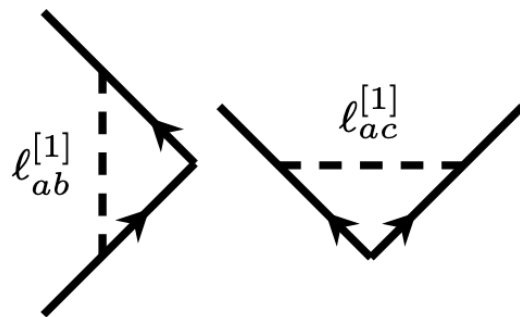
DOUBLE SOFT



+



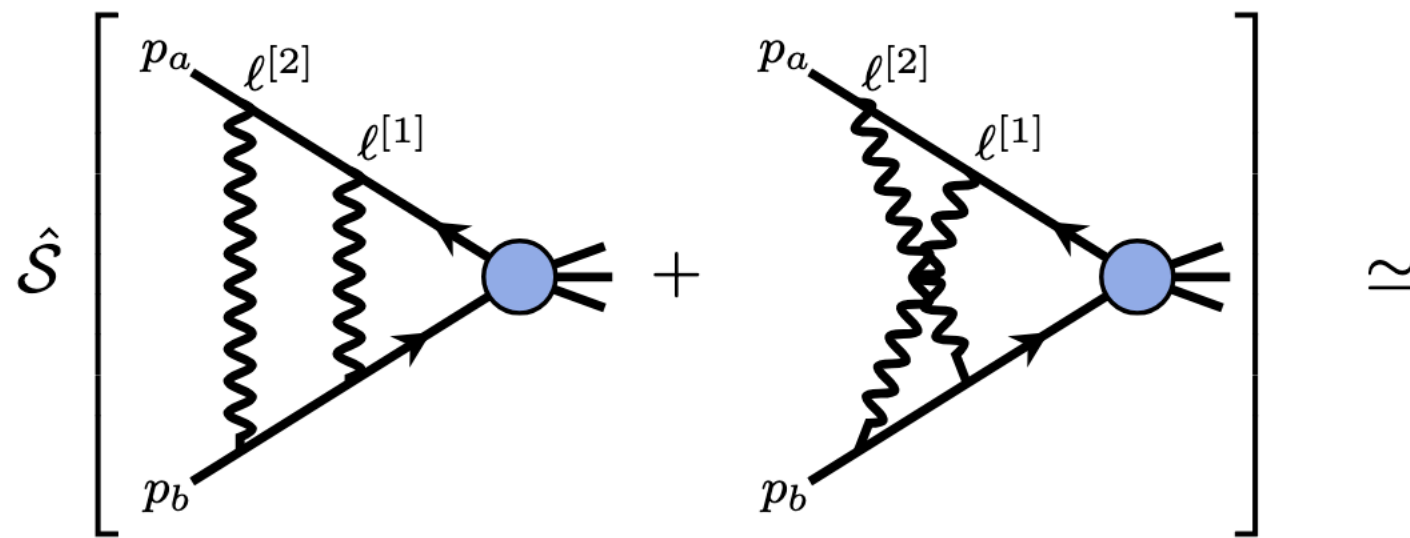
$\approx$



$$= \mathcal{S}_{ab}^{(1)}(\ell_{ab}^{[1]}) \mathcal{S}_{ac}^{(1)}(\ell_{ac}^{[1]}) \mathcal{A}^{(0)}$$

TWO LOOP

DOUBLE SOFT

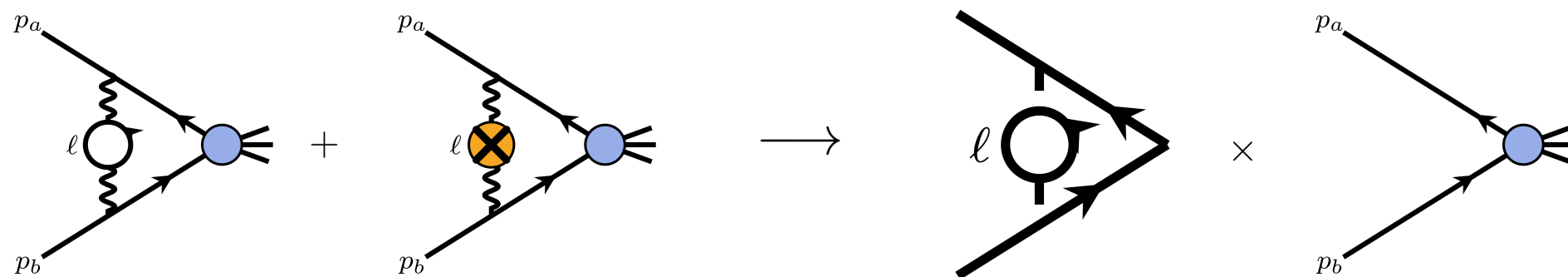
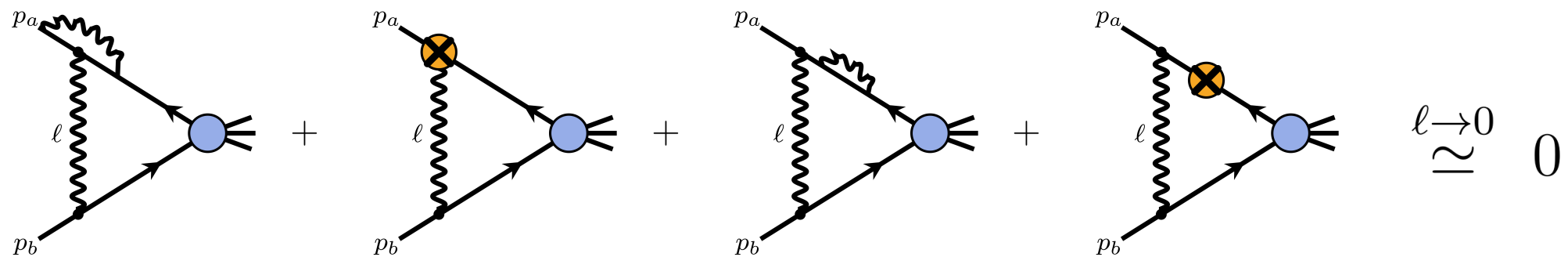


The amplitude is expressed as a product of soft factors and the tree-level amplitude:

$$\frac{1}{2!} \ell^{[1]} \ell^{[2]} = \frac{1}{2!} \mathcal{S}_{ab}^{(1)}(\ell_{ab}^{[1]}) \mathcal{S}_{ab}^{(1)}(\ell_{ab}^{[2]}) \mathcal{A}^{(0)}.$$

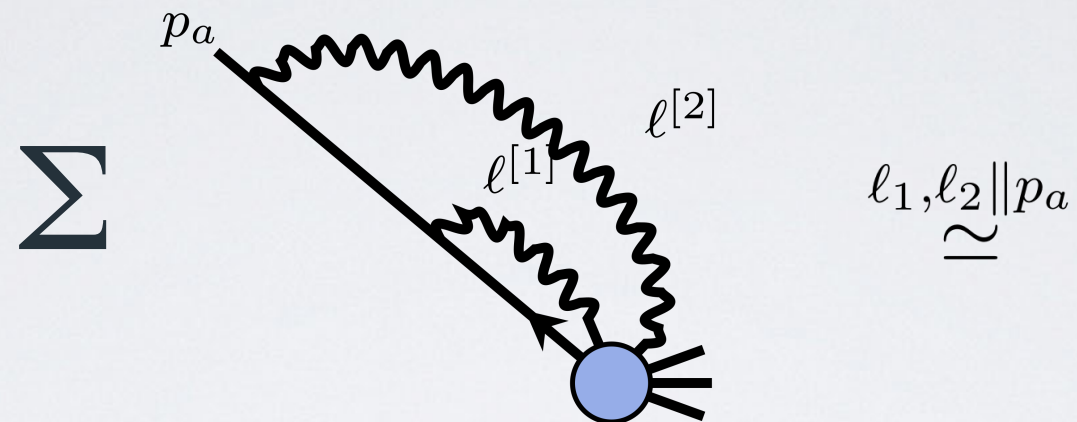
TWO LOOP

SOFT DIPOLE CORRECTIONS



Local manifestation of one-loop exactness

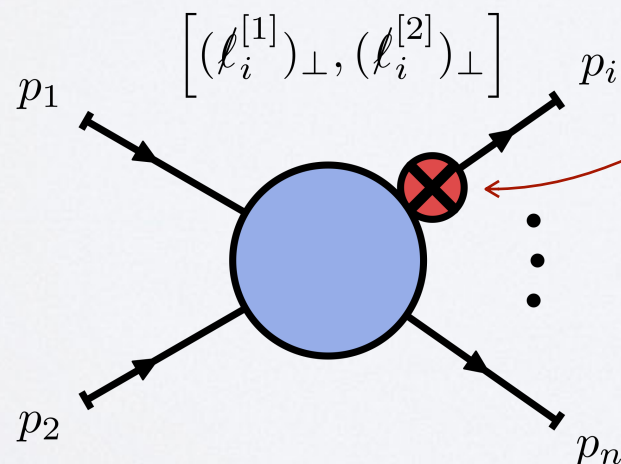
THE TRIPLE-COLLINEAR CONFIGURATION



$$\mathcal{K}_a^{(2)}(\ell_1, \ell_2) \mathcal{A}^{(0)} + C_{\perp}(\ell_1, \ell_2) \mathcal{A}_{u(p) \rightarrow [\ell_{1\perp}, \ell_{2\perp}]}^{(0)} u(p)$$

two-loop scalar
form factor

little-group
rotated spinor



TWO LOOP

TRIPLE COLLINEAR

