

Leading Singularities and Canonical Bases Beyond Polylogarithms

based on [arXiv:2604.25270] with C. C. Mella, C. Nega, L. Tancredi, F. J. Wagner

Felix Forner

QCD@LHC 2026, Oxford University



MODERN
FOUNDATIONS of
SCATTERING AMPLITUDES



Deutsche
Forschungsgemeinschaft
German Research Foundation

Scattering Amplitudes and Feynman Integrals

- perturbative corrections to scattering amplitudes $\rightarrow$ Feynman integrals
- increasing complexity with more loops and legs
- numerical evaluation not straightforward (UV & IR divergences, thresholds)
- direct analytical integration difficult

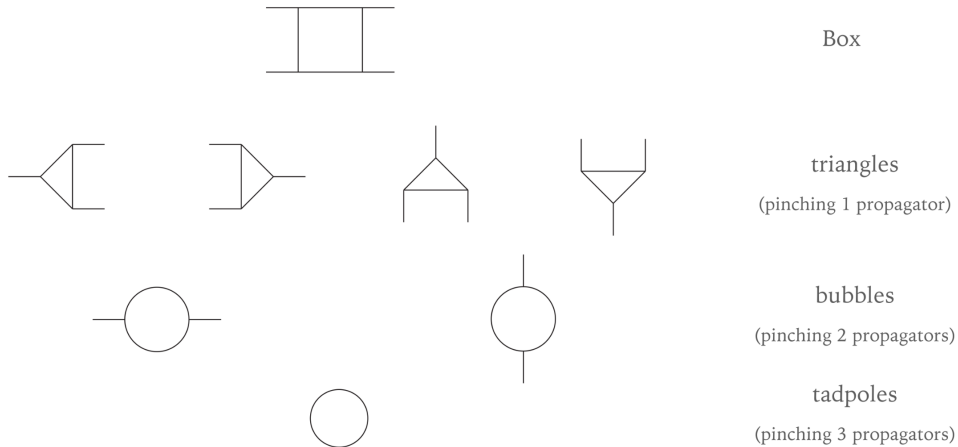
Scattering Amplitudes and Feynman Integrals

- perturbative corrections to scattering amplitudes → Feynman integrals
- increasing complexity with more loops and legs
- numerical evaluation not straightforward (UV & IR divergences, thresholds)
- direct analytical integration difficult
- integration-by-parts identities → master integrals
→ differential equations

[Kotikov, 1993] [Remiddi, 1997] [Gehrmann, Remiddi, 2000]

$$\frac{d\vec{\mathcal{I}}}{ds_i} = M(\epsilon; \vec{s}) \cdot \vec{\mathcal{I}}$$

The "Tree" of Master Integrals



Block-Triangularity

$$\mathbf{M}(\epsilon, \vec{s}) = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & 0 & \\ & & & & * \\ & * & & & \\ & & & * & \\ & & & & \ddots \\ & & & & & * \end{pmatrix}$$

But for arbitrary basis: higher poles in $\vec{s}$, complicated ϵ -dependence
How can we find a good basis?

$$\mathcal{A} \sim (\text{rational/square roots}) \times (\text{transcendental})$$

↑
polylogs, elliptic polylogs, ... ↔ geometry!

- poles ↔ single-particle states, branch cuts ↔ multi-particle states

$$\mathcal{A} \sim (\text{rational/square roots}) \times (\text{transcendental})$$

↑
polylogs, elliptic polylogs, ... ↔ geometry!

- poles ↔ single-particle states, branch cuts ↔ multi-particle states
- structure reflected in the differential equation, e.g.

$$f(x) = \frac{1}{x} \log(x-1) \quad \longrightarrow \quad \frac{df}{dx} = -\frac{1}{x} f(x) + \frac{1}{x(x-1)}$$

$$\mathcal{A} \sim (\text{rational/square roots}) \times (\text{transcendental})$$

↑
polylogs, elliptic polylogs, ... ↔ geometry!

- poles ↔ single-particle states, branch cuts ↔ multi-particle states
- structure reflected in the differential equation, e.g.

$$f(x) = \frac{1}{x} \log(x-1) \quad \longrightarrow \quad \frac{df}{dx} = -\frac{1}{x} f(x) + \frac{1}{x(x-1)}$$

- normalizing by the rational prefactor removes the homogeneous part:

$$g(x) = \log(x-1) \quad \longrightarrow \quad \frac{dg}{dx} = \frac{1}{x-1}$$

→ unipotence

Canonical Bases and Leading Singularities

- for $\epsilon \neq 0$, a canonical basis $\vec{\mathcal{I}}_C$ satisfies [\[Henn, 2013\]](#)

$$\frac{d\vec{\mathcal{I}}_C}{ds_i} = \epsilon M_C(\vec{s}) \cdot \vec{\mathcal{I}}_C$$

where M_C only has simple poles

- well-understood for multiple polylogarithms:
 - transcendental weight and purity

Canonical Bases and Leading Singularities

- for $\epsilon \neq 0$, a canonical basis $\vec{\mathcal{I}}_C$ satisfies [Henn, 2013]

$$\frac{d\vec{\mathcal{I}}_C}{ds_i} = \epsilon M_C(\vec{s}) \cdot \vec{\mathcal{I}}_C$$

where M_C only has simple poles

- well-understood for multiple polylogarithms:
 - transcendental weight and purity
 - leading singularities (LS): maximally iterated residues of the integrand

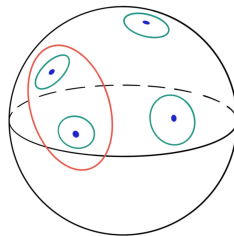
$$f(x) = \frac{1}{x} \log(x-1) = \int^x dy \frac{1}{x(y-1)} \rightarrow \text{Res}_{y=1} \left(\frac{1}{x(y-1)} \right) = \frac{1}{x}$$

- all LS constant $\leftrightarrow$ canonical

Beyond Polylogarithms

Polylogs:

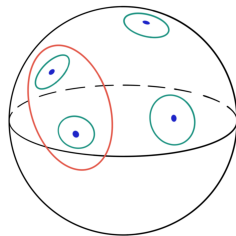
- geometry: Riemann sphere with poles
- contours: residues



Beyond Polylogarithms

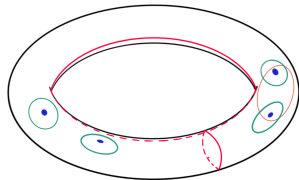
Polylogs:

- geometry: Riemann sphere with poles
- contours: residues



Beyond polylogs:

- elliptic curves, Calabi-Yau, higher-genus,...
- contours: residues + cycles of the geometry
- integrands with higher poles unavoidable



Leading Singularities Beyond Polylogarithms

Generalize concept of leading singularities:

- All (independent) closed contour integrals

[Bourjaily et al., 2021] [Bourjaily, Kalyanapuram, Langer, Patatoukos, 2021] [FF et al., 2026]

Then we can also generalize:

- canonical $\leftrightarrow$ properly normalized leading singularities

Higher Poles

- higher poles require integration-by-parts before setting $\epsilon = 0$
- also true for polylogs, but can be avoided
- Example:

$$\int dy \frac{1}{(y-x)^2} (y+1)^\epsilon \xrightarrow{\epsilon \rightarrow 0} \int dy \frac{1}{(y-x)^2} \xrightarrow{\text{Res}_{y=x}} 0$$

Higher Poles

- higher poles require integration-by-parts before setting $\epsilon = 0$
- also true for polylogs, but can be avoided
- Example:

$$\int dy \frac{1}{(y-x)^2} (y+1)^\epsilon \xrightarrow{\epsilon \rightarrow 0} \int dy \frac{1}{(y-x)^2} \xrightarrow{\text{Res}_{y=x}} 0$$

$$\begin{aligned} \int dy \frac{1}{(y-x)^2} (y+1)^\epsilon &\xrightarrow{\text{IBP}} - \int dy \frac{\epsilon}{(y-x)} (y+1)^{\epsilon-1} \xrightarrow{(\dots)^\epsilon \rightarrow 1} - \int dy \frac{\epsilon}{(y-x)(y-1)} \\ &\xrightarrow{\text{Res}_{y=x}} -\frac{\epsilon}{x-1} \end{aligned}$$

→ LS at higher order in ϵ

Some applications of canonical bases beyond polylogs

- gravitational scattering, 5PM, 1SF and 2SF (4 loops)

[Driesse et al., 2026] [Driesse et al., 2025] [Driesse et al., 2024]

- 2-loop 4-point integrals for diphoton

[Becchetti et al., 2025]

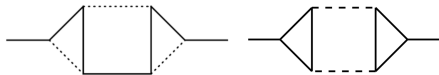
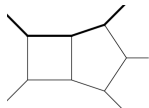
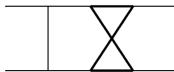
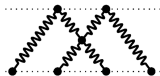
- 2-loop 5-point integrals for $t\bar{t}$ +jet

[Becchetti, Dlapa, Zoia, 2025]

- 3-loop QED self-energies

[Duhr et al., 2024] [FF, Nega, Tancredi, 2025]

⋮



Some applications of canonical bases beyond polylogs

- gravitational scattering, 5PM, 1SF and 2SF (4 loops)

[Driesse et al., 2026] [Driesse et al., 2025] [Driesse et al., 2024]

- 2-loop 4-point integrals for diphoton

[Becchetti et al., 2025]

- 2-loop 5-point integrals for $t\bar{t}$ +jet

[Becchetti, Dlapa, Zoia, 2025]

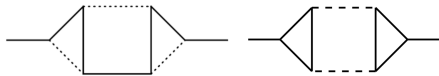
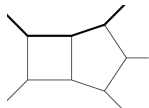
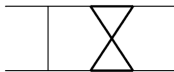
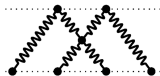
- 3-loop QED self-energies

[Duhr et al., 2024] [FF, Nega, Tancredi, 2025]

⋮

→ analytical control and insights, e.g. absence of some letters

→ numerical evaluation [Czakon, Tancredi, 2026]



Electron self-energy: towards four loops

Many of our new insights on LS came from here.

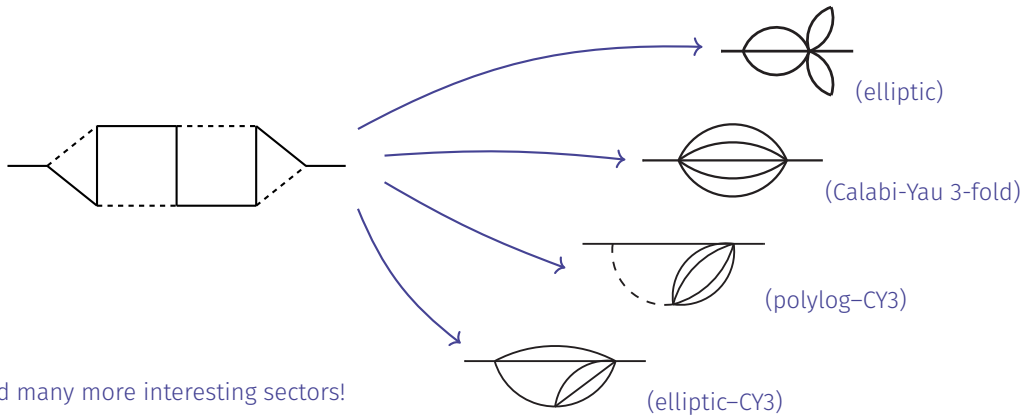
We studied this integral family with 160 master integrals in 72 sectors:



Electron self-energy: towards four loops

Many of our new insights on LS came from here.

We studied this integral family with 160 master integrals in 72 sectors:



And many more interesting sectors!

Three-loop self-energies from QED to QCD

[work in progress with M. Klann, S. Moch, D. Ossipov, K. Schönwald, L. Tancredi]

Generalizing QED to QCD with a heavy top quark, there are extra diagrams from the gluon self-coupling, e.g.



	# masters	geometry
electron	51	elliptic
heavy quark	56	elliptic
photon	36	K3
gluon	72	K3

Three-loop self-energies from QED to QCD

[work in progress with M. Klann, S. Moch, D. Ossipov, K. Schönwald, L. Tancredi]

Generalizing QED to QCD with a heavy top quark, there are extra diagrams from the gluon self-coupling, e.g.



	# masters	geometry
electron	51	elliptic
heavy quark	56	elliptic
photon	36	K3
gluon	72	K3

Up to three loops...

- QCD corrections are polylogarithmic
- QED contains leading complexity in terms of geometries ($\leftrightarrow$ massive lines)

Conclusion

Beyond polylogarithms...

- residues are not enough
- higher poles in the integrands are unavoidable

Conclusion

Beyond polylogarithms...

- residues are not enough
 - higher poles in the integrands are unavoidable
- generalized leading singularities as all closed contour integrals
- higher poles generate leading singularities at higher orders in ϵ
- obtain canonical differential equations on general geometries by constructing integrals with constant leading singularities
- broad applications in QFT and gravity

Conclusion

Beyond polylogarithms...

- residues are not enough
 - higher poles in the integrands are unavoidable
- generalized leading singularities as all closed contour integrals
- higher poles generate leading singularities at higher orders in ϵ
- obtain canonical differential equations on general geometries by constructing integrals with constant leading singularities
- broad applications in QFT and gravity

Thank you! ☺