

Two-loop Feynman integrals for $t\bar{t}$ hadron production in the leading color approximation

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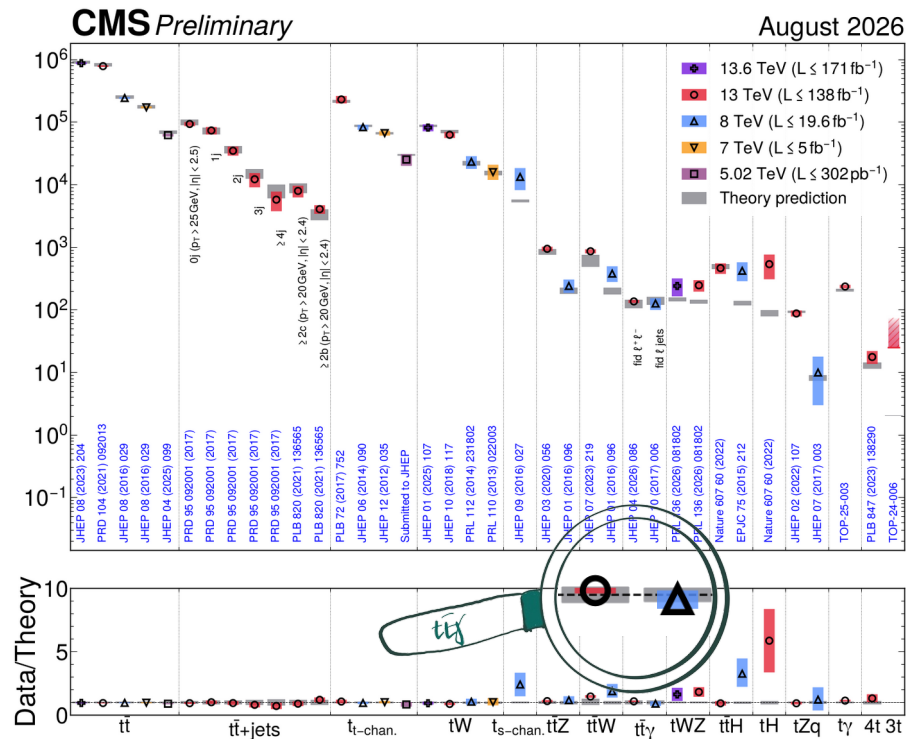
QCD@LHC 2026
Oxford, 1st October 2026



$t\bar{t}+X$ production at LHC – crucial tests of the SM

- $pp \rightarrow t\bar{t}+X$ (with $X=H,j,\gamma,Z,W,t\bar{t}$) are crucial processes for the LHC precision program

- $t\bar{t}\gamma$ production
 - precision **test of the SM**
 - direct measurement of the **top-quark electric charge**
 - possible hints of **new physics**:
 - Anomalous top-quark electric dipole moment
 - Top-quark charge asymmetry



As **experimental precision grows** , theory prediction must keep pace

$pp \rightarrow t\bar{t}\gamma$ theory status

Fixed order & matching

- NLO QCD and EW [Duan, Ma, Zhang, Han, Guo, Wang, Song, Li, 2009, 2011, 2017; Maltoni, Pagani, Tsinikos, 2016]
- Complete NLO corrections [Pagani, Shao, Tsinikos, Zaro, 2021]
- NLO QCD matched with the Pythia parton shower program [Kardos, Trócsányi, 2015]
- Top decays: narrow-width [Melnikov, Schulze, Scharf, 2011; Bevilacqua, Hartanto, Kraus, Weber, Worek, 2020, Stremmer, Worek, 2024]
- Top decays: full off-shell [Bevilacqua, Hartanto, Kraus, Weber, Worek, 2018, 2019; Stremmer, Worek, 2025]

The next frontier: NNLO QCD accuracy for $pp \rightarrow t\bar{t}\gamma$

The main bottleneck: two-loop Feynman integrals and scattering amplitudes

Towards exact NNLO

- Approx. NNLO cross-section: soft-gluon corrections added to the complete NLO result [Kidonakis, Toner, 2023]
- Approx. two-loop amplitudes: massification procedure [Wang, Xia, Yang, Ye, 2024]
- One-loop amplitudes up to $O(\epsilon^2)$ [Bera, Brancaccio, Canko, Hartanto, 2025]

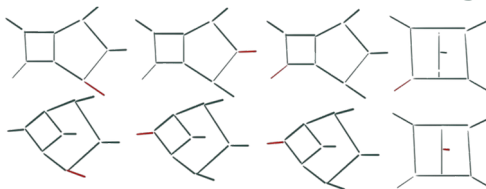
Two-loops five-points integrals - no internal masses

Massless external legs



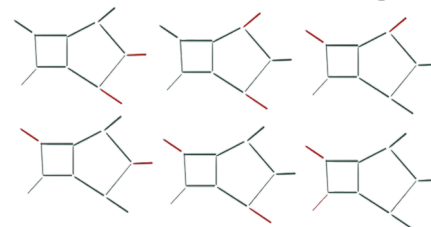
[Abreu, Chicherin, Dixon, Gehrmann, Henn, Herrmann, Lo Presti, Papadopoulos, Page, Sotnikov, Tomassini, Wasser, Wever, Zeng, Zhang, Zoia, 2015–2020]

One massive external leg



[Abreu, Canko, Chicherin, Ita, Kardos, Moriello, Page, Papadopoulos, Smirnov, Sotnikov, Syrrakos, Tomassini, Tschernow, Wever, Zeng, Zoia, 2015–2023]

Two massive external legs (planar)



[Jiang, Liu, Xu, Yang, 2024; Abreu, Chicherin, Sotnikov, Zoia, 2024]

Canonical dLog Differential equations. MIs expanded in terms of a **basis of special functions (pentagon functions)**, which enables:

- analytic ε -pole cancellations;
- simplifications at amplitude level;
- fast and stable numerical evaluation.

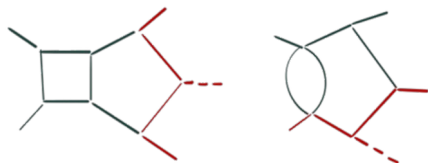
[Gehrmann, Henn, Lo Presti, 2018] [Chicherin, Sotnikov, 2020]
[Chicherin, Sotnikov, Zoia, 2022] [Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, 2024]

Many cross-section results available for processes sharing these topologies!

Two-loops five-points integrals – massive propagators

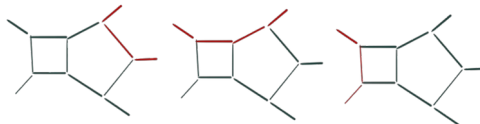
Increasing analytic complexity $\longrightarrow$ presence of complicated (elliptic) geometries

$pp \rightarrow ttH$ – leading colour, N_f part



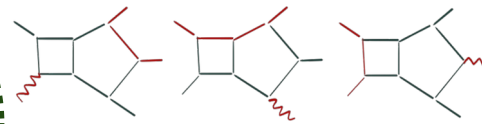
[Febres Cordero, Figueiredo, Kraus, Page, Reina, 2023]

$pp \rightarrow ttj$ – leading colour



[Badger, Becchetti, Chaubey, Marzucca, 2022; Badger, Becchetti, Giraudo, Zoia, 2024; Becchetti, Dlapa, Zoia, 2025]

$pp \rightarrow ttW$ – leading colour



[Becchetti, Canko, Chestnov, Peraro, Pozzoli, Zoia, 2025]

Also integrals of $pp \rightarrow tty$ – leading colour, light-quark electric charge contribution

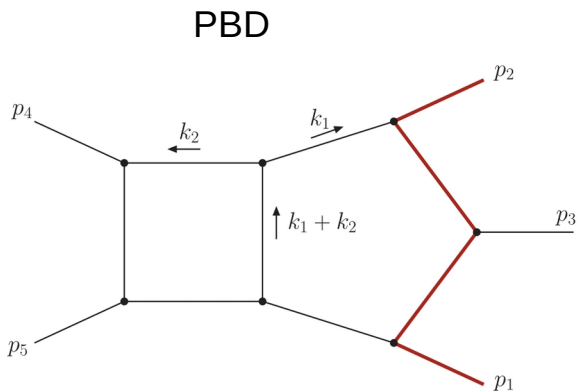
Few NNLO QCD cross-section results for processes involving these topologies are available.
Limited to the leading-colour approximation (or to other approximations of the double virtual contribution)

[Badger, Becchetti, Brancaccio, Czakon, Hartanto, Poncelet, Zoia, 2025]

[Devoto, Grazzini, Kallweit, Mazzitelli, Savoini '25]

[Becchetti, Canko, Chen, Chestnov, Delto, Ditsch, Peraro, Pozzoli, Savoini, Zoia, 2026] [Buonocore, Devoto, Grazzini, Kallweit, Mazzitelli, Rottoli, Savoini '25]

New Two-Loop Planar Topologies for Leading Colour (LC) tt



[Febres Cordero, Figueiredo, Kraus, Page, Reina '24]

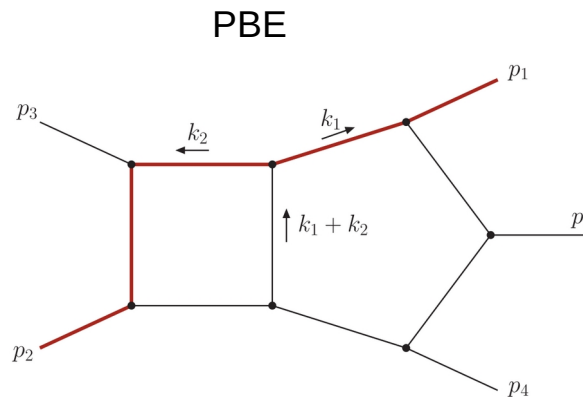
100 MIs

Canonical basis



[Henn '13]

d log alphabet



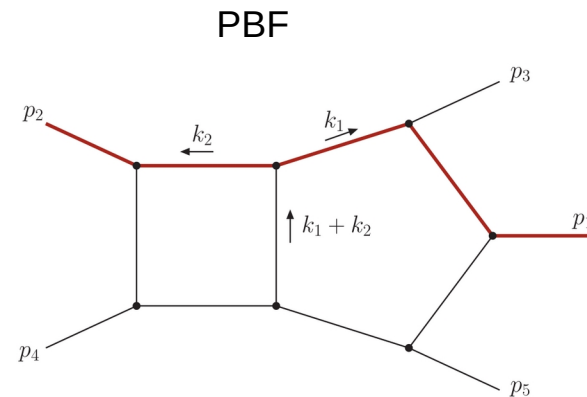
154 MIs

Canonical basis



2 Elliptic sectors

1 Nested square-root (NS)



142 MIs

Canonical basis



3 Elliptic sectors

2 NS

Differential equations: general structure

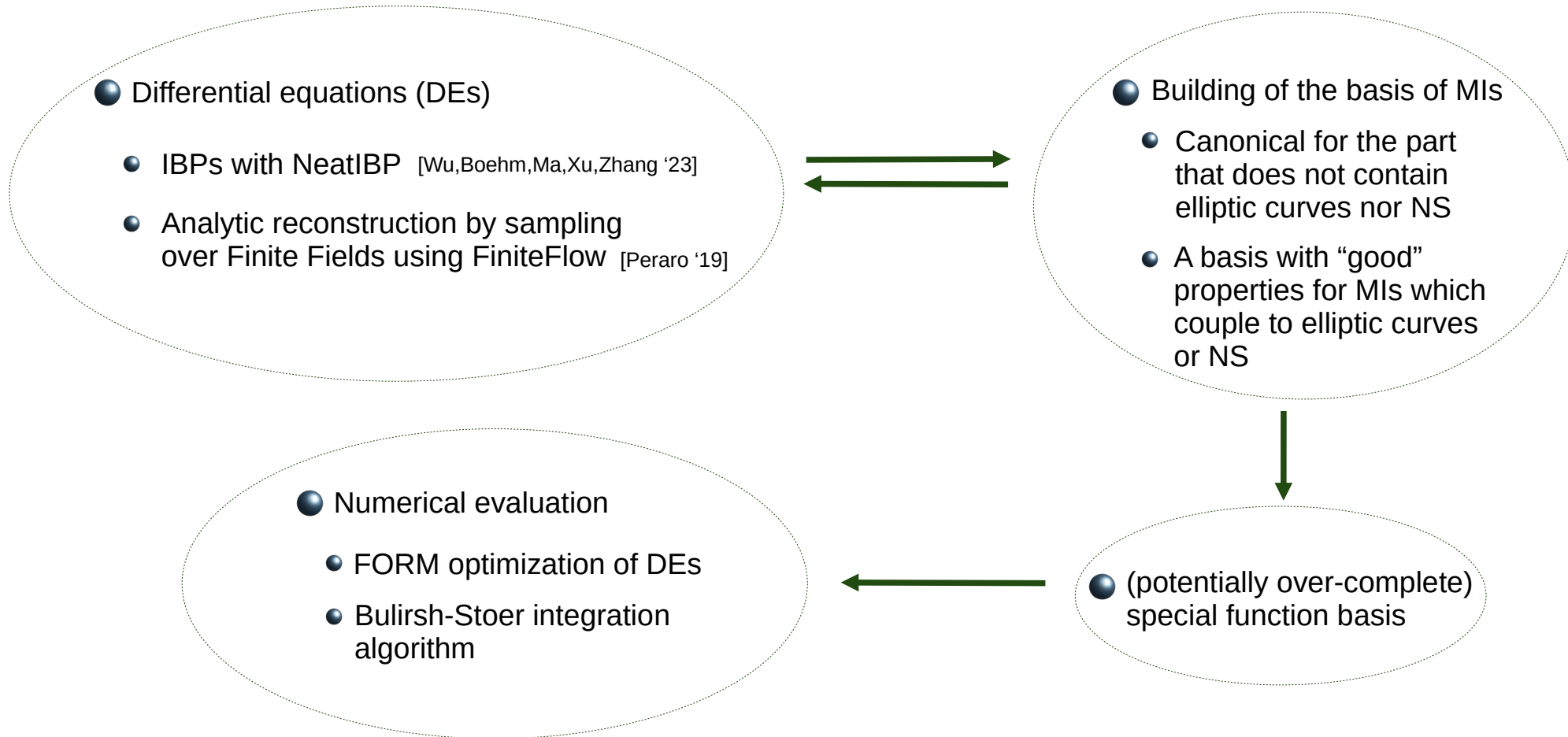
$$\frac{\partial}{\partial \xi} \vec{\mathcal{J}}^{(\text{F})}(\vec{x}, \varepsilon) = \mathcal{A}_{\xi}^{(\text{F})}(\vec{x}, \varepsilon) \cdot \vec{\mathcal{J}}^{(\text{F})}(\vec{x}, \varepsilon)$$

- Kinematics variables $\vec{x} = (d_{12}, d_{23}, d_{34}, d_{45}, d_{15}, m_t^2)$ $d_{ij} = p_i \cdot p_j$
- $\vec{\mathcal{J}}^{(\text{F})}(\vec{x}, \varepsilon)$ Basis of integrals
- $\mathcal{A}_{\xi}^{(\text{F})}(\vec{x}, \varepsilon)$ Connection matrices

The solution has not essential singularities (unphysical). So there is always a transformation of basis such that the connection matrix has only simple poles.

$$\tilde{\mathcal{A}}^{(\text{F})}(\vec{x}(\lambda), \varepsilon) \sim \frac{a_k(\varepsilon)}{\lambda - \lambda_k} \quad \lambda \sim \lambda_k \quad \forall \lambda_k \text{ singular point}$$

Strategy of the Computation



Differential equation structure

Canonical DEs: PBD

$$d\mathcal{A}^{(\text{PBD})}(\vec{x}, \varepsilon) = \varepsilon \sum_i \alpha_i d \log W_i(\vec{x})$$

ε -polynomial DEs: PBE, PBF

$$d\mathcal{A}^{(\text{F})}(\vec{x}, \varepsilon) = \sum_i (\beta_{0i} + \beta_{1i}\varepsilon + \beta_{2i}\varepsilon^2) \omega_i(\vec{x}) \\ + (\gamma_{0i} + \gamma_{1i}\varepsilon + \gamma_{2i}\varepsilon^2) d \log W_i(\vec{x})$$

- $W_i(\vec{x})$ Letters: rational and algebraic
- Algebraic letters have the form: $\frac{A + B\sqrt{C}}{A - B\sqrt{C}}$ found with the help of the algorithm described in [Matijasic '24 PhD Thesis]
- $\alpha_i, \gamma_{0i}, \gamma_{1i}, \gamma_{2i}, \beta_{0i}, \beta_{1i}, \beta_{2i}$ Matrices of rational numbers
- $\omega_i(\vec{x})$ Linearly independent one-forms
Algebraic one-forms: $\omega_i = \sqrt{C_i} \tilde{\omega}_i$ with $\tilde{\omega}_i$ rational

topology	square roots	d log letters	one-forms
PBD	6	77	0
PBE	9	113	177
PBF	11	129	222
2-Loop LC Q_t part	13	167	389

Differential equation structure

Canonical DEs: PBD

$$d\mathcal{A}^{(\text{PBD})}(\vec{x}, \varepsilon) = \varepsilon \sum_i \alpha_i d \log W_i(\vec{x})$$

ε -polynomial DEs: PBE, PBF

$$d\mathcal{A}^{(\text{F})}(\vec{x}, \varepsilon) = \sum_i \left(\beta_{0i} + \beta_{1i}\varepsilon + \beta_{2i}\varepsilon^2 \right) \omega_i(\vec{x}) \\ + \left(\gamma_{0i} + \gamma_{1i}\varepsilon + \gamma_{2i}\varepsilon^2 \right) d \log W_i(\vec{x})$$

● “Good” properties for ε -polynomial DEs:

- Canonical for blocks that do not couple to elliptic curves or NS

[Becchetti,Canko,Chestnov,Peraro,Pozzoli,Zoia '25]

- Non-canonical entries are polynomials, at worst quadratic, in ε (linear for NS)

[Badger,Becchetti,Giraud,Zoia '24]

- In the Laurent expansion of MIs, elliptic or NS part starts at $O(\varepsilon^4)$ $\longrightarrow$ Analytic cancellation of the poles in the amplitude

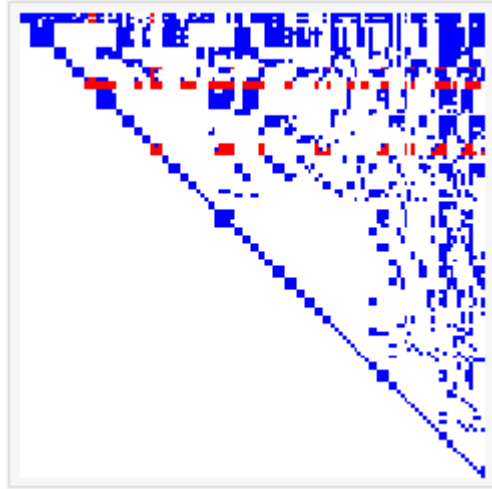
● Significant progress in defining and constructing canonical DEs beyond d Log. i) ε -factorised, ii) only simple poles, iii) one-forms independent up to total derivatives, are properties assumed to be valid for a possible definition of canonical DEs beyond d Log...see Felix talk!

[Adams,Chaubey,Weinzierl '17] [Adams,Weinzierl '18] [Frellesvig '22] [Görge,Nega,Tancredi,Wagner '23] [Dlapa,Henn,Wagner '23] [Pögel,Wang,Weinzierl '23] [Pögel,Wang,Weinzierl '23] [Frellesvig,Weinzierl '24] [Driesse,Jakobsen,Klemm,Mogull,Nega,Plefka,Sauer,Usovitsch '25] [Duhr,Porkert,Stawinski '25] [Duhr,Maggio,Nega,Sauer,Tancredi,Wagner '25] [Chen,Yang,Zhang '25] [Forner,Mella,Nega,Tancredi,Wagner '26]

● We DO NOT provide fully canonical DEs. Algebraic complexity renders this step challenging

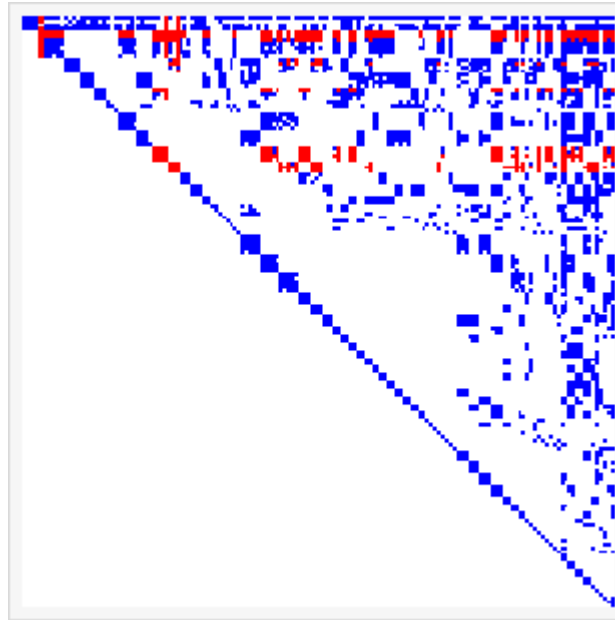
Differential equation structure

PBB 2 massive propagators

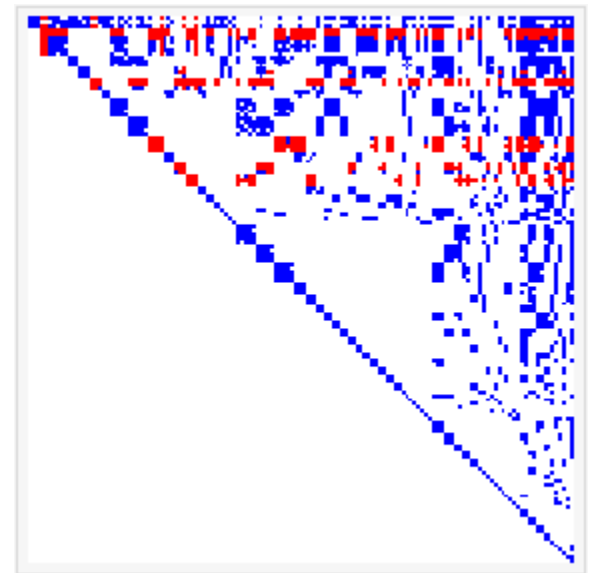


[Badger,Becchetti,Giraud,Zoia '24]

PBE 3 massive propagators



PBF 3 massive propagators



Masses make mathematical structure increasingly complex

■ Canonical ϵ d Log
■ Everything else

Elliptic sector $I_{11001111000}^{\text{PBE}}$

● Basis of integrals (3 MIs)

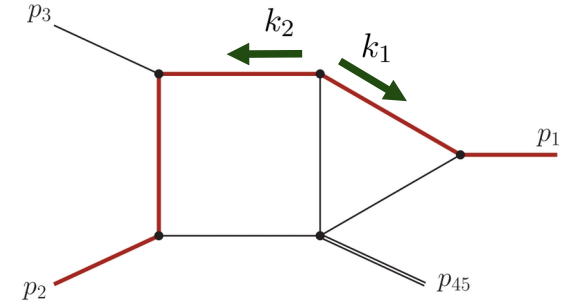
● simple pole: $\text{MC} \left(I_{11001111000}^{\text{PBE}}[k_2 \cdot p_1] \right)_{\varepsilon=0} \sim \int \frac{dz z}{\sqrt{P_4(z)}} + (\text{holomorphic})$

● double pole: $\text{MC} \left(I_{11001111000}^{\text{PBE}}[k_1 \cdot p_3] \right)_{\varepsilon=0} \sim \int \frac{dz}{(z - 4m_t^2)\sqrt{P_4(z)}}$

● holomorphic: $\text{MC} \left(I_{11001111000}^{\text{PBE}} \right)_{\varepsilon=0} \sim \int \frac{dz}{\sqrt{P_4(z)}}$

● Elliptic curve

$$y^2 = P_4(z) = z(z - 4m_t^2)(z - a_-)(z - a_+)$$



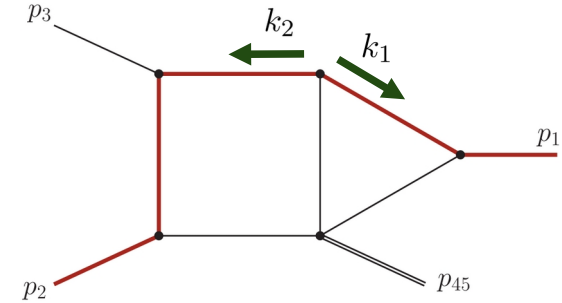
Elliptic sector $I_{11001111000}^{\text{PBE}}$

● Basis of integrals (3 MIs)

$$\mathcal{J}_{39}^{(\text{PBE})} = \varepsilon^4 d_{23} I_{11001111000}^{(\text{PBE})} [2 k_2 \cdot p_1]$$

$$\mathcal{J}_{40}^{(\text{PBE})} = \varepsilon^4 (1 - 2\varepsilon) d_{23} I_{11001111000}^{(\text{PBE})} [2 k_1 \cdot p_3]$$

$$\mathcal{J}_{41}^{(\text{PBE})} = \varepsilon^4 d_{23} (d_{12} + d_{23} - d_{45} + m_t^2) I_{11001111000}^{(\text{PBE})}$$



● DEs are quadratic in ε

$$d\mathcal{J} = \begin{pmatrix} * + *\varepsilon & * & *\varepsilon \\ (2\varepsilon - 1)(* + *\varepsilon) & * + *\varepsilon & (2\varepsilon - 1)*\varepsilon \\ * + *\varepsilon & * & *\varepsilon \end{pmatrix} \cdot \mathcal{J} + (\text{sub-sectors})$$

● Elliptic curve

$$y^2 = P_4(z) = z(z - 4m_t^2)(z - a_-)(z - a_+)$$



Nested square root sector $I_{11101111000}^{(\text{PBE})}$

● Next-to Maximal Cut of the corner integral

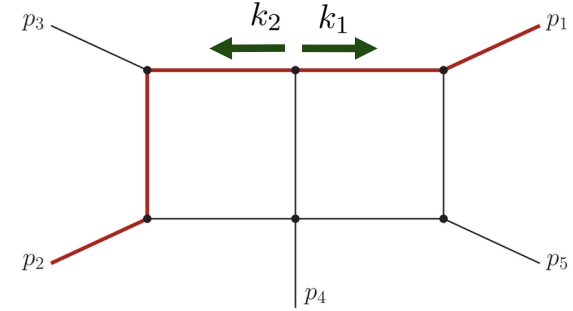
$$I_{11101111000}^{(\text{PBE})} \Big|_{\varepsilon \rightarrow 0}^{\text{MC} \setminus \{D_1\}} \propto \int \frac{dz_1 dz_4 dz_9}{z_1 \sqrt{\mathcal{P}_1 \mathcal{P}_2 \overline{\mathcal{P}}_2}} \quad \begin{array}{l} \mathcal{P}_1 \text{ quadratic polynomial} \\ \overline{\mathcal{P}}_2 = \mathcal{P}_2(r_1 \rightarrow -r_1) \end{array}$$

● 7 MIs

$$\left. \begin{array}{l} \mathcal{J}_5^{(\text{PBE})} = \frac{\varepsilon^4}{d_{45}} I_{1101111000}^{(\text{PBE})} \left[\frac{\mathcal{P}_2 - \overline{\mathcal{P}}_2}{2} \right] \\ \mathcal{J}_6^{(\text{PBE})} = \frac{\varepsilon^4}{d_{45}} I_{1101111000}^{(\text{PBE})} \left[\frac{\mathcal{P}_2 + \overline{\mathcal{P}}_2}{2} \right] \end{array} \right\} \text{Following procedure of [Becchetti, Canko, Chestnov, Peraro, Pozzoli, Zoia '25]}$$

$$\mathcal{J}_7^{(\text{PBE})} = \frac{1}{\varepsilon} \frac{d_{45} m_t^2 (d_{23} + d_{34})^2 - 2d_{13} d_{15} d_{23} d_{24}}{d_{15} r_2} \frac{\partial \mathcal{J}_{11}^{(\text{PBE})}}{\partial d_{34}} + \dots \left. \vphantom{\frac{1}{\varepsilon}} \right\} \text{linear combination of derivatives of other MIs to achieve } \varepsilon\text{-linear DEs}$$

$$\left. \begin{array}{l} \mathcal{J}_8^{(\text{PBE})} = \varepsilon^4 r_1 d_{15} I_{11101111000}^{(\text{PBE})} \left[\frac{\mu_{12}}{(k_1 - p_{15})^2} \right] \\ \mathcal{J}_9^{(\text{PBE})} = \varepsilon^4 r_1 I_{11101111000}^{(\text{PBE})} [\mu_{12}] \\ \mathcal{J}_{10}^{(\text{PBE})} = \varepsilon^4 d_{15} d_{23} I_{11101111000}^{(\text{PBE})} [(k_2 + p_1)^2] \\ \mathcal{J}_{11}^{(\text{PBE})} = \varepsilon^4 r_2 d_{15} I_{11101111000}^{(\text{PBE})} \end{array} \right\} \text{Numerators found requiring the integrands to have only simple poles}$$



Nested square root sector $I_{11101111000}^{(\text{PBE})}$

- Next-to Maximal Cut of the corner integral

$$I_{11101111000}^{(\text{PBE})} \Big|_{\varepsilon \rightarrow 0}^{\text{MC} \setminus \{D_1\}} \propto \int \frac{dz_1 dz_4 dz_9}{z_1 \sqrt{\mathcal{P}_1 \mathcal{P}_2 \overline{\mathcal{P}}_2}} \quad \begin{array}{l} \mathcal{P}_1 \text{ quadratic polynomial} \\ \overline{\mathcal{P}}_2 = \mathcal{P}_2(r_1 \rightarrow -r_1) \end{array}$$

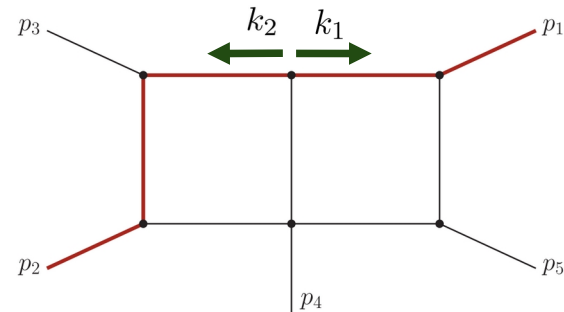
- DEs are linear in ε

$$d\mathcal{J} = \left[\begin{pmatrix} X & Y & \mathbf{0} \\ Y & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{5 \times 5} \end{pmatrix} + \mathcal{O}(\varepsilon) \right] \cdot \mathcal{J} + (\text{sub-sectors})$$

- DEs ε -factorised after the rotation

$$\begin{pmatrix} \mathcal{J}_5^{(\text{PBE})} \\ \mathcal{J}_6^{(\text{PBE})} \end{pmatrix} \mapsto \begin{pmatrix} n_2^+ & 0 \\ 0 & n_2^- \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{J}_5^{(\text{PBE})} \\ \mathcal{J}_6^{(\text{PBE})} \end{pmatrix}$$

$$n_2^\pm = \pm \sqrt{A_\pm \pm B_\pm \sqrt{G(p_1, p_2, p_3, p_4)}} \quad A_\pm, B_\pm \text{ rational}$$



- In the subsectors it couples to both elliptic curves of PBE
- For the numerical evaluation we use the basis without nested square-roots

(Potentially over-complete) basis of special functions

$$\vec{\mathcal{J}}^{(F)}(\vec{x}, \varepsilon) = \sum_{k=0}^4 \varepsilon^k \vec{\mathcal{J}}^{(F,k)}(\vec{x}) + \mathcal{O}(\varepsilon^5) \quad \text{O}(\varepsilon^4) \text{ is enough to compute the two-loop amplitude up to } \mathcal{O}(\varepsilon^0)!$$

$\vec{\mathcal{J}}^{(F,k)}(\vec{x})$ are polynomials with rational coefficients of $1, \underbrace{f_i^{(F,1)}, f_i^{(F,2)}, f_i^{(F,3)}, f_i^{(F,4)}}_{\text{iterated integrals of logarithmic one-forms}}, f_i^{(F,4*)}$

DEs for $f_i^{(F,k)}$:

- do not depend on ε
- have fewer one-forms (information at all order in ε truncated)



more efficient numerical evaluation!

Iterated integrals of logarithmic one-forms, pure functions, algebraic independent, algorithmic construction

[Gehrmann, Henn, Lo Presti, 2018]
 [Chicherin, Sotnikov, 2020]
 [Chicherin, Sotnikov, Zoia, 2022]
 [Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, 2024]

Elliptic and nested square root functions, appear at $\mathcal{O}(\varepsilon^4)$ in the MIs and $\mathcal{O}(\varepsilon^0)$ in the amplitude, have not a transcendental weight, defined by coupled DEs, potentially over-complete basis

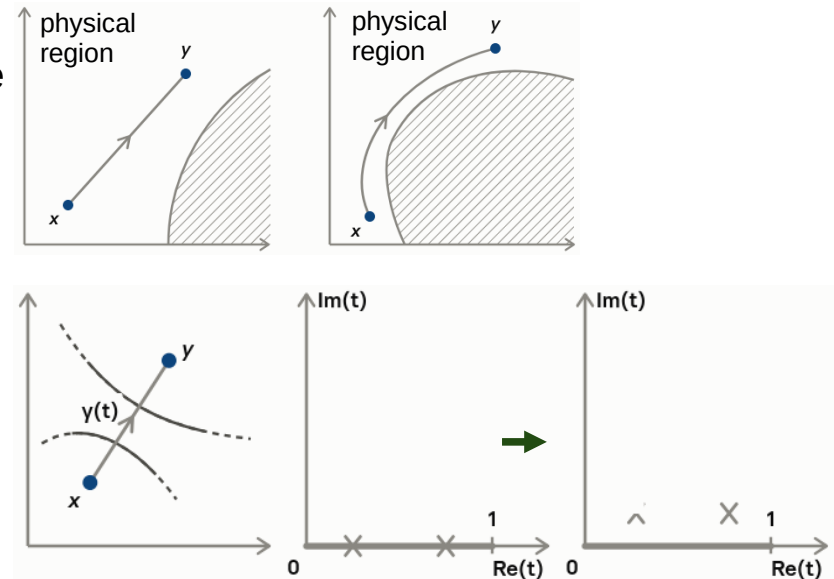
family	MI	$f^{(F,1)}$	$f^{(F,2)}$	$f^{(F,3)}$	$f^{(F,4)}$	$f^{(F,4*)}$	all $f^{(F,k)}$
PB _D	100	6	10	46	85	0	147
PB _E	154	6	10	53	110	13	192
PB _F	142	6	10	51	102	16	185

Numerical evaluation

- High-precision boundary values (70 digits) at a few random points done with AMFlow [Liu, Ma '22]
- Numerical solution of DEs with Bulirsh-Stoer algorithm implemented in BOOST, as done for NNLO LC ttj [Badger, Becchetti, Brancaccio, Czakon, Hartanto, Poncelet, Zoia '26]
The method has recently received attention [Rosas, Bobadilla '25] [Baur, Duhr '26] [Czakon, Tancredi '26]

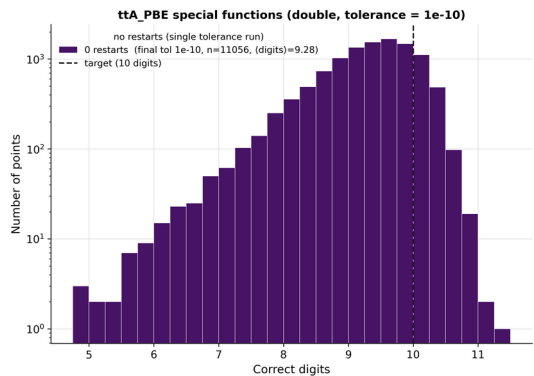
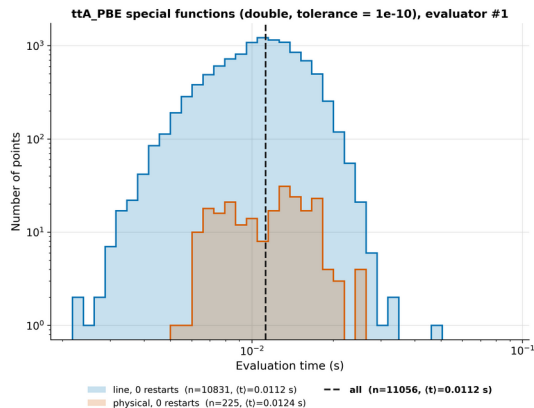
Main features of our numerical evaluation:

- The path connecting the boundary point to any given phase-space point is a straight line, unless the latter leaves the physical region
- Potential spurious singularities are avoided giving an imaginary part to the curve
- The error of the special functions is determined by performing the integration starting from two different points and comparing the results
- Derivatives are polynomials of one-forms and special functions. We optimise them and the expressions of all the objects they contain with an optimization algorithm (common subexpression elimination) implemented in FORM [Kuipers, Ueda, Vermaseren '13]



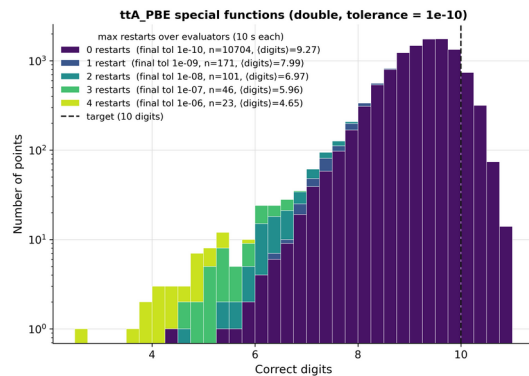
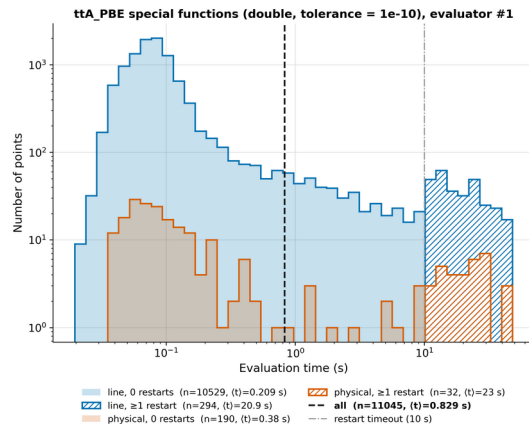
Numerical evaluation

PBD



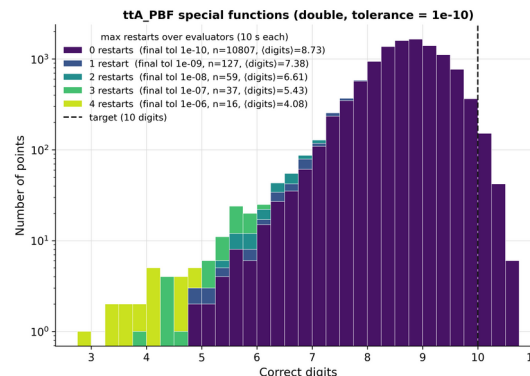
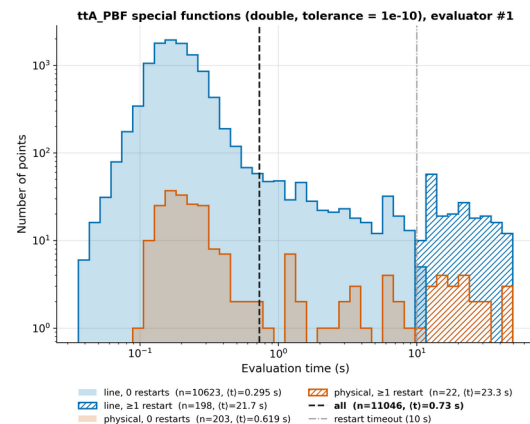
$$\langle t \rangle = 0.01 \text{ s}$$

PBE



$$\langle t \rangle = 0.83 \text{ s}$$

PBF



$$\langle t \rangle = 0.73 \text{ s}$$

Conclusion

- We have built (a basis of) functions where the ugly mathematical structure is pushed to a part of the finite remainder, where few (but more than ttj) elliptic and nested square root functions will appear.
- We are reasonably satisfied with the numerical evaluation, which we believe stable and fast enough for a phenomenological application.
- For the future a richer mathematical structure is waiting us...will this approach continue to be good enough for the phenomenological application? Can we do something better?
- This talk was not about rational coefficients in the amplitude, but their numerical evaluation is often slower than integral evaluation...can we choose bases of special functions which make them simpler?

Thank you!